MAXIMAL PARAHORIC ARITHMETIC TRANSFERS, RESOLUTIONS AND MODULARITY

ZHIYU ZHANG

Abstract. For any unramified quadratic extension of $p$-adic local fields $F/F_0$ ($p > 2$), we formulate several arithmetic transfer conjectures at any maximal parahoric level, in the context of Zhang’s relative trace formula approach to the arithmetic Gan–Gross–Prasad conjecture. The singularity is resolved via an arithmetic Atiyah flop using formal Balloon–Ground stratification. Then we do intersections and modify derived fixed points on the resolution.

By a local-global method and double induction, we prove these conjectures for $F_0$ unramified over $\mathbb{Q}_p$, including the arithmetic fundamental lemma for $p > 2$. We introduce the relative Cayley map and also establish Jacquet–Rallis transfers of vertex lattices. We also prove new modularity results for arithmetic theta series at maximal parahoric levels via modifications over $\mathbb{F}_q$ and $\mathbb{C}$. Along the way, we study the complex and mod $p$ geometry of Shimura varieties and special cycles through natural stratifications.

Contents

1. Introduction 1


2. Explicit transfer conjectures 8
3. Reductions via relative Cayley maps 8
4. Transfers for all vertex lattices 10

Part 2. Arithmetic transfer conjectures

5. Special cycles and modified derived fixed points 20
6. Arithmetic transfer conjectures 20
7. Reduction of intersection numbers 26
8. Bruhat–Tits strata 30
9. Local modularity 34
10. ATCs for unramified maximal orders 36

Part 3. Modularity of arithmetic theta series

11. Integral models and Balloon–Ground stratification 41
12. Kudla–Rapoport cycles and modified Hecke CM cycles 45
13. Modification over $\mathbb{C}$ and $\mathbb{F}_q$ 50
14. Global modularity via modification 55

Part 4. The proof of ATC for $p > 2$

15. The global analytic and geometric side 59
16. The end of proof 62

References 67

1. Introduction

The Gross–Zaiger formula [7] relates Néron–Tate heights of Heegner points on modular curves to central derivatives of certain $L$-functions of modular forms. The arithmetic Gan–Gross–Prasad...
**conjecture** (AGGP conjecture) [8][33] is a higher dimensional generalization of the Gross–Zaiger formula, which relates central derivatives of automorphic $L$-functions and heights of special cycles on unitary and orthogonal Shimura varieties.

In the relative trace formula (RTF) approach proposed by Wei Zhang [58] (inspired by the RTF approach of Jacquet–Rallis [19][60][61] to the original GGP conjecture [8]), the AGGP conjecture for unitary groups is reduced to local conjectures: the arithmetic fundamental lemma (AFL) at good places formulated in [58], and suitable arithmetic transfer conjecture (ATC) at any bad place (see [44][45] for some formulations). These conjectures relate arithmetic intersection numbers of special cycles on certain Rapoport–Zink spaces to central derivatives of orbital integrals of suitable test functions:

$$\partial \text{Orb}(\gamma, f_{\text{std}}) = -\text{Int}(g) \cdot \log q.$$ 

See Zhang’s ICM report [62] for details and some applications. In recent years, significant progress has been made in this approach: the AFL over $\mathbb{Q}_p$ (resp. any $p$-adic field $F_0$) for $p \geq n$ (resp. $q_{F_0} \geq n$) is proved via a global method in [59] (resp. [38]).

For arithmetic applications e.g. AGGP conjectures and their $p$-adic analogs, we need to allow mild ramifications at bad places and prove ATCs there. Unlike AFL, it is more difficult to formulate ATCs for at least two reasons: the relevant moduli spaces are no longer regular, and there is no natural test function on the analytic side. New ideas and techniques are needed for the formulation and proof of ATCs.

The first goal of this paper is to formulate and prove arithmetic transfer conjectures for all maximal parahoric levels at unramified places. We introduce a general method to resolve the singularity by blowing up along the local model diagram. Then we do intersections and modify derived fixed points on the resolution. And we construct explicit test functions on the analytic side. Our formulation is coordinate free.

To be more precise, let $F/F_0$ be an unramified quadratic extension of $p$-adic fields with residue fields $\mathbb{F}_q/\mathbb{F}_q$ where $p > 2$. Choose a uniformizer $\varpi$ of $F_0$. Let $V$ be a $F/F_0$ hermitian space of dimension $n \geq 1$, and $L \subseteq V$ be a vertex lattice of type $0 \leq t_0 \leq n$. Stabilizers of these vertex lattices give maximal parahoric subgroups of $U(V)(F_0)$ [55 Section 4]. Associated to $L$ is the “relative” Rapoport–Zink space $\mathcal{N} = \mathcal{N}_{U(L)} = \mathcal{N}_n^{[t_0]} \longrightarrow \text{Spf } O_{F_0}$.

For example, the space $\mathcal{N}_1^{[1]} \cong \mathcal{N}_1^{[0]} \cong \text{Spf } O_{F_0}$ is a point, and $\mathcal{N}_2^{[1]}$ is isomorphic to the Drinfeld half plane $\mathcal{H}_2^{[0]} \rightarrow \text{Spf } O_{F_0}$ by [29]. Using the computation of local models [10][5], we show the space $\mathcal{N}_n^{[t_0]} \rightarrow \text{Spf } O_{F_0}$ is formally smooth if $t_0 = 0$, and is of strictly semi-stable reduction if $0 < t_0 < n$. Hence the space $\mathcal{N} \times \mathcal{N}$ is singular if $0 < t_0 < n$. One question is how to resolve its singularity in a natural way. The group $G := U(V)$ could be non-quasi-split over $F_0$ and the geometry of the special fiber $\mathcal{N}_F$ is expected to be more complicated.

Inspired by recent work on Beilinson–Bloch–Kato conjectures [32] Section 5], we introduce the formal Balloon–Ground stratification on $\mathcal{N}_F$:

$$\mathcal{N}_F = \mathcal{N}^\bullet \cup \mathcal{N}^\circ.$$ 

The definition (5.3) is purely Lie algebra theoretic and indicates these strata are examples of Kottwitz–Rapoport strata. We blow up along products of formal Balloon strata $\mathcal{N}^\circ \times_F \mathcal{N}^\circ$ (a Weil divisor of $\mathcal{N} \times \mathcal{N}$) to resolve the singularity:

$$\widehat{\mathcal{N} \times \mathcal{N}} \longrightarrow \mathcal{N} \times \mathcal{N}$$

which may be thought as an arithmetic analog of the Atiyah flop [1]. Let $V := \text{Hom}_{O_F}(\mathbb{E}, X)$ be the space of special quasi-homomorphisms which realizes the nearby hermitian space of $V$. Any automorphism $g \in J_0(F_0) := U(V)(F_0)$ acts on $\mathcal{N}$ naturally by changing the framing.

Now we formulate a (group version) arithmetic transfer conjecture for any decomposition of vertex lattices $L = L^\circ \otimes O_F$ where $v((c, e)) = i \in \{0, 1\}$. We have $V = V^\circ \oplus Fe$ where $V^\circ := L^\circ \otimes \mathbb{Q}$. Choose a decomposition of the framing object for $\mathcal{N}$, we have a natural embedding

$$\mathcal{N}^\circ := \mathcal{N}_{U(L^\circ)} \longrightarrow \mathcal{N} := \mathcal{N}_{U(L)}$$
given via moduli interpretations by adding the factor $\langle \mathcal{E}^{[i]}, t_{E^{[i]}}, \lambda_{E^{[i]}} \rangle$ where $\langle \mathcal{E}^{[i]}, t_{E^{[i]}}, \lambda_{E^{[i]}} \rangle$ is a universal formal $O_{F_0}$ module over $O_{F_0}$ of relative height 2 and dimension 1 and $\lambda_{E^{[i]}} = [\varpi^i] \circ \lambda_{E}$ where $\lambda_{E} : E^{[i]} \cong (E^{[i]})^\vee$ is a principal polarization. Consider the induced GGP type embedding of Rapoport–Zink spaces $\mathcal{N}^\circ \to \mathcal{N}^\circ \times \mathcal{N}$ which lifts to an embedding of regular formal schemes via strict transforms inside the resolution $\mathcal{N} \times \mathcal{N}$:

$$\mathcal{N}^\circ \to \mathcal{N}^\circ \times \mathcal{N}.$$

On the geometric side, for $g' \in \mathcal{U}(V)(F_0)_{rs}$ we consider the derived intersection number as the Euler characteristic

$$\text{Int}(g') = \chi(\mathcal{N}^\circ \times \mathcal{N}, \mathcal{O}_{\mathcal{N}^\circ} \otimes L_{(1 \times g')\mathcal{N}^\circ}) \in \mathbb{Q}.$$

Choose an orthogonal basis of $L^2$ to endow $L^2$ (resp. $V^2$) with a $O_{F_0}(\text{resp. } F_0)$-structure $L_0$ (resp. $V_0^2$). We have a basis of $L$ (and $V$) by adding $e$. Consider the symmetric space over $F_0$:

$$S(V_0) = \{ \gamma \in \text{GL}(V) | \gamma \gamma^{-1} = \text{id} \}.$$

Write $S(L, L^\circ)$ as the stabilizer of the lattice chain $L \subseteq L^\circ$ inside $S(V_0)(F_0)$. On the analytic side, we consider the derived orbit integral $\partial \text{Orb}(\gamma, -)$ on $\gamma \in S(V_0)(F_0)_{rs}$ with respect to the natural action of $\text{GL}(V_0^2)(F_0)$, where we insert the transfer factor $\omega(\gamma') [2.7]$ for $L$.

**Conjecture 1.1.** (Group arithmetic transfer for $(L, e)$) (Conjecture 6.8) For regular semisimple element $g' \in \mathcal{U}(V)(F_0)_{rs}$ matching $g \in S(V_0)(F_0)_{rs}$, we have an equality in $\mathbb{Q} \log q$

$$\partial \text{Orb}(\gamma', 1_{S(L, L^\circ)}) = -\text{Int}(g') \log q. \quad (1.1)$$

In other words, for any natural embedding

$$\mathcal{N}^{[t-1]}_{n-1} \to \mathcal{N}^{[t-i]}_{n-1} \times \mathcal{N}^{[i]}_0,$$

we have formulated an arithmetic transfer conjecture which contains the arithmetic fundamental lemma conjecture (resp. arithmetic transfer conjecture) in [58] (resp. [45] Section 10) as the special case $t = 0, i = 0$ (resp. $t = 0, i = 1$).

Now we formulate two semi-Lie version arithmetic transfer conjectures for any vertex lattice $L$ using Kudla–Rapoport cycles $\mathcal{Z}(u)$ and $\mathcal{Y}(u)$ ($u \in V$) [6] [27] on $\mathcal{N} = \mathcal{N}_{\mathcal{U}(L)}$. The semi-Lie version arithmetic transfer conjecture generalizes the arithmetic fundamental lemma conjecture in the set up of [30] to maximal parahoric levels. For $g \in \mathcal{U}(V)(F_0)$, the naive fixed pointed locus $\text{Fix}(g) \to \mathcal{N}$ is poorly behaved. We introduce the modified derived fixed pointed locus $\tilde{\text{Fix}}^Z(g)$ as the derived fiber product of strict transforms of the graph $\Gamma_g = \tilde{\Gamma}_g$ and the diagonal $\mathcal{N} \to \mathcal{N}$ inside $\mathcal{N} \times \mathcal{N}$.

On the geometric side, for $(g, u) \in (\mathcal{U}(V) \times \mathcal{Y})(F_0)_{rs}$ we consider $\tilde{\text{Int}}^Z(g, u)$ (resp. $\tilde{\text{Int}}^Y(g, u)$) as the derived intersection number between $\mathcal{Z}(u)$ (resp. $\mathcal{Y}(u)$) and $\text{Fix}^Y(g)$ on $\mathcal{N}$. On the analytic side, we consider the derived orbit integral $\partial \text{Orb}((\gamma, u_1, u_2), -)$ on $(S(V_0) \times V_0 \times V_0^2)(F_0)$ with respect to the natural action of $\text{GL}(V_0^2)(F_0)$, where we insert the transfer factor $\omega(\gamma, u_1, u_2) [2.2]$ for $(L, e)$. Consider two test functions on $(S(V_0) \times V_0 \times V_0^2)(F_0)$:

$$f_{\text{std}} := 1_{S(L, L^\circ)} \times 1_{L_0} \times 1_{L_0^\circ}, \quad f'_{\text{std}} := 1_{S(L, L^\circ)} \times 1_{L_0^\circ} \times 1_{(L_0^\circ)^*}.$$

**Conjecture 1.2.** (Semi-Lie arithmetic transfers for $L$) (Conjecture 6.3) For any regular semisimple pair $(g, u) \in (\mathcal{U}(V) \times \mathcal{Y})(F_0)_{rs}$ matching $(\gamma, u_1, u_2) \in (S(V_0) \times V_0 \times V_0^2)(F_0)_{rs}$, we have equalities in $\mathbb{Q} \log q$:

1. $\partial \text{Orb}((\gamma, u_1, u_2), f_{\text{std}}) = -\tilde{\text{Int}}^Z(g, u) \log q$.
2. $\partial \text{Orb}((\gamma, u_1, u_2), f'_{\text{std}}) = -(\log 1) \tilde{\text{Int}}^Y(g, u) \log q$.

Our first main theorem in this paper is a proof of these arithmetic transfer conjectures for $p > 2$ under mild assumptions, which gives the first known example of arithmetic transfers in arbitrary higher dimension $\geq 4$.

**Theorem 1.3.** Assume $F_0$ is unramified over $\mathbb{Q}_p$ (e.g. $F_0 = \mathbb{Q}_p$) if $0 < t_0 < n$. Then the semi-Lie (resp. group) version arithmetic transfer conjectures hold for any vertex lattice $L$ (resp. any decomposition of vertex lattices $L = L^2 \oplus O_{F^e}$).
In particular, we prove the AFL over any $p$-adic field $F_0$ for small prime $p > 2$ which is not covered by \cite{hss,katz}. The proof is based on a local-global method and double induction, using new modularity results of arithmetic theta series at maximal parahoric levels proved in this paper.

1.1. Modularity of arithmetic theta series. For hermitian (resp. quadratic) lattices $L$ with suitable signatures, Kudla’s generating series of special cycles on unitary (resp. orthogonal) Shimura varieties for $L$ \cite{kudla} could be thought as arithmetic and geometric analogs of classical theta series for $L$, which are conjectured to be modular. The conjectured modularity on unitary Shimura varieties (in different set ups) has many arithmetic applications:

- In the work of Liu and Li–Liu \cite{katz,li-liu}, the modularity over the generic fiber is used to construct arithmetic theta liftings from certain automorphic forms to cycles on unitary Shimura varieties.

- In the work of Bruinier–Howard–Kudla–Rapoport–Yang \cite{bruinier}, the modularity in the arithmetic divisor case is proved when $F_0 = \mathbb{Q}$ and $L$ is self-dual, which applies to the average Colmez’s conjecture on the Faltings heights of abelian varieties with complex multiplication.

- In the work of Mihatsch–Zhang \cite{hss}, an almost modularity result is obtained in the arithmetic divisor case when $F_0 \neq \mathbb{Q}$ and $L$ is self-dual, which implies the AFL for $q_{F_0} \geq n$.

Moreover, the modularity on orthogonal Shimura varieties in the arithmetic divisor case is proved for maximal quadratic lattices in \cite{katz} Section 8-9, which has applications to exceptional jumps of Picard ranks of reductions of K3 surfaces over number fields \cite{liu-liu}. The second goal of this paper is to prove new modularity results for arithmetic theta series of parahoric hermitian lattices under some assumptions. More precisely, let $F/F_0$ be a CM quadratic extension over a totally real number field. We fix a CM type $\Phi \subseteq \text{Hom}(F, \mathbb{Q})$ of $F$ and a distinguished element $\varphi_0 \in \Phi$. Form the reflex field $E = \varphi_0(F)F_{\varphi} \subseteq \mathbb{Q}$, which is just $F$ if $F$ is Galois over $\mathbb{Q}$. Let $V$ be a $F/F_0$-hermitian space of dimension $n \geq 1$ and signature $\{(n-1,1), (n,0)\}_{x \in \Phi - \{\varphi_0\}}$. Choose a parahoric hermitian lattice $L$ in $V$, and a finite collection $\Delta$ of finite places for $F_0$. Assume that

- $F/F_0$ is unramified outside $\Delta$ and all 2-adic places are in $\Delta$.
- If $v \notin \Delta$ is a place of $F_0$ such that $L_v$ is not self-dual, then $v$ is inert in $F$ and $E \otimes_{F_0} F_{0, v}$ is unramified over $\mathbb{Q}_p$.

Consider $G = U(V)$ over $F_0$ and the level $K \subseteq G(\mathbb{A}_{0, f})$ associated to $L$ and $\Delta$. The level $K$ agrees with the stabilizer $U(L)$ away from $\Delta$, and could be chosen arbitrarily at $\Delta$. Then we have a natural integral model

$$\mathcal{M} = \mathcal{M}_{G, K} \longrightarrow \text{Spec } O_E[\Delta^{-1}]$$

of the RSZ-variant \cite{katz,li-liu} of the unitary Shimura variety $M_{G, K}$ over $F$. An advantage of the RSZ-variant is that $\mathcal{M}$ admits PEL type moduli interpretation, see Section 11.3.

In this paper, we consider the arithmetic theta series for $L$ as Kudla’s generating series $\tilde{\mathcal{Z}}(\phi_L)$ of arithmetic Kudla–Rapoport divisors $\tilde{\mathcal{S}}^B(\xi, \phi_L)$ \cite{kudla,rapoport} equipped with admissible automorphic Green functions. It has coefficients in the arithmetic Chow group $\tilde{\text{Ch}}^1(\mathcal{M})$. Consider the arithmetic intersection pairing \eqref{intersection} between arithmetic divisors and proper 1-cycles on $\mathcal{M}$:

$$(-, -) : \tilde{\text{Ch}}^1(\mathcal{M}) \times \tilde{\text{C}}^1(\mathcal{M}) \longrightarrow \mathbb{R}_\Delta$$

where $\mathbb{R}_\Delta$ is the $\mathbb{Q}$-vector space $\mathbb{R}_\Delta := \mathbb{R}/\sum_{p \in \Delta} \mathbb{Q} \log p$.

Our second main theorem in this paper is a numerical modularity result under mild assumptions.

**Theorem 1.4.** Assume that $F_0 \neq \mathbb{Q}$ and $E = F$ (e.g. if $F/\mathbb{Q}$ is Galois). Then for any 1-cycle $\mathcal{C} \rightarrow \mathcal{M}$, the generating series $(\tilde{\mathcal{Z}}(\phi_L), \mathcal{C}) \in \mathbb{R}_\Delta[[q]]$ is modular of weight $n$.

See Theorem 14.4 for precise statements. Now we briefly describe the proofs.

1.2. Proof of (double) modularity. We introduce a modification method to reduce the conjectured modularity above to the modularity over the generic fiber (known by \cite{katz}) and over the basic locus (see Section 11.4). The method uses results complex and mod $p$ geometry of Shimura varieties. And the modularity over basic locus is related via basic uniformization to some local modularity results on relevant Rapoport–Zink spaces in Section 9.
To be precise, for any \( v \notin \Delta \) such that \( L_v \) is not self-dual and a place \( \nu \) of \( E \) above \( v \), write the special fiber \( M_{k_v} = \bigcup_{i=1}^{m_v} X_i \) as unions of irreducible components. If the 1-cycle \( C \) has degree 0 over \( \mathbb{C} \) and \( (X_i,C) = 0 \) for any such \( v \) and \( 1 \leq i \leq m_v \), then \((-C)\) factors through \( \text{Ch}^1(M_E) \) therefore Theorem 1.4 for \( C \) follows from the known modularity on \( M_E \). To deal with general 1-cycle \( C \), we modify \( C \) to 1-cycles above by subtracting certain 1-cycles \( C' \) for which we can show the modularity of \( \widetilde{Z}(\phi_L), C' \) directly:

1. By studying the Galois and Hecke action on connected components of Shimura varieties over \( \mathbb{C} \), we show the degree of \( C \) over \( \mathbb{C} \) could be modified to 0 after subtracting rational scalars of (non-empty) Hecke CM cycle \( CM(\alpha, \mu_\Delta) \) introduced in Section 12.

For suitable maximal order \( \alpha_0 \), we could compute \((\widehat{Z}(\phi_L), CM(\alpha_0, \mu_\Delta))\) directly, using simple cases of AFLs and semi-Lie ATCs proved in Section 10. It corresponds to \( \alpha_0 \)-part of the analytic generating function \( \partial(h, \Phi') \) in Section 15 which is known to be modular.

2. For \( v \notin \Delta \) such that \( L_v \) is not self-dual, consider the nearby hermitian space \( V^{(v)} \) of \( V \) at \( v \). For 1-cycle \( C' \) in the basic locus of \( M_{k_v} \), by basic uniformization (Section 12.6) we find that \((\widehat{Z}(\phi_L), C')\) is a theta series for certain function \( \phi' \) on \( V^{(v)}(k_{0,f}) \). Its prime-to-\( v \) part \( \phi'' \) is a translation of \((\phi_L)^v \) and \( (\phi'_L)^v(u) := (\phi''(u), C'^v) \) for any \( u \in V^{(v)}(k_{0,f}) \) on the Rapoport–Zink space \( \mathcal{N} \) for \( L_v \). The modularity of \((\widehat{Z}(\phi_L), C')\) follows immediately from the modularity of classical theta series (in the adele language) if \( \phi'' \) extends to a Schwartz function on \( V^{(v)} \).

For very special 1-cycles \( C' \subseteq M_{k_v} \), we deduce the extension hence the modularity for \( C' \) from some local modularity results (Theorem 9.1) using Bruhat–Tits stratification and explicit computations on the Drinfeld half plane. The Deligne–Lusztig varieties appearing in the Bruhat–Tits stratification are not of Coxeter type in general \( \text{[31]} \), Section 6) hence more subtle to deal with.

3. We are done by showing for any 1-cycle \( C \) on \( M \), there exists a very special 1-cycle \( C' \) such that \((X_i,C) = (X_i,C')\) for any irreducible component \( X_i \) of \( M_{k_v} \). Introduce the Balloon–Ground stratification

\[
M_{k_v} = M^{\bullet}_{k_v} \cup M^\circ_{k_v}.
\]

Then irreducible components of \( M_{k_v} \) are given by irreducible components of \( M^{\bullet}_{k_v} \) and \( M^\circ_{k_v} \).

Consider the type \( t_v = \dim_k L_v/L_v \in [1,n] \). The case \( t_v = n \) is identical to the self-dual case \( t_v = 0 \) by duality. If \( t_v = 1 \), \( M_{k_v} \) is almost self-dual, then irreducible components of the Balloon stratum \( M^\circ_{k_v} \) are isomorphic to \( \mathbb{P}^{n-1} \) and contained in the basic locus of \( M_{k_v} \). And the Ground stratum \( M^{\bullet}_{k_v} \) is irreducible in any given connected component of \( M_{k_v} \). Therefore, we can construct \( C' \) as a finite linear combination of \( \mathbb{P}^1 \) in different irreducible components \( \mathbb{P}^{n-1} \) of the Balloon stratum. The case \( t_v = n-1 \) is identical to the case \( t_v = 1 \) by duality.

![Figure 1. “Balloons” (with grey colors) on the “ground” (the bottom line) for almost self dual \( L_v \)](image)

If \( 1 < t_v < n-1 \), the level \( K_v \) is a non-special parahoric. For dimension reasons, the Balloon and Ground stratum are not in the basic locus. Nevertheless, we show \( M^\circ_{k_v} \) and \( M^{\bullet}_{k_v} \) are both irreducible in any given connected component of \( M_{k_v} \). This is related to the irreducibility of non-basic Kottwitz–Rapoport strata at parahoric levels (Conjecture 13.20 (2)) studied in [13]. Hence any very special 1-cycle \( C' \) lying in the basic locus but not in \( M^{\bullet}_{k_v} \cap M^\circ_{k_v} \) works. The result for \( 1 < t_v < n-1 \) follows.

In fact we obtain a double modularity result (i.e., for both \( \mathcal{V} \) and \( \mathcal{Z} \) special divisors) as “vector-valued” modular forms, see Section 14. As in [51], the modularity of arithmetic theta series plays a similar role in the proof of ATCs as the “perverse continuation principle” in the geometric approach (see [11, 56]) to fundamental lemmas over function fields.
1.3. Proof of ATCs via modularity and double induction. For the proof of ATCs, we globalize the local data to cycles on suitable global RSZ unitary Shimura varieties as above. Even in the AFL case, our proof is simpler as we avoid the use of elementary CM cycles in [35].

The original induction in the proof of AFL [59] doesn’t apply directly at parahoric levels. We introduce a method of double induction at parahoric levels in Section 16.1 to prove one ATC of dimension \( n \) using two ATCs of dimension \( n-1 \). To prove the group version conjecture, we need to work with semi-Lie version conjecture for both \( \mathcal{Z} \)-cycles and \( \mathcal{Y} \)-cycles (hence the name of “double induction”). The method uses the new (double) modularity result above as an input.

Besides the modularity, we need a reduction from group version ATC [6.8] of dimension \( n \) to semi-Lie version ATC [6.3] of rank \( n-1 \). Then we may focus on semi-Lie version arithmetic transfer conjectures. We introduce (variants of) the relative Cayley map as a more natural reduction tool (3.1) for a decomposition of \( F \)-vector spaces \( V = V^\vee \oplus F_e \). We construct unitary (3.2) and symmetric (3.4) variants, which behaves well for all primes \( p > 2 \). We expect the relative Cayley map will be useful in other settings.

Although we modify the derived fixed points, we establish the reduction of intersection numbers in Section 7 using smallness of the resolution \( \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \rightarrow \mathcal{N} \times \mathcal{N} \) just as smallness of the Atiyah flop. Given a decomposition of vertex lattices \( L = L' \oplus O_F e \) where \( \nu((e,e)) = i \in \{0,1\} \), consider the unitary relative Cayley map for the decomposition \( \mathcal{V} = \mathcal{V}^\vee \oplus F e[i] \)

\[
\begin{align*}
\text{cu} & : U(\mathcal{V}) \longrightarrow U(\mathcal{V}^\vee) \times \mathcal{V}^\vee \\
g' = \begin{pmatrix} t & u \\ w & d \end{pmatrix} & \longmapsto (g = t + \frac{wu}{1-a}, u_1 = \frac{u}{1-a}).
\end{align*}
\]

If \( i = 0 \), then for \( g' \in U(\mathcal{V})(F_0)_{1S} \) with \( 1-d \in O_F^\times \) and \( u_1 \neq 0 \) we have the reduction equality:

\[
\tilde{\text{Int}}(g') = \tilde{\text{Int}} \left( g, u_1 \right).
\]

From this, we show the group ATC for \( \mathcal{N}_{n-1}^{[t_0]} \rightarrow \mathcal{N}_{n-1}^{[t_0]} \times \mathcal{N}_{n}^{[t_0]} \) (resp. \( \mathcal{N}_{n-1}^{[t_0-1]} \rightarrow \mathcal{N}_{n-1}^{[t_0-1]} \times \mathcal{N}_{n}^{[t_0]} \)) follows from the semi-Lie ATC for \( \mathcal{Z} \)-cycles (resp. \( \mathcal{Y} \)-cycles) on \( \mathcal{N}_{n}^{[t_0]} \) (resp. \( \mathcal{N}_{n}^{[t_0-1]} \)). Moreover, we show semi-Lie ATCs for \( \mathcal{Z} \)-cycles on \( \mathcal{N}_{n}^{[t_0]} \) and \( \mathcal{Y} \)-cycles on \( \mathcal{N}_{n}^{[n-t_0]} \) are equivalent using the dual isomorphism \( \lambda : \mathcal{N}_{n}^{[t_0]} \rightarrow \mathcal{N}_{n}^{[n-t_0]} \) sending \((X,\iota,\lambda,\rho)\) to its dual \((X',\iota',\lambda',\rho')\). These reductions and equivalences show base cases of our double induction, hence finish the proof of ATCs.

Along the way, we study the geometry of relevant Rapoport–Zink spaces and reconstruct the Bruhat–Tits stratification ([5], [55]) on the reduced locus via special cycles. The phenomenon that there are two kinds of Bruhat–Tits strata at our parahoric levels is observed by the formal Balloon–Ground stratification, see Remark 5.14. We also use Bruhat–Tits strata to study fixed points of \( g \) on \( \mathcal{N} \) in simple cases, see e.g. Example 10.6. The behavior of \( \text{Fix}(g) \) in general depends on the embedding \( O_F[g] \rightarrow \text{End}(\mathcal{V}) \) rather than just the structure of \( O_F[g] \).

We also prove explicit transfer conjectures (Conjecture 2.1) for unitary groups at all maximal parahoric levels (including \( p = 2 \)), see Theorem 4.1. The proof is via a pure local method and similar double induction above, inspired by the work of Beuzart-Plessis [4]. The local Weil representation descends to the space of orbit integral functions from two sides. Our coordinate-free formulation allows us to establish the reduction and induction of orbit integrals.

1.4. Further directions. We comment on the assumptions. The assumption that \( F_0 \) is unramified over \( \mathbb{Q}_p \) is used to show the relevant Rapoport–Zink spaces in our ATCs appear in the basic uniformization of certain unitary Shimura varieties. We hope to establish a general comparison theorem as in [63] for general \( F_0 \) at parahoric levels in the future. Applications include the proof of ATCs and understanding the cohomology of local Shimura varieties (e.g. [12]) for any \( p \)-adic field \( F_0 \) via globalization.

Our methods may be further developed to formulate and to prove AFL and ATC for more general parahoric levels (e.g. the Iwahori level) and other set ups (e.g. [16] [35] [48] [63]). In general, the singularity of suitable integral model is controlled by certain sections of universal vector bundles (or “Sheulas” as in [12]) on them. We may blow up along the local model diagram (using Kottwitz–Rapoport strata) to resolve the singularity. We plan to pursue the topic in the
near further, and use these results to prove AGGP conjectures in higher dimensions for certain number fields.

Finally, we give an overview of the paper. We formulate the set up of Jacquet–Rallis transfers in Section 2. In Section 3 we introduce the relative Cayley map and do reductions for orbits and (derived) orbit integrals. In Section 4 we establish the Jacquet–Rallis transfer for vertex lattices using a double induction and local uncertainty principles. In Section 5 we introduce relevant Rapoport–Zink spaces and two kinds of Kudla–Rapoport divisors, and the formal Balloon–Ground stratification to modify derived fixed points. In Section 6 we formulate semi-Lie and group version arithmetic transfer conjectures. In Section 7 we use relative Cayley map and moduli interpretation to modify derived fixed points. In Section 8 we formulate semi-Lie and group Rapoport–Zink spaces and two kinds of Kudla–Rapoport divisors, and the formal Balloon–Ground using a double induction and local uncertainty principles. In Section 9, we prove the local interpretation to show the reduction of intersection numbers. In Section 10, using moduli interpretation and Bruhat–Tits stratification on reduced locus of Rapoport–Zink spaces. In Section 11, we prove the local modularity for very special 1-cycles. In Section 12, we introduce relevant special cycles on Shimura varieties for parahoric hermitian lattices and study the singularity using Balloon–Ground modularity for very special 1-cycles. In Section 13, we introduce Bruhat–Tits stratification and local models. In Section 13 and 14, we prove global modularity by modification over \( C \) stratification and local models. In Section 15, we prove the global analytic side and geometric side. In Section 16, we do a double induction and prove our arithmetic transfer conjectures by globalization.

Acknowledgments. I heartily thank my advisor Wei Zhang for introducing me into this beautiful area and his constant encouragement. I enjoy discussions with him during different stages of this project. I thank P. van Hoften, Y. Liu, A. Mihatsch, L. Xiao and Z. Yun for interesting discussions. I also thank C. Li and M. Rapoport for comments on a draft.

Notations and Conventions.

• Unless otherwise stated, any hermitian space \( V \) over a field \( F \) in this article is assumed to be non-degenerate. We always denote by \( \langle \cdot, \cdot \rangle_V \) the hermitian form on \( V \), which is by definition is \( F \)-linear on the first factor and conjugate-linear on the second factor. The \( F_0 \)-valued bilinear form \( (x, y) \to \text{tr}_{F/F_0}(x, y)_V \) makes \( V \) a quadratic space over \( F_0 \).

• Let \( X \) be an affine variety with an action of a reductive group \( G \) over a field \( F_0 \). Then an element \( x \in X(F_0) \) is called regular-semisimple if and only if the stabilizer of \( x \) inside \( G \) is trivial, and the orbit of \( x \) is closed in \( X \) under the Zariski topology. If \( F_0 \) is a p-adic field, this is equivalent to that \( G(F_0)x \) is closed in \( X(F_0) \) under the analytic topology.

Local notations.

• In local set up, we denote by \( F/F_0 \) an unramified quadratic extension of \( p \)-adic fields with residue fields \( \mathbb{F}_p^2/F_0 \), where \( p \geq 2 \). Denote by \( \varpi = \varpi_{F_0} \) a uniformizer of \( F_0 \), and write the non-trivial Galois involution on \( F \) by \( x \mapsto \bar{x} \). The standard valuation map is denoted by \( v = v_F : F^\times \to \mathbb{Z} \), where \( v_F(\varpi) = 1 \). Choose a unit \( \delta \in O_{F_0} \) such that \( O_F = O_{F_0}[\sqrt{\delta}] \) and \( \delta = -\delta \).

• We denote by \( \eta : F_0^\times \to F_0^\times/NF_0^\times \cong \{ \pm 1 \} \subseteq \mathbb{C}^\times \) the quadratic character associated to \( F/F_0 \) by local class field theory. We extend it to \( \eta : F^\times \to \mathbb{C}^\times \) by \( \eta(x) = (-1)^{v_F(x)} \).

• For a \( O_F \)-lattice \( \Lambda \subseteq V \) (of full rank), denote \( \Lambda^\vee = \{ x \in V | (x, \Lambda)_V \subseteq O_F \} \) to be its dual lattice. A lattice \( \Lambda \) is called a vertex lattice if \( \Lambda \subseteq \Lambda^\vee \subseteq \varpi^{-1}\Lambda \). The type of a vertex lattice \( \Lambda \) is \( t(\Lambda) := \dim_F \Lambda^\vee/\Lambda \in [0, \dim_F V] \). A self-dual lattice is a vertex lattice of type 0. An almost self-dual lattice is a vertex lattice of type 1.

• For a \( O_{F_0} \)-lattice \( \Lambda \subseteq V_0 \) (of full rank) in a \( F_0 \)-vector space \( V_0 \), denote by \( \Lambda^* = \{ y \in (V_0)^* | g(\Lambda) \subseteq O_{F_0} \} \) its linear dual lattice in \( (V_0)^* \). Let \( \tilde{F}_0 \) be the \( p \)-adic completion of the maximal unramified extension of \( F_0 \), \( W = O_{\tilde{F}_0} \) be its ring of integers, and \( \sigma \in \text{Gal}(\tilde{F}_0/F_0) \) be the Frobenius element \( x \to x^\sigma \).

• We fix a non-trivial unramified additive character \( \psi : F_0 \to \mathbb{C} \), which induces a character \( \psi_F : F \to \mathbb{C} \) given by \( \psi_F := \psi \circ \text{tr}_{F/F_0} \).

• For a smooth affine algebraic variety \( X \) over \( F_0 \), let \( \mathcal{S}(X(F_0)) \) be the space of Schwartz functions on \( X(F_0) \). Here all Schwartz functions on totally disconnected topological spaces are \( \mathbb{Q} \)-valued locally constant functions.
For a $F/F_0$-hermitian space $V$, denote by $\omega$ the Weil representation of $\text{SL}_2(F_0)$ on $S(V(F_0))$, which commutes with the natural right translation action of $U(V)(F_0)$.

We use covariant version of the relative Dieudonné theory.

For any $\xi \in \mathbb{R}$ and $n \in \mathbb{Z}$, let $W^{(n)}_\xi(h)$ be the standard weight $n$ Whittaker function on $h \in \text{SL}_2(\mathbb{R})$. Concretely, for the Iwasawa decomposition $h = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{1/2} & 0 \\ 0 & a^{-1/2} \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \in \text{SL}_2(\mathbb{R})$, we have $W^{(n)}_\xi(h) := |a|^{n/2} e^{2\pi i (b + a) \xi} e^{i k \theta}.$

Global notations.

In global set up, we always denote by $F/F_0$ a CM quadratic extension over a totally real number field. Let $F_{0,+}$ be the subset of totally positive elements in $F_0$.

We denote by $A$, $A_0$, and $A_F$ for the adele rings of $Q$, $F_0$, and $F$, respectively. We use the subscript $(-)_f$ to denote the terms of finite adeles. We use the subscript $(-)_p$, $(-)_v$, and $(-)_w$ to denote the local term at a place $p$ of $Q$, a place $v$ of $F_0$, and a place $w$ of $F$, respectively.

Let $\Delta$ be a finite collection of places of a number field $F_0$. We use the lower index $(-)_\Delta$ to denote the product of $(-)_v$ over all places $v \in \Delta$.

We denote by $\Phi$ for a chosen CM type of $F$. A CM type $\Phi$ is called unramified at $p$, if $\Phi \otimes Q_p : F \otimes Q_p \to \overline{\mathbb{Q}}_p$ is induced from a CM type of the maximal subalgebra $(F \otimes Q_p)^u$ of $F \otimes Q_p$ that is unramified over $Q_p$.

For a smooth affine algebraic variety $X$ over $F_0$, let $S(X(A_0))$ be the space of Schwartz functions on $X(A_0)$.

We use $\text{Hom}^c(-, -)$ to mean $\text{Hom}(-, -) \otimes Q$.

All $K$-groups and Chow groups have $Q$-coefficients.

We use the standard additive character $\psi_Q : Q \backslash A \to \mathbb{C}$, which induces an additive character $\psi_{F_0} : F_0 \backslash A_0 \to \mathbb{C}$ given by $\psi_{F_0} := \psi_Q \circ \text{tr} F_0/Q$.

Denote by $N^+ \leq \text{SL}_2$ the subgroup of upper triangular unipotents. For any continuous function $f : \text{SL}_2(A_0) \to \mathbb{C}$ that is left $N^+(F_0)$-invariant and $\xi \in F_0$, the $\xi$-th Fourier coefficient of $f$ is the following function on $h \in \text{SL}_2(A_0)$:

$$W_{\xi,f}(h) = \int_{F_0 \backslash A_0} f \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} h \psi_{F_0}(-\xi b) db. \quad (1.2)$$

For a quadratic space $V$ over $F_0$, and $\xi \in F_0$, we denote by $V_{\xi}$ the $F_0$-subscheme of $V$ defined by $(x,x)_V = \xi$.

For a $F/F_0$-hermitian space $V$, denote by $\omega$ the Weil representation of $\text{SL}_2(A_{0,f})$ on $S(V(A_{0,f}))$, which commutes with the natural right translation action of $U(V)(A_{0,f})$.

We denote by $A_{00}(\text{SL}_2(A_0), K_0, n)$ the space of automorphic forms on $\text{SL}_2(A)$ (left invariant under $\text{SL}_2(F_0)$) that is of parallel weight $n$, holomorphic and right invariant under a chosen compact open subgroup $K_0 \subseteq \text{SL}_2(A_{0,f})$.


2. Explicit transfer conjectures

Let $F/F_0$ be an unramified quadratic extension of $p$-adic local fields, where $p \geq 2$. Consider the quadratic character $\eta : F_0^* \to \{ \pm 1 \}$ associated to $F/F_0$ by local class field theory, which extends to $F^\times$ by $\eta(x) := (-1)^{v_F(x)}$. In this section, we recall the set up of Jacquet–Rallis transfers and formulate explicit transfer conjectures for suitable hermitian lattices.

2.1. Semi-Lie version transfers. Let $L$ be a hermitian lattice in an $n$-dimensional $F/F_0$-hermitian space $V$. Choose an orthogonal basis of $L$ (which exists for $p \geq 2$ by [24, Section 7]) to endow $L$ (resp. $V$) with a $O_F(n)$ (resp. $F_0$)-structure $L_0$ (resp. $V_0$).

The unitary group $U(V)$ acts on the space $U(V) \times V$ by

$$h.(g,u) = (h^{-1}gh, h^{-1}u).$$
For \( f \in S((U(V) \times V)(F_0)) \), define its orbit integral at \((g, u) \in (U(V) \times V)(F_0)_{rs} \):

\[
\text{Orb}((g, u), f) := \int_{h \in U(V)} f(h.(g, u)) dh.
\]

Consider the symmetric space over \( F_0 \):

\[
S(V_0) = \{ \gamma \in GL(V) | \gamma | = id \}.
\]  
(2.1)

The general linear group \( GL(V_0) \) acts on the space \( S(V_0) \times V_0 \times V_0^* \) by

\[
h.(\gamma, u_1, u_2) = (h^{-1} \gamma h, h^{-1} u_1, u_2 h).
\]

Define the transfer factor on \((S(V_0) \times V_0 \times V_0^*) (F_0)_{rs} \) with respect to \( L \) by

\[
\omega(\gamma, u_1, u_2) := \eta(\det(\gamma u_1))^{-1} \in \{ \pm 1 \}.
\]  
(2.2)

For \( f' \in S((S(V_0) \times V_0 \times V_0^*) (F_0)_{rs} ) \), define orbit integrals at \((\gamma, u_1, u_2) \in (S(V_0) \times V_0 \times V_0^*) (F_0)_{rs} \):

\[
\text{Orb}((\gamma, u_1, u_2), f', s) = \omega(\gamma, u_1, u_2) \int_{h \in GL(V_0)} f'(h.(\gamma, u_1, u_2)) \eta(h)|h|^s dh, \quad s \in \mathbb{C}.
\]  
(2.3)

\[
\text{Orb}((\gamma, u_1, u_2), f') := \text{Orb}((\gamma, u_1, u_2), f', 0).
\]  
(2.4)

\[
\partial \text{Orb}((\gamma, u_1, u_2), f') := \left. \frac{d}{ds} \right|_{s=0} \text{Orb}((\gamma, u_1, u_2), f', s).
\]  
(2.5)

Here we write \( \eta(h) := \eta(\det h) \) and \( |h|^s := |\det h|^s \) for short.

These integrals converge absolutely for regular semi-simple elements. We say \((g, u) \in (U(V) \times V)(F_0)_{rs} \) and \((\gamma, u_1, u_2) \in (S(V_0) \times V_0 \times V_0^*) (F_0)_{rs} \) matches if they are conjugated by \( GL(V) \) in \( \text{End}(V) \times V \times V^* \), where we use the embedding

\[
U(V) \times V \longrightarrow \text{End}(V) \times V \times V^*,
\]

\[
(g, u) \longmapsto (g, u, u^*: (x \mapsto (x, u)V)).
\]

The matching relation is equivalent to matching of the following invariants [19]:

\[
\text{det}(T_{id} + g) = \text{det}(T_{id} + \gamma) \in F[T], \quad (g^i u, u) = u_2(\gamma^i u_1), \quad 0 \leq i \leq n - 1.
\]

The matching relation gives a natural bijection of orbit spaces of regular semisimple elements:

\[
[(U(V) \times V)(F_0)]_{rs} \boxtimes [(U(V) \times V)(F_0)]_{rs} \longrightarrow [(S(V_0) \times V_0 \times V_0^*)(F_0)]_{rs}
\]  
(2.6)

where \( V \) is the nearby hermitian space of \( V \) and the action of \( U(V) \) on \( U(V) \times V \) is defined similarly.

From now on, we normalize the Haar measure on \( U(V)(F_0) \) (resp. \( GL(V_0)(F_0) \)) such that the stabilizer \( U(L) \) (resp. \( GL(L_0, L_0^\vee) \)) is of volume 1. Here \( L_0^\vee := L^\vee \cap V_0 \).

We say \( f' \in S((S(V_0) \times V_0 \times V_0^*)(F_0)) \) and \( f \in S((U(V) \times V)(F_0)) \) are transfers to each other, if for any regular semisimple \( (\gamma, u_1, u_2) \in (S(V_0) \times V_0 \times V_0^*) (F_0)_{rs} \) we have

\[
\text{Orb}((\gamma, u_1, u_2), f') = \begin{cases} \text{Orb}((g, u), f) & \text{if } (\gamma, u_1, u_2) \text{ matches some } (g, u) \in (U(V) \times V)(F_0)_{rs}, \\ 0 & \text{else.} \end{cases}
\]

2.2. **Group version transfers.** Consider an orthogonal decomposition \( L = L^\perp \oplus O_F e \) of hermitian lattices where \( (e, e)_{V^*} \neq 0 \).

Choose an orthogonal basis of \( L^\perp \) which gives a basis of \( L \) by adding \( e \). Consider the conjugacy action of \( U(V^*) \) (resp. \( GL(V^*_0) \)) on \( U(V) \) (resp. \( S(V_0) \)) similar to the semi-Lie set up. Form the matching relation and orbit integrals as in [19]. In particular, for \( f' \in S((S(V_0)(F_0)) \) and \( \gamma \in (S(V_0)(F_0)_{rs} \), we define \( (s \in \mathbb{C} \):

\[
\text{Orb}(\gamma, f', s) := \omega(\gamma) \int_{h \in GL(V_0^*)} f'(h^{-1} \gamma h) \eta(h)|h|^s dh,
\]

where the transfer factor on \( S(V_0)(F_0) \) with respect to \( L \) is defined by

\[
\omega(\gamma) := \eta(\det(\gamma e))^{n-1} \in \{ \pm 1 \}.
\]  
(2.7)

We normalize the Haar measure on \( U(V^*)(F_0) \) (resp. \( GL(V^*_0)(F_0) \)) such that the stabilizer \( U(L^\perp) \) (resp. \( GL(L_0^\perp, L_0^{\perp\vee}) \)) is of volume 1.
2.3. Conjectures. If $L$ is a hermitian lattice in $V$, define the valuation of $L$ as

$$v(L) := \min_{x \in L, y \in L} v((x, y)_V).$$

Recall that $L$ is called a vertex lattice if $L \subseteq L' \subseteq \omega^{-1}L$, and the type of $L$ is $t(L) := \dim_{\mathbb{Q}_p} L'/L$.

We say $L$ is maximal parahoric if $v(L) + v(L') \geq -1$. The condition is stable under taking dual of $L$ and scaling $L$. Using an orthogonal basis for $L$, we see $L$ is maximal parahoric if and only if some scalar of $L$ or $L'$ is a vertex lattice. We have the following explicit Jacquet–Rallis transfer conjectures (TC).

**Conjecture 2.1.** (1) (Semi-Lie transfer for $L$) Let $L$ be a maximal parahoric hermitian lattice, then $f_L := 1_{U(L)} \times 1_L$ and $f'_L := 1_{S(L, L'_{\mathbb{C}})} \times 1_{L'_{\mathbb{C}}} \times 1_{L'_{\mathbb{C}}}$ are transfers.

(2) (Group transfer for the pair $(L, e)$) Let $L = L' \oplus O_{F_0}$ be an orthogonal decomposition of maximal parahoric hermitian lattices where $(e, e)_V \neq 0$. Then $f_{(L, e)} := 1_{U(L)}$ and $f'_{(L, e)} := 1_{S(L, L'_{\mathbb{C}})}$ are transfers.

Here $L^{\vee}_0 := L' \cap V_0$, and $L^{\vee}_{0*}$ is the linear dual of $L^{\vee}_0$ defined as $L^{\vee}_{0*} := \{ f \in V_{0*} | f(L^{\vee}_0) \subseteq O_{F_0} \}$. By definition, the group TC for the pair $(L, e)$ is equivalent to the group TC for the pair $(L', e'(e, e)_V)$.

**Remark 2.2.** Any $F$-basis of $V$ gives a trivialization $\wedge^n V \cong F$ hence a transfer factor $\omega(\gamma, u_1, u_2) := \eta(\det(\gamma u_1 u_2)_{n-1}) \in \{ \pm 1 \}$. For different $F$-basis of $V$, the associated transfer factors may differ up to a constant sign. In Conjecture 2.1, the transfer factors are defined using a basis of $L$ and depends only on $L$. We have

$$\omega_L = \gamma_V \omega_L$$

where $\gamma_V = \eta(\det V) \in \{ \pm 1 \}$ is the Weil constant of $V$.

3. Reductions via relative Cayley maps

In this section, we introduce the relative Cayley map and do reductions of orbits and orbit integrals from group version to semi-Lie version. We have the reductions for $s$-variable orbit integrals, which is used for the reduction of arithmetic transfers conjectures in Section 7.

3.1. The relative Cayley map. Let $V$ be a $F$-vector space with a decomposition $V = V^p \oplus F e$. For any endomorphism $A' \in \End(V)$, we write

$$A' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \End(V^p) & V^p \\ (V^p)^* & F \end{pmatrix} = \End(V).$$

**Definition 3.1.** The relative Cayley map is the $GL(V^p)$-equivariant rational map

$$c : \End(V) \longrightarrow \End(V^p) 	imes V^p \times (V^p)^* \quad \text{A'} \mapsto (A = a + \frac{b}{1-t}, b_1 = \frac{b}{1-t}, c_1 = \frac{c}{1-t}).$$

(3.1)

3.2. Reduction for unitary groups. Let $V$ be a $F/F_0$-hermitian space with an orthogonal decomposition $V = V^p \oplus F e$ where $(e, e)_V \neq 0$. For any

$$g' = \begin{pmatrix} t & u \\ \bar{w} & d \end{pmatrix} \in \mathbb{U}(V),$$

we have $d = \frac{(g', e, e)}{(e, e)}$ and $w(x) = \frac{(g', x, e)}{(e, e)}$ for $x \in V^p$.

**Proposition 3.2.** Assume that $1 - d \neq 0$, consider the well-defined relative projection

$$g = t + \frac{uw}{1-d} : V^p \longrightarrow V^p.$$

(1) We have $g \in \mathbb{U}(V^p)$. If $x \in V^p$ such that $w(x) = 0$, then $g(x) = g'(x)$.

(2) We have $\frac{w}{1-d}(x) = \frac{(g, x, e)}{(e, e)}$ for $x \in V^p$.

Moreover, these properties uniquely characterize $g$ in terms of $g'$. Given $g \in \mathbb{U}(V^p)$, $u \in V^p$ and $d \neq 1$, there exists a unique $g' \in \mathbb{U}(V)$ with these invariants.
Remark 3.7. We don’t assume that λ. The Definition 3.3. g is non-zero. Conversely, we can reverse above computations and reconstruct

Proof. Using $d = \frac{(g',e)}{(e,e)}$ and $w(x) = \frac{(g',e)}{(e,e)}$ we compute for $x \in V^\flat$

$$gx - g'x = -w(x)e + \frac{w(x)}{1-d}u = -\lambda(x)(g'e - e).$$

where $\lambda(x) := \frac{(g',e)}{(e,e)}$. For $x, y \in V$, using $(g',g'y) = (e, y) = 0 = (g', g'e) = (x, e)$ we find

$$(gx, gy) - (x, y) = (gx, gy) - (g'x, g'y) = \lambda(x)(e, g'y) + \lambda(y)(g'x, e) + \lambda(x)\lambda(y)(g'e - e, g'e - e)$$

which is $\lambda(x)\lambda(y)(g'e - e, e) + (g'e - e, e) + (g'e - e, g'e - e) = 0$. Hence $g \in U(V^\flat)$.

If $w(x) = 0$, then by definition $gx = tx = g'x$. For $x \in V^\flat$, we compute $(gx, u)$

$$(g'x, e) - \frac{(g'x, e)}{(g'e, e)}(g'e - e, e) = (g'x, e)(0 - \frac{\overline{d}}{d - 1} - \overline{d} + \overline{d}) = \frac{1-\overline{d}}{1-d}(g'x, e).$$

As $w(x) = \frac{(g',e)}{(e,e)}$, we get the equality $\frac{w(x)}{1-d}(x) = \frac{(gx, u)}{(e,e)}$.

These functions determine $g^{-1}u$ and $g|_{ker w}$ from $g'$, hence $g$ itself as $w(g^{-1}u) = \frac{(w,e)}{(e,e)} = 1 - d$ is non-zero. Conversely, we can reverse above computations and reconstruct $g'$.

□

Definition 3.3. The unitary relative Cayley map is the $U(V^\flat)$-equivariant rational map

$$c_U : U(V) \longrightarrow U(V^\flat) \times V^\flat$$

$$g' \longmapsto (g = t + \frac{w}{1-d}, u_1 = \frac{u}{1-d}). \quad (3.2)$$

From Proposition 3.2 we have

Corollary 3.4. We have $c(g') = (g, u_1, w_1)$ where $w_1(x) = \frac{(gx, u_1)}{(e, e)}$. The map $c_U$ induces an isomorphism over the locus where $d \neq 1$ is fixed. We have equalities of invariants ($i \in \mathbb{Z}$):

$$(g^iu_1, u_1) = (e, e)w_1(g^{-1}u_1).$$

Proposition 3.5. The element $g'$ is regular semi-simple if and only if $c_U(g')$ is regular semi-simple.

Proof. By induction on $i$ we see for any $1 \leq i \leq n$, the element $g^iu - t^iu$ is in the $F$-span of $\{u, tu, \ldots, t^{-1}u\}$. We have $g'e = u + de$ and $g'x = tx + w(x)e, (x \in V^\flat)$. So $\{g^iu_1\}_{i \geq 0}$ generates $V^\flat \Rightarrow \{g^iu\}_{i \geq 0}$ generates $V^\flat \Rightarrow \{t^iu\}_{i \geq 0}$ generates $V$.

So we have a natural reduction of $(U(V^\flat))$-orbits. Now we do reduction for $U(V^\flat)$-orbit integrals. Consider a hermitian lattice $L \subset V = V^\flat \oplus Fe$ with an orthogonal decomposition

$$L = L^\perp \oplus Fe \subset L$$

where $e_L = \lambda_L e$ for some $\lambda_L \in F_0^\times$.

Proposition 3.6. Let $g' \in U(V)(F_0)$ with $c_U(g') = (g, u_1)$. Assume that $1 - d \in O_F^\times$, then for any $h \in U(V^\flat)$, we have

$$h^{-1}g'h \in U(L)$$

if and only if

$$h^{-1}gh \in U(L^\flat), \quad h^{-1}u_1 \in \lambda^{-1}_L L^\flat, \quad h^{-1}u \in \lambda^{-1}_e (\lambda^{-1}_L L^\flat)^{\vee}.$$

Proof. The map $c_U$ is $U(V^\flat)$-equivariant and $d$ is $U(V^\flat)$ conjugacy invariant, so we may assume $h = id$. As $\det(g') \in F_0^{Nm_F/\alpha}$ is a unit, the condition $g' \in U(L)$ is the same as $g'L \subset L$.

Under the decomposition, we see $g'L \subset L$ if and only if $t(L^\flat) \subset L$ as $1 - d \in O_F^\times$.

$$t(L^\flat) \subset L^\flat, \quad u \in \lambda^{-1}_L L^\flat, \quad t^1(L) \subset \lambda_L Pf.$$ The last two conditions imply $u = \frac{w}{1-d}(L^\flat) \subset L^\flat$. As $g = t + \frac{w}{1-d} \in U(V^\flat)$, above is equivalent to $g(L^\flat) = \lambda_L L^\flat, \quad u_1 \in \lambda^{-1}_L L^\flat, \quad w_1(L^\flat) \subset \lambda_L Pf$. We have $w_1(x) = \frac{(g', u_1)}{(e, e)}$ by Proposition 3.2 hence $w_1(L^\flat) \subset \lambda_L Pf$ is equivalent to $(L^\flat, u_1) \subset \lambda^{-1}_L (e, e) \lambda_L Pf$. The result follows.

□

Remark 3.7. We don’t assume that $g'$ is regular semi-simple. We have similar reduction result for intersection numbers on unitary Rapoport–Zink spaces in Section 7.
Corollary 3.8. For \( g' \in \text{U}(V)(F_0)_{rs} \) with \( 1 - d \in O_F^\times \) and any Haar measure on \( \text{U}(V')(F_0) \), we have
\[
\text{Orb}(g',1_{U(L)}) = \text{Orb}(c_U(g'),1_{U(L')} \times 1_{(\lambda_L^{-1}L')^\vee}).
\]

Definition 3.9. We say the lattice \( L = L^\ast \oplus O_F e_L \) satisfies the reduction if
\[
v_F(e_L,e_L) = \min_{x,y \in L} v_F((x,y)).
\]
By definition, \( L \) (resp. \( L^\ast \)) satisfies the reduction if and only if \( \lambda_L^{-1}L^\ast \subseteq (e,e)(\lambda_L^{-1}L')^\vee \) (resp. \( (e,e)(\lambda_L^{-1}L')^\vee \subseteq \lambda_L^{-1}L^\ast \)).

Lemma 3.10. If \( L \) is maximal parahoric, then either \( L \) or \( L^\ast \) satisfies the reduction.

Proof. Otherwise, we have \( v_F(e_L,e_L) \geq v(L) + 1 \) and \( v_F(e_L,e_L) = -v_F(e_L,e_L) \geq v(L^\ast) + 1 \), which sum together to give \( 0 \geq v(L) + v(L^\ast) + 2 \geq 1 \), a contradiction. \( \Box \)

Corollary 3.11. If \( L \) satisfies the reduction, then for any \( g' \in \text{U}(V)(F_0)_{rs} \) with \( 1 - d \in O_F^\times \) we have
\[
\text{Orb}(g',1_{U(L)}) = \text{Orb}(c_U(g'),1_{U(\lambda_L^{-1}L^\ast)} \times 1_{(\lambda_L^{-1}L')^\vee}).
\]
For \( \xi \in F^{\text{Nm}_{F/F_0}=1} \), we define the \( \xi \)-twisted unitary Cayley map by
\[
c_{U,\xi}(g') := c_U(\xi g').
\]
For any element \( g' \in \text{U}(V)(F_0)_{rs} \), we have \( 1 - d\xi \in O_F^\times \) for infinitely many \( \xi \in F^{\text{Nm}_{F/F_0}=1} \) because \( \#\{x \in F_{\rho}\mid \text{Nm}_{F/F_0}(x) = 1\} = q + 1 > 1 \). As \( \xi \) is a unit, the elements \( g' \) and \( \xi g' \) have same orbit integrals for any \( f \in \text{S}(U(V)(F_0)) \). Therefore, the reduction of orbit integrals holds for any regular semi-simple orbit in \( U(V)(F_0) \) under suitable \( \xi \)-twisted unitary Cayley maps.

3.3. Reduction for symmetric spaces. Consider a decomposition \( V = V^\ast \oplus F e \), \( F \)-vector spaces. Choose a basis of \( V^\ast \) to endow \( V^\ast \) (resp. \( V \)) a \( F_0 \)-rational structure \( V_0^\ast \) (resp. \( V_0 \)) by adding \( e \), which induces a \( F/F_0 \) semi-linear involution \((\bar{-})\) on \( V \) with fixed subspace \( V_0 \). Denote also by \((\bar{-})\) the induced involutions on \( V^\ast \) and \( \text{End}(V) \).

Consider the embedding of symmetric spaces:
\[
S(V_0^\ast) = \{ \gamma \in \text{GL}(V^\ast) | \gamma \gamma = \text{id}) \leftrightarrow S(V_0) = \{ \gamma' \in \text{GL}(V) | \gamma' \gamma' = \text{id} \}.
\]
Write any \( \gamma' \in S(V_0) \) under the decomposition \( V = V^\ast \oplus F e \) as
\[
\gamma' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

Proposition 3.12. Assume that \( 1 - d \neq 0 \), consider the well-defined relative projection
\[
\gamma = a + \frac{bc}{1-d} : V^\ast \rightarrow V^\ast.
\]
(1) We have \( \gamma \in S(V_0^\ast) \). If \( x \in V^\ast \) such that \( c(x) = 0 \), then \( \gamma(x) = \gamma'(x) \).
(2) We have
\[
\gamma \frac{b}{1-d} = \frac{b}{1-d}, \quad \gamma \frac{c}{1-d} = \frac{c}{1-d}.
\]
Moreover, these properties uniquely characterize \( \gamma \) in terms of \( \gamma' \). Given \( \gamma \in S(V_0^\ast) \), \( b \in V, c \in V^\ast \) and \( d \neq 1 \) satisfying these properties, there exists a unique \( \gamma' \in S(V_0) \) with these invariants.

Proof. By definition, if \( x \in V^\ast \) such that \( c(x) = 0 \), then we have \( \gamma(x) = a(x) = \gamma'(x) \). From the equality \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} = \text{id} \), we have
\[
a\bar{a} = \text{id} - b\bar{c}, \quad a\bar{b} = -d\bar{b}, \quad c\bar{a} = -d\bar{c}, \quad c\bar{b} = 1 - d\bar{a}.
\]
We find \( \gamma\bar{\gamma} = a\bar{a} + \frac{b\bar{c}}{1-d} + \frac{bc}{1-d}a + \frac{bc}{1-d}b \) is equal to \( \text{id} + b\bar{c}(-1-\frac{b}{1-d}) - d + \frac{-d}{1-d} + \frac{1-a\bar{a}}{(1-d)(1-d)} \) = \text{id}.
Therefore we have \( \gamma \in S(V_0^\ast) \). The second property follows from
\[
\gamma\bar{b} = a\bar{b} + c\bar{b} \frac{b}{1-d} = -d\bar{b} + (1-d\bar{d}) \frac{b}{1-d} = \frac{1-d}{1-d} b,
\]
\[ \tau \gamma = \tau a + \frac{\tau b c}{1 - d} = -dc + (1 - d) \frac{c}{1 - d} = \frac{1 - d}{1 - d} c. \]

Conversely, the uniqueness and reconstruction of \( g' \) follows by reversing the above computations. \( \square \)

**Definition 3.13.** The *untwisted symmetric relative Cayley map* is the rational map

\[
\begin{array}{ccc}
\varepsilon_S : (V_0^a) & \longrightarrow & (V_0^b) \times V^a \times (V^b)^* \\
\gamma & \longmapsto & (\gamma, b_1 = \frac{b}{a}, c_1 = \frac{c}{a}).
\end{array}
\] (3.4)

It is \( GL(V_0) \)-equivariant. Now we modify \( \varepsilon_S \) to make the image lying in \( (V_0^a) \times V_0^a \times (V_0^a)^* \).

**Definition 3.14.** A *twisting element* for \( \gamma \in (V_0^a)(F_0) \) is any \( B_\gamma \in \text{GL}(V_0^a)(F_0) \) such that

\[ \gamma = B_\gamma B_{\gamma}^{-1} = B_{\gamma}^{-1} B_\gamma. \]

By Hilbert 90 for \( F/F_0 \), there is always some \( B \in \text{GL}(V_0^a) \) such that \( \gamma = B B_{\gamma}^{-1} \). Here the condition is slightly stronger as we require \( \gamma B_\gamma = B_\gamma \gamma \) and \( \gamma B_{\gamma} = B_{\gamma} \gamma \).

**Example 3.15.** Choose any \( \xi \in F_{\text{Nm}(F_0)} = F' \) such that \( \det(1 + \xi \gamma) \neq 0 \). Write \( \xi = a/\bar{a} \) for some \( a \in F' \), then \( a^{-1}(1 + \xi \gamma) \) is a twisting element for \( \gamma \).

Let \( \gamma' \in (V_0^a)(F_0) \) with \( 1 - d \neq 0 \) and \( \varepsilon_S(\gamma') = (\gamma, b_1, c_1) \). Choose a twisting element \( B_\gamma \) for \( \gamma \), set

\[ b_2 := B_\gamma^{-1} b_1, \quad c_2 := c_1 B_{\gamma}. \]

**Proposition 3.16.** Choose any twisting element \( B_\gamma \) for \( \gamma \) as above.

1. We have \( (b_2, c_2) \in V_0^a \times (V_0^a)^* \).
2. We have following equalities of invariants \((i \in \mathbb{Z})\):

\[ c_2 \gamma^{i+1} b_2 = c_1 \gamma^{i} b_1. \]
3. The \( GL(V_0^a) \)-orbit of \( (\gamma, b_2, c_2) \) doesn’t depend on the choice of \( B_\gamma \) and only depends on the \( GL(V_0^a) \)-orbit of \( \gamma' \).

**Proof.** By Proposition 3.12, we have \( \gamma \bar{b}_1 = b_1, \gamma \bar{c}_1 = c_1 \). From \( \gamma = B_{\gamma} B_{\gamma}^{-1} \), we see

\[ B_{\gamma}^{-1} b_1 = B_{\gamma}^{-1} b_1, \quad c_1 B_{\gamma} = c_1 B_{\gamma}, \]

which shows \( (b_2, c_2) \in V_0^a \times (V_0^a)^* \). Since \( \gamma \) commutes with \( B_{\gamma} \) and \( B_{\gamma} \), we get \( c_2 \gamma^{i+1} b_2 = c_1 \gamma^{i} b_1 \).

Finally, if \( B_{\gamma}, B_{\gamma}' \) are two twisting elements for \( \gamma \), then \( h_0 := B_{\gamma}^{-1} B_{\gamma}' \) is in \( GL(V_0^a)(F_0) \) and commutes with \( \gamma \). Hence \( h_0(\gamma, b_2, c_2) = (\gamma, b_2', c_2') \). And for any \( h \in GL(V_0^a)(F_0) \), the element \( h^{-1} B_{\gamma} h \) is a twisting element for \( h^{-1} \gamma h \). Property (3) follows. \( \square \)

**Definition 3.17.** The *relative symmetric Cayley map* on \( GL(V_0^a) \)-orbits is the map between \( GL(V_0^a)(F_0) \)-orbits

\[
\begin{array}{ccc}
\nu_S : [S(V_0^a)(F_0)] & \longrightarrow & [(S(V_0^a) \times V_0^a \times (V_0^a)^*)(F_0)] \\
\gamma' & \longmapsto & (\gamma, b_2 := B_{\gamma}^{-1} b_1, c_2 := c_1 B_{\gamma})
\end{array}
\] (3.5)

where \( B_{\gamma} \) is any twisting element of \( \gamma \). This is well-defined by Proposition 3.16. We also write \( \nu_S(\gamma') = (\gamma, b_2, c_2) \) with respect to a chosen twisting element \( B_{\gamma} \).

**Proposition 3.18.** The element \( \gamma' \) is regular semisimple if and only if \( \nu_S(\gamma') = (\gamma, b_2, c_2) \) is regular semisimple.

**Proof.** By Proposition 3.16 we have \( c_2 \gamma^{i+1} b_2 = c_1 \gamma^{i} b_1 \). So by the same argument in Proposition 3.5 we see the equivalences: \( \nu_S(\gamma') = (\gamma, b_2, c_2) \) is regular semisimple \( \Leftrightarrow \nu_S(\gamma') = (\gamma, b_1, c_1) \) is regular semisimple \( \Leftrightarrow \gamma' \) is regular semisimple. \( \square \)

An element \( \gamma \in S(V_0^a)(F_0) \) is called *integral* if its characteristic polynomial \( \text{char}(\text{Id}_{V_0} - \gamma) \) lies in \( O_F[T] \). Recall that \( F/F_0 \) is unramified and \( p \geq 2 \).
Proposition 3.19. (Integral Hilbert 90) Let $\gamma \in S(V_0^\otimes)(F_0)$ be an integral element and $R = O_F[\gamma] \subseteq \End(V)$ be the commutative $O_F$-algebra generated by $\gamma$.

Then there exists an element $B_\gamma \in (O_F[\gamma])^\times$ such that $\gamma = B_\gamma \overline{\gamma}$.

Proof. As $\gamma$ is integral and $\det(\gamma) \in F^{Nm_{F/F_0} \gamma^{-1}}$ is a unit, we see $R$ is finite flat over $O_F$. As $\gamma^{-1}$ lies in $R$ by Cayley-Hamilton theorem, we see $R$ is stable under the involution $(-)$ on $\End(V)$.

The ring map $R_0 \otimes_{O_{F_0}} O_F \to R$ is an isomorphism: we base change along the étale morphism $O_{F_0} \to O_F$ where things are trivial. If $p > 2$, we can see this by observing any $x \in R$ is in the image by the formula $x = x^\pm + \frac{x^2}{2} \delta$ for chosen $\delta \in O_F^\times$ such that $\delta = -\delta$. We are done by the commutative algebra lemma 3.20 below.

Lemma 3.20. For any finite flat commutative $O_{F_0}$-algebra $A$, consider the induced involution $(-)$ on $A = A_0 \otimes_{O_{F_0}} O_F$ given by $a \otimes b = a \otimes \overline{b}$. If $x \in A^\times$ satisfies $x\overline{x} = 1$, then there exists $y \in A^\times$ such that $x = y\overline{y}^{-1}$.

Proof. This is proved by vanishing of certain first Galois cohomology in [21, Lem. 8.6].

Now we do reduction for orbit integrals. Consider any lattice $L$ in $V$ with a decomposition $L = L^\flat \oplus O_F e_L$ that is stable under $(-)$. Here $e_L = \lambda_L e$ for some $\lambda_L \in F_0^\times$. So $L_0 = \{x \in L| x = x\}$ is an $O_F$-lattice in $V_0$ such that $L = L_0 \otimes_{O_{F_0}} O_F$. Consider the stabilizer for $L$:

$$S(L) = \{\gamma \in S(V_0)(F_0)| \gamma L = L\}.$$ 

Proposition 3.21. Let $\gamma' \in S(V_0)(F_0)$ with $c_S(\gamma') = (\gamma, b_1, c_1)$. Assume that $1 - d \in O_F^\times$, then

1. Assume that $\gamma$ is integral. Choose the twisting element $B_\gamma \in O_F[\gamma]^\times$ as in Proposition 3.19 and set

$$c'_S(\gamma') = (\gamma, b_2 = B_\gamma^{-1} b_1, c_2 = c_1 B_\gamma).$$

For any $h \in GL(V_0^\otimes)(F_0)$, we have $h^{-1}\gamma' h \in S(L)$ if and only if

$$h^{-1}\gamma h \in S(L^\flat), \quad h^{-1}b_2 \in \lambda_L^{-1} L_0^\flat, \quad c_2 h \in (\lambda_L^{-1} L_0^\flat)^\ast.$$ 

2. If $\gamma$ is not integral, then there is no $h \in GL(V_0^\otimes)(F_0)$ such that $h^{-1}\gamma' h \in S(L)$ or $h^{-1}\gamma h \in S(L^\flat)$.

Proof. Because $\det(h^{-1}\gamma h)$ is a unit, the condition $h^{-1}\gamma h \in S(L)$ is equivalent to $h^{-1}\gamma' h \subseteq L$. Under the decomposition, we see (using $1 - d \in O_F^\times$) that $h^{-1}\gamma h \subseteq L$ if and only if

$$h^{-1}b_1(l') \subseteq L^\flat, \quad h^{-1}b_1 \in \lambda_L^{-1} L^\flat, \quad c_1 h(l') \subseteq \lambda_L O_F.$$ 

The latter two conditions imply $\frac{bc}{1-\lambda_L^{-1}}(l') \subseteq L^\flat$. As $h^{-1}\gamma = h^{-1}a h + \frac{bc}{1-\lambda_L^{-1}}$, above is equivalent to

$$h^{-1}\gamma h(l') \subseteq L^\flat, \quad h^{-1}b_1 \in \lambda_L^{-1} L^\flat, \quad c_1 h(l') \subseteq \lambda_L O_F.$$

In particular $\gamma$ must be integral, the second part follows.

Assume that $h^{-1}\gamma h(l') \subseteq L^\flat$ holds. By Proposition 3.19 $B_\gamma \in O_F[\gamma]$ in particular $\det(B_\gamma) \in O_F^\times$. Hence we have $h^{-1}B_\gamma h(l') = L^\flat$ and $h^{-1}B_\gamma h(l') = L^\flat$.

So the condition $h^{-1}b_1 \in \lambda_L^{-1} L^\flat$ (resp. $c_1 h(l') \subseteq \lambda_L O_F$) is equivalent to the condition $h^{-1}B_\gamma^{-1} h^{-1}b_1(h^{-1}b_1) = h^{-1}b_2 \in \lambda_L^{-1} L^\flat$ (resp. $c_2 h(l') = c_1 h(h^{-1}B_\gamma)(l') = c_1 h(l') \subseteq \lambda_L O_F$).

Now the result follows by noting $(h^{-1}b_2, c_2 h) \in V_0^\otimes \times (V_0^\otimes)^\ast$.

□

Corollary 3.22. For $\gamma' \in S(V_0)(F_0), s \in \mathbb{C}$ we have

$$\text{Orb}(\gamma', 1_{S(L)}) = \text{Orb}(c'_S(\gamma'), 1_{S(L^\flat)}) \times 1_{\lambda_L^{-1} L_0^\flat} \times 1_{(\lambda_L^{-1} L_0^\flat)^\ast}, \quad s).$$

In general, for any collection of lattices $\{L_i\}$ of the form $L_i = L_i^\flat \oplus O_F \lambda_{L_i} e$, we have

$$\text{Orb}(\gamma', 1_{S(L_i)}) = \text{Orb}(c'_S(\gamma'), 1_{S(L_i^\flat)}) \times 1_{\lambda_{L_i}^{-1} L_i^\flat} \times 1_{(\sum_i \lambda_{L_i}^{-1} L_i^\flat)^\ast}, \quad s).$$
Theorem 3.26. For \( \omega(\gamma, b_2, c_2) = \omega(\gamma') \).

From \( \gamma^*b_2 = B^{-1}\gamma^*b_1 \) and \( \det(B_\gamma) \in O_F^\times \), we have
\[
\omega(\gamma, b_2, c_2) = \omega(\gamma, b_1, c_1) = \omega(\gamma, b, c).
\]

By induction on \( i \geq 0 \), we see \( \gamma^*b - a^*(b) \) is in the \( F \)-span of \( \{b, ab, \ldots, a^{i-1}b\} \) so \( \omega(\gamma, b, c) = \omega(a, b, c) \). As \( \gamma(x) = a(x) + c(x) \) \( x \in V^g \), we are done by
\[
\omega(\gamma') = \det(e \wedge \gamma' e \cdots \wedge \gamma^{n-1} e) = \det(e \wedge b \wedge \gamma b \cdots \wedge \gamma^{n-2} b) = \omega(\gamma, b, c).
\]

The reduction for any collection of lattices follows along the same line as in Proposition 3.21 and above arguments.

For any \( \xi \in F^{Nm_F} = 1 \), define the \( \xi \)-twisted symmetric Cayley map by
\[
c_{S,\xi}(\gamma') := c_S(\xi\gamma'), \quad c'_{S,\xi}(\gamma') := c_S'(\xi\gamma').
\]

For \( \gamma' \in S(V_0)(F_0)_{rs} \), we have \( 1 - d\xi \in O_F^\times \) for infinitely many \( \xi \in F^{Nm_F} \) as \# \{ \( x \in k_F \mid Nm_{F/k_F} (x) = 1 \} = q + 1 > 1 \). The orbit integral is the same for \( \gamma' \) and \( \xi\gamma' \). Therefore, the reduction of orbit integrals holds for any regular semi-simple \( \gamma' \), using suitable \( \xi \)-twisted symmetric Cayley maps.

3.4. Matching and the equivalence of two transfer conjectures. Return to our transfer conjectures 2.1. Choose an orthogonal \( F \)-basis \( \{e_i\} \) of \( V^g \), and extend it to an orthogonal basis of \( V \) by adding \( e \).

Proposition 3.23. If \( g' \in U(V)(F_0)_{rs} \) matches \( \gamma' \in S(V_0)(F_0)_{rs} \), then \( c'_U(g') = (g, u_1) \in (U(V^g) \times V^g)(F_0) \) matches
\[
(e, e) c'_S(\gamma') := (\gamma, b_2, (e, e)c_2) \in (S(V_0^g) \times V_0^g) \times (V_0^g)')(F_0).
\]

Proof. We only need to check invariants of \( (g, u_1) \) and \( (\gamma, b_2, (e, e)c_2) \). If \( g' \) matches \( \gamma' \), then there exist \( h \in GL(V^g) \) such that \( h^{-1}g'h = \gamma' \). This gives
\[
h^{-1}u = b, \quad wh = c, \quad h^{-1}th = a, \quad d_{g'} = d_{\gamma'} = d.
\]

As \( \gamma = a + b \), \( g = t + u \), we see \( h^{-1}gh = \gamma \). So we have
\[
char(Tid_{V^g} - g) = char(Tid_{V^g} - \gamma)
\]
and \( c_1 \gamma^*b_1 = w_1 g' u_1 \) for any \( i \in \mathbb{Z} \). By Proposition 3.22 we have \( (g' u_1, u_1) = (e, e) w_1(g^{-1} u_1) \).

By Proposition 3.16 \( c_{g} \gamma^*b_2 = c_1\gamma^*b_1 \), so \( (e, e)c_2 \gamma^*b_2 = (e, e)c_1\gamma^*b_2 \). Therefore we have \( (g' u_1, u_1) = (e, e)c_2 \gamma^*b_2 \) for any \( i \in \mathbb{Z} \). So \( (g, u_1) \) matches \( (\gamma, b_2, (e, e)c_2) \).

Consider any \(-\)-stable hermitian lattice \( L \) in \( V \) with an orthogonal decomposition \( L = L^\vee \otimes O_F e_L, e_L = \lambda_L e \) for some \( \lambda_L \in F_0^\times \). We have \( L = L_0 \otimes_{O_{F_0}} O_F \) for the lattice \( L_0 = \{ x \in L | \bar{x} = x \} \). Apply Corollary 3.22 to the collection of \( L \) and \( L^\vee \), we see

Proposition 3.24. For \( \gamma' \in S(V_0)(F_0)_{rs} \) with \( 1 - d \in O_F^\times \), we have
\[
\text{Orb}(\gamma', 1_{S(L, L^\vee)}) = \text{Orb}((e, e), c'_{S}(\gamma'), f')
\]
where \( (e, e), c'_{S}(\gamma') := (\gamma, b_2, (e, e)c_2) \) and
\[
f' = 1_{S(L, (L^\vee)')} \times 1_{\lambda_L^\vee L^\vee \cap (e, e)(\lambda_L^\vee L^\vee)'} \times 1_{(e, e)(\lambda_L^\vee L^\vee)\cap (\lambda_L^\vee L^\vee)'}.
\]

Corollary 3.25. If \( L \) satisfies the reduction (Definition 3.3), then for \( \gamma' \in S(V_0)(F_0)_{rs} \) with \( 1 - d \in O_F^\times \) we have
\[
\text{Orb}(\gamma', 1_{S(L, L^\vee)}) = \text{Orb}((e, e), c'_{S}(\gamma'), f')
\]
where
\[
f' = 1_{S(\lambda_L^\vee L^\vee, (\lambda_L^\vee L^\vee)')} \times 1_{\lambda_L^\vee L^\vee} \times 1_{(\lambda_L^\vee L^\vee)'}.
\]

Theorem 3.26. (1) If \( L \) satisfies the reduction, then the group transfer conjecture 2.1 for \( L = L^\vee \otimes O_F e_L \) is equivalent to the semi-Lie transfer conjecture 2.1 for \( L_{new} := \lambda_L^\vee L^\vee \).
If $L'$ satisfies the reduction, then the group transfer conjectures \ref{2.1} for $L = L' \oplus O_F e_L$ is equivalent to the semi-Lie transfer conjectures \ref{2.1} for $L_{\text{new}} := \lambda_{L'} L' \oplus O_F$.

Proof. By duality, we only need to show Part 1. By Corollary \ref{3.25} Corollary \ref{3.11} and Proposition \ref{5.22} semi-Lie version TC implies group version TC. Moreover, as relative Cayley maps are surjective on orbits (for given $d$) by Proposition \ref{3.6} and \ref{3.12} we can reverse the process. □

In particular, if $L$ is maximal parahoric then either (1) or (2) holds. The new lattice $L_{\text{new}}$ is still maximal parahoric, so we can continue the process once we could relate (boundary) cases of semi-Lie version transfer conjectures to lower rank group version transfer conjectures.

4. Transfers for all vertex lattices

In this section, we give the proof of explicit transfer conjectures \ref{2.1} for any $p$-adic field $F_0$.

Theorem 4.1. The semi-Lie version transfer conjectures \ref{2.1} hold for all maximal parahoric lattice $L$ i.e., $v(L) + v(L') \geq -1$ e.g. any vertex lattice $L$. Hence by Theorem \ref{3.26} the group version conjectures \ref{2.1} also hold for any pair $(L, c)$.

For self-dual $L$, the method is similar to \cite{H} which gives a pure local proof for the Lie algebra Jacquet–Rallis fundamental lemma (FL). We handle the group version FL for $p = 2$ which is not covered by \cite{H}. In general, our strategy is to apply double uncertainty principle and do induction on two boundary cases.

Consider the space of regular semisimple $GL(V_0)$-orbits

$$B = [S(V_0) \times V_0 \times V_0]_{1s}(F_0) \longrightarrow [U(V) \times V]_{1s}(F_0) \prod [U(V) \times V]_{1s}(F_0). \quad (4.1)$$

where $V$ is the nearby hermitian space of $V$.

Endow the $n$-dimensional hermitian space $V$ with the $F_0$-valued quadratic form $q(u) := (u, u)_V$, and the space $V_0 \times V_0^*$ with the $F_0$-valued quadratic form $q'(b, c) := c(b)$. They induced a map on orbit space $B$:

$$q = q' : B \longrightarrow F_0, \quad [(\gamma, b, c)] \longmapsto c(b).$$

Fix an unramified quadratic character $\psi$ of $F_0$. We denote by

- $F_U$ the Fourier transform on $V$ with respect to the additive character $\psi \circ q$,
- $F_S$ the Fourier transform on $V_0 \times V_0^*$ with respect to the additive character $\psi \circ q'$.

We always use self-dual Haar measure on $V$ and $V_0 \times V_0^*$ for the Fourier transforms. Then for a $(\cdot)$-stable hermitian lattice $L \subseteq V$ of type $t$, we have

$$\text{vol}(L) = q_F^{-t/2} = 1 \times q^{-t} = \text{vol}(L_0 \times (L_0')^*).$$

4.1. Double uncertainty principle on distributions

We have the Weil representation of $S\text{L}_2(F_0)$ on $S(S(V_0) \times V_0 \times V_0^*)$ and $S(U(V) \times V)$ respectively, by acting on the linear factors $V_0 \times V_0^*$ and $V$ respectively. Concretely, the action of $S\text{L}_2(F_0)$ on $f \in S(U(V) \times V)$ is given by

$$\begin{pmatrix} 1 & t \\ 1 \end{pmatrix} \cdot f(g, u) = \psi(t(u, u)_V)f(g, u), \quad (4.2)$$

$$\begin{pmatrix} 1 & -1 \\ 1 \end{pmatrix} \cdot f(g, u) = \gamma_V F_U f(g, u), \quad (4.3)$$

where $\gamma_V := \eta(\det(V)) \in \{\pm 1\}$ is the Weil constant of $V$. The action of $S\text{L}_2(F_0)$ on $f' \in S(S(V_0) \times V_0 \times V_0^*)$ is given by

$$\begin{pmatrix} 1 & t \\ 1 \end{pmatrix} \cdot f'(\gamma, b, c) = \psi(t(\bar{c}b))f'(\gamma, b, c), \quad (4.4)$$

$$\begin{pmatrix} 1 & -1 \\ 1 \end{pmatrix} \cdot f'(\gamma, b, c) = F_S f'(\gamma, b, c). \quad (4.5)$$

By pushforward, we define two space of orbit integral functions $\text{Orb}_U, \text{Orb}_S \subseteq C^\infty(B)$:

$$\text{Orb}_U = \{\text{Orb}(\cdot, f) : (\gamma, b, c) = (g, u) \in U(V) \times V \mapsto \text{Orb}((g, u, f) | f \in S(U(V) \times V))\},$$

$$\text{Orb}_S = \{\text{Orb}(\cdot, f') : (\gamma, b, c) \mapsto \text{Orb}((\gamma, b, c, f') | f' \in S(S(V_0) \times V_0 \times V_0^*))\}.$$
Here $\text{Orb}(\gamma, bc, f)$ means the orbit integral of $f \in \mathcal{S}(U(V) \times V)$ at any regular semisimple $(g, u)$ matching $(\gamma, b, c)$. If there is no such element $(g, u)$, then we define $\text{Orb}(f, \gamma, b, c) = 0$.

By [59, Theorem A.1], the Weil representations of $\text{SL}_2(F_0)$ on $\mathcal{S}(S(V_0) \times V_0 \times V_0^*)$ and $\mathcal{S}(U(V) \times V)$ descend to $\text{Orb}_L$ and $\text{Orb}_S$, and agree on the intersection $\text{Orb}_L \cap \text{Orb}_S$. In other words,

**Proposition 4.2.** There is an action of $\text{SL}_2(F_0)$ on the linear subspace

$$\text{Orb}_L + \text{Orb}_S \subseteq C^{\infty}(B)$$

compatible with the Weil representation on $V$ and $V_0 \times V_0^*$. And $f \in \mathcal{S}(U(V) \times V), f' \in \mathcal{S}(S(V_0) \times V_0 \times V_0^*)$ are transfers, if and only if $\text{Orb}(-f) = \text{Orb}(-f') \in \text{Orb}_L + \text{Orb}_S$ is zero.

**Remark 4.3.** The proof of [59, Theorem A.1] is via local trace formula, which is eventually reduced to the fact that orbit integral function is locally integrable and Fourier transform on a quadratic space preserves the $L^2$-norm. Therefore, it also holds for $p = 2$.

The following double uncertainty principle says the supports of a function and its “Fourier transform” can’t be both too small, unless the function is zero.

**Proposition 4.4.** Let $a_1, a_2$ be two integers with $a_1 + a_2 \geq -1$. Assume that $\Phi \in \text{Orb}_L + \text{Orb}_S$ satisfies

1. $\Phi(x) = 0$ for all $x \in B$ with $v(q(x)) \leq a_1$.
2. $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \Phi(x) = 0$ for all $x \in B$ with $v(q(x)) \leq a_2$.

Then $\Phi = 0$.

**Proof.** Let $k = -a_1 - 1$ and $k' = -a_2 - 1$ so $k + k' \leq -1$. By assumption, we have $\begin{pmatrix} 1 & t \\ 1 & 1 \end{pmatrix} \Phi = \Phi$ for all $t \in \mathbb{Z}^k O_{F_0}$ and $\begin{pmatrix} 1 & t \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \Phi = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \Phi$ for all $t \in \mathbb{Z}^k O_{F_0}$. The subgroups $\begin{pmatrix} 1 & \mathbb{Z}^k \cdot O_{F_0} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & \mathbb{Z}^k \cdot O_{F_0} \\ 1 \end{pmatrix}$ generates the whole group $\text{SL}_2(F_0)$. Hence $\Phi$ is fixed by $\text{SL}_2(F_0)$.

As $\begin{pmatrix} 1 & t \\ 1 & 1 \end{pmatrix} \Phi([\gamma, b, c]) = \psi(t(c(b)) \Phi([\gamma, b, c])$ for any $t \in F_0$, we see $\Phi = 0$. \hfill $\square$

Return to the lattice $L$. Using an orthogonal basis for $L$, we see the valuation $v(L) := \min_{u \in L} v\left(\frac{1}{u} (u, u)\right)$ is equal to the valuation $v(L_0, L_0^*) := \min_{u \in L_0, c \in (L_0^*)} v(c(b))$.

By proposition 4.4, to prove Theorem 4.1 for $L$ we only need to show vanishing of the function

$$\Phi_L = \text{Orb}(-1, 1_U(L) \times 1_L) - \text{Orb}(-1, 1_{S(L_0, L_0^*)} \times 1_{L_0} \times 1_{L_0^*}) \in C^{\infty}(B) \quad (4.6)$$

**Proposition 4.5.** We have

1. $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \Phi_L = \text{vol}(L) \Phi_L$.
2. $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \Phi_L(x) = 0$ for all $x \in B$ with $v(q(x)) \leq v(L) - 1$.
3. $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \Phi_L(x) = 0$ for all $x \in B$ with $v(q(x)) \leq v(L^*) - 1$.

**Proof.** The first part follows from that $q(u) = q'(u_1, u_2)$ is an invariant for an orbit $x = (\gamma, u_1, u_2) \in B$, and that $v(L) = v(L_0, L_0^*)$. By definition, $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \Phi_L$ is given by

$$\text{vol}(L) \text{Orb}(1_{U(L)} \times 1_{L^*}, -) - \text{vol}(L_0 \times (L_0^*)^*) \gamma_V \text{Orb}(1_{S(L_0, L_0^*)} \times 1_{L_0} \times 1_{L_0^*}, -)$$

By Remark 2.2 we have $\omega_{L^*} = \gamma_V \omega_L$. And $\text{vol}(L) = \text{vol}(L_0 \times (L_0^*)^*) = |L^*/L|^{1/2}$, hence $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \Phi_L = \text{vol}(L) \Phi_L$. \hfill $\square$
4.2. Proof of TC assuming inductions on boundary cases.

**Theorem 4.6.** Let $L$ be a rank $n$ maximal parahoric hermitian $O_F$-lattice i.e., $v(L) + v(L^\vee) \geq -1$. Then the semi-Lie transfer Conjecture \[\text{Conj. 2.1}\] for $L$ holds.

**Proof.** We do induction on the rank of $L$. The case $n = 1$ is standard and works for any hermitian lattice $L$ (see also the computation in section 10.1). Now assume that Conjecture \[\text{Conj. 2.1}\] hold for all maximal parahoric lattices of rank $\leq n - 1$.

As $v(L) + v(L^\vee) \geq -1$, by Proposition 4.5 and Proposition 4.4 to show the conjecture for $L$ we just need to prove $\Phi_L(x) = 0$ if $v(q(x)) = v(L)$, and $\Phi_{L^\vee}(x) = 0$ if $v(q(x)) = v(L^\vee)$.

By symmetry, we only need to deal with $L$. Let $e \in L$ be a vector with maximal length i.e., $v((e,e)_V) = v(L)$. We will show TC for $L$ in the boundary case $v(q(x)) = v(L)$ can be reduced to the transfer conjecture for the pair $(L,e)$ in the following three subsections below.

By Proposition 3.26, the group version conjecture for $(L,e)$ is equivalent to the semi-Lie version conjecture for the rank $n - 1$ lattice $L^b$, hence is true by induction. The result follows. \[\square\]

4.3. Integral transitivity on maximal length vectors. Now we show two integral transitivity lemmas for the purpose of induction on boundary cases. Recall that $F/F_0$ is unramified, the norm map is surjective on units i.e., $\text{Nm}_{F/F_0}(O_F^x) = O_F^x$.

**Lemma 4.7.** For any $O_F$-hermitian lattice $L$, $v(L) = \min_{x \in L} v_F((x,x)) = \min_{x,y \in L} v_F((x,y))$.

**Proof.** The result follows once we can find an orthogonal basis of $L$ by splitting. If $p > 2$, there still exists an orthogonal basis of $L$ by \[\cite{20}\] Section 7.

Let $L$ be a $O_F$-hermitian lattice. Let $e \in L$ be a vector with maximal length i.e., $v((e,e)) = v(L)$. By Lemma 4.7 we have

$$\frac{(x,e)}{(e,e)} \in O_F$$

for all $x \in L$. Hence there is an orthogonal decomposition $L = L^b \oplus O_F e$ for $L^b := \{x \in L | (x,e) = 0\}$.

**Lemma 4.8.** (Integral transitivity) Choose a vector $x_0 \in L$ with maximal length. Then the compact open subgroup $U(L)$ of $U(V)$ acts transitively on the sphere $S_{x_0} = \{x \in L | (x,x) = (x_0,x_0)\}$.

**Proof.** For any $x \in S_{x_0}$, we get a splitting $L = L_1 \oplus O_F x_0$. Choose a maximal vector of $L_1$ and keep doing splitting, finally we extend $x$ to an orthonormal basis $\{e_1, \ldots, e_{n-1}\}$ of $L$. For any other $x' \in S_{x_0}$, we get another orthonormal basis $\{e_{i_1}', \ldots, e_{i_{n-1}}', x'\}$ of $L$. Multiply $e_i$ by units in $O_F^x$ and rearrange the order, we can assume that $(e_{i}, e_i) = (e_{i}', e_{i}')$, $1 \leq i \leq n - 1$. Then the endomorphism $h \in \text{GL}(V)$ sending $e_i$ to $e_i'$ and $x$ to $x'$ is inside the compact open subgroup $U(L)$. The result follows. \[\square\]

**Remark 4.9.** We don’t have integral transitivity on “smaller spheres”: consider the natural action of $O_F^x$ acts on $\{(x,y) \in O_F \oplus O_F | yx = t\}$, then the number of orbits is $v_F(t) + 1$. It is interesting to classify orbits on “smaller spheres” and consider similar problems for ramified $F/F_0$.

Now let $L_1, L_2$ be two lattices inside a $F_0$-vector space $V_0$. We say a pair $(u_1, u_2) \in L_1 \times L_2^\circ$ of maximal length if we have $v(u_2 u_1) = v(L_1, L_2) := \min_{x \in L_1, y \in L_2^\circ} v_F(y(x))$.

**Lemma 4.10.** Choose a pair $(u_1, u_2) \in L_1 \times L_2^\circ$ of maximal length. Let $V_0^b = \text{Ker}(u_2)$. Then

- $V_0 = V_0^b \oplus F_0 u_1$;
- $L_1 = L_1^0 \oplus O_F u_1$ and $L_2 = L_2^0 \oplus O_F \varpi^{-v(L_1,L_2)} u_1$, where $(-)^b$ means the projection to $V_0^b$;
- $L_2^0 = (L_2^0)^* \oplus O_F u_2$.

**Proof.** For any $x \in V_0$, we have

$$x = (x - \frac{u_2(x)}{u_2(u_1)} u_1) + \frac{u_2(x)}{u_2(u_1)} u_1 \in V_0^b \oplus F_0 u_1$$

and the first property follows.
Now set $k = -v(L_1, L_2)$. For any $f_2 \in L_2^*$, we have $f_2(\omega^k u_1) \in \omega^{k+v(L_1, L_2)} O_{F_0} = O_{F_0}$, hence $\omega^k u_1 \in L_2$. For $x \in L_2$, there exists $a \in O_{F_0}$ such that $u_2 x - a \omega^k u_1 = 0$. So we have the decomposition $L_2 = L_2^* \oplus O_{F_0} \omega^k u_1$. Taking its dual, we see $L_2^* = L_2^* \oplus O_{F_0} u_2$.

**Lemma 4.11.** (Integral transitivity) Choose a pair $(u_1, u_2) \in L_1 \times L_2^*$ of maximal length. Then the compact open subgroup $S(L_1, L_2)$ of $GL(V_0)$ acts transitively on the sphere

$$S_{u_1, u_2} = \{(x_1, x_2) \in L_1 \times L_2^* | u_2 x_1 u_1 = x_2 x_1 \}.$$ 

**Proof.** Apply the above Lemma 4.10 several times to extend $u_1$ to a basis $\{ e_i \}$ of $L_0$ (hence of $V_0$) such that $u_2$ is a multiple of the dual basis of $u_1$ and $L_2$ has a basis $\{ \omega^k e_i^* \}$. Similarly, we extend $(x_1, x_2)$ to another basis $\{ e'_i \}$ of $L_0$. Then the automorphism $h$ sending the basis $\{ e'_i \}$ to $\{ e_i \}$ lies in $S(L_1, L_2)$ and we have $h^{-1} u_1 = x_1, u_2 h = x_2$. \qed

4.4. **Induction on boundary cases: unitary groups.** Recall that $e$ is a fixed vector in $L$ of maximal length, and we have a decomposition $L = L' \oplus O_F e$. On the unitary side, we now show orbit integrals for $L$ in the boundary case $v(q(x)) = v(L)$ can be reduced to orbit integrals for $(L, e)$. Consider $y = (g_0, u_0) \in [U(V) \times V]_{\text{res}}(F_0)$ such that $v((u_0, u_0)_V) = v(L)$. Conjugate $(g_0, u_0)$ by an element in $U(V)$, we can assume that $u_0 \in L_0 \subseteq V_0$. By Lemma 4.8 we can even assume $u_0 = e$.

**Proposition 4.12.** Assume that $u_0 \in L_0$ and $v((u_0, u_0)_V) = v(L)$, we have

$$\text{Orb}((g_0, u_0), 1_{U(L)} \times 1_L) = \text{Orb}(g_0, 1_{U(L)})$$

where the right hand side is group version orbit integral for the pair $(L, u_0)$.

**Proof.** Let $h \in U(V)$ be any element with $1_L(h^{-1} u_0) \neq 0$. By Lemma 4.8 there exists $h_1 \in U(L)$ such that

$$h^{-1} u_0 = h_1 u_0.$$ 

Hence the support of such $h$ is in $U(L) U(V_0)$. Recall that the volume of $U(L)$ is normalized to be 1. As $1_{U(L)} \times 1_L$ is invariant under $U(L)$, we see

$$\text{Orb}((g_0, u_0), 1_{U(L)} \times 1_L) = \int_{h \in U(V_0)} 1_{U(L)}(h, g_0) 1_L(h^{-1} u_0) dh$$

$$= \int_{h \in U(V_0)} 1_{U(L)}(h, g_0) 1_L(u_0) dh = \text{Orb}(g_0, 1_{U(L)}).$$

The result follows. \qed

4.5. **Induction on boundary cases: symmetric spaces.** Now we show on the general linear side, orbit integrals for $L$ in the boundary case $v(q(x)) = v(L)$ can be reduced to orbit integrals for $(L, e)$. Consider $x = (\gamma, u_1, u_2) \in [S(V_0) \times V_0 \times V_0]_{\text{res}}(F_0)$ such that $v((u_1, u_2)_V) = v(L, L_0^*) = v(L)$. Conjugate $(\gamma, u_1, u_2)$ by an element in $GL(V_0)$, we can assume that $u_1 \in L_0, u_2 \in (L_0^*)^*$. By Lemma 4.8 we can even assume $(u_1, u_2) = (e, (u_2 u_1) e^*)$.

**Proposition 4.13.** With above assumption, we have an equality for all $s \in \mathbb{C}$:

$$\text{Orb}((\gamma, u_1, u_2), 1_{S(L_0, L_0^*)} \times 1_{L_0} \times 1_{L_0^*}, s) = \text{Orb}(\gamma, 1_{S(L_0, L_0^*)}, s),$$

where the right hand side is the group version orbit integral for the pair $(L, u_1)$.

**Proof.** Let $h \in GL(V_0)$ be any element with $1_{L_0}(h^{-1} u_1) 1_{L_0^*}, (u_2 h) \neq 0$. By Lemma 4.11 there exists $h_1 \in GL(L_0, L_0^*)$ such that

$$h^{-1} u_1 = h_1 u_1, u_2 h = u_2 h_1.$$ 

Hence the support of such $h$ is in $GL(L_1, L_2) GL(V_0)$. Recall that the volume of $GL(L_1, L_2)$ is normalized to be 1. As $1_{S(L_0, L_0^*)} \times 1_{L_0} \times 1_{L_0^*}$ is invariant under $GL(L_1, L_2)$ and $\eta(\det(GL(L_1, L_2))) \in \eta(O_{F_0}) = \{1\}$, we see

$$\text{Orb}((\gamma, u_1, u_2), 1_{S(L_0, L_0^*)} \times 1_{L_0} \times 1_{L_0^*}, s) =$$

$$\omega(\gamma, u_1, u_2) \int_{h \in GL(V_0)} 1_{S(L_0, L_0^*)}(h, \gamma) 1_{L_0}(h^{-1} u_1) 1_{L_0^*}, (u_2 h) \eta(h) |h|^s dh$$
\[= \omega(\gamma) \int_{h \in \text{GL}(V_h)} 1_{S(L_0, L^\nu)}(h, \gamma) \eta(h)|h|^s dh = \text{Orb}(\gamma, 1_{S(L_0, L^\nu)}, s).\]

Here the right hand side is group version orbit integral for the pair \((L, e = u_1)\). Note the transfer factor \(\omega(\gamma, u_1, u_2)\) for \(L\) and \(\omega(\gamma)\) for \((L, u_1)\) agree by Definition \(2.2\) and \(2.7\).

Now we finish the proof of induction on the boundary case \(v(q(x)) = v(L)\). Assume that \(y = (g_0, e)\) matches \(x = (\gamma, e, (u_2u_1)e^*)\), where \(e^*\) is the dual basis of \(e\) under the chosen basis of \(V\). Then there exists \(h \in \text{GL}(V)\) such that

\[h.(g_0, e) = (\gamma, e, (u_2u_1)e^*)\]

inside \(\text{End}(V) \times V \times V^*\). In particular \(h \in \text{GL}(V^*)\), and we see \(g\) matches \(\gamma\). Hence from TC for \((L, e)\) which holds by induction, we see

\[\text{Orb}(g_0, u_0, 1_{U(L)} \times 1_L) = \text{Orb}(g_0, 1_{U(L)}) \]

\[= \text{Orb}(\gamma, 1_{S(L_0, L^\nu)}) = \text{Orb}(\gamma, u_1, u_2), 1_{S(L_0, L^\nu)} \times 1_L \times 1_{L^\nu^*}.\]

Therefore, on two boundary cases (for \(L\) and \(L^\nu\)), the semi-Lie TC reduces to two lower rank group TCs. Hence \(\Phi_L\) (see \(4.6\)) satisfies the conditions in Corollary \(4.4\) and Theorem \(4.1\) is proved.

Now we turn to arithmetic transfer conjectures at maximal parahoric levels.

### Part 2. Arithmetic transfer conjectures

Let \(F/F_0\) be an unramified quadratic extension of \(p\)-adic local fields \((p > 2)\). Fix an embedding \(O_F \to O_{F_0}\) and a uniformizer \(\varpi\) of \(F_0\). Denote the residue field of \(F_0\) by \(\mathbb{F}_q\) (resp. \(\mathbb{F}\)). Denote \(\sigma = (\varpi)\) the non-trivial Galois involution in \(\text{Gal}(F/F_0)\).

Let \(V\) be a \(F/F_0\)-hermitian space of dimension \(n \geq 1\). Choose a vertex lattice \(L\) in \(V\) (i.e., \(L \leq L^\nu \leq \varpi^{-1}L\)) of type \(\kappa(L) = t \in [0, n]\).

#### 5. Special cycles and modified derived fixed points

In this section, we study the “relative” Rapoport–Zink space \(N\) for the lattice \(L\) and two kinds of Kulda–Rapoport cycles on it introduced in [5] Section 2. We use the dual isomorphism \(\lambda : N \to N^\circ\) as the tool of “taking dual” on geometric side.

Then we introduce the formal Balloon–Ground stratification on the special fiber \(N_F = N^\bullet \cup N^\circ\). We resolve the singularity of \(N \times N\) by blowing up along \(N^\circ \times N^\circ\), which is an arithmetic analog of the Atiyah flop [1]. This allows a (derived) modification of the (unbounded) derived fixed point TCs. Hence \(\Phi_L\) satisfies the conditions in Corollary 4.4 and Theorem 4.1 is proved.

Now we turn to arithmetic transfer conjectures at maximal parahoric levels.

For any \(\text{Spf} O_{F_0}\)-scheme \(S\), a triple \((X, \iota, \lambda)\) of dimension \(n\) and type \(t\) (and signature \((1, n-1)\)) over \(S\) is the following data:

- \(X\) is a strict formal \(O_{F_0}\)-module over \(S\) of relative height \(2n\) and dimension \(n\). Here strictness means the induced action of \(O_{F_0}\) on \(X\) is via the structure morphism \(O_{F_0} \to OS\).
- \(\iota : O_F \to \text{End}(X)\) is an action of \(O_F\) on \(X\) that extends the action of \(O_{F_0}\). We require that the Kottwitz condition of signature \((1, n-1)\) holds for all \(a \in O_F\):
  \[\text{char}(\iota(a) \mid X) = (T - a)(T - \overline{a})^{n-1} \in OS[T],\]
  \[(5.1)\]
- \(\lambda\) is a polarization on \(X\), which is \(O_{F}/O_{F_0}\) semi-linear in the sense that the Rosati involution \(\text{Ros}_\lambda\) induces the non-trivial involution \((\lambda) \in \text{Gal}(F/F_0)\) on \(\iota : O_F \to \text{End}(X)\).
- We require that the finite flat group scheme \(\text{Ker} \lambda\) over \(S\) lies in \(X[\varpi]\) and is of order \(q^{2t}\).

An isomorphism \((X_1, \iota_1, \lambda_1) \sim (X_2, \iota_2, \lambda_2)\) between two such triples is an \(O_F\)-linear isomorphism \(\varphi : X_1 \to X_2\) such that \(\varphi^*(\lambda_2) = \lambda_1\).

Up to \(O_F\)-linear quasi-isogeny compatible with the polarization, there exists a unique such triple \((X, \iota, \lambda)\) over \(F\). Fix one choice of \((X, \iota, \lambda)\) as the framing object.

**Definition 5.1.** The Rapoport–Zink space for \(L\) is the functor

\[N_{U(L)} = N_n^{[t]} \to \text{Spf} O_{F_0}\]

sending \(S\) to the set of isomorphism classes of tuples \((X, \iota, \lambda, \rho)\), where
• $(X, t, \lambda)$ is a triple of dimension $n$ and type $t$ over $S$.
• $\rho : X \times_S \mathcal{F} \to \mathbb{X} \times \mathcal{F}$ an $O_F$-linear quasi-isogeny of height 0 over the reduction $\mathcal{F} := S \times_{O_F} \mathbb{F}$.
• $\rho^*(\lambda_X, \mathcal{F}) = \lambda_{\mathcal{F}}$.

From [17 Theorem 2.16], the functor $\mathcal{N}_{U(L)} = \mathcal{N}_{n}^{[\ell]}$ is representable by a formal scheme locally formally of finite type over $\text{Spf}O_{F_0}$ of relative dimension $n - 1$.

**Example 5.2.** Let $\mathbb{E}$ be a formal $O_{F_0}$-module over $\mathbb{F}$ of relative height 2 and dimension 1. Complete it into a triple $(\mathbb{E}, \iota_{\mathbb{E}}, \lambda_{\mathbb{E}})$ of signature $(0, 1)$, dimension 1 and type 0 over $\mathbb{F}$, and still denote it by $\mathbb{E}$. Changing the $O_F$-action on $\mathbb{E}$ by Galois involution $(-)$, we obtain a framing object $(\mathbb{E}, \iota_{\mathbb{E}} \circ \sigma, \lambda_{\mathbb{E}})$ for $\mathcal{N}_{1}^{[\ell]}$, which we denote by $\mathbb{E}$.

The theory of canonical lifting gives an isomorphism $\mathcal{N}_{1}^{[\ell]} \simeq \text{Spf}O_{F_0}$, where the universal triple $\mathcal{E}$ over $\mathcal{N}_{1}^{[\ell]} \simeq \text{Spf}O_{F_0}$ is the canonical lifting of $\mathbb{E}$. Denote by $\mathcal{E}$ the triple over $O_{F_0}$ obtained from the canonical lifting of $\mathbb{E}$.

The space of special quasi-homomorphisms is the $F$-vector space $\mathbb{V} = \text{Hom}_F^\sigma(\mathbb{E}, \mathbb{X})$ equipped with the hermitian form $(x, y)_{\mathbb{V}} := \lambda^{-1}_{\mathbb{X}} \circ y^\sigma \circ \lambda_{\mathbb{X}} \circ x \in \text{Hom}_F^\sigma(\mathcal{E}, \mathcal{E}) \cong F$. (5.2)

The hermitian space $\mathbb{V}$ is split, if and only if the type $t$ is odd.

The automorphism group of quasi-isogenies $\text{Aut}^\circ(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ acts on $\mathbb{V}$ naturally. By Dieudonné theory, we have $\text{Aut}^\circ(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}}) \cong \text{U}(\mathbb{V})(F_0)$ and the action of $\text{U}(\mathbb{V})(F_0)$ on $\mathcal{N}_{U(L)}$ is given by

$$g.(X, t, \lambda, \rho) = (X, t, \lambda, g \circ \rho).$$

From now on, we fix $n$ and $t$ and write $\mathbb{N} = \mathcal{N}_{U(L)} = \mathcal{N}_{n}^{[\ell]}$ for simplicity.

5.1. **The dual isomorphism** $\lambda_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}^\vee$. By taking dual, the framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ gives a framing object $(\mathbb{X}^\vee, \iota_{\mathbb{X}^\vee}, \lambda_{\mathbb{X}^\vee})$ of dimension $n$ and type $n - t$, where $\lambda_{\mathbb{X}^\vee} : \mathbb{X}^\vee \to \mathbb{X}$ is the dual polarization of $\lambda_{\mathbb{X}}$ such that $\lambda_{\mathbb{X}^\vee} \circ \lambda_{\mathbb{X}} = [\varpi] : \mathbb{X} \to \mathbb{X}$.

Let $\mathbb{V}^\vee := \text{Hom}_F^\sigma(\mathbb{E}, \mathbb{X}^\vee)$ be the space of special quasi-homomorphisms for $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$. The polarization $\lambda_{\mathbb{X}}$ induces a $F$-linear map

$$\lambda_{\mathbb{V}} : \mathbb{V} \to \mathbb{V}^\vee, \quad x \mapsto \lambda_{\mathbb{X}} \circ x.$$ (5.4)

**Proposition 5.3.** The map $\lambda_{\mathbb{V}} : \mathbb{V} \to \mathbb{V}^\vee$ is a bijection. For any $x, y \in \mathbb{V}$, we have

$$(\lambda_{\mathbb{V}} \circ x, \lambda_{\mathbb{V}} \circ y)_{\mathbb{V}^\vee} = -\varpi(x, y)_{\mathbb{V}}.$$ (5.5)

**Proof.** Any polarization is an anti-symmetric isogeny. Unraveling the definitions we obtain the comparison between hermitian forms on $\mathbb{V}$ and $\mathbb{V}^\vee$. $\square$

Let $\mathbb{V}^\vee$ be the hermitian space with same underlying $F$-vector space as $\mathbb{V}$, but equipped with the hermitian form $(x, y)_{\mathbb{V}^\vee} := -\varpi(x, y)_{\mathbb{V}}$.

Denote by $\lambda_{\mathbb{V}} : \mathbb{V} \cong \mathbb{V}^\vee$ the identification of underlying vector spaces. Then the lattice $\lambda_{\mathbb{V}} \circ L^\vee$ is a vertex lattice in $V^\vee$ of type $n - t$.

The dual Rapoport–Zink space of $\mathbb{N}$ is the Rapoport–Zink space $\mathbb{N}^\vee := \mathcal{N}_{U(\lambda_{\mathbb{V}} \circ L^\vee)} = \mathcal{N}_{n}^{[n-t]}$ for the framing object $(\mathbb{X}^\vee, \iota_{\mathbb{X}^\vee}, \lambda_{\mathbb{X}^\vee})$. We have the dual isomorphism

$$\lambda_{\mathbb{N}^\vee} : \mathbb{N}^\vee \to \mathbb{N}^\vee$$

$$(X, t_{\mathbb{X}}, \lambda_{\mathbb{X}}, \rho_{\mathbb{X}}) \mapsto (X^\vee, t_{\mathbb{X}^\vee}, \lambda_{\mathbb{X}^\vee}, \rho_{\mathbb{X}^\vee})$$

where $\lambda_{\mathbb{X}^\vee}$ is the dual polarization of $\lambda_{\mathbb{X}}$, and $\rho_{\mathbb{X}^\vee}$ is the quasi-isogeny $((\rho_{\mathbb{X}})^\vee)^{-1} : \mathbb{X}^\vee \times_S \mathcal{F} \to \mathbb{X}^\vee \times_S \mathcal{F}$. By definition, we have

$$(\mathbb{N}^\vee)^\vee = \mathbb{N}, \quad \lambda_{\mathbb{N}^\vee} \circ \lambda_{\mathbb{N}} = id_{\mathbb{N}}, \quad \lambda_{\mathbb{N}} \circ \lambda_{\mathbb{N}^\vee} = id_{\mathbb{N}^\vee}.$$
5.2. Kudla–Rapoport cycles. In particular, the dual of $\mathcal{E}$ is a framing object $\mathcal{E}^\vee$ for $\mathcal{N}^{[1]}$. For $i \in \{0, 1\}$, denote by $\mathcal{E}^{[i]}$ the universal triple over $\mathcal{N}^{[i]} \cong \text{Spf} \ O_{F_0}$, and by $\mathcal{E}^{[i]}$ the triple over $O_{F_0}$ obtained from $\mathcal{E}^{[i]}$ via changing the $O_F$-action on $\mathcal{E}^{[i]}$ by the Galois involution $\sigma$. So $\mathcal{E} = \mathcal{E}^{[0]}$.

**Definition 5.4.** For any non-zero $u \in \mathcal{V}$,

(1) The Kudla–Rapoport cycle $Z(u) \to N$ is the closed formal subscheme of $N$ sending each $\text{Spf} \ O_{F_0}$-scheme $S$ to the subset $(X, \iota_X, \lambda_X, \rho_X) \in N(S)$ such that

$$ (\rho_X)^{-1} \circ u : \mathcal{E} \times_{\mathcal{F}} \mathcal{S} \to X \times_S \mathcal{S} $$

lifts to a homomorphism from $\mathcal{E}$ to $X$ over $S$.

(2) The Kudla–Rapoport cycle $Y(u) \to N$ is the closed formal subscheme of $N$ sending each $\text{Spf} \ O_{F_0}$-scheme $S$ to the subset $(X, \iota_X, \lambda_X, \rho_X) \in N(S)$ such that

$$ (\rho_X^\vee)^{-1} \circ \lambda_X \circ u : \mathcal{E} \times_{\mathcal{F}} \mathcal{S} \to \mathcal{V} \times_S \mathcal{S} $$

lifts to a homomorphism from $\mathcal{E}$ to $\mathcal{V}$ over $S$.

**Proposition 5.5.** The functors $Z(u)$ and $Y(u)$ are representable by Cartier divisors of $N$.

**Proof.** See [5, Prop. 5.9].

**Remark 5.6.** Unlike the case $t = 0$ [27], in general $Z(u)$ and $Y(u)$ may not be relative Cartier divisors on $N$.

**Proposition 5.7.** (1) We have natural inclusions of formal schemes:

$$ Z(u) \hookrightarrow Y(u) \hookrightarrow Z(\varpi u). \quad (5.6) $$

(2) Under the dual isomorphism $\lambda_X : N \to N^\vee$, we have

$$ \lambda_X(Z(u)) = Z(\lambda_X \circ u), \quad \lambda_X(Y(u)) = Y(\lambda_X \circ \varpi u). \quad (5.7) $$

(3) $Z(u)$ is empty unless $v((u, u)_{\mathcal{V}}) \geq 0$; $Y(u)$ is empty unless $v((u, u)_{\mathcal{V}}) \geq -1$.

**Proof.** We use the moduli interpretation. If $(X, \iota_X, \lambda_X, \rho_X) \in Z(u)(S)$, then $\lambda_X \circ \rho_X^{-1} \circ u : E \to X^\vee \times_S \mathcal{S}$ lifts to a homomorphism from $\mathcal{E}$ to $X^\vee$ over $S$. By compatibility of $\lambda_X$ and $\lambda_X$ under the quasi-isogeny $\rho_X$, we have $\lambda_X \circ \rho_X^{-1} \circ u = \rho_X^{-1} \circ \lambda_X \circ u$ hence $(X, \iota_X, \lambda_X, \rho_X) \in Y(u)(S)$. Similarly, from $\lambda_X \circ \rho_X^{-1} \circ u = [\lambda_X(\varpi)] \circ \rho_X^{-1} \circ u$ we see that $Y(u) \subseteq Z(\varpi u)$. The second property holds by the definition of $Y(u)$ and $\lambda_X$.

For the last property, if $(X, \iota_X, \lambda_X, \rho_X) \in Z(u)(S)$, then

$$ \lambda_X^{-1} \circ (\rho_X^{-1} \circ u)^\vee \circ \lambda_X \circ (\rho_X^{-1} \circ u) \in \text{Hom}_{\mathcal{O}_{F_0}}(E \times_{\mathcal{F}} \mathcal{S}, E \times_{\mathcal{F}} \mathcal{S}) \simeq O_F $$

which exactly gives $(u, u)_{\mathcal{V}} \in O_F$ by the compatibility $\lambda_X \circ \rho_X^{-1} \circ u = \rho_X^{-1} \circ \lambda_X \circ u$. Using the dual isomorphism $\lambda : N \to N^\vee$ and Proposition 5.5, the assertion for $Y(u)$ follows by duality. 

5.3. Formal Balloon-Ground stratification.

**Proposition 5.8.** The formal scheme $N_{U(L)} \to \text{Spf} \ O_{F_0}$ is formally smooth if $t = 0$ or $n$, and is of strictly semi-stable reduction if $0 < t < n$. More precisely, for any closed point $x$ of $N$, the completed local ring of $N$ at $x$ is either isomorphic to

$$ O_{F_0}[[x_0, y_0, t_1, \ldots, t_{n-1}]]/(x_0y_0 - \varpi) $$

or isomorphic to

$$ O_{F_0}[[t_0, t_1, \ldots, t_{n-1}]]. $$

**Proof.** This follows from the local model computation as in [5, Prop 3.33], which reduces to the $\text{GL}_n$ case in [10].

For any $F$-scheme $S$ and any point $(X, \iota, \lambda, \rho) \in N(S)$, we have a short exact sequence of locally free $\mathcal{O}_S$-modules

$$ 0 \to \omega_{X^\vee} \to \mathbb{D}(X)(S) \to \text{Lie}X \to 0. \quad (5.8) $$

where $\mathbb{D}(X)$ is the covariant relative Dieudonné crystal of $X$ over $S$ and $\omega_{X^\vee} = (\text{Lie}X^\vee)^\vee$.

We have a natural identification $\mathbb{D}(X)(S)^\vee \cong \mathbb{D}(X^\vee)(S)$ via the perfect pairing

$$ (-, -) : \mathbb{D}(X)(S) \times \mathbb{D}(X^\vee)(S) \to \mathcal{O}_S. \quad (5.9) $$
The induced morphism $D(\lambda) : D(X)(S) \to D(X^\vee)(S)$ produces an antisymmetric bilinear form
\[
(\cdot, \cdot)_\lambda : D(X)(S) \times D(X)(S) \to O_S.
\] (5.10)
by $(x, y)_\lambda := (x, D(\lambda)(y))$.

The $O_F$-action on $X$ induces a decomposition
\[
\mathbb{D}(X) = \mathbb{D}(X)_0 \oplus \mathbb{D}(X)_1
\] (5.11)
such that $\mathbb{D}(X)_0$ and $\mathbb{D}(X)_1$ are isotropic under $(-, -)_\lambda$. From the Kottwitz signature condition $(1, n-1)$, we see $(\omega_{X^\vee})_0$ (resp. $(\omega_{X^\vee})_1$) is locally free of rank $n-1$ (resp. 1). Consider the induced maps between line bundles
\[
\text{Lie} \lambda : (\text{Lie}X)_0 \to (\text{Lie}X^\vee)_1, \quad \text{Lie} \lambda^\vee : (\text{Lie}X^\vee)_1 \to (\text{Lie}X)_0.
\]
where $\lambda^\vee : X^\vee \to X$ is the dual polarization of $\lambda$.

We introduce the formal Balloon–Ground stratification on the special fiber $N_\mathbb{F}$:
- The formal Balloon stratum $N^\circ$ is the vanishing locus of universal Lie $\lambda$.
- The formal Ground stratum $N^\bullet$ is the vanishing locus of universal Lie $\lambda^\vee$.
- The formal linking stratum $N^\dagger$ is the intersection $N^\circ \cap N^\bullet$.

As $\text{Lie} \lambda^\vee \circ \text{Lie} \lambda = \varpi$, we have the stratification $N_\mathbb{F} = N^\bullet \cup N^\circ$.

**Proposition 5.9.** The formal subschemes $N^\circ$ and $N^\bullet$ are Cartier divisors of $N$ and are formally smooth over $\mathbb{F}$. The structure map $N - N^\dagger \to O_F$ is formally smooth. And for any closed point $x$ of $N^\dagger$, the completed local ring of $N$ at $x$ is isomorphic to
\[
O_F[[x_0, y_0, t_1, \ldots, t_{n-1}]]/(x_0y_0 - \varpi).
\]

**Proof.** By proposition 5.8, $N$ is regular and $\varpi$ is locally non-zero. As vanishing locus of sections of line bundles, $N^\circ \subseteq N_\mathbb{F}$ and $N^\bullet \subseteq N_\mathbb{F}$ are Weil divisors of $N$. As $\text{Lie} \lambda \circ \text{Lie} \lambda = \varpi$, the local generators of $N^\circ$ and $N^\bullet$ are also locally non-zero, hence $N^\circ$ and $N^\bullet$ are in fact Cartier divisors of $N$.

To show formally smoothness, we pass to completed local rings of $N$ at a closed point $x$. As $\text{Lie} \lambda \circ \text{Lie} \lambda = \varpi$, the local equation $x^\varpi$ (resp. $x^\dagger$) of $N^\circ$ (resp. $N^\bullet$) at any can be chosen such that $x^\varpi x^\dagger = \varpi$. By the description of completed local rings in Proposition 5.8, we see that $x^\varpi$ (resp. $x^\dagger$) can be arranged as local generators $x^0$ (resp. $y^0$) in 5.8 after applying an automorphism on $\hat{O}_N$.

To see the structure map $N - N^\dagger \to O_F$ is formally smooth, we apply Grothendieck–Messing theory. We only need to deform the Hodge filtration. As $(\omega_X)_1 = (\text{Lie}X^\vee)_1$ and $(\omega_{X^\vee})_1 = (\text{Lie}X)_1$ are orthogonal complements under the perfect pairing $(-, -)$ hence determine each other, we only need to deform the quotient lines $(\text{Lie}X)_0$ and $(\text{Lie}X^\vee)_1$ in the Hodge filtration. If $x \in N - N^\dagger$, then one of $(\text{Lie}X)_0$ and $(\text{Lie}X^\vee)_1$ determines the other using the non-zero section $\text{Lie} \lambda$ or $\text{Lie} \lambda^\vee$, so we only need to deform one quotient line, which forms a unobstructed deformation problem. \hfill $\square$

**Definition 5.10.** Let $f : \hat{N} \times \hat{N} \to \hat{N} \times \hat{N}$ be the (formal) blow up of $N \times_{O_F} N$ along the Weil divisor $N^\circ \times_{\mathbb{F}} N^\circ$.

**Theorem 5.11.** (i) The formal scheme $\hat{N} \times \hat{N}$ is of strictly semi-stable reduction over Spf $O_{F_0}$ if $0 < t < n$. In particular, $\hat{N} \times \hat{N}$ is regular.

(ii) The map $f : \hat{N} \times \hat{N} \to \hat{N} \times \hat{N}$ is an isomorphism away from $N^\dagger \times_{\mathbb{F}} N^\dagger$. The geometric fiber of $f$ is either a point or a projective line $\mathbb{P}^1$. In particular $f$ is a small resolution.

**Proof.** By formal smoothness of $N - N^\dagger \to O_F$, we know that $N \times N$ is regular away from $N \times N^\dagger$ or $N^\dagger \times N$. Hence the center $N^\circ \times_{\mathbb{F}} N^\circ$ is a Cartier divisor of $N \times N$ away from $N^\dagger \times_{\mathbb{F}} N^\dagger$, where blow up changes nothing.

Blow up commutes with flat base change in particular open immersions, so we can pass to completed local rings. By proposition 5.9 we can choose local generators of $N^\circ$ (resp. $N^\bullet$) as $x^0$ (resp. $y^0$) in the isomorphism
\[
\hat{O}_N \simeq O_{F_0}[[x_0, y_0, t_1, \ldots, t_{n-1}]]/(x_0y_0 - \varpi).
\]
for any closed point \( x \in \mathcal{N}^1 \). By direct computation, we see the blow up of

\[
O_\mathcal{F}_0 [[x_0, y_0, x_1, y_1, t_1, \ldots, t_{n-1}, s_1, \ldots, s_{n-1}]]/(x_0y_0 - \zeta, x_1y_1 - \zeta)
\]

along \((x_0, x_1)\) is of strictly semi-stable reduction over \(O_\mathcal{F}_0\). As the ideal \((x_0, x_1)\) is generated by two elements, the fiber of \( f \) at \( x \in \mathcal{N}^1 \times \mathbb{P}^1 \) can be embedded into \( \mathbb{P}_2^1 \). By local computations, it is isomorphic \( \mathbb{P}_2^1 \). Hence the resolution \( f \) has only \( \dim \leq 1 \) fibers and is an isomorphism outside the codim \( \geq 3 \) locus \( \mathcal{N}^1 \times \mathbb{P}^1 \mathcal{N}^o \), hence a small resolution.

**Remark 5.12.** Consider the quadratic cone \( X := V(x_0y_0 = x_1y_1) \subseteq \mathbb{A}^4_2 = \text{Spec} \mathbb{F}[x_0, y_0, x_1, y_1] \) which is singular at the origin. The resolution \( f \) above is similar to the classical Atiyah flop

\[
\mathbb{P}_2^1 \xrightarrow{f_{x_0x_1}} X \xrightarrow{f_{x_0y_1}} X \xrightarrow{f_{x_0y_1}} \mathbb{P}_2^1 .
\]

We always have \((\text{Lie}X_0^1, (\omega X_0^1)_1)_\lambda = 0 \in \mathcal{O}_S\).

Similarly, the space \( \mathcal{N}^o \subseteq \mathcal{N}_\mathcal{F} \) is defined by the condition

\[
(\mathbb{D}(X)(S)_1) \perp \subseteq (\omega X_0^1)_0.
\]

Here \((-)^\perp\) means the (right) orthogonal complement under \((-,-)_\lambda \) inside \(\mathbb{D}(X)(S)_0\).

**Proof.** We always have \((\omega X_0^1, (\omega X_0^1)_1)_\lambda = 0 \), so \(\mathbb{D}(X)(S)_0, (\omega X_0^1)_1)_\lambda = 0 \) is equivalent to \((\text{Lie}X_0^1, (\omega X_0^1)_1)_\lambda = 0 \ i.e., (\text{Lie}X_0^1, (\text{Lie}X_0^1))_\lambda = 0 \) i.e., \text{Lie}\lambda = 0.

On the other hand, \(\mathbb{D}(X)(S)_1\) is the same as the orthogonal complement of \(\text{Im}(\lambda))(\mathbb{D}(X)(S)_1) \subseteq \mathbb{D}(X^1)(S)_0\) under the perfect pairing \((-,-)_0\) between \(\mathbb{D}(X)(S)_0\) and \(\mathbb{D}(X^1)(S)_0\). By perfectness, we see the condition \((\mathbb{D}(X)(S)_1)^\perp \subseteq (\omega X_0^1)_0\) is equivalent to the following inclusion inside \(\mathbb{D}(X^1)(S)_0\):

\[
(\text{Lie}X_0^1)_0^\perp = (\omega X_0^1)_0 \subseteq \text{Im}(\lambda))(\mathbb{D}(X)(S)_1).
\]

As \( S \) is of characteristic \( p \), we have \(\text{Im}(\lambda))(\mathbb{D}(X)(S)_1) = \text{Ker}(\lambda)(\mathbb{D}(X^1))_{(\omega X_0^1)(S)_0}\) by \([32]\) Lem. 3.4.12. So the inclusion is the same as the condition that

\[
\mathbb{D}(\lambda^1)((\text{Lie}X_0^1))_0^\perp = 0
\]

which is exactly \(\text{Lie}\lambda^1 = 0\) i.e., \(\text{Lie}^1 = 0\).

**Remark 5.14.** We describe the induced Balloon-Ground stratification on the reduced locus \(\mathcal{N}^\text{red} \) using Bruhat–Tits strata in Section\([8]\). For example, the stratum \((\mathcal{N}^o)^\text{red}\) is defined by the condition \( A \subseteq A^\perp \) where \( A = M_0 \) is 0-part of the relative Dieudonne module of \( X \), and the dual is with respect to the bilinear form \((-,-)_0\) \([8, 6]\).

If \( t = 1 \), then \((\mathcal{N}^o)^\text{red}\) is the union of all type 0 Bruhat–Tits strata \(\mathcal{Y}((L^1)^\text{red}) \simeq \mathbb{P}^{n-1}_{L^1} \) in \(\mathcal{N}^\text{red} \), and \(\mathcal{N}^1 \subseteq (\mathcal{N}^o)^\text{red} \) is the disjoint union of Fermat hypersurfaces in \(\mathbb{P}^{n-1}_{L^o} \) of degree \( q + 1 \).
5.4. Modifying derived fixed points. We recall some well-known facts about blow-ups.

**Proposition 5.15.** Let \( Z \hookrightarrow X \) be a closed immersion of formal schemes that are locally formally of finite type over \( \mathcal{O}_{F_0} \), and \( \text{Bl}_Z X \to X \) be the blow up morphism of \( X \) along \( Z \). For any closed formal subscheme \( Y \to X \), denote by \( \widetilde{Y} \) the strict transform of \( Y \) in \( X \). Then

1. There is a natural isomorphism \( \widetilde{Y} \cong \text{Bl}_{\mathcal{O}_{\mathcal{X} \times \mathcal{Y}}} \mathcal{Y} \) over \( Y \). Here \( \mathcal{Z} \cap \mathcal{Y} \) is the fiber product of \( Y \) and \( \mathcal{Z} \) in \( X \).
2. Either \( \widetilde{Y} \) is empty, or \( \dim_{\mathcal{Y}} \widetilde{Y} = \dim_{\mathcal{Y}} Y \) for any closed point \( \widetilde{y} \) of \( \widetilde{Y} \) with image \( y \in Y \).
3. Blow up commutes with flat base change \( X' \to X \) of the base.

For a closed formal scheme \( Y \to \mathcal{N} \times \mathcal{N} \), write \( \mathcal{Y} \) its strict transform under the resolution \( 5.10 \).

**Remark 5.16.** For the embedding \( \delta_{(L,e)} : \mathcal{N}^\circ \hookrightarrow \mathcal{N} \) in Section \( 5.5 \), we have \( \mathcal{N}^\circ \cap \mathcal{N}^\circ = (\mathcal{N}^\circ)^\circ \). Hence the notation \( \mathcal{Y} \) is compatible for closed formal subscheme \( Y \hookrightarrow \mathcal{N} \times \mathcal{N} \).

**Proposition 5.17.** For any automorphism \( g \in U(\mathcal{V})(F_0) \), the strict transform of its graph \( \Gamma_g \hookrightarrow \mathcal{N} \times \mathcal{N} \) is an isomorphism \( \mathcal{Y}_g \cong \Gamma_g \).

**Proof.** The formal Balloon stratum \( \mathcal{N}^\circ \) is stable under \( g \) by moduli interpretation. So the pullback of \( \mathcal{N}^\circ \times_{\mathcal{F}} \mathcal{N}^\circ \) to \( \mathcal{Y}_g \) is isomorphic to \( \mathcal{N}^\circ \hookrightarrow \mathcal{N} \) via projection to the first factor \( \mathcal{N} \). As \( \Gamma_g \cong \mathcal{N} \) is regular, the Weil divisor \( i_g : \mathcal{N}^\circ \to \mathcal{N} \) is Cartier hence the blow up changes nothing.

Therefore, we have a natural lifting \( \Gamma_g = \mathcal{Y}_g \to \mathcal{N} \times \mathcal{N} \). Take \( g = \mathrm{id} \), we get a natural lifting of the diagonal embedding

\[ \hat{\Delta} : \mathcal{N} \to \mathcal{N} \times \mathcal{N} \].

**Definition 5.18.** For \( g \in U(\mathcal{V})(F_0),r,s \), the **fixed point locus** of \( g \) inside \( \mathcal{N} \) is the formal scheme \( \text{Fix}(g) := \mathcal{N} \cap_{\mathcal{N} \times \mathcal{N}} \Gamma_g \). The **modified fixed locus** of \( g \) is the fibered product

\[ \text{Fix}(g) := \Gamma_g \times_{\mathcal{N} \times \mathcal{Y}} \hat{\Delta} \]

in the category of formal schemes over \( \mathcal{O}_{F_0} \). As \( \Gamma_g \to \mathcal{N} \times \mathcal{N} \) is a closed immersion, \( \text{Fix}(g) \to \mathcal{N} = \mathcal{N} \) is a closed immersion. The **modified derived fixed locus** of \( g \) is the derived fibered product

\[ \widetilde{\text{Fix}} \xrightarrow{L} (g) \xrightarrow{\square} \Gamma_g \]

\[ \xrightarrow{\square} \]

\[ \mathcal{N} \xrightarrow{\hat{\Delta}} \mathcal{N} \times \mathcal{N} \]

viewed as an element in the \( K \)-group \( K_0(\mathcal{N})_{\text{Fix}(g)} \) (of \( \mathbb{Q} \)-coefficients) of coherent sheaves on \( \mathcal{N} \) with support in \( \text{Fix}(g) \).

From Theorem \( 5.11 \), \( f : \mathcal{N} \times_{\mathcal{F}} \mathcal{N} \to \mathcal{N} \times \mathcal{N} \) is an isomorphism away from \( \mathcal{N} \cap_{\mathcal{N} \times \mathcal{N}} \mathcal{N} \). So the inclusion of closed formal subscheme

\[ \text{Fix}(g) \subseteq \widetilde{\text{Fix}}(g) \]

is an isomorphism if \( \text{Fix}(g) \cap \mathcal{N} = \emptyset \).

5.5. **Embeddings.** Let \( L = L^\wedge \oplus \mathcal{O}_{\mathcal{F}e} \) be an orthogonal decomposition of vertex lattices where \( v_F((e,e)_V) = i \in \{0,1\} \). We have \( t(L^\wedge) = t(L) - i \). Choose a decomposition of framing objects:

\[ (X^\wedge_i,t^\wedge_i,\lambda^\wedge_i) = (X^\wedge_i,t^\wedge_i,\lambda^\wedge_i) \times (E^\wedge_i,t^\wedge_i,\lambda^\wedge_i). \]  

\[ (5.12) \]

Set \( \mathcal{N}^\circ := \mathcal{N}^\circ_{U(L^\wedge)} \) and \( \mathcal{N} := \mathcal{N}^\circ_{U(L)} \) with above framing objects. There is a natural embedding:

\[ \delta := \delta_{(L,e)} : \mathcal{N}^\circ \to \mathcal{N} \]

\[ (5.13) \]

\[ (X^\wedge_i,t^\wedge_i,\rho^\wedge_i) \mapsto (X^\wedge_i,t^\wedge_i,\lambda^\wedge_i) := (X^\wedge_i,t^\wedge_i,\lambda^\wedge_i) \times (E^\wedge_i,t^\wedge_i,\lambda^\wedge_i,\rho^\wedge_i). \]

Taking dual, we have decompositions of vertex lattices and framing objects:

\[ \lambda^\vee \circ L^\vee = \lambda^\vee \circ ((L^\wedge)^\vee) \oplus \lambda^\vee \frac{e}{(e,e)_V}, \]

\[ \lambda^\vee \circ L^\vee = \lambda^\vee \circ ((L^\wedge)^\vee) \oplus \lambda^\vee \frac{e}{(e,e)_V}. \]
Proposition 5.19. (1) If $i = 0$, then $\mathcal{N}^0 = \mathcal{Z}(e^{[0]})$ under the embedding $\delta(\mathcal{L}, e)$.

(2) If $i = 0$, then $\mathcal{N}^0 = \mathcal{Y}(e^{[1]})$ under the embedding $\delta(\mathcal{L}, e)$.

Proof. If $i = 0$, then the valuation of $e^{[0]}$ is a unit and $e^{[0]}$ is an idempotent up to an isomorphism. Hence for any point $(X, tX, \lambda_X, \rho_X)$ of $\mathcal{Z}(e^{[0]})(S)$, the lifting of $\rho_X \circ e^{[0]}$ to $X$ gives a splitting of $X$ to identify it with a point in $\mathcal{N}^0(S)$. The first property follows. The second property follows by duality using the dual isomorphism (5.14).

6. Arithmetic transfer conjectures

6.1. Semi-Lie version. For any bounded complex $\mathcal{F}$ in the $K$-group of coherent sheaves on the regular formal scheme $\mathcal{N}$, recall $\chi(h, \mathcal{F}) := \sum_{i \in \mathbb{Z}} (-1)^i \text{length}_{\mathcal{O}_{F_0}} H^i(\mathcal{N}, \mathcal{F})$ is the Euler characteristic of $\mathcal{F}$.

Definition 6.1. For any regular semisimple pair $(g, u) \in (U(V) \times V)(F_0)_{rs}$, we consider the following derived intersection numbers

$$\text{Int}^{\mathcal{Z}}(g, u) := \chi(\mathcal{N}, \mathcal{O}_\mathcal{Z}(u) \otimes^L \overline{\text{Fix}}^L(g)).$$

$$\text{Int}^{\mathcal{Y}}(g, u) := \chi(\mathcal{N}, \mathcal{O}_\mathcal{Y}(u) \otimes^L \overline{\text{Fix}}^L(g)).$$

By regular-semisimplicity, the intersection $\mathcal{Z}(u) \cap \Gamma_g$ is a proper scheme over $O_{F_0}$. As the blow up morphism is proper, $\mathcal{Z}(u) \cap \Gamma_g$ is also a proper scheme, so is its closed formal subscheme $\overline{\mathcal{Z}}(u) \cap \overline{\Gamma}_g$. Moreover, $\overline{\text{Fix}}^L(g)$ is of bounded degree by the regularity of $\overline{\mathcal{N}} \times \mathcal{N}$. Hence $\text{Int}^{\mathcal{Z}}(g, u)$ and $\text{Int}^{\mathcal{Y}}(g, u)$ are well-defined and finite.

Remark 6.2. We warn the reader that in general $\overline{\mathcal{Z}}(u) \cap \overline{\Gamma}_g \neq \overline{\mathcal{Z}}(u) \cap \overline{\Gamma}_g$.

Recall that in Section 2, we have set up orbit integrals on the analytic side. Choose an orthogonal basis of $L$ to endow $L$ (resp. $V$) with an $O_{F_0}$ (resp. $F_0$)-structure $L_0$ (resp. $V_0$). Consider the symmetric space over $F_0$

$$S(V_0) = \{ \gamma \in \text{GL}(V)|\gamma \tau = id \}.$$

The general linear group $\text{GL}(V_0)$ acts on $S(V_0) \times V_0 \times V_0^*$ by

$$h.(\gamma, u_1, u_2) = (h^{-1} \gamma h, h^{-1} u_1, u_2 h).$$

For $f' \in S(S(V_0) \times V_0 \times V_0^*)(F_0)$, recall the derived orbit integral (2.2).

$$\partial \text{Orb}((\gamma, u_1, u_2), f') := \omega_L(\gamma, u_1, u_2) \frac{d}{ds}|_{s=0} \int_{h \in \text{GL}(V_0)} f'(h.(\gamma, u_1, u_2)) \eta(h)|h|^s dh$$

We consider the following test functions on $(S(V_0) \times V_0 \times V_0^*)(F_0)$:

$$f_{std} := 1_{S(L,v^*)} \times 1_{L_0} \times 1_{(L_0)^*}, \quad f_{std}' := 1_{S(L,v^*)} \times 1_{L_0} \times 1_{(L_0)^*}.$$

Now we introduce the explicit arithmetic transfer conjectures at maximal parahoric levels.

Conjecture 6.3. (Semi-Lie ATC for L) For any regular semisimple pair $(g, u) \in (U(V) \times V)(F_0)_{rs}$ matching $(\gamma, u_1, u_2) \in (S(V_0) \times V_0 \times V_0^*)(F_0)_{rs}$, we have equalities in $\mathbb{Q} \log q$: 
\(1\) \(\partial\text{Orb}((\gamma, u_1, u_2), f_{\text{std}}) = -\int_{N}^{\text{g}} (g, u) \log q.\)

\(2\) \(\partial\text{Orb}((\gamma, u_1, u_2), f'_{\text{std}}) = -(-1)^{d} \int_{N}^{\text{g}} (g, u) \log q.\)

**Proposition 6.4.** The part (2) of semi-Lie ATC for \(L \subseteq V\) is equivalent to part (1) of conjecture for the dual vertex lattice \(\lambda_V \circ L' \subseteq V'.\)

**Proof.** By Proposition 5.7, we have

\[f'_{\text{std}, L}(\gamma, u_1, u_2) = f_{\text{std}, \lambda_V \circ L'}(\gamma, u_1, u_2).\]

On the analytic side, we have

\[f'_{\text{std}, L}(\gamma, u_1, u_2) = f_{\text{std}, \lambda_V \circ L'}(\gamma, u_1, u_2).\]

We are done by noting the transfer factor \(\omega_L\) and \(\omega_{L'}\) for \(L\) and \(L'\) differ by the multiplication of the Weil constant \(\gamma_V = (-1)^{d}.\)

---

**6.2. Group version.** We have group version arithmetic transform conjectures for any orthogonal decomposition of vertex lattices \(L = L^{\circ} \oplus O_{Fe}\) where \(v((e, e)V) = i \in \{0, 1\}\).

**Definition 6.5.** With respect to the embedding \(\delta_{L, e} : N^{\circ} \to N\) (5.13), we consider the derived intersection number of \(g \in U(V)(F_0)_{\text{rs}}\) as the following Euler characteristic number

\[\tilde{\text{Int}}(g) := \chi(N, O_{N}^{\circ} \otimes \tilde{L}^{\text{g}} \text{Fix}(g)).\]

By Proposition 5.19 we have

\[\tilde{\text{Int}}(g) = \begin{cases} \int_{N}^{\text{g}} (g, e^{[0]}), & \text{if } i = 0. \\ \int_{N}^{\text{g}} (g, e^{[1]}), & \text{if } i = 1. \end{cases}\]

Consider the GGP type embedding \(N^{\circ} \to N^{\circ} \times N\), which lifts to an embedding of formal schemes

\[N^{\circ} \to N^{\circ} \times N\]

via strict transforms inside \(N \times N\).

**Proposition 6.6.** The natural commutative diagram

\[
\begin{array}{ccc}
N^{\circ} \times N^{\circ} & \longrightarrow & N \times N \\
\downarrow & & \downarrow \\
N^{\circ} \times N^{\circ} & \longrightarrow & N \times N \\
\end{array}
\]

is cartesian. Moreover, the commutative diagram for the embedding \(N^{\circ} \times N \to N \times N\) over \(N^{\circ} \times N \to N \times N\) is also cartesian.

**Proof.** Denote the fiber product by \(X\), then we have a closed immersion \(i : N^{\circ} \times N^{\circ} \to X\) over \(N^{\circ} \times N^{\circ}\). It sufficient to show \(i\) is an isomorphism when localizing at every closed point \(P \in N^{\circ} \times N^{\circ}\). As the blow up is an isomorphism outside \(N^{\circ} \times N^{\circ}\), we can assume \(P \in N^{\circ} \times N^{\circ}\).

From moduli interpretation, the formulation of Balloon–Ground strata is compatible with the embedding \(\delta_{L, e}\). Hence by Proposition 5.9 and Theorem 5.11 we can choose local generators such that the embedding \(N^{\circ} \times N^{\circ} \to N \times N\) at \(P\) is isomorphic to the standard embedding from

\[O_{\tilde{P}}[[x_0, y_0, x_1, y_1, t_1, \ldots, t_{n-1}, s_1, \ldots, s_{n-1}]]/(x_0y_0 - \varpi, x_1y_1 - \varpi)\]

\[\to O_{\tilde{P}}[[x_0, y_0, x_1, y_1, t_1, \ldots, t_{n-2}, s_1, \ldots, s_{n-2}]]/(x_0y_0 - \varpi, x_1y_1 - \varpi)\]

where \(x_0\) (resp. \(y_0\)) is a local generator for \(N^{\circ}\) (resp. \(N^{\circ}\)). Then the compatibility follows from direct computations. The claim for the embedding \(N^{\circ} \times N \to N \times N\) follows similarly.

The formal scheme \(N^{\circ} \times N\) is regular by compatibility of blow ups in Proposition 6.6.

**Proposition 6.7.** For \(g \in U(V)(F_0)_{\text{rs}},\) we have

\[\tilde{\text{Int}}(g) = \chi(N^{\circ} \times N^{\circ}, N^{\circ} \otimes \tilde{L}(1 \times g)N^{\circ}).\]
Proof. By definition, \( \tilde{\text{Int}}(g) \) is the derived intersection number between \( N^0 \) and \((1 \times g)N\) inside the resolution \( \tilde{N} \times \tilde{N} \). By projection formula, we only need to show the (derived) pullback of \((1 \times g)N\) along the embedding \( \tilde{N}^0 \times \tilde{N} \to \tilde{N} \times \tilde{N} \) agrees with \((1 \times g)N^0\). This is true before taking blow ups, and we are done by compatibility of blow ups in Proposition 6.6. □

Consider the conjugacy action of \( GL(V_0^0) \) on \( S(V_0) \). For \( f' \in S(S(V_0)(F_0)) \) and \( \gamma' \in S(V_0)(F_0)_\text{rs} \), recall the derived orbit integral

\[
\partial \text{Orb}(\gamma', f') = \omega_L(\gamma') \frac{d}{ds} \int_{h \in GL(V_0^0)} f'(h^{-1} \gamma' h) \eta(\det(h)) |\det h|^s dh
\]

where \( \omega_L(\gamma') := \eta(\det(\gamma' e)_{i=0}^{n-1}) \in \{ \pm 1 \} \) is the transfer factor for the basis of \( L \) \[2.7\].

**Conjecture 6.8.** (Group \( ATC \) for \((L, e)\)) For regular semisimple element \( g' \in U(\mathcal{V})(F_0)_{rs} \) matching \( \gamma' \in S(V_0)(F_0)_{rs} \), we have an equality in \( \mathbb{Q} \log q \)

\[
\partial \text{Orb}(\gamma', 1)_{S(L, L')} = -\tilde{\text{Int}}(g') \log q. \tag{6.6}
\]

The group version arithmetic transfer conjecture includes the case \[45, \text{Section 10}\] as a special case i.e., \( L = L^d \oplus O_F e \) where \( L^d \) is self dual and \((e, e)_V = \omega \).

**7. Reduction of intersection numbers**

Consider any orthogonal decomposition of vertex lattices \( L = L^d \oplus O_F e \) where \( v_F((e, e)_V) = i \in \{ 0, 1 \} \). In this section, we show the reduction from Conjecture 6.8 for \((L, e)\) to part \((i + 1)\) of Conjecture 6.3 for \( L^d \). Thus we generalize the reduction lemma \[59, \text{Lem. 4.4}\] to vertex lattices.

Assume that \( i = 0 \) firstly. Consider the map \( * : \text{Hom}_{O_F}(\mathbb{E}, \mathbb{X}^0) \to \text{Hom}_{O_F}(\mathbb{X}, \mathbb{E}) \) defined by \( f \mapsto \lambda_{e}^{-1} f' \circ \lambda_{X} \). It induces an isomorphism of \( F\)-vector spaces

\[
* : \mathbb{V}^0 = \text{Hom}_{O_F}(\mathbb{E}, \mathbb{X}^0) \simeq (\mathbb{V})^* = \text{Hom}_{O_F}(\mathbb{X}^0, \mathbb{E}). \tag{7.1}
\]

Write any element \( g' \in U(\mathcal{V})(F_0) \) in the matrix form

\[
\begin{pmatrix}
a & u \\
w & d
\end{pmatrix} \in \mathbb{X}^0 \times \mathbb{E}
\]

where \( a \in \text{End}_{O_F}(\mathbb{X}^0) = \text{End}(\mathcal{V}^0), \ u \in \mathbb{V}^0, \ w \in (\mathbb{V})^* \) and \( d \in \text{End}_{O_F}(\mathbb{E}) = F \).

For the decomposition \( \mathcal{V} = \mathbb{V}^0 \oplus F e[i] \), consider its unitary relative Cayley map \[3.2\]:

\[
\mathcal{U} : U(\mathcal{V}) \to U(\mathbb{V}^0) \times \mathbb{V}^0,
\]

\[
g' = \begin{pmatrix} a & u \\ w & d \end{pmatrix} \mapsto (g = a + \frac{uw}{1-d}, u_1 = \frac{u}{1-d}). \tag{7.4}
\]

For \( g' \in U(\mathcal{V})(F_0)_{rs} \), let \( \mathcal{U}(g) = (g, u_1) \), consider the graph \( \Gamma_g \) (resp. \( \Gamma_g' \)) of \( g' \) (resp. \( g \)) inside \( N \times N \) (resp. \( N^0 \times N^0 \)).

**Lemma 7.1.** If \( i = 0 \) and \( 1 - d \in O_F^2 \), then we have

\[
(\mathbb{N}^0 \times \mathbb{Z}(u)) \cap N^0 \times N^0 \cap \Gamma_g = (\mathbb{N}^0 \times \mathbb{N}^0) \cap N \times N \cap \Gamma_g'.
\]

**Proof.** By Proposition 3.2 and the assumption \( i = 0 \), we have \((w^{\gamma'})^* = w\) for the element

\[
w^{\gamma'} = g^{-1} \epsilon_d u
\]

in \( \mathbb{V}^0 \) where \( \epsilon_d := \frac{1-\gamma}{d} \). For any test scheme \( S \), consider a \( S\)-point \((X_1, X_2)\) of \( N^0 \times N^0 \).

On the one hand, if \((X_1, X_2)\) is on the graph \( \Gamma_g' \) via the embedding \( \delta : N^0 \to N \), then we have a homomorphism \( \varphi' : X_1 \times E \to X_2 \times E \) lifting \( g' \). Write \( \varphi' \) in the matrix form

\[
\varphi' = \begin{pmatrix} \varphi & \psi_u \\ \psi_w & d \end{pmatrix}
\]

lifting the diagram \[7.2\] (after applying framings of \( X_1 \) and \( X_2 \)). Here \( \varphi : X_1 \to X_2, \psi_u : E \to X_2, \psi_w : X_1 \to E \) are all homomorphisms, in particular \( X_2 \in \mathbb{Z}(u) \). As \( 1 - d \in O_F^2 \), the quasi-isogeny

\[
\tilde{\varphi} = \varphi + (1 - d)^{-1} \psi_u \circ \psi_w
\]
is a homomorphism lifting $g$ \((7.4)\). Hence \((X_1, X_2)\) is on the intersection \((\mathcal{N}^\circ \times Z(u)) \cap_{\mathcal{N}^\circ \times \mathcal{N}^\circ} \Gamma_g\).

On the other hand, we start with homomorphisms $\psi_u : \mathcal{E} \to X_2$ (resp. $\varphi_g : X_1 \to X_2$) lifting $u$ (resp. $g$). By the formula $w^\vee = g^{-1} \epsilon_d u$ above, the homomorphism $\epsilon_d$ is a unit in $O_{\mathcal{E}}$

\[
\psi_{w^\vee} := [\epsilon_d] \circ \varphi_g^{-1} \circ \psi_u : \mathcal{E} \to X_2
\]
lifting the vector $w^\vee \in \mathcal{V}$. As $i = 0$, the polarization $\lambda_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}^\vee$ is an isomorphism. Then the homomorphism $\psi_w := \lambda_{\mathcal{E}}^{-1} \circ (\psi_{w^\vee})^\vee \circ \lambda_{\mathcal{E}}$ gives the desired lifting of $w$. The homomorphism $\varphi := \varphi_g - (1 - d)^{-1} \psi_u \circ \psi_w$ lifts $a$, hence we get the desired lifting of $g'$. Therefore, \((X_1, X_2)\) is on the intersection \((\mathcal{N}^\circ \times \mathcal{N}^\circ) \cap_{\mathcal{N}^\circ \times \mathcal{N}^\circ} \Gamma_{g'}\). \(\square\)

**Proposition 7.2.** We have an identification of two fiber products as closed formal subschemes of $\mathcal{N}^\circ \times \mathcal{N}^\circ$:

\[
(\mathcal{N}^\circ \times \mathcal{N}^\circ) \cap_{\mathcal{N}^\circ \times \mathcal{N}^\circ} \Gamma_{g'} = (\mathcal{N}'^\circ \times \mathcal{N}^\circ) \cap_{\mathcal{N}'^\circ \times \mathcal{N}^\circ} \Gamma_{g'}.
\]

**Proof.** We denote by $\tilde{\gamma}$ the fiber product of the diagram:

\[
\begin{array}{ccc}
\mathcal{N}^\circ \times \mathcal{N}^\circ & \xrightarrow{f} & \mathcal{N}'^\circ \times \mathcal{N}^\circ \\
\downarrow & & \downarrow \\
\mathcal{N} \times \mathcal{N} & \xrightarrow{f} & \mathcal{N}' \times \mathcal{N} \\
\downarrow & & \downarrow \\
\mathcal{N} \times \mathcal{N} & \xrightarrow{f} & \mathcal{N} \times \mathcal{N} \\
\end{array}
\]

The bottom is cartesian by Proposition \[6.6\]. The strict transform of $\Gamma_{g'}$ equals to itself by Proposition \[5.17\]. The result follows. \(\square\)

**Proposition 7.3.** We have an identification of two fiber products as closed formal subschemes of $\mathcal{N}^\circ \times \mathcal{N}^\circ$:

\[
(\mathcal{N}^\circ \times Z(u)) \cap_{\mathcal{N}^\circ \times \mathcal{N}^\circ} \Gamma_g = (f')^{-1}(\mathcal{N}^\circ \times Z(u)) \cap_{\mathcal{N}'^\circ \times \mathcal{N}^\circ} \Gamma_{g'},
\]

where $f' : \mathcal{N}'^\circ \times \mathcal{N}^\circ \to \mathcal{N}^\circ \times \mathcal{N}^\circ$ is the small resolution for $\mathcal{N}^\circ$.

**Proof.** By Proposition \[5.17\], the graph $\Gamma_g = \Gamma_{g'} \to \mathcal{N}^\circ \times \mathcal{N}^\circ$ factors through $f$. The equality follows by base change. \(\square\)

**Proposition 7.4.** Assume that $u \in \mathcal{V}$ is non-zero, then we have equalities in the $K$-group of $\mathcal{N}^\circ \times \mathcal{N}^\circ$:

\[
(\mathcal{N}^\circ \times \mathcal{N}^\circ) \cap_{\mathcal{N}^\circ \times \mathcal{N}^\circ} \Gamma_{g'} = (\mathcal{N}^\circ \times \mathcal{N}^\circ) \cap_{\mathcal{N}^\circ \times \mathcal{N}^\circ} \Gamma_g,
\]

\[
(f')^{-1}(\mathcal{N}^\circ \times Z(u)) \cap_{\mathcal{N}'^\circ \times \mathcal{N}^\circ} \Gamma_{g'} = (f')^{-1}(\mathcal{N}^\circ \times Z(u)) \cap_{\mathcal{N}'^\circ \times \mathcal{N}^\circ} \Gamma_g.
\]

**Proof.** As $u \neq 0$, $Z(u)$ is a Cartier divisor of $\mathcal{N}^\circ$. By Lemma \[7.1\], Proposition \[7.2\] and Proposition \[7.3\], the intersection $\mathcal{N}^\circ \times \mathcal{N}^\circ \cap_{\mathcal{N}^\circ \times \mathcal{N}^\circ} \Gamma_{g'}$ has expected dimension $\dim Z(u) = n - 2$, and $f'^{-1}(\mathcal{N}^\circ \times Z(u)) \cap_{\mathcal{N}'^\circ \times \mathcal{N}^\circ} \Gamma_{g'}$ has expected dimension $\dim Z(u) = n - 2$.

As $\mathcal{N}^\circ \times \mathcal{N}^\circ$ and $\mathcal{N} \times \mathcal{N}$ are regular, we conclude by applying \[59\] Lem. B.2. The assumptions in loc. cit. are satisfied: $\mathcal{N}^\circ \times Z(u)$ is a Cartier divisor of $\mathcal{N}^\circ \times \mathcal{N}^\circ$, so the pullback $(f')^{-1}(\mathcal{N}^\circ \times Z(u))$ is a Cartier divisor in the regular formal scheme $\mathcal{N}'^\circ \times \mathcal{N}^\circ$. Hence $(f')^{-1}(\mathcal{N}^\circ \times Z(u))$ is Cohen-Macaulay and of pure dimension $2n - 4$. \(\square\)

Now we finish the reduction of intersection numbers.

**Theorem 7.5.** Consider any regular semi-simple element $g' \in U(\mathcal{V})(F_0)_{rs}$ and $c_U(g') = (g, u_1) \in (U(\mathcal{V}) \times \mathcal{V}')(F_0)_{rs}$. Assume that $u \neq 0$.

1. If $i = 0$, then we have

\[
\text{Int}(g') = \text{Int}^Z(g, u).
\]
(2) If \( i = 1 \), then we have
\[
\widehat{\text{Int}}(g') = \int_Y (g, u).
\]

(3) Conjecture 6.3 for \((L, e)\) is implied by part \((i + 1)\) of Conjecture 6.3 for \(L^\flat\).

Proof. Assume that \( i = 0 \) firstly. Twisting by elements in \(F^{\text{red}}_{\ell}\), we can assume \( 1 - d \in O_{\ell}^\flat\).

By Proposition 7.4, we have
\[
\int_Y (g, u) = 0 \text{ firstly.}
\]

If \( i \) was non-zero by approximation. We have the reduction on the analytic side by [53, Lem. B.2], we have
\[
\int_Y (g, u) = 0 \text{ using the compatibility [5, 14].}
\]

Therefore, \( \int_Y (g', u) = \int_Y (g, u) \).

If \( i = 1 \), we apply the dual isomorphism \( \lambda_N : \mathcal{N} \rightarrow \mathcal{N}^\vee \) to reduce to the case \( i = 0 \) using the compatibility [5, 14].

For the last part, consider matching elements \( g' \in U(V)(F_0)_{\text{red}} \) and \( \gamma' \in S(V_0)(F_0)_{\text{red}} \). Apply relative Cayley maps to \( g' \) and \( \gamma' \). By locally constancy of intersection numbers \([53]\), we can always assume \( u \) is non-zero by approximation. We have the reduction on the analytic side by Corollary 3.22 which concludes the proof. \( \square \)

Remark 7.6. Although \([53]\) only lists as examples the unitary Rapoport–Zink spaces in the self-dual case, the argument in loc. cit. on locally constancy of intersection numbers is quite general and applies to our current situation.

8. Bruhat–Tits strata

In this section, we study Bruhat–Tits stratification on the reduced locus \( \mathcal{N}^{\text{red}} \) based on works of [5, 24, 52], which is useful for explicit computations. We redefine Bruhat–Tits strata via the Kudla–Rapoport cycles, which is used later to prove ATCs in the unramified maximal order case and some local modularity results.

For any \( p \)-adic field \( F_0 \), consider the associated Rapoport–Zink space \( \mathcal{N} = \mathcal{N}_{U(L)} \rightarrow \text{Spf} O_{\ell} \) for a vertex lattice \( L \subseteq V \) of rank \( n \) and type \( t \). Recall the hermitian space \( \mathcal{V} = \text{Hom}_{\mathcal{O}_{\ell}}(E, \mathcal{X}) \).

Definition 8.1. For a vertex lattices \( L' \subseteq \mathcal{V} \), the Bruhat–Tits strata \( BT(L') \) is the closed subscheme of \( \mathcal{N}^{\text{red}} \) over \( \mathbb{F} \) defined by
\begin{enumerate}
\item If \( L' = L^\flat \) has type \( \leq t - 1 \), then \( BT(L') := \mathcal{V}(L^\flat)^{\text{red}} \).
\item If \( L' = L^\flat \) has type \( \geq t + 1 \), then \( BT(L') := \mathcal{V}(L^\flat)^{\text{red}} \).
\end{enumerate}

In this section, we study Bruhat–Tits stratification using relative Dieudonne theory.

Remark 8.2. Compared to general results for quasi-split groups [18], for our (non-quasi-split) set up the reduced locus \( \mathcal{N}^{\text{red}} \) may not be equidimensional and stabilizers of its top dimensional irreducible components may not be a special parahoric.
Proposition 8.3. Consider an orthogonal decomposition $L = L^0 \oplus O_F e$ with $v((e,e)_{\mathbb{V}}) = i \in \{0,1\}$, with induced decomposition $\mathbb{V} = \mathbb{V}^0 \oplus F e[1]$ and the embedding $\delta_{(L,e)} : \mathbb{N}^0 \to \mathbb{N}$. If $L'$ is a vertex lattice in $\mathbb{V}$ containing $e[0]$, then

$$BT(L') \cap \mathbb{N}^0 = BT(L^0).$$

(8.1)

Proof. We may assume $i = 0$ using the dual isomorphism. Identify $\mathbb{N}^0$ with $\mathbb{Z}(e[0])$. The result follows from moduli definition of $\mathbb{Z}(L')$ or $\mathbb{Y}(L^0')$.

8.1. Relative Dieudonné theory. Denote by $\sigma$ the $F_0$-linear Frobenius automorphism on $\bar{F}_0$. We have $O_F \otimes_{O_{F_0}} O_{\bar{F}_0} \simeq O_{\bar{F}_0} \times O_{\bar{F}_0}$ where $O_F$ acts on the first (resp. second) factor $O_{\bar{F}_0}$ by the fixed embedding $i_0 : O_F \to O_{\bar{F}_0}$ (resp. $\sigma \circ i_0$).

Let $M(E)$ (resp. $M(X)$) be the (covariant) relative Dieudonné module of the framing object $E$ (resp. $X$), which is a free $O_{\bar{F}_0}$-module of rank 2 (resp. rank $2n$) with the relative Frobenius morphism $F_E$ (resp. $F_M$) and the relative Verschiebung $V_E$ (resp. $V_M$).

Remark 8.4. See [47] Prop. 3.56, Defn. 3.57 for backgrounds on relative Dieudonné theory. Let $f$ be the inertial degree of $F_0$ over $Q_p$. The absolute Dieudonné module of $X$ is a free $W(F)$ module of rank $2n[F_0 : Q_p]$, which decomposes into direct summands under the action of $O_F \otimes_{Z_p} W(F) \cong \prod_{i=1}^f O_{\bar{F}_0}$. The 0-th summand has $O_{F_0}$-rank $\frac{2n[F_0 : Q_p]}{f} \times \frac{1}{p^{k_X}} = 2n$ and is exactly the relative Dieudonné module $M(X)$ of $X$.

Denote by $N(X) = M(X) \otimes Q$ the associated isocrystal. The $O_F$-action on $X$ induces $\mathbb{Z}/2$-gradings:

$$M(X) = M(X)_0 \oplus M(X)_1, \quad N(X) = N(X)_0 \oplus N(X)_1. \quad (8.2)$$

The polarization $\lambda_X$ induces a nondegenerate $\bar{F}_0$-bilinear alternating form $\langle -,- \rangle_X$ on $N(X)$. For $x,y \in N(X)$ and $a \in O_F$, we have

$$\langle F_x x, y \rangle_X = \sigma(\langle x, F_y y \rangle_X), \quad \langle \iota(a)x, y \rangle_X = \langle x, \iota(\bar{y})y \rangle_X. \quad (8.3)$$

For any lattice $\Lambda_1 \subseteq N(X)_1$, write $(\Lambda_1)^{\perp} \subseteq N(X)_0$ as its dual inside $N(X)_0$ under the pairing $\langle -,- \rangle$. Consider the $\sigma^2$-linear operator $\tau = V_X^{-1} F_X$ on $N(X)_0$. Let $C(X) = N(X)_0^{-1}$ be the $F$-vector space of invariant elements in $N(X)_0$.

By the (relative) supersingular assumption on $X$, we have

$$N(X)_0 \simeq C(X) \otimes_F \bar{F}_0. \quad (8.4)$$

Definition 8.5. Let $\langle -,- \rangle$ be the nondegenerate form on $N(X)_0$ given by

$$\langle x, y \rangle := \langle x, F_y y \rangle_X$$

which is linear in the first variable and $\sigma$-linear in the second variable. The restriction of the form $\langle -,- \rangle|_{C(X)}$ to $C(X)$ is skew-hermitian.

We have similar decompositions and forms on the isocrystal $N(E) = M(E) \otimes Q$. Choose a unit $\delta \in O_F^*$ such that $\overline{\delta} = -\delta$. By [27] Remark. 2.5], we can find generators $1_i$ of $M(E)_i$ ($i = 0,1$) such that

$$F_{\delta} 1_1 = 1_0, \quad F_{\delta} 1_0 = \varpi 1_1, \quad \{1_0,1_0\}_E = \delta. \quad (8.5)$$

A vector $x \in \mathbb{V} = \text{Hom}_{O_{\bar{F}_0}}^c(E,X)$ induces a $F \otimes_{\bar{F}_0} \bar{F}_0$-linear map from $N(E)$ to $N(X)$. As $F_\mathbb{V}(1_0) = V_{\mathbb{V}}(1_0)$, we have $x(1_0) \in C(X)$. From [27] Lem. 3.9] we obtain:

Proposition 8.6. The association $x \mapsto x(1_0)$ gives a bijection

$$\mathbb{V} \cong C(X)$$

under which the hermitian form $\langle -,- \rangle_\mathbb{V}$ on $\mathbb{V}$ is identified with $\frac{1}{\varphi} \langle -,- \rangle$ on $C(X)$.

Consider the following non-degenerate $\sigma$-sesquilinear form on $N(X)_0$ and $C(X)$:

$$\langle x,y \rangle_N := \frac{1}{\varphi \delta} \langle x, y \rangle. \quad (8.6)$$

For $x,y \in N(X)_0$, we have

$$\sigma(\langle x,y \rangle_N) = \tau(\langle y,x \rangle_N), \quad \langle \tau(x), \tau(y) \rangle_N = \sigma^2(\langle x,y \rangle_N). \quad (8.7)$$
8.2. F-points of \( \mathcal{N} \).

For any \( O_{F_0} \) lattice \( A \) in \( \mathbb{N}(X)_0 \), denote by \( A^\vee \) its dual under \((-,-)_N\):
\[
A^\vee = \{ x \in \mathbb{N}(X)_0 | (x, A)_N \subseteq O_{F_0} \}. \tag{8.8}
\]

For two \( O_{F_0} \) lattices \( A_1, A_2 \) in \( \mathbb{N}(X)_0 \) and \( m \geq 0 \), write \( A_1 \subseteq A_2 \) if \( A_1 \subseteq A_2 \) and \( A_2 / A_1 \) is a \( \mathbb{F} \)-vector space of dimension \( m \). Note we have
\[
(A^\vee)^\vee = \tau(A). \tag{8.9}
\]

We regard \( O_F \) lattices in \( \mathbb{V} \simeq C(X) \) the same as \( \tau \) stable \( O_{F_0} \) lattices in \( \mathbb{N}(X)_0 \). For any \( O_F \) lattice \( L' \) in \( \mathbb{V} \simeq C(X) \), we denote by \( L'^\vee \) its dual under the form \((-,-)_V \simeq (-,-)_N \). We have
\[
(L'^\vee)^\vee = L'. \tag{8.10}
\]

Consider a point \((X, \iota, \lambda, \rho) \in \mathcal{N}(\mathbb{F})\). The framing gives an isomorphism \( \mathbb{N}(X) \simeq \mathbb{N}(X)_0 \). Hence the relative Dieudonne module of \((X, \iota, \lambda, \rho)\) gives an \( O_{F_0} \) lattice \( M = M_0 \oplus M_1 \) inside \( \mathbb{N}(X) \). Consider two \( O_{F_0} \) lattices inside \( \mathbb{N}(X)_0 \) given by
\[
A = M_0, \quad B = \sigma^{-1}(F_{X}M_1)^\vee. \tag{8.11}
\]

By definition, we have \( \sigma^{-1}(F_{X}M_1)^\vee = (M_1)_{1} \). A pair of \( O_{F_0} \) lattices \((A, B)\) in \( \mathbb{N}(X)_0 \) is call special if we have
\[
B^\vee \subseteq A, \quad A^\vee \subseteq B. \tag{8.12}
\]

**Proposition 8.7.** We have the following description of \( \mathbb{F} \)-points of \( \mathcal{N} \):
\[
\mathcal{N}(\mathbb{F}) = \{ \text{special pairs of } O_{F_0} \text{-lattices } (A, B) \subseteq \mathbb{N}(X)_0 \}. \tag{8.13}
\]

**Proof.** The lattice \( M_1 \) and \( B \) determines each other by \( B^\vee = F_{X}M_1 \). In the definition of \( \mathcal{N} \), the condition on the polarization is equivalent to \( A \subseteq B \), and the Kottwitz signature condition is equivalent to \( B^\vee \subseteq A, A^\vee \subseteq B \). Hence we conclude by relative Dieudonne theory. See also [5, Prop. 2.4]. \( \square \)

Consider a non-zero vector \( u \in \mathbb{V} \) and a point \((X, \iota, \lambda, \rho) \in \mathcal{N}(\mathbb{F}) \). By Grothendieck-Messing theory, \( \rho_X^{-1} \circ u \) lifts to \( \text{Hom}_{O_F}(E, X) \) if and only if
\[
u(1) \in M_0, \quad u(1) \in M_1. \tag{8.14}
\]

We have \( F_{X}1_1 = 1_0 \), \( F_{X}1_0 = \omega 1 _1 \) by \((8.3)\). Hence the condition \((8.13)\) is equivalent to that \( u(1) \in F_{X}M_1 = B^\vee \), and we have the following description of \( \mathcal{Z}(u)(\mathbb{F}) \) (see also [5, Prop. 5.5]):
\[
\mathcal{Z}(u)(\mathbb{F}) = \{(A, B) \in \mathcal{N}(\mathbb{F}) | u \in B^\vee \}. \tag{8.15}
\]

The relative Dieudonne module of \( X^\vee \) can be identified with \( M^\perp = B \oplus A^\perp \), where \((-)^\perp \) means the dual with respect to \((-,-)_X \). We obtain similar description for \( \mathcal{Y}(u) \):
\[
\mathcal{Y}(u)(\mathbb{F}) = \{(A, B) \in \mathcal{N}(\mathbb{F}) | u \in A^\vee \}. \tag{8.16}
\]

**Remark 8.8.** The inclusion \( \mathcal{Z}(u) \subseteq \mathcal{Y}(u) \subseteq \mathcal{Z}(\omega^{-1}u) \) on \( \mathbb{F} \)-points corresponds the inclusion
\[
B^\vee \subseteq A^\vee \subseteq \omega^{-1}B^\vee. \tag{8.17}
\]

Moreover, identify \( \mathbb{V} \cong \mathbb{V}^\vee \) by the dual isomorphism \( \lambda \). By Proposition \( \ref{5.3} \), we have \((-,-)_V = -\omega(-,-)_V \). So on \( \mathbb{F} \)-points, the dual isomorphism \( \lambda : \mathcal{N} \rightarrow \mathcal{N}^\vee \) is given by
\[
\lambda : \mathcal{N}(\mathbb{F}) \rightarrow \mathcal{N}(\mathbb{F}^\vee), \quad (A, B) \mapsto (A', B') := (B, \omega^{-1}A). \tag{8.18}
\]

Given a special pair \((A, B)\), denote by \( L_A \) (resp. \( L_B \)) the smallest \( \tau \)-stable lattice containing \( A \) (resp. \( B \)). By definition,
\[
A \subseteq L_A, \quad (L_A)^\vee \subseteq A^\vee, \quad \tag{8.19}
\]
\[
L_B^\vee \subseteq B^\vee, \quad B \subseteq L_B. \tag{8.20}
\]

**Proposition 8.9.** For a special pair \((A, B)\), at least one of the two following cases holds:

- \( \omega L_A^\perp \subseteq L_A \subseteq L_A^\perp \), in particular \( t^{-1}A \subseteq A^\vee \). Hence \( L^0 = L_A \) is a vertex lattice of type \( \leq \dim_F(A^\vee /A) = t-1 \).
\* \( \varpi L_B \subseteq L_B' \subseteq L_B \), in particular \( B^\vee \overset{t+1}{\subseteq} B \). Hence \( L^* = L_B' \) is a vertex lattice of type \( \geq \dim(B/B^\vee) = t + 1 \).

**Proof.** See [5, Lem. 2.7]. \( \square \)

For a special pair \((A, B) \in N^\text{red}(F)\), we define

1. \((A, B) \in N^\text{red}\) if and only if \(A \leq A^\vee\).
2. \((A, B) \in N^\text{red}\) if and only if \(B^\vee \subseteq B\).

Any automorphism \(g \in U(V)(F_0)\) preserves the stratum \(N^\text{red} \) and \(N^\text{red}\). By Proposition 8.9 we can stratify \(N^\text{red}\) into

\[ N^\text{red} = N^\text{red} \cup N^\text{red}. \]

### 8.3. KR cycles and BT strata.

For a vertex lattice \( L^* \subseteq V \) of type \( \geq t + 1 \), from above we have

\[ Z(L^*)(F) = \{ (A, B) \in N(F) | L^* \subseteq B^\vee \subseteq A \leq B \leq (L^*)^\vee, L^* \subseteq B^\vee \subseteq A \leq B \subseteq (L^*)^\vee \}. \]

Consider the \( F_q \)-vector space \( V(L^*)_0 := (L^*)^\vee / L^* \) of dimension \( t(L^*) \). Here we regard \( L^* \) as a finite rank \( O_F \)-lattice. Equip \( V(L^*)_0 \) with the perfect \( \sigma \)-hermitian form

\[ (x, y) = \varpi(x, y)\gamma. \]

Set \( U_1 \) (resp. \( U_2 \)) as the reduction of \( B^\vee \) (resp. \( A \)) inside \( V(L^*)_0 \otimes F \). Then \( Z(L^*)(F) \) can be identified with the set

\[ \{ (U_1, U_2) \subseteq V(L^*)_0 \otimes F | U_1 \subseteq U_2, U_2 \subseteq U_1, U_1 \subseteq U_1^\perp, U_1 \subseteq U_1^\perp, \dim U_1 = \frac{t(L^*) - t - 1}{2}, \dim U_2 = \dim U_1 + 1 \}. \quad (8.18) \]

Here by definition \( U^\perp \) is the subspace \( \{ x \in V(L^*)_0 \otimes F | (x, y) = 0, \forall y \in U \} \).

Similarly, we have

\[ V((L^*)^\vee)(F) = \{ (A, B) \in N(F) | \varpi(A^\vee) \subseteq \varpi A^\vee \subseteq \varpi B \subseteq A \leq L^\vee, \varpi(L^*) \subseteq \varpi A^\vee \subseteq B^\vee \subseteq A \subseteq L^\vee \}. \quad (8.19) \]

Consider the \( F_q \)-vector space

\[ V(L^*_0) := L^\vee / \varpi(L^*_0)^\vee \]

of dimension \( n - t(L^*_0) \). Here we regard \( L^*_0 \) as a finite rank \( O_F \)-lattice. Equip \( V(L^*_0)_0 \) with the perfect \( \sigma \)-hermitian form

\[ (x, y) : V(L^*_0)_0 \times V(L^*_0)_0 \longrightarrow F_q^\ast, \]

\[ (x, y) = (x, y)^\gamma. \]

Set \( U_3 \) (resp. \( U_4 \)) as the reduction of \( \varpi A^\vee \) (resp. \( \varpi B \)) inside \( V(L^*_0)_0 \otimes F \). Then \( V((L^*_0)^\vee)(F) \) can be identified with the set

\[ \{ (U_3, U_4) \subseteq V(L^*_0)_0 \otimes F | U_3 \subseteq U_4, U_3 \subseteq U_4, U_4 \subseteq U_3^\perp, U_4 \subseteq U_3^\perp, \dim U_3 = \frac{t - 1 - t(L^*_0)}{2}, \dim U_4 = \dim U_3 + 1 \}. \quad (8.20) \]

Here \( U^\perp \) is by definition the subspace \( \{ x \in V(L^*_0)_0 \otimes F | (x, y) = 0, \forall y \in U \} \).

In conclusion, we have (see also [5, Defn. 2.9, Prop. 2.18]):

**Proposition 8.10.** (1) For two vertex lattices \( L^*_1, L^*_2 \) of type \( \leq t - 1 \), we have

\[ \text{BT}(L^*_1) \cap \text{BT}(L^*_2) = \begin{cases} \text{BT}(L^*_1 \cap L^*_2) & \text{if } L^*_1 \cap L^*_2 \text{ is a vertex lattice} \\ \emptyset & \text{else} \end{cases} \]

(2) For two vertex lattices \( L^*_1, L^*_2 \) of type \( \geq t + 1 \), we have

\[ \text{BT}(L^*_1) \cap \text{BT}(L^*_2) = \begin{cases} \text{BT}(L^*_1 + L^*_2) & \text{if } L^*_1 + L^*_2 \text{ is a vertex lattice} \\ \emptyset & \text{else} \end{cases} \]
Theorem 8.11. The collection \( \{ \text{BT}(L') \}_{L'} \) indexed by vertex lattices \( L' \) in \( V \) gives a natural stratification on \( N^{\text{red}} \). Each stratum \( \text{BT}(L') \) is projective, smooth, and irreducible over \( \mathbb{F} \), and isomorphic to a (closed) Deligne–Lusztig variety for \( U(\mathbb{F}) \) (which may not be of Coxeter type). We have

\[
\dim \text{BT}(L') = \begin{cases} 
\frac{t-1-t(L')}{2} + n - t & \text{if } t(L') \leq t - 1 \\
\frac{t(L') - t - 1}{2} + t & \text{if } t(L') \geq t + 1
\end{cases}
\]

More precisely,

(1) For any perfect field \( k \) over \( \mathbb{F}_q \), there is a natural bijection from \( \text{BT}(L')(k) \) to

\[
\{(U_1, U_2) \subseteq V(L^*)_0 \otimes k | U_1 \subseteq U_2, U_2 \subseteq U_1^\perp, U_1 \subseteq U_2^\perp, \dim U_1 = \frac{t(L^*) - t - 1}{2}, \dim U_2 = \dim U_1 + 1\}
\]

which is a union of two Deligne–Lusztig varieties, depending on whether \( U_2 \subseteq U_1^\perp \).

(2) For any perfect field \( k \) over \( \mathbb{F}_q \), there is a natural bijection from \( \text{BT}(L')(k) \) to

\[
\{(U_3, U_4) \subseteq V(L^*)_0 \otimes k | U_3 \subseteq U_4, U_4 \subseteq U_3^\perp, U_3 \subseteq U_4^\perp, \dim U_3 = \frac{t - 1 - t(L^*)}{2}, \dim U_4 = \dim U_3 + 1\}
\]

which is a union of two Deligne–Lusztig varieties, depending on whether \( U_4 \subseteq U_3^\perp \).

(3) \( \text{BT}(L^o) \cap \text{BT}(L^*) \) is non-empty if and only if \( L^* \subseteq L^o \). In this case,

\[
\text{BT}(L^o) \cap \text{BT}(L^*)(\mathbb{F}) = \{(A, B) \in N(\mathbb{F})L^* \subseteq B^\perp \subseteq A \subseteq L^o\}.
\]

is equal to \( \mathbb{F} \)-points of the flag variety \( F_l(L^o, L^o) \) parameterizing flags \((B^\perp, A)\) of dimension \( \frac{t(L^o) - t - 1}{2}, \frac{t(L^o) - t + 1}{2} \) in the \( \frac{t(L^o) - t(L^o)}{2} \) dimensional \( \mathbb{F} \)-vector space \( L^o/L^* \).

(4) In particular if \( t(L^*) = t + 1 \) and \( t(L^o) = t - 1 \), then \( \text{BT}(L^o)(\mathbb{F}) \cong \mathbb{P}^{t-1}(\mathbb{F}) \) and \( \text{BT}(L^*)(\mathbb{F}) \cong \mathbb{P}^{t}(\mathbb{F}) \). And \( \text{BT}(L^o) \cap \text{BT}(L^*)(\mathbb{F}) \) is exactly one point if \( L^* \subseteq L^o \), and empty otherwise.

Proof. This is [5] Thm. 1.1 after identifying the strata in loc. cit. with our definitions via special cycles. Proposition 8.9 shows the collection \( \{ \text{BT}(L') \}_{L'} \) covers \( N^{\text{red}} \) which forms a stratification. In loc. cit., we could identify \( \text{BT}(L') \) with (closed) Deligne–Lusztig varieties [8.18, 8.20]. In particular, we have natural bijections in the theorem by applying relative Dieudonné theory over \( k \). By the identifications, we obtain dimensions, projectivity, smoothness, and irreducibility of \( \text{BT}(L') \).

9. Local modularity

In this section, we prove some local modularity results, which is used in Section 14.4 to show (double) modularity of arithmetic theta series over the basic locus, see also Proposition 16.6.

Let \( L \) be a vertex lattice of rank \( n \) and type \( t \). Consider the intersection pairing between divisors \( Z \) on \( N = \mathcal{N}_U(L) \) and \( 1 \)-cycles \( C \) in \( N^{\text{red}} \):

\[
(Z, C) := \chi(N, \mathcal{O}_Z \otimes^L \mathcal{O}_C).
\]

Consider the following functions on \( u \in V - 0 \):

\[
f_L(C)(u) := (Z(u), C), \quad f_{L^o}(C)(u) := (\mathcal{Y}(u), C)
\]

Consider the Weil representation of \( \text{SL}_2(F_0) \) on \( S(V) \) with respect to a fixed unramified additive character \( \psi : F_0 \to \mathbb{C} \). Then for \( f \in S(V) \), we have

\[
(1 - 1)f(u) = \gamma_V F_V(f)(u)
\]  

(9.1)

where \( \gamma_V = \eta(\det(V)) \in \{ \pm 1 \} \) is the Weil constant of \( V \), and \( F_V \) is the Fourier transform on \( V \) with respect to \( \psi_F \circ \eta_V \):

\[
F_V(f)(y) := \int_{x \in V} f(x) \psi_F((x, y) \gamma_V) dx, \quad y \in V.
\]  

(9.2)

Here the Haar measure is the self-dual one with respect to \( \psi_F \).

A 1-cycle \( C \) in \( N^{\text{red}} \) is called very special, if it is a finite linear combination of 1-cycles from standard embeddings of the form

\[
\delta : N^{[1]}_2 \to N^{[t]}_n = \mathcal{N}_n
\]
Let \( U(\mathbb{R}) \) (resp. type 2) if \( m \) induced by a decomposition of framing objects \( X \). Theorem 9.1. Choose any \( Z(1) \) in the K-group of \( f \) and \( \lambda \) to vertex lattice in \( t \) and \( t \). We can decompose the embedding into the base case 9.1. The space \( \delta \) of type 0 \( \sim \) \( h \). By locally constancy of intersection numbers [39], we can always work with non-isotropic vector \( u = u_1 + u_2 + u_3 \in \mathcal{V} = \mathcal{V}^0 \oplus (\oplus_{i=1}^{t-1} \mathcal{F}_c^1) \oplus (\oplus_{j=1}^{n-t-1} \mathcal{F}^{[0]}_c) \). (9.5)

Proposition 9.2. After restriction to the embedding \( \delta \), for any \( u \in \mathcal{V} \) we have

1. \( \mathcal{Z}(u)|_\delta = \mathcal{Z}(u_1)|_{\mathcal{O}_{F_0}|_{\mathcal{V}}} = \mathcal{O}_{F_0} = \mathcal{O}_{F_0} \delta^{-1}(u_3) \).
2. \( \mathcal{Y}(u)|_\delta = \mathcal{Y}(u_1)|_{\mathcal{O}_{F_0} = \mathcal{O}_{F_0} \delta^{-1}(u_3)} \).

Proof. We can decompose the embedding into \( \delta : \mathcal{N}_{2}^{[1]} \delta \rightarrow \mathcal{N}_{t+1}^{[1]} \delta \rightarrow \mathcal{N}_{n}^{[1]} = \mathcal{N} \), then the result follows from moduli definitions of Kudla– Rapoport cycles.

So by the projection formula for \( \delta \), Theorem 9.1 for the very special 1-cycle \( \mathcal{C} \) lives on a chosen embedding \( \delta : \mathcal{N}_{2}^{[1]} \delta \rightarrow \mathcal{N}_{n}^{[1]} = \mathcal{N} \), write \( u \in \mathcal{V} \) as

\[
u((u_1, u_2)) = \begin{cases} 0 & m \text{ is even}, \\ -1 & m \text{ is odd.} \end{cases} \tag{9.6}
\]

By [34] Lem. 3.8 (the notion of vertex lattices in loc.cit. is dual to us) which works for any \( F_0 \), there exists a unique vertex lattice \( L_u \) such that \( u_1 \in L_u - \mathcal{V}L_u \). Moreover, \( L_u \) is of type 0 (resp. type 2) if \( m \) is even (resp. odd). We call \( L_u \) the central lattice of \( u \).

Let \( d(L_1, L_2) \) be the distance of two vertex lattices \( L_1, L_2 \subseteq \mathcal{V}_2 \), in the Bruhat–Tits tree of \( U(\mathcal{V}_2)(F_0) \).

Theorem 9.3. Let \( u \in \mathcal{V}_2 \) with \( m = \nu((u, u)_{\mathcal{V}}) \geq 0 \), then we have a decomposition

\[
\mathcal{Z}(u) = \mathcal{Z}(u)^h + \sum_{u \in L} m(u, L)\mathcal{P}_L \tag{9.7}
\]
in the K-group of \( \mathcal{N}_{2}^{[1]} \) with supports on \( \mathcal{Z}(u) \) where

1. \( \mathcal{Z}(u)^h \cong \mathcal{O}_{F_0} \) is the horizontal part of \( \mathcal{Z}(u) \). It meets \( \mathcal{N}_{\text{red}} \) at an ordinary point \( x_u \) on \( \mathcal{P}_{L_u} \), corresponding to the non-isotropic line given by \( u_1 \) in \( L_u - \mathcal{V}L_u \).
(2) A vertex lattice \( L \) contains \( u \) iff \( d(L, L_u) \leq m \), in which case we have

\[
m(u, L) = \max(\max\{r \in \mathbb{Z}| \mod{\omega^{-r}u} \in L\}, 0) = \begin{cases} \frac{1}{2}(m - d(L, L_u)), & m = d(L, L_u) \mod 2, \\ \frac{1}{2}(m + 1 - d(L, L_u)), & m + 1 = d(L, L_u) \mod 2. \end{cases}
\]

(9.8)

Proof. The decomposition follows from explicit equations of \( Z(u) \) on each Bruhat–Tits stratum, see [51] Thm. 3.14. The multiplicity formula follows from [51] Lem. 3.12. \( \square \)

Remark 9.4. The case \( m = 0 \) and \( m = 1 \) form the base shapes of \( Z \)-cycles. Note \( m(u, L) = 0 \) if \( u \in L \sim \omega L \). If \( v((u, u)_{\gamma}) \geq 0 \), we have \( Z(\omega u) - Z(u) = \sum_{u \in L} p_L \).

Theorem 9.5. Let \( u \in V_2 \) with \( v((u, u)_{\gamma}) = m \geq -1 \). Then we have a decomposition in the K-group of \( N_{21} \) with supports on \( Y(u) \):

\[
Y(u) = Y(u) + \sum_{u \in L} m(u, L)p_L \quad (9.9)
\]

(1) Here \( Z(u) = \text{Spf} O_{\mathcal{E}_u} \) is the horizontal part of \( Z(u) \). It meets \( N_{21, u} \) at an ordinary point \( x_u \) on \( \mathbb{P}_{L_{u, 1}} \) corresponding to the non-isotropic line given by \( u_1 \) in \( L_u/L_{u, 1} \).

(2) If \( u \in L \), the multiplicity \( m(u, L) \) is equal to

\[
m(u, L) = \begin{cases} \frac{1}{2}(m + 2 - d(L, L_u)), & m = d(L, L_u) \mod 2, \\ \frac{1}{2}(m + 1 - d(L, L_u)), & m + 1 = d(L, L_u) \mod 2. \end{cases}
\]

(9.10)

Proof. We are done by applying the dual isomorphism \( \Lambda : \mathcal{N} \to \mathcal{N}^\vee \). Note if \( d(L, L_u) = m + 1 \), then \( m(u, L) = 0 \). Hence we can write the summation \( \sum m(u, L)p_L \) over vertex lattices \( L \) such that \( d(L, L_u) \leq m \) i.e., \( u \in L \) instead of \( d(L, L_u) \leq m + 1 \).

\( \square \)

Theorem 9.6. Consider a vertex lattice \( L \subseteq V_2 \).

(1) If \( L \) has type 0, then for any \( u \neq 0 \) we have

\[
(Z(u), p^1_L) = l_1(u), \quad (Y(u), p^1_L) = -qL(u).
\]

(9.11)

(2) If \( L \) has type 2, then for any \( u \neq 0 \) we have

\[
(Z(u), p^1_L) = -qL(u), \quad (Y(u), p^1_L) = l_1(u).
\]

(9.12)

Proof. The number \( (Z(u), p_L) \) is computed in [52] Lem 2.10 explicitly. The result depends on the parity of \( d(L, L_u) - v((u, u)_{\gamma}) \). But it is equivalent to our formulation, as if \( L \) is of type 0 (resp. 2), then \( d(L, L_u) - v((u, u)_{\gamma}) \) is always even (resp. odd). We could also (re)prove the theorem using the Remark 9.4 and the case \( m = 0 \) and \( m = 1 \).

Applying the dual isomorphism \( \mathcal{N} \to \mathcal{N}^\vee \), we have \( (Y(u), p^1_L) = (\lambda_{\gamma} \circ u, p^1_{\lambda(L)}) \). The result for \( Y \)-cycles follows.

The following corollary finishes the proof of Theorem 9.1

Corollary 9.7. Assume that \( n = 2 \), \( t = 1 \), in particular \( V = V_2 \) is split and \( \gamma_{\gamma} = 1 \). Then for any 1-cycle \( C \subseteq N_{21, u} \), we have

\[
\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}(Z(u), C) = -q^{-1}(Y(u), C).
\]

(9.13)

Proof. Any vertical 1-cycle in \( N_{21, u}^{\text{red}} \) is a finite linear combination of \( BT(L) \) for vertex lattices \( L \subseteq V_2 \). By self-duality of the Haar measure on \( V \), we have \( F_{\gamma} l_1 = \gamma(\gamma_{\gamma} \circ u) \gamma(u) = (\#(L_{\gamma}/L))^{-1/2} \gamma^{-1}(\gamma) \). So the result follows from Theorem 9.6 \( \square \)

10. ATCs for unramified maximal orders

In this section, we show Conjecture 6.3 for \( (g, u) \in (U(V) \times V)(F_0)_{\text{rs}} \) holds in the case \( O_F[g] \) is a maximal order that is étale over \( O_F \). By duality, we only need to work with Part (1) i.e., for \( Z \)-cycles. Then we generalize the thesis of Mihatsch [37] Section 9 on AFLs in the maximal order case to our set up under an unramifiedness condition.
10.1. Semi-Lie ATC for $n = 1$. Assume that $n = 1$, and the type of $L \subseteq V$ is $t_0 \in \{0, 1\}$. Consider the function $f_{\text{std}} := 1_S(L, L^\vee) \times 1_{L_0} \times 1_{(L_0^\vee)^*}$, and a regular semisimple pair $(\gamma, b, c) \in (S(V_0) \times V_0 \times (V_0)^*)_{\text{rs}}$.

**Proposition 10.1.** (1) If $v(cb) = t_0 \mod 2$, then
\[ \text{Orb}((\gamma, b, c), f_{\text{std}}) = \begin{cases} 1 & v(cb) \geq t_0, \\ 0 & \text{else.} \end{cases} \quad (10.1) \]

(2) If $v(cb) = t_0 + 1 \mod 2$, then $\text{Orb}((\gamma, b, c), f_{\text{std}}) = 0$ and
\[ \partial \text{Orb}((\gamma, b, c), f_{\text{std}}) = \begin{cases} -(\log q)^{v(cb) - t_0 + 1} & v(cb) \geq t_0 + 1, \\ 0 & \text{else.} \end{cases} \quad (10.2) \]

**Proof.** Choose a generator of $L$, then we are computing orbit integrals for $f_{\text{std}} = 1 \times 1_{O_{F_0}} \times 1_{\omega_o O_{F_0}}$. And the transfer factor is $\omega_L((\gamma, b, c), f_{\text{std}}) = (-1)^{v(b)}$. So $\text{Orb}((\gamma, b, c), f_{\text{std}}, s)$ is
\[ (-1)^{v(b)} \int_{h \in \text{GL}_1(F_0)} 1_{O_{F_0}}(h^{-1}b) 1_{\omega_o O_{F_0}}(ch)(-1)^{v(b)}|h|^s dh = (-1)^{v(b)} \sum_{k = -v(c) + t_0} v(b)(-1)^k q^{-ks}. \]

Take $s = 0$, then the first property follows. Now assume that $v(cb) = t_0 + 1 \mod 2$. Then we obtain
\[ \partial \text{Orb}((\gamma, b, c, 0) = (-1)^{v(b)} \sum_{k = -v(c) + t_0} (-1)^k |\log q|(-k) = -\log q^{v(cb) - t_0 + 1 \mod 2}. \]

\[ \square \]

**Proposition 10.2.** For any regular semisimple pair $(g, u) \in (U(V) \times V)(F_0)_{\text{rs}}$, we have identifications $\text{Fix}(g) = N_{1}^{[t_o]}$ and $Z(u) \simeq \text{Spf} O_{F_0}/\omega^k$ where $k := \max\{0, \frac{u((u, u)) - t_0 + 1}{2}\}$.

**Proof.** By the theory of canonical lifting, we have $N_{1}^{[t_o]} = \text{Spf} O_{F_0}$. As $N_{1}^{[t_o]} O_{F_0} N_{1}^{[t_o]} = N_{1}^{[t_o]}$, the fixed point locus $\text{Fix}(g) = N_{1}^{[t_o]}$ is the whole space. The lifting condition for $Z(u)$ doesn’t depend on the polarization. Only the hermitian form on $V$ changes when we change $t_0 = 0$ to $t_0 = 1$ by scaling. So we may assume $t_0 = 0$. Again by the theory of canonical lifting, we have $Z(u) \simeq \text{Spf} O_{F_0}/\omega^k$. \[ \square \]

10.2. Reduction in the maximal order case. Return to the semi-Lie Conjecture [6.3] for a vertex lattice $L \subseteq V$ of type $t$ and rank $n$. Recall the hermitian space $V = \text{Hom}_G(F_0(E, X))$.

Consider a maximal order $O_F[g] = \prod_{i=1}^m O_{F_i} \rightarrow \text{End}(V)$ which is étale over $O_F$ i.e., all $F_i$ are unramified field extensions of $F$. Then each $F_i$ is an unramified quadratic extension of its fixed subfield $F_0$. Let $f_i$ be the residual degree of $F_i$ over $F_0$. As $F_i = F_0 \otimes_{F_0} F$, we have a decomposition of $F/F_0$-hermitian spaces
\[ V = \prod_{i=1}^m V_i, \quad V_i \cong F_i. \quad (10.3) \]

Lift the $F/F_0$-hermitian form on $V_i$ to a $F_i/F_0$-hermitian form on $V_i$ along the trace morphism $\text{tr}: F_i \rightarrow F_0$. As $F_i$ is unramified over $F$, the hermitian dual of any $O_{F_i}/O_{F_0}$ hermitian lattice agrees with the hermitian dual of its underlying $O_F/O_{F_0}$ hermitian lattice.

**Remark 10.3.** The structure of $O_F[g]$ itself doesn’t determine the structure of $\text{Fix}(g)$, which depends on the isomorphism classes of $V_i$ hence the embedding $O_F[g] \rightarrow \text{End}(V)$. For $n = 2$, any étale maximal order $O_F[g] \simeq F \times F$ is totally split as $f_i$ are odd.

10.3. Fixed points via Bruhat–Tits strata. We can compute $F$-points of $\text{Fix}(g)$ for $g \in U(V)(F_0)_{\text{rs}}$ using the Bruhat–Tits strata in Section [5].

**Proposition 10.4.** Assume that $O_F[g]$ is an étale maximal order. Then there exists a unique vertex lattice $L \subseteq V$ that is $O_F[g]$-stable. Moreover, its type $t(L) = \sum_i f_i$ where the index runs over $i$ such that $V_i$ is non-split as a $F/F_0$-hermitian space.
Hence \( \text{Fix}(L) = L_i \) also fixed by reversing the above process.

\[ \square \]

Assume that \( \tau \) happens if and only if \( V \) is unramified, the following generalization of [36, Prop. 4.14].

Proof. Consider any \( O_F[X]\)-stable vertex lattice \( L \subseteq \mathbb{V} \). Then \( L = \prod_i L_i \) as \( O_F/O_{F_0}\)-hermitian lattices. As \( L \subseteq L_i \subseteq \mathbb{V}^{-1}L_i \), we have inclusions of \( O_F\)-stable lattices \( L_i \subseteq L_i' \subseteq \mathbb{V}^{-1}L_i \).

We know that there is a unique \( O_F\)-stable lattices in \( \mathbb{V} \) up to \( O_F\)-scaling. As \( F_i/F_0 \) is unramified, \( \mathbb{V} \) is a uniformizer for \( F_i \). We have two cases: either \( L_i' = L_i \) thus \( t(L_i) = 0 \), which happens if and only if \( V \) is a split \( F/F_0\)-hermitian space; or \( L_i' = \mathbb{V}^{-1}L_i \) thus \( t(L_i) = f_i \), which happens if and only if \( V \) is a non-split \( F/F_0\)-hermitian space.

Hence \( L_i \) are all uniquely determined. Conversely, we can construct a \( O_F[X]\)-stable lattice by reversing the above process.

If \( x \in \mathcal{N}(\mathbb{F}) \) is a fixed point of \( g \), then the associated special pair \((A_x, B_x)\) by Proposition 8.7 is also fixed by \( g \). Recall \( L_A \) (resp. \( L_B \)) the smallest \( \tau \)-stable lattice containing \( A \) (resp. \( B \)) (8.16) is generated by \( \tau^i(A) \) (resp. \( \tau^i(B) \)) \((i = 0, 1, 2, \ldots)\). So \( L_A, L_B \) are \( O_F[X]\)-stable. Therefore, from Proposition 8.9 and Proposition 10.4 we see that

\[ \text{Corollary 10.5. Assume that } O_F[X] \text{ is an étale maximal order. Let } L_o \text{ be the unique } O_F[X]-\text{stable vertex lattice in } \mathbb{V}. \text{ Then we have an inclusion} \]

\[ \text{Fix}(g)(\mathbb{F}) \subseteq \text{BT}(L_o). \]

Hence \( \text{Fix}(g)(\mathbb{F}) \) agrees with fixed points of \( \mathcal{g} \in U(L'_o/L_o)(\mathbb{F}) \) on the smooth projective variety \( \text{BT}(L_o) \).

Example 10.6. Consider the case \( \sum_{V_i} \) is not quasi-split \( f_i = t - 1 \). Then \( \text{Fix}(g)(\mathbb{F}) \) is equal to the fixed points of \( g \in U(L'_o/L_o)(\mathbb{F}) \) on \( \mathcal{Y}(L'_o) = \mathbb{F}(L_o/\mathbb{V}(L'_o)). \) It corresponds to \( O_F[X] = \prod_{i=1}^{n} O_F\)-stable lines in \( L_o/\mathbb{V}(L'_o) \). It is the number of fixed points of \( g \) is the number of index \( i \) such that \( f_i = 1 \) and \( V_i \) is split.

If \( n = 2 \) and \( t = 1 \), then \( F[g] = F \times F \) and \( \mathbb{V} = \mathbb{V}_{1} \times \mathbb{V}_{2} \). We see any unramified maximal étale \( g \in U(V_2)(F_0) \) has exactly two fixed points on \( (\mathcal{V}_2)_{\text{red}} \), which lies in a type 0 (resp. 2) Bruhat–Tits stratum \( \mathbb{P}^{1}_{L} \) if \( V_1 \) and \( V_2 \) are both split (resp. non-split).

10.4. Fixed points via moduli interpretation. By definition, the subscheme \( \text{Fix}(g) \to \mathcal{N} \) parameterizes tuples \((X, \iota, \lambda, \rho) \in \mathcal{N}\) with a \( O_F\)-linear isomorphism \( \varphi : X \to X \) such that \( \varphi^* \lambda = \lambda \) and that we have a commutative diagram

\[ \begin{array}{ccc}
X \times_{\mathcal{S}} \mathcal{S} & \xrightarrow{g} & X \times_{\mathcal{S}} \mathcal{S} \\
\rho \downarrow & & \downarrow \rho \\
X \times_{\mathcal{F}} \mathcal{F} & \xrightarrow{g} & X \times_{\mathcal{F}} \mathcal{F}
\end{array} \]

For any point \((X, \iota, \lambda, \rho) \in \text{Fix}(g)(\mathcal{S})\), using the idempotents in \( O_F[X] \), we have a decomposition

\[ (X, \iota, \lambda, \rho) = \prod_{i=1}^{m} (X_i, \iota_i, \lambda_i, \rho_i) \quad (10.4) \]

with induced decomposition \( \ker \lambda = \prod \ker \lambda_i \).

Here \( X_i \) is a supersingular hermitian \( O_{F_i} - O_{F_0} \)-module of dimension \( n_i = \dim \mathcal{V}_i \) and \( O_{F_0}\)-type \( t_i \) for some integer \( t_i \). See [36] Defn. 3.1, [5] Defn. 3.21) for the notion. In particular, the \( O_{F_i} - O_{F_0} \) rank of \( X_i \) is 1 and \( \ker \lambda \subseteq X_i \) has order \( q^{2t_i} \). By our assumption, \( n_i = \dim \mathcal{V}_i = f_i \).

As \( X \) satisfies Kottwitz condition of signature \((1, n - 1)\), there is a unique index \( i_0 \) such that \( X_{i_0} \) has signature \((1, n_{i_0} - 1)\) and other \( X_i \) \((i \neq i_0)\) has signature \((0, n_i)\).

Fix \( i_0 \) and \( t_i \) \((i = 1, \ldots, m)\). The \( O_F[X] \) action on \( \mathcal{X} \) induces a decomposition (depending on \( i_0 \) and \( t_i \))

\[ \mathcal{X} = \prod_{i \neq i_0} X_i \]

where \( \mathcal{X} \) is a supersingular hermitian \( O_{F_i} - O_{F_0} \)-module of \( O_{F_i} - O_{F_0} \) rank 1 and \( O_{F_0}\)-type \( t_i \).

Consider the universal object \( X \) on \( \text{Fix}(g) \). As \( F_i/F_0 \) is unramified, we have the comparison between hermitian \( O_{F_i} - O_{F_0} \) modules and hermitian \( O_F \) modules, see [5] Prop. 3.29. We obtain the following generalization of [39] Prop. 4.14].
Theorem 10.7. There is a natural identification of formal schemes over $O_{F_0}$:
\[
\text{Fix}(g) = \prod_{(i_0, \{t_i\}_{i=1}^m) \in \mathcal{N}^{[0]}_{F_0, (1, 0)} \times \prod_{i \neq i_0} \mathcal{N}^{[t_i/f_i]}_{F_i, (0, 1)}
\]
where $(i_0, \{t_i\}_{i=1}^m)$ runs over integers $1 \leq i_0 \leq n, 0 \leq t_i \leq n$ such that
\begin{enumerate}[(1)]
  \item $f_i | t_i$ and $\sum_{i=1}^m t_i = t$.
  \item $t_{i_0}$ is odd if $\forall_{i_0}$ is split, and is even if $\forall_{i_0}$ is non split.
  \item For $i \neq i_0$, $t_i$ is even if $\forall_i$ is split, and is odd if $\forall_i$ is non split.
\end{enumerate}

Note $\sum_{i=1}^m f_i = n \geq t = \sum_{i=1}^m t_i = t$, so condition (1) implies $t_i = 0$ or $f_i$. From this, we see
\[
t_i = \begin{cases} f_i & \text{if } i = i_0, \forall_{i_0} \text{ is split}, \\ 0 & \text{if } i = i_0, \forall_{i_0} \text{ is not split}, \\ f_i & \text{if } i \neq i_0, \forall_i \text{ is split}, \\ 0 & \text{if } i \neq i_0, \forall_i \text{ is not split}. \end{cases}
\]

Corollary 10.8. Assume that $O_F[g]$ is an étale maximal order. Then
\begin{enumerate}[(1)]
  \item $\text{Fix}(g)$ is formally smooth over $\text{Spf } O_{F_0}$.
  \item We have $\text{Fix}(g) = \sqrt[1]{\text{Fix}(g)}$.
\end{enumerate}

Proof. As $\mathcal{N}^{[i]}_1$ and $\mathcal{N}^{[0]}_1$ are isomorphic to $\text{Spf } O_{F_i} = \text{Spf } O_{F_0}$, Fix$(g)$ is formally smooth over $\text{Spf } O_{F_0}$.

For any fixed point $(A, B) \in \mathcal{N}(\mathcal{F})$ of $g$, both $L_A$ and $L_B$ in Proposition 8.9 are $O_F[g]$-stable, so can’t both be vertex lattices by the uniqueness in Proposition 10.4. So Fix$(g)$ doesn’t intersect with the link stratum $\mathcal{N}^1$, hence Fix$(g) = \text{Fix}(g)$. As Fix$(g)$ is purely 1-dimensional, by [59, Lem. B.2.] we have $\sqrt[1]{\text{Fix}(g)} = \text{Fix}(g)$ (using $\mathcal{N}$ and $\mathcal{N} \times \mathcal{N}$ are regular). \hfill \Box

Example 10.9. Assume that $t_0 = 1$, $f_i = 1$ and $\forall_i$ is split for all $i$, then Fix$(g) = \prod_{i=1}^n \text{Spf } O_{F_0}$ which matches Example 10.6.

10.5. Geometric side. Consider a non-zero vector $u = (u_i) \in \mathcal{V} = \prod_i \mathcal{V}_i$.

Proposition 10.10. On the $(i_0, \{t_i\}_{i=1}^m)$-th copy of $\text{Spf } O_{F_0}$ of Fix$(g)$, we have
\[
\text{Z}(u)|_{\text{Spf } O_{F_0}} \cong \text{Z}(u_{i_0}) \hookrightarrow \mathcal{N}^{[t_{i_0}/f_{i_0}]}_{F_0, (1, 0)}.
\]
The $O_{F_0}$-length of $\text{Z}(u)|_{\text{Spf } O_{F_0}}$ is
\[
\max \left\{ \frac{v_F((u_i, u_{i_0}v))}{2} \right\} \quad \text{if } \forall_{i_0} \text{ is split}
\]
\[
\max \left\{ \frac{v_F((u_i, u_j)v + 1)}{2} \right\} \quad \text{if } \forall_{i_0} \text{ is not split}.
\]

Proof. We observe that
\[
\text{Hom}_{O_{F_i}}(\mathcal{E}, X) = \prod_i \text{Hom}_{O_{F_i}}(\mathcal{E} \otimes_{O_{F_i}} O_{F_i}, X_i).
\]
Therefore, the lifting of $u$ corresponds to liftings of each $u_i$ on the right hand side of (10.5).

If $i \neq i_0$ then $X_i$ has signature $(0, 1)$, hence $u_i$ lifts to the whole space $\mathcal{N}^{[t_i/f_i]}_{F_i, (0, 1)}$ by Lubin–Tate theory. So we have $\text{Z}(u)|_{\text{Spf } O_{F_0}} \cong \text{Z}(u_{i_0})$. The computation of length($\text{Z}(u)|_{\text{Spf } O_{F_0}}$) follows from the case $n = 1, t \in \{0, 1\}$, see Proposition 10.2. \hfill \Box

Theorem 10.11. For any regular semisimple pair $(g, u) \in (U(\mathcal{V}) \times \mathcal{V})(F_0)_\mathbb{A}$ such that $O_F[g]$ is an étale maximal order, we have
\[
\text{Int} (g, u) = \sum_{i=1}^m \max \left\{ \frac{v_F((u_i, u_{i_0}v) + a_i)}{2} \right\}.
\]
where $a_i = 0$ if $V_i$ is split and $a_i = 1$ if $V_i$ is not split.

Proof. As $(u, u)_{\mathcal{V}} = \sum_i (u_i, u_i)_{\mathcal{V}}$, the result follows from Corollary 10.8 and Proposition 10.2. \hfill \Box
10.6. Analytic side via counting lattices. Recall that $L \subseteq V$ is a vertex lattice of rank $n$ and type $t$. Choose an orthogonal basis of $L$ to endow $L$ (resp. $V$) with a $O_{F_0}$ (resp. $F_0$)-structure $L_0$ (resp. $V_0$), which induces a $F/F_0$ semi-linear involution $(-)$ on $V$ with fixed subspace $V_0$. Recall the Haar measure on $GL(V_0)$ is normalized so that the volume of $GL(L_0, V_0)$ is 1.

Under the basis, we identify $L_0$ (resp. $L_0^\vee$) with $O_{F_0}$ (resp. $\varpi^{-1}O_{F_0} \oplus O_{F_0}^{-\varpi}$). The $GL(V_0)$-orbit of $(L, L^\vee)$ in the set of pairs of $O_F$-lattices $(L_1, L_2) \subseteq V$ can be identified with the set

$$\text{Lat}_{n, F}^t := \{(L_1, L_2) \subseteq V| L_1 \subseteq L_2 \subseteq \varpi^{-1}L_1, \ [L_2 : L_1] = t, \ \tau(L_1) = L_1, \ \tau(L_2) = L_2\}.$$ 

Set

$$M_{n, F}^t := \{(L_1, L_2) \in \text{Lat}_{n, F}^t| u_1 \in L_1, \ u_2 \in L_2^\vee, \ \gamma L_1 = L_1, \ \gamma L_2 = L_2\},$$

$$M_{n, F, \gamma}^t := \{(L_1, L_2) \in M_{n, F}^t| [L_1 : L] = i\}.$$ 

For the function $f_{\text{std}} = 1_{s(L, L)^\vee} \times 1_{L_0} \times 1_{(L_0^\vee)^*}$ and a regular semisimple pair $(\gamma, u_1, u_2) \in (S(V_0) \times V_0 \times V_0^*) (F_0)_m$, write $(s \in \mathbb{C})$

$$\text{Orb}_F((\gamma, u_1, u_2), s, t) := \text{Orb}((\gamma, u_1, u_2), f_{\text{std}}).$$

Consider the lattice

$$L_{\gamma, u_1} = \sum_i O_F \gamma^i u_1.$$ 

Write $\ell = [L_{\gamma, u_1} : L]$.

**Proposition 10.12.** We have

$$\text{Orb}_F((\gamma, u_1, u_2), s, t) = \sum_{i \in \mathbb{Z}} (-1)^i (\# M_{n, F, \gamma}^t) q^{-i + \ell}.$$ 

**Proof.** By our choice of Haar measure, we have

$$\text{Orb}_F((\gamma, u_1, u_2), s, t) = \omega_L(\gamma, u_1, u_2) \int_{h \in GL(V_0)/GL(L_0, L_0^\vee)} f_{\text{std}}(h, (\gamma, u_1, u_2)) \eta(\det(h)) |\det h|^s dh.$$ 

By definition, $f_{\text{std}}(h, (\gamma, u_1, u_2))$ has value 1 if $(hL, hL^\vee) \in M_{n, F}^t$ and value 0 else. For $(L_1, L_2) = (hL, hL^\vee) \in M_{n, F}^t$, we have $v(\det(h)) = [hL : L] = [L_1 : L] = [L_1 : L_{\gamma, u_1}] + \ell$.

We conclude by noting that the parity of $i = [L_1 : L_{\gamma, u_1}]$ is equal to the transfer factor $\omega_L(\gamma, u_1, u_2).$ \hfill $\square$

Now we assume $(\gamma, u_1, u_2)$ matches $(g, u)$ such that $O_F[g] = \prod_{i=1}^m O_{F_i}$ is an étale maximal order. As the characteristic polynomial of $\gamma$ agrees with $g$, the embedding $O_F[\gamma] \subseteq \text{End}(V)$ induces a compatible decomposition $V = \prod_{i=1}^m V_i$. Here $V_i$ is a $F_i/F_0$ hermitian space such that $V_i \not\subseteq V_i$ and $\dim F V_i = f_i$. Consider the induced decomposition

$$(\gamma, u_1, u_2) = \prod_{i=1}^m (\gamma_i, u_{i1}, u_{i2}).$$

**Proposition 10.13.** We have

$$\text{Orb}_F((\gamma, u_1, u_2), s, t) = \sum_{\{t_i\}_{i=1}^m} \prod_{\{t_i\}_{i=1}^m} \text{Orb}_{F_i}((\gamma_i, u_{i1}, u_{i2}), s/f_i, t_i/f_i)$$

(10.6)

where $\{t_i\}_{i=1}^m$ runs over integers $0 \leq t_i \leq n$ such that $t_i \in \{0, f_i\}$ and $\sum_{i=1}^m t_i = t$.

**Proof.** From definition, we have

$$M_{n, F}^{t_i} = \prod_{\{t_i\}_{i=1}^m} \prod_{\{t_i\}_{i=1}^m} M_{n, F_i}^{t_i/f_i}.$$ 

where $\{t_i\}_{i=1}^m$ runs over integers $0 \leq t_i \leq n$ such that $f_i|t_i$ and $\sum_{i=1}^m t_i = t$. Note $\sum_{i=1}^m f_i = n \geq t$, we see $t_i \leq f_i$ hence $t_i \in \{0, f_i\}$. By definition, $\omega_L(\gamma, u_1, u_2) = \ell = \prod_i t_i = \prod_{i=1}^m \omega_{L_i}(\gamma_i, u_{i1}, u_{i2})$. So the result follows from Proposition 10.12 \hfill $\square$
Corollary 10.14. We have an equality in $\mathbb{Q}\log q$:

$$
\partial_{\mathcal{O}_F}(\{\gamma, u_1, u_2\}, t) = \sum_{i_0, \{t_{1, i_0, i=1}^n} \frac{1}{f_{i_0}} \partial_{\mathcal{O}_F}(\{\gamma_{i_0, u_1, u_2, i_0}\}, t_0/f_{i_0}) \prod_{i \neq i_0} \mathcal{O}_F(\{\gamma, u_1, u_2\}, t_i/f_{i_i}).
$$

where $i_0$ runs over integers $1 \leq i_0 \leq n$ with

$$
t_i = \begin{cases} f_i & \text{if } i = i_0, \forall i_0 \text{ is split}, \\ 0 & \text{if } i = i_0, \forall i_0 \text{ is not split}, \\ 0 & \text{if } i \neq i_0, \forall i_0 \text{ is split}, \\ f_i & \text{if } i \neq i_0, \forall i_0 \text{ is not split}. \\
\end{cases}
$$

Proof. The equality follows from above proposition by taking derivative. We only need to explain the conditions on $(i_0, \{t_{1, i_0, i=1}^n}$. The pair $(\gamma_i, u_{i_1, u_{i_2}})$ matches the pair $(g, u_1, u_2) \in \langle U_i \rangle \times \mathbb{V}_i \rangle (F_0)_{rs}$. Consider any $i \neq i_0$. From Section 10.1, we have the vanishing of $\mathcal{O}_F(\{\gamma_i, u_{i_1, u_{i_2}}\}, 0, t_i/f_{i_i})$ unless $t_i = 0$ if $\forall i_0$ is split and $t_i = f_i$ if $\forall i_0$ is non-split.

Then $t_{i_0}$ is determined by the formula $\sum_{i=1}^n t_i = t$. Compare $\sum_{i=1}^n t_i = t$ and $\mathbb{V} = \otimes_{i=1}^n \mathbb{V}_i$ (using $f_i$ is odd), we see $t_{i_0} = f_i$ (resp. $t_{i_0} = 0$) if $\forall i_0$ is split (resp. not split).

Theorem 10.15. Let $L$ be a vertex lattice of dimension $n$ and type $t$. Consider a regular semisimple pair $(g, u) \in \langle U \rangle \times \mathbb{V}(F_0)_{rs}$ matching $(\gamma, u_1, u_2, \mathbb{V}) \in \langle S(V_0) \times V_0 \times V_0 \rangle (F_0)_{rs}$. Assume that $O_F[g]$ is an étale maximal order, then Conjecture 6.3 holds for $(g, u)$ and $(\gamma, u_1, u_2)$.

Proof. This follows from Theorem 10.11 and Corollary 10.14 and the computation in the case $n = 1$ in Section 10.3.

Part 3. Modularity of arithmetic theta series

11. Integral models and Balloon–Ground stratification

In this section, we introduce the RSZ-variant of unitary Shimura varieties [46, 43] for parahoric Shimura varieties as a globalization of Rapoport–Zink spaces studied in Part 2 which admit natural integral model $\mathcal{M}$ with PEL type moduli interpretation. A general principle [6] is that a Shimura variety of abelian type often admits an isogenous Shimura variety of Hodge type after possibly increasing the reflex field. Then we globalize the formal Balloon–Ground stratification in Section 5.3 to the Balloon–Ground stratification on mod $p$ fiber of $\mathcal{M}$.

11.1. The Shimura data. Let $F/F_0$ be a totally imaginary quadratic extension of a totally real number field. Write $\Sigma_F := \text{Hom}(F, \mathbb{Q})$ and choose a CM type $\Phi \subseteq \Sigma_F$ of $F$ with a distinguished element $\phi_0 \in \Phi$. Let $V$ be an $F/F_0$-hermitian space of dimension $n \geq 1$ with signature $\{(r_{\varphi}, s_{\varphi}) \in \Phi \times (0, 0), (n, 0), (n, 0), \varphi \in \Phi \}$ under a fixed embedding $\mathbb{Q} \hookrightarrow \mathbb{C}$.

Consider the reductive group $G := U(V)$ over $F_0$, and the following reductive groups over $\mathbb{Q}$:

$Z_\mathbb{Q} := \{ c \in \text{Res}_{F_0/\mathbb{Q}} \text{GU}(V) \in \mathbb{G}_m \}$,

$G_\mathbb{Q} := \{ g \in \text{Res}_{F_0/\mathbb{Q}} \text{GU}(V) \in \mathbb{G}_m \}$,

where $c : \text{Res}_{F_0/\mathbb{Q}} \text{GU}(V) \rightarrow \text{Res}_{F_0/\mathbb{Q}} \mathbb{G}_m$ denotes the similitude character.

For any $\varphi \in \Phi$, choose a basis of $V_{\varphi}$ over $\mathbb{C}$ under which the hermitian form is given by $\text{diag}(1, 1, -1)$. It induces a homomorphism $h_{G_\mathbb{Q}, \varphi} : \text{Res}_{F_0/\mathbb{Q}} \mathbb{G}_m \rightarrow \text{Res}_{F_0/\mathbb{Q}} \text{GU}(V_{\varphi})$, $z \mapsto \text{diag}(z, 1, 1)$. The homomorphism

$h_{G_\mathbb{Q}} : \text{Res}_{F_0/\mathbb{Q}} \mathbb{G}_m \rightarrow G_\mathbb{Q} \rightarrow \prod_{\varphi \in \Phi} \text{Res}_{F_0/\mathbb{Q}} \text{GU}(V_{\varphi})$, $z \mapsto (h_{G_\mathbb{Q}, \varphi}(z))_{\varphi \in \Phi}$

(11.1)

gives a Shimura datum $(G_\mathbb{Q}, \{h_{G_\mathbb{Q}}\})$ whose reflex field is $\varphi_0(F) \subseteq \mathbb{Q}$. The homomorphism

$h_{\Phi} : \text{Res}_{F_0/\mathbb{Q}} \mathbb{G}_m \rightarrow Z_\mathbb{Q} \rightarrow \prod_{\varphi \in \Phi} \text{Res}_{F_0/\mathbb{Q}} \mathbb{G}_m$, $z \mapsto (\varphi(z))_{\varphi \in \Phi}$

(11.2)
gives a Shimura datum \((Z^\mathcal{O}, h_\Phi)\) whose reflex field \(F^\text{Ref} \subseteq \overline{Q}\) is the subfield fixed by \(\{\sigma \in \text{Gal}(\overline{Q}/Q) | \sigma \circ \Phi = \Phi\}\). Consider the fibre product

\[ \tilde{G} = Z^\mathcal{O} \times_{G_m} G^\mathcal{O} \]

along projection maps \(\text{Nm}_{F/F_0} : Z^\mathcal{O} \to G_m\) and \(c : G^\mathcal{O} \to G_m\). The homomorphism

\[ h_\mathcal{G} = (h_\Phi, h_\mathcal{G}_0) : \text{Res}_{C/R} G_m \longrightarrow \tilde{G}_R \]

gives a Shimura datum \((\tilde{G}, \{h_\mathcal{G}\})\) whose reflex field \(E\) is the composition \(\varphi_0(F)F^\text{Ref} \subseteq \overline{Q}\). Note there is a natural isomorphism

\[ Z^\mathcal{O} \times_{G_m} G^\mathcal{O} \cong Z^\mathcal{O} \times \text{Res}_{F_0/Q} G, \quad (z, g) \mapsto (z, z^{-1}g). \] (11.3)

For neat compact open subgroups \(K_{Z^\mathcal{O}} \subseteq Z^\mathcal{O}(\mathbb{A}_f)\) and \(K_G \subseteq G(\mathbb{A}_{0,f})\), the RSZ unitary Shimura variety \([46]\) with level \(K_G = K_{Z^\mathcal{O}} \times K_G\) is the Shimura variety associated to \((G, \{h_\mathcal{G}\})\) (over \(C\)):

\[ \text{Sh}_{K_G}(\tilde{G}, \{h_\mathcal{G}\})(C) \cong \text{Sh}_{K_{Z^\mathcal{O}}}(Z^\mathcal{O}, h_\Phi)(C) \times \text{Sh}_{K_G}(\text{Res}_{F_0/Q} G, \{h_\mathcal{G}\})(C). \] (11.4)

From now on, we assume that \(F_0 \neq Q\). Denote by

\[ M = M_{\tilde{G}, \tilde{K}} \longrightarrow \text{Spec } E \]

the canonical model of \(\text{Sh}_{\tilde{K}}(\tilde{G}, h_\mathcal{G})\), which is a \(n-1\) dimensional smooth projective (as \(F_0 \neq Q\)) variety over \(E\).

11.2. The level for \(L\) and \(\Delta\). Choose a finite collection \(\Delta\) of finite places for \(F_0\) such that \(F/F_0\) is unramified outside \(\Delta\) and all \(2\)-adic places are in \(\Delta\).

Let \(L\) be a hermitian lattice in \(V\). If \(v \not\in \Delta\) is a finite place of \(F_0\) such that the localization \(L_v\) is not self-dual, then we assume \(v\) is inert in \(F\) and moreover

- \(L_v \subseteq L_v^c \subseteq \varpi^{-1}_v L_v\) and \(L_v^c/L_v\) has size \(q^{2t_v}\) for some \(0 \leq t_v \leq n\) i.e., \(L_v\) is a vertex lattice in \(V_v\) of type \(t_v\).
- If \(0 < t_v < n\), then \(E \otimes Q_{p_v}\) decomposes into unramified extensions of \(Q_{p_v}\). Here \(p_v > 2\) is the underlying prime of \(v\). In particular, \(F_v\) is unramified over \(Q_{p_v}\).

Let \(K_{Z^\mathcal{O}, \Delta}\) (resp. \(K_{G, \Delta}\)) be a compact open subgroup of \(Z^\mathcal{O}(F_0, \Delta)\) (resp. \(U(V)(F_0, \Delta)\)). Let \(K_G(L)^\Delta\) be the (completed) stabilizer of \(L\) in \(U(V)(\mathbb{A}_{0, f}^\Delta)\), and \(K_{Z^\mathcal{O}, \Delta}^\Delta\) be the unique maximal compact open subgroup of \(Z^\mathcal{O}(\mathbb{A}_{0, f}^\Delta)\). Consider the level structure for \(L\) and \(\Delta\):

\[ K_\mathcal{G} = K_{Z^\mathcal{O}, \Delta} \times K_{Z^\mathcal{O}, \Delta}^\Delta \times K_{G, \Delta} \times K_G(L)^\Delta \subseteq \tilde{G}(\mathbb{A}_{0, f})\].

11.3. The PEL type moduli. Now we define the integral model for \(M\) with level for \(L\) and \(\Delta\) generalizing the construction in \([34]\) Section 14.1, \([46]\) Section 6.1. Consider the functor

\[ M_0 \longrightarrow \text{Spec } O_E[\Delta^{-1}] \]

sending a locally noetherian \(O_E[\Delta^{-1}]-\text{scheme } S\) to the groupoid \(M_0(S)\) of tuples \((A_0, t_0, \lambda_0, \varpi_0)\) where

- \(A_0\) is an abelian scheme over \(S\) of dimension \([F_0 : Q]\).
- \(t_0 : O_F[\Delta^{-1}] \to \text{End}(A_0)[\Delta^{-1}]\) is an \(O_F[\Delta^{-1}]-\text{action satisfying the Kottwitz condition of signature }\{\langle 1, 0 \rangle_{\varphi \in \Phi}\}:

\[ \text{char}(t_0(a)) | \text{Lie } A_0 = \prod_{\varphi \in \Phi} (T - \varphi(a)) \in O_S[T]. \] (11.5)

- \(\lambda_0 : A_0 \to A_0^c\) is an away-from-\(\Delta\) principal polarization such that for all \(a \in O_F[\Delta^{-1}]\) we have

\[ \lambda_0^{-1} \circ t_0(a)^c \circ \lambda_0 = t_0(\varpi). \]

- \(\varpi_0\) is a \(K_{Z^\mathcal{O}, \Delta}\)-level structure as in \([34]\) Section C.3].

An isomorphism between two tuples is a quasi-isogeny preserving the polarization and \(K_{Z^\mathcal{O}, \Delta}\)-level structure. The functor \(M_0 \to \text{Spec } O_E[\Delta^{-1}]\) is representable, finite and étale \([14]\) Prop. 3.1.2.

From now on, we assume that \(M_0\) is non-empty which holds if \(F/F_0\) is ramified at some place, see \([15]\) Rem. 3.5 (ii)]. The generic fiber \(M_0\) of \(M_0\) is a disjoint union of copies of \(\text{Sh}_{K_{Z^\mathcal{O}}}(Z^\mathcal{O}, h_\Phi)\), see \([43]\) Lem 3.4. We work with one copy and still denote its étale integral model by \(M_0\).
Remark 11.1. We allow $L$ to be non-self dual at some $v \not\in \Delta$, so $M_0$ can be non-empty even if $F/F_0$ is unramified everywhere.

Definition 11.2. The integral RSZ Shimura varieties with level $K$ for $L$ and $\Delta$ is the functor

$$M_{G, K} \to \text{Spec } OE[\Delta^{-1}]$$

sending a locally noetherian $OE[\Delta^{-1}]$-scheme $S$ to the groupoid of tuples $(A_0, t_0, \lambda_0, \eta_0, A, t, \lambda, \eta_\Delta)$ where

- $(A_0, t_0, \lambda_0, \eta_0)$ is an object of $M_0(S)$.
- $A$ is an abelian scheme over $S$ of dimension $n[F_0 : \mathbb{Q}]$.
- $t : O_F[\Delta^{-1}] \to \text{End}(A_0)[\Delta^{-1}]$ is an $O_F[\Delta^{-1}]$-action satisfying the Kottwitz condition of signature $\{(n-1)_{\nu, \varphi}, (n, 0)_{\varphi \in \Phi - \{\varphi_0\}}\}$.
- $\lambda : A \to A^\vee$ is a polarization such that for all $a \in O_F[\Delta^{-1}]$ we have
  $$\lambda^{-1} \circ (a)^\vee \circ \lambda = t(a),$$
- $\Phi$ is a $K_{G, \Delta}$-orbit of isometries of hermitian modules
  $$\eta_\Delta : V_\Delta(A_0, A) = \prod_{v \in \Delta} V_v(A_0, A) \cong V(F_0, \Delta)$$
as smooth $F_0, \Delta$-sheaves on $S$. Here $V_v(A_0, A) = \text{Hom}_{F \otimes_{\mathbb{Q}_p} F_0}(V_v A_0, V_v A)$ is the Hom space between rational Tate modules of $A_0$ and $A$, with the hermitian form
  $$(x, y) := \lambda_0^{-1} \circ y^\vee \circ \lambda \circ x \in \mathcal{O}_S.$$
- For any finite place $v \not\in \Delta$ of $F_0$ such that $L_v$ is not self-dual, the induced map $\lambda_v : A[\varpi_v^{\infty}] \to A^\vee[\varpi_v^{\infty}]$ has kernel $\text{Ker} \lambda_v \subseteq A[\varpi_v]$ which has order $\#L_v/L_v = q_v^{2n_v}$.

Remark 11.1. An isomorphism between two tuples is a pair of quasi-isogenies $\langle \phi_0, \phi \rangle : (A_0, A) \to (A_0', A')$ preserving polarizations and the $K_{G, \Delta} \times K_{G, \Delta}$-orbit of level structures.

Theorem 11.3. [39] Thm. 6.2] The functor $M_{G, K}$ is representable by a separated scheme flat and of finite type over $\text{Spec } OE[\Delta^{-1}]$. Moreover, $M_{G, K}$ is smooth over $OE \otimes_{\mathbb{Q}_p} O_{F, v}$ for any finite place $v \not\in \Delta$ such that $L_v$ is self-dual.

From now on, for simplicity we write $\mathcal{M} = M_{G, K}$ and write $(A_0, A, \eta)$ for a $S$-point of $\mathcal{M}$. By the assumption $F_0 \not= \mathbb{Q}$, $\mathcal{M}$ is in fact projective over $OE[\Delta^{-1}]$.

11.4. The singularity and Balloon–Ground stratification. Now we analyze the singularity of $\mathcal{M}$ at bad places. Let $v \not\in \Delta$ be a finite place of $F_0$ such that $L_v$ is not self-dual. Choose a place $w/v$ of the reflex field $E$. Denote by $k_w = \mathbb{Q}_w$ (resp. $k_v$) the residue field of $F_v$ (resp. $E_v$). Denote by $\mathcal{M}_{k_v}$ the special fiber of $\mathcal{M}$ over $k_v$.

Theorem 11.4. [5] Prop. 4.2.] If $t_v = n$, then $\mathcal{M} \to \text{Spec } OE, v$ is smooth. If $0 < t_v < n$, then $\mathcal{M} \to \text{Spec } OE, v$ is of strictly semi-stable reduction.

The proof is based on computations of local models. For our need, we give a different proof using Grothendieck–Messing theory, which contains finer information and relates the singularity of $\mathcal{M} \to \text{Spec } OE, v$ with the Balloon-Ground stratification on $\mathcal{M}$.

Let $S$ be a locally noetherian scheme over $k_v$. For any point $(A_0, A, \eta) \in M(S)$, the Hodge filtration induces a short exact sequence of locally free $\mathcal{O}_S$-modules $(\omega_{A^\vee} := (\text{Lie } A^\vee)^\vee)$:

$$0 \to \omega_{A^\vee} \to H^1_{\text{K}}(A) \to \text{Lie } A \to 0. \qquad (11.6)$$

which decomposes under the $O_F$-action into $\varphi$-parts for $\varphi \in \Phi \cup \Phi^\vee = \text{Hom}(F, \overline{\mathbb{Q}})$:

$$0 \to (\omega_{A^\vee})_{\varphi} \to H^1_{\text{K}}(A)_{\varphi} \to (\text{Lie } A)_{\varphi} \to 0.$$
By the Kottwitz signature condition, \((\omega_A)_{\varphi_0}\) (resp. \((\omega_A)_{\varphi_0}'\)) is a vector bundle on \(S\) of rank 1 (resp. \(n - 1\)).

Consider the alternating bilinear form
\[
(\cdot, \cdot): H^1_{dR}(A) \times H^1_{dR}(A) \to O_S
\]
induced by the polarization \(\lambda: A \to A\). As the kernel \(\text{Ker} \chi_{|A_v}\) has order \(q^{2v}\), the orthogonal complements \((H^1_{dR}(A)_{\varphi_0})^\perp \subseteq H^1_{dR}(A)_{\varphi_0}\) and \((H^1_{dR}(A)_{\varphi_0}')^\perp \subseteq H^1_{dR}(A)_{\varphi_0}'\) are vector bundles of rank \(t_v\) over \(S\).

**Definition 11.5.** (1) The Balloon stratum \(M^b_{k_v}\) is the closed locus of \(M_{k_v}\) where \((\omega_A)_{\varphi_0} \subseteq (H^1_{dR}(A)_{\varphi_0})^\perp\).

(2) The Ground stratum \(M^g_{k_v}\) is the closed locus of \(M_{k_v}\) where \((H^1_{dR}(A)_{\varphi_0})^\perp \subseteq (\omega_A)_{\varphi_0}'\).

(3) The Linking stratum \(M^l_{k_v} := M^b_{k_v} \cap M^g_{k_v} \to M_{k_v}\).

The Balloon–Ground stratification \(M_{k_v} = M^b_{k_v} \cup M^g_{k_v}\) globalizes the formal Balloon–Ground stratification in Section 5.3.

**Proposition 11.6.** The Balloon stratum \(M^b_{k_v}\) is the vanishing locus of the universal section \(\text{Lie} \lambda : A \to A\) inside \(M_{O_{E_v}}\). The Ground stratum \(M^g_{k_v}\) is the vanishing locus of the universal section \(\text{Lie} \lambda^V : A^V \to A\) inside \(M_{O_{E_v}}\). Here \(\lambda^V\) is the dual polarization of \(\lambda\) at \(v\) such that \(\lambda^V \circ \lambda = [\omega_v]\).

**Proof.** This is proved in the same way as Proposition 5.13. \(\square\)

**Theorem 11.7.** If \(0 < t_v < n\), then \(M \to \text{Spec} O_{E_v}\) is of strictly semi-stable reduction. Moreover,

(1) For a geometric point \(x\) of \(M^l_{k_v}\), the completed local ring of \(M\) at \(x\) admits a smooth map to \(W(k_v)[[x_1, y_1]]/(x_1 y_1 - p)\) such that \(M^b_{k_v}\) (resp. \(M^g_{k_v}\)) is the pullback of the Cartier divisor \(x_1 = 0\) (resp. \(y_1 = 0\)).

(2) The Balloon stratum \(M^b_{k_v}\) and the Ground stratum \(M^g_{k_v}\) are smooth projective varieties of dimension \(n - 1\) over \(k_v\).

**Proof.** The proof is similar to Proposition 5.9. We compute the completed local rings via Grothendieck–Messing theory. By our assumption on \(L\), the local ring \(O_{E_v} \cong W(k_v)\) is the ring of Witt vectors for \(k_v\). Let \(k\) be any perfect field over \(k_v\) and \(A\) be a \(k\)-point of \(M\).

Denote by \(D(A)\) the covariant Dieudonné crystal of \(A\). Consider the Dieudonné module \(M = D(A)(W(k))\) which is finite free over \(W(k)\) of rank \(2n[F_0 : \mathbb{Q}]\). By the signature condition, we only need to consider deformations \(\varphi_0\) and \(\varphi_0'\) part of the Hodge Filtration of \(A\). Now \(M_{\varphi_0}\) and \(M_{\varphi_0}'\) are finite free over \(W(k)\) of rank \(n\). The short exact sequence \(0 \to \text{Lie}(A^V) \to H^1_{dR}(A) \to \text{Lie}(A) \to 0\) can be identified by Dieudonné theory with \((V_M, \text{the Verschiebung operator on } M)\)\(\to VM/M/\mathbb{Z}_v M \to M/M/\mathbb{Z}_v M \to M/V_M M \to 0\).

Let \(V = K^n\) be an \(n\)-dimensional quadratic space over \(K := W(k)[1/p]\) with the bilinear form
\[
(x, y) = \sum_{i=1}^{t_v} \mathbb{Z}_v x_i y_i + \sum_{i=t_v+1}^n x_i y_i.
\]

Let \(\text{Ker}\) be the radical of this bilinear form. We can identify \(M_{\varphi_0} \otimes K\) and \(M_{\varphi_0}' \otimes K\) with \(V\) such that the pairing between them induced by \(\lambda\) can be identified with the quadratic form above. By Grothendieck–Messing theory we see that the local model for \(M \to \text{Spec} O_{E_v}\) is the following moduli problem \((\ell, H)\) for \(W(k_v)\)-algebras \(R\) where \(p \in R\) is locally nilpotent:

\(\ell\) is a line in \(V \otimes R\) and \(H\) is a hyperplane in \(V \otimes R\) such that \((\ell, H) = 0\).

The Hodge filtration of \(A\) defines such a pair \((\ell_0, H_0)\) for \(R = k\). And we can compute the completed local ring of \(M\) at \(x\) by deforming \((\ell_0, H_0)\) in the local model. If \(\text{Ker} \not\subseteq H\), then \(\text{Ker} + H = V \otimes R\), hence \((\ell, V \otimes R) = 0\) i.e., \(\ell \subseteq \text{Ker}\). So we have the following three cases:

- \(\ell \subseteq \text{Ker}\), and \(\text{Ker} \subseteq H\), which corresponds to \(x \in M^l_{k_v}\).
- \(\ell \not\subseteq \text{Ker}\), and \(\text{Ker} \not\subseteq H\), which corresponds to \(x \in M^g_{k_v} = M^l_{k_v}\).
\[ \ell \subseteq \text{Ker}, \text{ and Ker} \not\subseteq H, \text{ which corresponds to } x \in \mathcal{M}_{k_0} - \mathcal{M}_{k_0}^{\text{reg}}. \]

Standard computations show the semi-stable reduction results (1) and (2).

**Remark 11.8.** If \( t_0 = 1 \) i.e., \( L_0 \) is almost-self dual, the Balloon–Ground stratification is introduced in [32, Section 5] under the assumption that \( p \) is a special inert prime in particular \( F_{0,v_0} = \mathbb{Q}_p \). In this case, the Balloon stratum \( \mathcal{M}_0^\text{reg} \) lies in the basic locus of \( \mathcal{M}_{k_0} \). Hence the basic locus of \( \mathcal{M}_{k_0} \) contains some irreducible components of \( \mathcal{M}_{k_0} \). Such phenomenon also appears in Stamm’s example [30] for Hilbert modular surfaces at Iwahori level.

If \( 1 < t_0 < n - 1 \), we will show in Section 13.5 that the Balloon and Ground strata of \( \mathcal{M}_{k_0} \) are irreducible (in any given connected component of \( \mathcal{M}_{k_0} \)), which matches the conjecture that any non-basic Kottwitz–Rapoport stratum is “irreducible”, see [13].

12. KUDLA–RAPOPORT CYCLES AND MODIFIED HECKE CM CYCLES

In this section, we globalize the construction in Section 13. Choose a distinguished place \( v_0 \not\in \Delta \) of \( F_0 \) such that \( L_{v_0} \) is not self-dual. We introduce two kinds of Kudla–Rapoport cycles on the integral model with testing functions \( 1_{L_{v_0}} \) and \( 1_{L_{v_0}^0} \) at \( v_0 \) respectively. Then we desingularize the self-product \( \mathcal{M} \times \mathcal{M} \) and define (derived) modified Hecke CM cycles on \( \mathcal{M} \). Finally, we prove basic uniformization results to connect with Section 5.

12.1. The Kudla–Rapoport divisors over the generic fiber. Consider any neat compact open subgroup \( K_G = K_{2,0} \times K_G \) of \( \tilde{G}(k_f) \) (not necessarily from a lattice \( L \)).

For any \((A_0, A, \eta) \in M_{\tilde{G}, \tilde{K}}(S)\), endow the \( F \)-vector space \( \text{Hom}_{\tilde{G}}^\vee(A_0, A) \) with the hermitian form

\[ \langle u_1, u_2 \rangle := \lambda_0^{-1} \circ u_2 \circ \lambda \circ u_1 \in \text{End}_F^\vee(A_0) \cong F. \]  

**Definition 12.1.** Let \( \xi \in F_0 \) and \( \mu \in V(\mathbb{A}_{0,f})/K_G \). The Kudla–Rapoport cycle \( Z(\xi, \mu) \) is the functor sending any locally noetherian \( E \)-scheme \( S \) to the groupoid of tuples \((A_0, A, \eta, u)\) where

- \((A_0, A, \eta) \in M_{\tilde{G}, \tilde{K}}(S)\);
- \( u \in \text{Hom}_{\tilde{G}}^\vee(A_0, A) \) such that \( \langle u, u \rangle = \xi \).
- \( \eta(u) \) is in the \( K_G \)-orbit of \( \mu \).

An isomorphism between two tuples \((A_0, A, \eta, u)\) and \((A'_0, A', \eta', u')\) is an isomorphism of tuples \((\phi_0, \phi) : (A_0, A, \eta) \cong (A'_0, A', \eta') \) such that \( \phi \circ u = u' \circ \phi_0 \).

By positivity of the Rosati involution, \( Z(\xi, \mu) \) is empty unless \( \xi \in F_0^\times \) is totally positive definite or \( \xi = 0 \). The natural forgetful morphism \( i : Z(\xi, \mu) \to M \) is finite and unramified [30, Prop. 4.22]. It is étale locally a Cartier divisor if \( \xi \neq 0 \).

By taking the image of \( Z(\xi, \mu) \) (with multiplicities) inside \( M_{\tilde{G}, \tilde{K}} \), we view \( Z(\xi, \mu) \) as elements in the Chow group \( \text{Ch}^1(M_{\tilde{G}, \tilde{K}}) \). For a Schwartz function \( \phi \in S(V(\mathbb{A}_{0,f}))^{K_G} \) and \( \xi \in F_0^\times \), the \( \phi \)-averaged Kudla–Rapoport divisor is the finite summation

\[ Z(\xi, \phi) := \sum_{\mu \in V_\xi(\mathbb{A}_{0,f})/K_G} \phi(\mu) Z(\xi, \mu) \in \text{Ch}^1(M_{\tilde{G}, \tilde{K}}). \]  

Here \( V_\xi = \{ x \in V | \langle x, x \rangle = \xi \} \) is the hyperboloid in \( V \) of length \( \xi \). We put

\[ Z(0, \phi) := -\phi(0)c_1(\omega) \in \text{Ch}^1(M_{\tilde{G}, \tilde{K}}) \]

where \( \omega \) is the automorphic Hodge line bundle as in [22].

12.2. Generating series. For \( \xi \in F_0 \), consider the weight \( n \) Whittaker function on \( \text{SL}_2(F_{0,\infty}) \):

\[ W_\xi^{(n)}(h_\infty) = \prod_{v|\infty} W_\xi^{(n)}(h_v). \]

Consider the Weil representation \( \omega \) of \( \text{SL}_2(\mathbb{A}_{0,f}) \) on \( S(V(\mathbb{A}_{0,f})) \) which commutes with the natural action of \( U(V)(\mathbb{A}_{0,f}) \). For a Schwartz function \( \phi \in S(V(\mathbb{A}_{0,f}))^{K_G} \), we form the generating series of Kudla–Rapoport divisors on \( M_{\tilde{G}, \tilde{K}} \) by

\[ Z(h, \phi) := W_0^{(n)}(h_\infty) Z(0, \omega(h_f) \phi) + \sum_{\xi \in F_{0,\infty}} W_\xi^{(n)}(h_\infty) Z(\xi, \omega(h_f) \phi) \in \text{Ch}^1(M_{\tilde{G}, \tilde{K}}). \]  

(12.3)
for any \( h = (h_\infty, h_f) \in SL_2(\mathbb{A}_0) = SL_2(F_{0, \infty}) \times SL_2(\mathbb{A}_{0, f}) \).

For any \( h_f \in SL_2(\mathbb{A}_{0, f}) \), by definition we have

\[
Z(h, \omega(h_f)\phi) = Z(hh_f, \phi). \quad (12.4)
\]

The known modularity over \( E \) \[31\, 64\] is one main input of our modularity result.

**Proposition 12.2.** \[59\] Thm. 8.1] The function \( Z(h, \phi) \) is a weight \( n \) holomorphic automorphic form valued in \( Ch^1(M_{G, R}) \) i.e., we have

\[
Z(h, \phi) \in A_{hol}(SL_2(\mathbb{A}_0), K_0, n) \otimes \mathbb{Q} Ch^1(M_{\tilde{G}, \tilde{R}}) \mathbb{Q}
\]

where \( K_0 \subseteq SL_2(\mathbb{A}_{0, f}) \) is any compact open subgroup that fixes \( \phi \) under the Weil representation.

12.3. The Kudla–Rapoport divisors on the integral model \( M \). Consider the level structure for \( L \) and \( \Delta \) in section 11.2.

\[
K_{\tilde{G}} = K_{\tilde{G}_0} \times K_{Z, \Delta} \times K_{G, \Delta} \times K_G(L) \Delta \leq G(\mathbb{A}_f)
\]

So \( K_{G, v_0} = U(L_{v_0}) \) and \( L_{v_0} \) is a vertex lattice of type 0 \( \leq t_0 \leq n \).

For any point \((A_0, A, \eta) \in \mathcal{M}_{G, R}(S)\), consider the hermitian form \( \langle \cdot, \cdot \rangle \) on \( \text{Hom}_{O_E}(A_0, A)[1/\Delta] \) given by

\[
\langle u_1, u_2 \rangle = h_0^{-1} u_2^T \sigma \lambda \circ u_1 \in \text{End}_{O_E}(A_0)[1/\Delta] \simeq O_F[1/\Delta].
\]

The polarization \( \lambda : A \rightarrow A^\vee \) gives an injection

\[
\lambda \circ - : \text{Hom}_{O_E}(A_0, A)[1/\Delta] \rightarrow \text{Hom}_{O_E}(A_0, A^\vee)[1/\Delta]
\]

**Definition 12.3.** Let \( \xi \in F_{0, +} \) and \( \mu_{\Delta} \in V(F_{0, f})/K_{G, \Delta} \). The Kudla–Rapoport cycle \( \mathcal{Z}(\xi, \mu_{\Delta}) \) (resp. \( \mathcal{Y}(\xi, \mu_{\Delta}) \)) is the functor sending any locally noetherian scheme \( S \) over \( O_E[1/\Delta] \) to the groupoid of tuples \((A_0, A, \eta, u)\) where

- \((A_0, A, \eta) \in \mathcal{M}_{G, R}(S)\);
- \( u \in \text{Hom}_{G}(A_0, A) \) with \( \langle u, u \rangle = \xi \).
- \( u \in \text{Hom}_{O_E}(A_0, A)[1/\Delta] \) (resp. \( \lambda \circ u \in \text{Hom}_{O_E}(A_0, A^\vee)[1/\Delta] \)).
- \( \eta(u) \) is in the \( K_{G, \Delta} \)-orbit \( \mu_{\Delta} \).

**Proposition 12.4.** The maps \( \mathcal{Z}(\xi, \mu_{\Delta}) \rightarrow \mathcal{M}_{G, R} \) and \( \mathcal{Y}(\xi, \mu_{\Delta}) \rightarrow \mathcal{M}_{G, R} \) are representable, finite and unramified, and \( \text{étale} \) locally Cartier divisors on \( \mathcal{M}_{G, R} \).

**Proof.** This follows from the proof of \[30\, \text{Prop. 4.22} \]. \( \square \)

**Remark 12.5.** We have a natural inclusion \( \mathcal{Z}(\xi, \mu_{\Delta}) \rightarrow \mathcal{Y}(\xi, \mu_{\Delta}) \). In general, the Cartier divisors \( \mathcal{Z}(\xi, \mu_{\Delta}) \) and \( \mathcal{Y}(\xi, \mu_{\Delta}) \) may not be flat over \( O_E \), see the Shimura curve case \[25\].

Choose \( \phi_{\Delta} \in S(V(\mathbb{A}_f)) \) as the indicator function of the completion of \( L \) inside \( V(\mathbb{A}_f) \).

For \( i \in \{1, 2\} \), consider the Schwartz function \( \phi_i = \phi_\Delta \otimes \phi_{v_0} \otimes \phi_{\Delta, v_0} \in S(V(\mathbb{A}_f))^K \mathbb{A} \) where \( \phi_\Delta \in S(V(F)_{K, \Delta}) \).

\[
\phi_{v_0} = 1_{L_{v_0}}, \quad \phi_{2, v_0} = 1_{L_{v_0}}
\]

For \( \xi \in F_{0, +} \), the \( \phi_i \)-averaged Kudla–Rapoport divisor is the finite summation

\[
\mathcal{Z}(\xi, \phi_i) = \sum_{\mu_{\Delta} \in V_i(F_{0, f})/K_{G, \Delta}} \phi_\Delta(\mu_{\Delta}) \mathcal{Z}(\xi, \mu_{\Delta}), \quad (12.6)
\]

\[
\mathcal{Z}(\xi, \phi_2) = \sum_{\mu_{\Delta} \in V_i(F_{0, f})/K_{G, \Delta}} \phi_\Delta(\mu_{\Delta}) \mathcal{Y}(\xi, \mu_{\Delta}) \quad (12.7)
\]

viewed as elements in the Chow group \( Ch^1(M_{G, R}) \).

**Proposition 12.6.** For \( \phi_i \in S(V(\mathbb{A}_f))^K \mathbb{A} \) above \((i = 1, 2) \), the generic fiber of \( \mathcal{Z}(\xi, \phi_i) \) over \( E \) agrees with \( \mathcal{Z}(\xi, \phi_1) \) defined by \[12.2 \].

**Proof.** This follows from PEL type moduli interpretation of the canonical model \( M_{G, R} \) similar to Definition 11.2 \( \square \)
12.4. **The Hecke CM cycles on the integral model** \( \mathcal{M} \). Fix a degree \( n \) monic polynomial \( \alpha \in F[t] \) that is conjugate self-reciprocal i.e.,
\[
t^n \alpha(t^{-1}) = \alpha(0)\alpha(\overline{t}).
\]

**Definition 12.7.** For any \( \mu_{G,\Delta} \in K_{G,\Delta}(G(F_0,\Delta))/K_{G,\Delta} \), the naive Hecke CM cycle \( \mathcal{CM}(\alpha, \mu_{G,\Delta}) \to \mathcal{M} \)
is the functor sending any locally noetherian scheme \( S \) over \( O_E[1/\Delta] \) to the groupoid of tuples \((A_0, A, \eta, \varphi)\) where
1. \( (A_0, A, \eta) \in \mathcal{M}(S) \).
2. \( \varphi \in \text{End}_{O_E}(A)[1/\Delta] \) is a prime-to-\( \Delta \) endomorphism such that \( \text{char}(\varphi) = \alpha \in F[t] \).
3. \( \varphi^* \lambda = \lambda_A \) and \( \eta_1 \varphi \eta_2^{-1} \in \mu_{G,\Delta} \) for some \( \eta_1, \eta_2 \in \eta \).

**Definition 12.8.** The Hecke correspondence
\[
\text{Hk}_{\mu_{G,\Delta}} : \mathcal{M} \to \mathcal{M} \times_{O_E[1/\Delta]} \mathcal{M}
\]
is the functor sending any locally noetherian scheme \( S \) over \( O_E[1/\Delta] \) to the groupoid of tuples \((A_0, A_1, \eta_1 A_1, A_2, \eta_2 A_2, \varphi, \lambda)\) where
1. \( (A_0, A_1, \eta_1 A_1, A_2, \eta_2 A_2) \in \mathcal{M}(S) \).
2. \( \varphi \in \text{Hom}_{O_E}(A_1, A_2)[1/\Delta] \) is a prime-to-\( \Delta \) homomorphism such that \( \varphi^* \lambda_B = \lambda_A \).
3. We have \( \eta_2 A_2 \varphi \eta_1^{-1} \in \mu_{G,\Delta} \) for some \( \eta_1, \eta_2 \in \eta \).

They generalize the constructions in [59, Section 7.5].

**Proposition 12.9.** The pullback of \( \text{Hk}_{\mu_{G,\Delta}} \) along the diagonal morphism \( \Delta : \mathcal{M} \to \mathcal{M} \times_{O_E[1/\Delta]} \mathcal{M} \)
has the following decomposition as schemes over \( O_E[\Delta^{-1}] \):
\[
\text{Hk}_{\mu_{G,\Delta}}|\mathcal{M} = \coprod_{\alpha \in \mathcal{M}} \mathcal{CM}(\alpha, \mu_{G,\Delta}).
\](12.8)

where the index \( \alpha \) runs over all degree \( n \) conjugate self-reciprocal monic polynomials \( \alpha \in O_E[\Delta^{-1}][t] \).

**Proof.** This follows from the moduli interpretation directly similar to in [59, Section 7.5]. \( \square \)

Assume that \( \alpha \in F[t] \) is irreducible. Then the map \( \mathcal{CM}(\alpha, \mu_{G,\Delta}) \to \mathcal{M} \) is finite and unramified as in [59, Section 7.5]. Hence \( \mathcal{CM}(\alpha, \mu_{G,\Delta}) \) is proper over \( O_E[1/\Delta] \) by the assumption \( F_0 \neq \mathbb{Q} \).

12.5. **Derived Hecke CM cycles and blow up.** Assume that \( \alpha \in O_E[\Delta^{-1}][t] \) is irreducible.

Then \( F_{\alpha,0} := F[t]/\alpha(t) \) is a CM number field. The involution on \( F \) extends to \( F_{\alpha} \) by sending \( t \) to \( t^{-1} \). The subalgebra \( F_{\alpha,0} \) fixed by the involution on \( F_{\alpha} \) is a product of totally real fields over \( F_0 \).

We have an isomorphism \( F_{\alpha} \cong F_{\alpha,0} \otimes_{F_0} F \). Consider the \( O_E[1/\Delta] \)-order generated by \( t \) in \( F_{\alpha} \):
\[
O_{F}(\alpha) := O_E[1/\Delta][t]/(\alpha(t)).
\]
We say \( O_{F}(\alpha) \) is a maximal order at a finite place \( v \notin \Delta \) of \( F_0 \), if \( O_{F}(\alpha) \otimes_{O_{F_0}} O_{F_{\alpha,v}} \) is a product of discrete valuation rings.

**Definition 12.10.** The modified self product of \( \mathcal{M} \) is the blow up of \( \mathcal{M} \times_{O_E[1/\Delta]} \mathcal{M} \) along union of Weil divisors \( \mathcal{M}_{L_v} \times_{L_v} \mathcal{M}_{L_v} \) for all finite places \( v \) of \( E \) over \( v \) such that \( L_v \) is not self-dual:
\[
\mathcal{M} \times_{O_E[1/\Delta]} \mathcal{M} \to \mathcal{M} \times_{O_E[1/\Delta]} \mathcal{M}.
\]

**Proposition 12.11.** Then the scheme \( \mathcal{M} \times_{O_E[1/\Delta]} \mathcal{M} \to \mathcal{M} \times_{O_E[1/\Delta]} \mathcal{M} \) is regular and of strictly semi-stable reduction over \( O_E[1/\Delta] \).

**Proof.** We have assumed that \( E_v \) is unramified over \( \mathbb{Q}_p \). This follows from Theorem [11.7] and computations on completed local rings. \( \square \)

**Proposition 12.12.** The Hecke correspondence \( \text{Hk}_{\mu_{G,\Delta}} \) lifts naturally along \( \mathcal{M} \times_{O_E[1/\Delta]} \mathcal{M} \to \mathcal{M} \times_{O_E[1/\Delta]} \mathcal{M} \) via strict transforms.
Proof. This is a global analog of Proposition 5.17. We only put the level structure \( \mu_G, \Delta \) on \( \Delta \), so the first projection \( \text{pr}_1 : H_k\mu_G, \Delta \to M \) is smooth and \( H_k\mu_G, \Delta \) is regular.

From moduli interpretation, the pullback of \( \mathcal{M}_k^2 \times \kappa \eta \mathcal{M}_k^2 \) to \( H_k\mu_G, \Delta \) is the preimage of \( \mathcal{M}_k^0 \) under \( \text{pr}_1 \), hence a divisor in \( H_k\mu_G, \Delta \). Due to regularity of \( H_k\mu_G, \Delta \), \( \text{pr}_1^{-1}(\mathcal{M}_k^0) \) is a Cartier divisor hence the strict transform of \( H_k\mu_G, \Delta \) is an isomorphism.

In particular, we get a natural lifting of the diagonal morphism

\[
\hat{\Delta} : M \longrightarrow \widehat{M \times_{\mathcal{O}_E[1/\Delta]} M}. \tag{12.9}
\]

**Definition 12.13.** The derived fixed point locus of the Hecke correspondence \( H_k\mu_G, \Delta \) is the derived tensor product

\[
\hat{1}^\text{Fix}(\mu_G, \Delta) := \mathcal{O}_H\mu_G, \Delta \otimes \hat{1}^\text{M \times_{\mathcal{O}_E[1/\Delta]} M} \mathcal{O}_M,
\]

viewed as an element in the \( K \)-group \( K_0^G(M) \) of coherent sheaves on \( M \) with \( \mathbb{Q} \)-coefficients.

The modified derived Hecke CM cycle

\[
\hat{1}^\mathcal{C}M(\alpha, \mu_G, \Delta) \in K_0^G(\mathcal{C}M(\alpha, \mu_G, \Delta))
\]

is the restriction of \( \hat{1}^\text{Fix}(\mu_G, \Delta) \) to \( \mathcal{C}M(\alpha, \mu_G, \Delta) \) under the decomposition \( \hat{1}^G \mathcal{C}. \)

By [58, Appendix. B.3] and regularity of the modified self product of \( M \), \( \hat{1}^\mathcal{C}M(\alpha, \mu_G, \Delta) \) is a virtual 1-cycle in the sense that it lies in the subgroup \( F_1K_0^G(\mathcal{C}M(\alpha, \mu_G, \Delta)) \) generated by coherent sheaves supported on 1-dimensional subschemes of \( \mathcal{C}M(\alpha, \mu_G, \Delta) \).

**Remark 12.14.** By the theory of complex multiplication, the generic fiber \( CM(\alpha, \mu_G, \Delta) \) over \( E \) of \( \mathcal{C}M(\alpha, \mu_G, \Delta) \) is 0-dimensional. Hence \( \hat{1}^G \mathcal{C}M(\alpha, \mu_G, \Delta) \) is the preimage of \( G(F_0, \Delta) \)-invariant Schwartz functions on \( G(F_0, \Delta) \).

**Definition 12.15.** For any function \( \phi_{CM, \Delta} \in S(K_0^G(\mathcal{C}M(\alpha, \mu_G, \Delta))) \), the \( \phi_{CM, \Delta} \)-averaged version of modified derived Hecke CM cycle is the finite summation

\[
\hat{1}^\mathcal{C}M(\alpha, \phi_{CM}) = \sum_{\mu_G, \Delta \in K_0^G(\mathcal{C}M(\alpha, \mu_G, \Delta))} \phi_{CM, \Delta}(\mu_G, \Delta) \hat{1}^\mathcal{C}M(\alpha, \mu_G, \Delta) \in F_1K_0^G(M). \tag{12.10}
\]

Here \( \phi_{CM} := \phi_{CM}^\Delta \otimes \phi_{CM, \Delta} \) with \( \phi_{CM}^\Delta := 1_{K_0^G} \).

12.6. Basic uniformization and local-global compatibility. In the rest of this section, we establish basic uniformizations of special cycles at inert places of \( F_0 \), see also [13, Thm. 8.15]. We show local-global compatibility under blow ups and relate (modified) local and global intersection numbers.

Let \( \nu \) be a finite place of \( E \) with \( \nu|_F = w_0 \), \( \nu|_{F_0} = v_0 \). Assume that \( v_0 \) is inert in \( F \). Consider the RSZ Shimura variety

\[
\mathcal{M} \longrightarrow \text{Spec} \mathcal{O}_E[1/\Delta]
\]

associated to \( L \) in Section [11] and the (relative) unitary Rapoport–Zink space

\[
\mathcal{N} \longrightarrow \text{Spf} \mathcal{O}_{F_0}
\]

associated to \( L_{v_0} \) in Section [5].

Denote by \( \mathcal{M}_{O_{F_0}}^\omega, \) the formal completion of \( \mathcal{M}_{O_{F_0}} \) along the basic locus of the geometric fiber \( \mathcal{M} \otimes \hat{F}_\nu \). The formal scheme \( \mathcal{M}_{O_{F_0}}^\omega \) is a regular and formally finite type over \( O_{F_0} \).

Let \( V(v_0) \) be the nearby \( F/F_0 \)-hermitian space of \( V \) at \( v_0 \), which is positively definite at all \( v|\infty \), and locally isomorphic to \( V \) at all finite places \( v \neq v_0 \) of \( F_0 \).

Consider reductive groups \( \tilde{G}^{(v_0)} := U(V(v_0)) \) over \( F_0 \) and \( \tilde{G}^{(v_0)} := Z^0 \times \text{Res}_{F_0/\mathbb{Q}} U(V(v_0)) \) over \( \mathbb{Q} \). We have the following basic uniformization theorem.

**Proposition 12.16.** There is a natural isomorphism of formal schemes over \( O_{F_0} \):

\[
\Theta : \mathcal{M}_{O_{F_0}}^\omega \cong \tilde{G}^{(v_0)}(\mathbb{Q})/\mathcal{N}' \times \tilde{G}^{(v_0)}(\mathbb{Q})/K_{G, v_0}.
\]

where

\[
\mathcal{N}' = (Z^0(\mathbb{Q}_p)/K_{0, p}) \times \mathcal{N}_{O_{F_0}} \times \prod_{v|p, v \neq v_0} U(V)(F_{0, v})/K_{G, v}.
\]
And there is a natural projection map
\[ M_{O_{E^0}}^\wedge, \text{basic} \rightarrow \mathbb{Z}^G(\mathbb{Q}) \backslash (\mathbb{Z}^G(\mathbb{A}_{0,f})/K_{\mathbb{Q}}) \]
(12.12)
with fibers isomorphic to
\[ \Theta_0 : M_{O_{E^0}, 0}^\wedge, \text{basic} \rightarrow G^{(\nu_0)}(F_0) \backslash [N_{O_{E^0}} \times G(\mathbb{A}_{0,f}^{v_0})/K_{\mathbb{Q}}] \].

**Proof.** As \( E^0 \) is unramified over \( \mathbb{Q}_p \), we have the comparison between absolute and relative Rapoport–Zink spaces [5, Prop. 3.30]. The result follows from [5, Thm. 4.3]. \( \square \)

12.7. **Basic uniformization for Kudla–Rapoport cycles.** In Definition 12.3, we defined Kudla–Rapoport cycles on \( M \) for suitable \( \phi_i \in \mathcal{S}(V(\mathbb{A}_{0,f}))^{K_G} \) (i = 1, 2). Denote by \( Z(\xi, \phi_i)_{\wedge} \) any fiber of the formal completion \( Z(\xi, \phi_i) \) under the projection map \( \Theta \) constructed in [5, Thm. 4.3].

**Proposition 12.17.** For any \( \xi \in F_{0,+} \), we have
\[ Z(\xi, \phi_1)_{\wedge} = \sum_{(u,g) \in G^{(\nu_0)}(F_0) \backslash V^{(\nu_0)}(F_0) \times G(\mathbb{A}_{0,f}^{v_0})/K_{\mathbb{Q}}} \phi_1^{v_0}(g^{-1}u)[Z(u,g)]_{K_{\mathbb{Q}}^{v_0}}, \]
\[ Z(\xi, \phi_2)_{\wedge} = \sum_{(u,g) \in G^{(\nu_1)}(F_0) \backslash V^{(\nu_1)}(F_0) \times G(\mathbb{A}_{0,f}^{v_1})/K_{\mathbb{Q}}^{v_1}} \phi_2^{v_1}(g^{-1}u)[Z(u,g)]_{K_{\mathbb{Q}}^{v_1}}. \]
Here \( [Z(u,g)]_{K_{\mathbb{Q}}^{v_0}} \) is the descent of \( \sum Z(u') \times 1_{g'}K_{\mathbb{Q}}^{v_0} \) to \( M_{O_{E^0}, 0}^\wedge, \text{basic} \). And \( [Z(u,g)]_{K_{\mathbb{Q}}^{v_1}} \) is the descent of \( \sum Z(u') \times 1_{g'}K_{\mathbb{Q}}^{v_1} \) over the same index set.

**Proof.** This follows from checking moduli definitions on both sides using the uniformization map \( \Theta \) constructed in [5, Thm. 4.3]. \( \square \)

12.8. **Comparison of Balloon–Ground stratification.** Return to the local set up in Section 5.3 and consider \( \widetilde{N} \times_{O_{E^0}} N \rightarrow \widetilde{N} \times_{O_{E^0}} N \) the formal blow up morphism along the Weil divisor \( N^0 \times_{k_{\mathbb{Q}}} N^0 \).

**Proposition 12.18.** The uniformization map
\[ \Theta_0 : M_{O_{E^0}, 0}^\wedge, \text{basic} \rightarrow G^{(\nu_0)}(F_0) \backslash [N_{O_{E^0}} \times G(\mathbb{A}_{0,f}^{v_0})/K_{\mathbb{Q}}^{v_0}] \]
induces an isomorphism
\[ (M_{O_{E^0}, 0} \times M_{O_{E^0}, 0})^\wedge \rightarrow (G^{(\nu_0)}(F_0) \times G^{(\nu_0)}(F_0)) \backslash [N_{O_{E^0}} \times N_{O_{E^0}} \times G(\mathbb{A}_{0,f}^{v_0})/K_{\mathbb{Q}}^{v_0} \times G(\mathbb{A}_{0,f}^{v_0})/K_{\mathbb{Q}}^{v_0}]. \]

**Proof.** Blow up commutes with flat base change \( O_{E^0} \rightarrow O_{E^0} \) so we can work over \( O_{E^0} \). From Proposition 5.13 and the definition of \( \Theta_0 \), the base change of the Balloon stratum \( M^\wedge_{k_{E^0}, 0} \) along the faithfully flat morphism
\[ G^{(\nu_0)}(F_0) \backslash N_{O_{E^0}} \times G(\mathbb{A}_{0,f}^{v_0})/K_{\mathbb{Q}}^{v_0} \stackrel{\Theta}{\rightarrow} M_{O_{E^0}, 0}^\wedge, \text{basic} \rightarrow M_{O_{E^0}, 0}^\wedge, \text{basic} \rightarrow M_{O_{E^0}, 0}^\wedge, \text{basic} \]
(12.13)
is equal to the formal subscheme \( G^{(\nu_0)}(F_0) \backslash N_{O_{E^0}} \times G(\mathbb{A}_{0,f}^{v_0})/K_{\mathbb{Q}}^{v_0} \). The result follows. Note we used Kottwitz signature \((1, n-1)\) locally and \((n-1, 1)\) globally, so implicitly we use the identification of Rapoport–Zink spaces of signature \((n-1, 1)\) and \((1, n-1)\). \( \square \)

**Remark 12.19.** The local model diagram for global Shimura variety
\[ M_{\text{basic}, \wedge} \rightarrow M \rightarrow [G/M^{loc}] \]
is compatible with the local one
\[ N \rightarrow [G/M^{loc}] \]
under the basic uniformization. For general Shimura varieties, we may expect a blow up along the local model diagram (using Kottwitz–Rapoport strata) to resolve the singularity:

\[ \text{Sh}_{G} \rightarrow \mathcal{G}|M_{\text{loc}} \]

12.9. Basic uniformization for Hecke CM cycles. Let \( H_{\mu G, \Delta}^{\wedge} \) be the locally noetherian formal scheme over \( O_{E_v} \) obtained as pullback of \( H_{\mu G, \Delta}^{\wedge} \) along the map

\[ \mathcal{O}_{E_v} \times O_{E_v}[\Delta^{-1}] \rightarrow \mathcal{O}_{E_v} \rightarrow \mathcal{O}_{E_v}[\Delta^{-1}] \rightarrow \mathcal{M}. \]

There is a natural projection

\[ H_{\mu G, \Delta}^{\wedge} \rightarrow Z^G(Q) \setminus Z^G(\mathbb{A}_0, f)/K_{Z^G}. \] (12.14)

Write \( H_{\mu G, \Delta, 0}^{\wedge} \) as any fiber of \( H_{\mu G, \Delta}^{\wedge} \) of the projection.

**Proposition 12.20.** We have an isomorphism:

\[ H_{\mu G, \Delta, 0}^{\wedge} \cong G^i(F_0)\backslash [\mathcal{N}_{O_{E_v}} \times H_{\mu G, \Delta, 0}^{\wedge}] \]

compatible with the projection under the isomorphism \( \Theta \) (12.11). Here \( H_{\mu G, \Delta, 0}^{\wedge} \) is the discrete set

\[ H_{\mu G, \Delta, 0}^{\wedge} = \{(g_1, g_2) \in G(K_{F_0}^G)/K_{G}^G \times G(K_{F_0}^G)/K_{G}^G | g_1^{-1}g_2 \in K_{G, \mu G, \Delta}K_{G} \}. \]

**Proof.** This follows from Proposition 59 Prop. 7.16. \( \square \)

For \((\delta, h) \in G((\nu_0))(F_0) \times G((\nu_0)/(\mathbb{A}_0)_{f}^G)/K_{G}^G \), consider the Hecke CM cycle

\[ C\mathcal{M}(\delta, h)_{K_G^G} := [\mathcal{N}_{O_{E_v}} \times 1_{K_G^G}] \rightarrow [\mathcal{N}_{O_{E_v}} \times G((\nu_0)/(\mathbb{A}_0)_{f}^G)/K_{G}^G]. \]

The summation

\[ \sum_{(\delta, h) \in G((\nu_0))(F_0) \setminus G((\nu_0)/(\alpha)(F_0)) \times G((\nu_0)/(\mathbb{A}_0)_{f}^G)/K_{G}^G} C\mathcal{M}(\delta, h)_{K_G^G} \]

over the \( G((\nu_0))(F_0) \)-orbit of \((\delta, h) \) in \( G((\nu_0))(F_0) \times G((\nu_0)/(\mathbb{A}_0)_{f}^G)/K_{G}^G \) descends to a cycle \( [C\mathcal{M}(\delta, h)]_{K_G^G} \) on the quotient \( \mathcal{M}_{O_{E_v}} \). Consider \( \varphi_{CM} = \varphi_{CM}^G \otimes \varphi_{CM, \Delta} \in \mathcal{S}(U(\mathbb{A}_0, f), K) \) where \( \varphi_{CM}^G = 1_{K_G^G} \) (including \( v_0 \)). As in the proof of 59 Prop. 7.17, we have

**Corollary 12.21.** The restriction of the formal completion \( C\mathcal{M}(\alpha, \varphi_{CM})^{\wedge} \) to any fiber of the projection \( \Theta \) (12.11) is the sum

\[ \sum_{(\delta, h) \in G((\nu_0))(F_0) \setminus G((\nu_0)/(\alpha)(F_0)) \times G((\nu_0)/(\mathbb{A}_0)_{f}^G)/K_{G}^G} \varphi_{CM}^G(h^{-1} \delta h)[C\mathcal{M}(\delta, h)]_{K_G^G}. \] (12.16)

Plus the comparison in Proposition 12.18 we get

**Corollary 12.22.** The restriction of the basic uniformization \( C\mathcal{M}(\alpha, \varphi_{CM})^{\wedge} \) to any fiber of the projection \( \Theta \) (12.11) is the sum

\[ \sum_{(\delta, h) \in G((\nu_0))(F_0) \setminus G((\nu_0)/(\alpha)(F_0)) \times G((\nu_0)/(\mathbb{A}_0)_{f}^G)/K_{G}^G} \varphi_{CM}^G(h^{-1} \delta h)[C\mathcal{M}(\delta, h)]_{K_G^G}. \] (12.17)

13. Modification over \( \mathbb{C} \) and \( \mathbb{F}_q \)

In this section, we use basic geometric results on the generic fiber and special fibers of our Shimura varieties to understand 1-cycles on our integral model by modification.

Firstly, we show that the degree function of any Hecke CM cycle \( C\mathcal{M}(\alpha, \mu) \rightarrow E \) is equidistributed on geometric connected components of \( \mathcal{M}_{G, \bar{K}} \) under mild assumptions. This shows we can use suitable Hecke CM cycles to modify general 1-cycle to a cycle of degree zero over \( \mathbb{C} \).

Secondly, we use Balloon–Ground stratification and basic uniformization to approximate 1-cycles on bad fibers by 1-cycles of special shapes in the basic locus.
13.1. Modification over $\mathbb{C}$.

13.2. Hecke and Galois actions on $\pi_0$. Let $(G, X)$ be a Shimura datum, $G^{ab} = G/G^{der}$ be the maximal abelian quotient of $G$ which is a torus over $\mathbb{Q}$, and $Z$ be the center of $G$.

For any neat compact open subgroup $K \leq G(\mathbb{A})$, consider the complex uniformization:

$$\text{Sh}_K(G, X)(\mathbb{C}) = (G(\mathbb{Q}) \backslash X \times G(\mathbb{A}))/K.$$ 

Assume that $G^{der}$ is simply connected.

**Proposition 13.1.** The natural projection $\det_G : G \to G^{ab}$ induces an isomorphism of finite sets

$$\pi_0(\text{Sh}_K(G, X)(\mathbb{C})) \cong G^{ab}(\mathbb{Q})^1 \backslash G^{ab}(\mathbb{A})/\det_G(K),$$  \hspace{1cm} (13.1)

where $G^{ab}(\mathbb{Q})^1 = G^{ab}(\mathbb{Q}) \cap \text{Im}(\det_G|_\mathbb{Z} : Z(\mathbb{R}) \to G^{ab}(\mathbb{R}))$.

**Proof.** This follows from the strong approximation theorem for non-compact type, simply connected semisimple algebraic groups over number fields, see [40, Thm. 5.17]. \qed

**Remark 13.2.** The Hecke action of $G(\mathbb{A})$ on the tower $\{\text{Sh}_K(G, X)\}_K$ is compatible with the projection $\det_G : G(\mathbb{A}) \to G^{ab}(\mathbb{A})$, hence is transitive on $\lim_K \pi_0(\text{Sh}_K(G, X)(\mathbb{C}))$.

Let $E$ be the reflex field of $(G, X)$. The conjugacy class of the Hodge cocharacter $\mu_h : \mathbb{G}_{m, \mathbb{C}} \to G_{\mathbb{C}}$ is defined on $E$. Take its composition with $\det_G : G \to G^{ab}$, we get a homomorphism

$$r_{G, E} : (\mathbb{G}_m)_E \to (G^{ab})_E.$$ 

between tori over $E$. The reflex norm map is the norm of $r_{G, E}$:

$$r_G : \text{Res}_{E/\mathbb{Q}} \mathbb{G}_m \to G^{ab}.$$ 

We describe the Galois action on $\pi_0(\text{Sh}_K(G, X)(\mathbb{C}))$ via the theory of Shimura reciprocity law as in [40, Section 12-13]. By global class field theory, there is a continuous surjective homomorphism $\text{Art}_E : \text{Res}_{E/\mathbb{Q}} \mathbb{G}_m(\mathbb{A}) \to \text{Gal}_E$. By [40, p. 119], we have

**Proposition 13.3.** For any Galois automorphism $\sigma \in \text{Gal}_E$, choose some $s = (s_{\infty}, s_f) \in \mathbb{A}_E^\times = \text{Res}_{E/\mathbb{Q}} \mathbb{G}_m(\mathbb{A})$ such that $\text{Art}_E(s) = \sigma|_{E^\times}$. Then $\sigma$ acts on the finite abelian group $\pi_0(\text{Sh}_K(G, X)(\mathbb{C})) \cong G^{ab}(\mathbb{Q})^1 \backslash G^{ab}(\mathbb{A})/\det_G(K)$ via multiplication with $r_G(s_f)$:

$$\sigma[t] = [r_G(s_f)t].$$  \hspace{1cm} (13.2)

**Remark 13.4.** We can compute connected components of PEL type Shimura varieties using moduli interpretation. For the Siegel modular variety $\mathcal{A}_g,N$ with level structure $T(N)$, we have $\pi_0(\mathcal{A}_g,N)(\mathbb{C}) \cong (\mathbb{Z}/N\mathbb{Z})^\times$ given $(A, \lambda) \in \mathcal{A}_g,N(\mathbb{C})$, its image in $(\mathbb{Z}/N\mathbb{Z})^\times$ is the ratio of a fixed perfect pairing on $(\mathbb{Z}/N\mathbb{Z})^{2g}$ and the Weil pairing on $A[N]$ under $\eta : A[N] \cong (\mathbb{Z}/N\mathbb{Z})^{2g}$.

Return to our set up. We have $\tilde{G} = Z^Q \times \text{Res}_{F_0/\mathbb{Q}} G$ where $G = U(V)$ is a reductive group over $F_0$. Recall $M_{G,K}$ (resp. $M_{G,K}$) is the canonical model of $\text{Sh}_K(\tilde{G}, \{h_G\})$ (resp. $\text{Sh}_K(\text{Res}_{F_0/\mathbb{Q}} G, \{h_G\})$) over the reflex field $E$ (resp. $\varphi_0(F) \subseteq \overline{Q}$). We have

$$T^1 := (\text{Res}_{F_0/\mathbb{Q}} G)^{ab} = \text{Res}_{F/\mathbb{Q}} \mathbb{G}_m^{N_{M_{F_0/F_0}} = 1}.$$ 

As $T^1(\mathbb{R})$ is connected, we have $T^1(\mathbb{Q})^+ = T^1(\mathbb{Q})$.

**Lemma 13.5.** The reflex norm map $r_{\text{Res}_{F_0/\mathbb{Q}} G}$ for the datum $(\text{Res}_{F_0/\mathbb{Q}} G, \{h_G\})$ is given by

$$r_{\text{Res}_{F_0/\mathbb{Q}} G} : \text{Res}_{F/\mathbb{Q}} \mathbb{G}_m^{N_{M_{F_0/F_0}} = 1} \to \text{Res}_{F/\mathbb{Q}} \mathbb{G}_m^{N_{M_{F_0/F_0}} = 1}, \quad z \mapsto z/\ell(z).$$  \hspace{1cm} (13.3)

**Proof.** By definition, the Hodge cocharacter is given by $z \mapsto \text{diag}\{1_{n-1}, \ell/2\}$. Take its composition with $\det_G : \text{Res}_{F_0/\mathbb{Q}} U(V) \to \mathbb{G}_m^{N_{M_{F_0/F_0}} = 1}$, we get the result. \qed

**Corollary 13.6.** The Galois action of $\text{Gal}_F$ on $\pi_0(M_{G,K}(\mathbb{C}))$ is transitive.

**Proof.** By above lemma and Hilbert 90, the reflex norm map $r_{\text{Res}_{F_0/\mathbb{Q}} G}$ induces a surjection $\text{Res}_{F_0/\mathbb{Q}} \mathbb{G}_m(A_f) \to T^1(\mathbb{A})$. Since $\pi_0(M_{G,K}(\mathbb{C})) \cong T^1(\mathbb{Q})\backslash T^1(\mathbb{A})/\text{det}_{\text{Res}_{F_0/\mathbb{Q}} G}(K)$, we conclude by Proposition 13.3.
13.3. Complex uniformization of Hecke CM cycles. Denote by $X$ the Grassmannian of negative definite $C$-lines in $V \otimes_{F,F_0} C$. Consider any neat compact open subgroup $\bar{K} = K_{Z^2} \times K$. We have the following complex uniformization \[ M_{\bar{G}, \bar{K}} \cong \Sh_{K_{Z^2}}(Z\bar{G}, \{h_{Z^2}\})_C \times M_{G,K} \otimes_{F,F_0} C \] (13.4)

Each fiber of the projection $M_{\bar{G}, \bar{K}}(\mathbb{C}) \to Z^2(\mathbb{Q}) \setminus Z^2(\mathbb{A}_f)/K_{Z^2}$ is isomorphic to $M_{\bar{G}, \bar{K}}(\mathbb{C}) = U(V)(F_0) \times U(V)(\mathcal{A}_0,f)/K_G$.

For any $(\delta, g) \in U(V)(\alpha)(F_0) \times U(V)(\mathcal{A}_0,f)/K_G$, define $[CM(\alpha, g)]_{K_G} \to M_{G,K}(\mathbb{C})$ as the descent of the summation

$$\sum_{(\delta', g')} X_{\delta'=\text{id}} \times 1_{g'K_G}$$

where $(\delta', g')$ runs over the $G(F_0)$-orbit of $(\delta, g)$ in $U(V)(\alpha)(F_0) \times U(V)(\mathcal{A}_0,f)/K_G$.

Let $\mu_{G, \Delta} \in K_{G, \Delta} \setminus G(F_0, \Delta)/K_{G, \Delta}$ and $\alpha \in F[t]$ be an irreducible degree $n$ conjugate self-reciprocal monic polynomial. Similar to [59] Prop. 7.17, the complex points of Hecke CM cycles have the following description.

Proposition 13.8. The restriction of $CM(\alpha, \mu_{G, \Delta})(\mathbb{C})$ on each fiber of the projection $M_{\bar{G}, \bar{K}}(\mathbb{C}) \to Z^2(\mathbb{Q}) \setminus Z^2(\mathbb{A}_f)/K_{Z^2}$ is the disjoint union

$$\prod [CM(\alpha, g)]_{K_G} \to M_{G,K}(\mathbb{C})$$

where the index runs over the set

$$A_{\mu_{G, \Delta}} := \{(\delta, g) \in U(V)(\alpha)(F_0) \setminus U(V)(\alpha)(F_0) \times U(V)(\mathcal{A}_0,f)/K_G | g^{-1}\delta g \in K_{G, \mu_{G, \Delta}}K_G\}.$$
Proposition 13.12. Assume that $E = F$ or Conjecture 13.10 holds. For any 1-cycle $C \to M_{G, K}$ over $E$ stable under the Hecke action of $Z^G(A_f)$, there exists some $C' = C_M(\alpha, \mu_G, \Delta)$ such that
\[
\tilde{\deg}_{G} = a(\tilde{\deg}_{C'})
\]
for some $a \in \mathbb{Q}$. Moreover, we can assume $\alpha$ is maximal order at any finite place $v$ of $F_0$ outside $\Delta$, and is unramified at $v_0$.

Proof. The degree function $\tilde{\deg}_{G}$ is Gal-$F$-invariant and stable under Hecke action of $Z^G(A_f)$, hence also a constant. It suffices to show $C' = C_M(\alpha, \mu_G, \Delta)$ can be chosen to be non-empty. This follows from the complex uniformization 13.8 above.

Remark 13.13. As Kudla–Rapoport divisors are invariant under Hecke action of $Z^G(A_f)$, the assumption that $C$ is invariant under $Z^G(A_f)$ is harmless for the application to our modularity results.

13.5. Modification over $F_p$. Let $v \not\in \Delta$ be a finite place of $F_0$ such that $L_v$ is not self-dual. So $L_v$ is a vertex lattice of type $0 < t_0 \leq n$. The case $t_0 = n$ is same to the case $t_0 = 0$ by duality, so assume $0 < t_0 < n$.

By assumption, $v$ is inert in $F$ and $E_v$ is unramified over $\mathbb{Q}_p$. Hence $M \to \text{Spec } O_{E_v}$ is of strictly semi-stable reduction by Theorem 11.17.

For a 1-cycle $C$ on the special fiber $M_{k_v}$, consider its degree function on irreducible components $X_i$ of $M_{k_v}$ which is obtained via the intersection pairing $(\cdot, \cdot)_M$ between $C$ and $X_i$ inside the regular scheme $M$. Recall the Balloon–Ground stratification in Section 11.14:
\[
M_{k_v} = M_{k_v}^\circ \cup M_{k_v}^\bullet.
\]
Note $M_{k_v}^\circ$ and $M_{k_v}^\bullet$ are closed smooth subvariety of $M_{k_v}$ over $k_v$ of dimension $n - 1$. So irreducible components of $M_{k_v}$ are exactly connected components of the Balloon and Ground stratum:
\[
M_{k_v} = \bigcup_{i=1}^a X_i^\circ \cup \bigcup_{j=1}^b X_j^\bullet.
\]
Consider the basic uniformization in Proposition 12.16:
\[
\Theta_0 : M_{O_{E_v}, 0}^{\wedge, \text{basic}} \to G^{(v)}(F_0) \backslash [\mathcal{N}_{E_v}] \times G(A_{0, f})/K_{G}^{(v)}.
\]
Here $M_{O_{E_v}, 0}$ is any fiber of the projection $M_{O_{E_v}}^{\wedge, \text{basic}} \to Z^G(\mathbb{Q}) \backslash (Z^G(A_{0, f})/K_{G})$.

Call a 1-cycle $C'$ in $M_{k_v}^{\wedge, \text{basic}}$ very special, if via basic uniformization it can be generated from the projection of a cycle in an embedding $\mathcal{N}_{E_v}^\circ \to \mathcal{N}_{E_v}$ for a decomposition of vertex lattice $L_v = L_v^0 \oplus M_v$ where $L_v^0$ has rank 2 and type 1.

13.6. The case $t_0 = 1$ or $n - 1$.

Proposition 13.14. If $t_0 = 1$, then $M_{k_v}^\circ$ is in the basic locus of $M_{k_v}$.

Proof. This follows either by the computation of slopes of associated Dieudonné modules or by the construction of basic isogenies, see [32 Thm. 5.3.4].

Proposition 13.15. Any geometric irreducible component $X_i^\circ$ in $M_{k_v}^\circ$ is isomorphic to the $n - 1$ dimensional projective space $\mathbb{P}^{n-1}$. Moreover, $X_i^\circ \cap M_{k_v}^\bullet$ is isomorphic to the degree $q + 1$ Fermat hypersurface in $\mathbb{P}^{n-1}$.

Proof. As $M_{k_v}^\circ$ is in the basic locus, we can use basic uniformization (Proposition 12.16) to prove similar statements in the basic locus. Then this follows from explicit descriptions of Bruhat–Tits strata in Theorem 8.11 see also Remark 5.14.

Proposition 13.16. In any given connected component of $M_{k_v}$, the Ground stratum is geometrically irreducible.
Proposition 13.17. If \( t_0 = 1 \) or \( n - 1 \), then for any 1-cycle \( C \) in \( M_{k_v} \), there exists a very special 1-cycle \( C' \) in \( M^\text{basic}_{k_v} \) such that
\[
(C',X)_M = (C,X)_M
\]
for any irreducible component \( X_i \) of \( M_{k_v} \).

Proof. We can work in any connected component of \( M_{k_v} \), hence in some fiber \( M_{O_{E_v,0}} \). Assume \( t_0 = 1 \). Then there is only one irreducible component \( X^*_i \) in \( M^*_{k_v,0} \), and many possible irreducible components \( X^*_i (i = 1, \ldots, r) \) in \( M^*_{k_v,0} \). Then \( (C,X)_M \) is given by a collection of intersection numbers \( \{l_j(C), l_1(C), \ldots, l_r(C)\} \) such that \( l_i(C) = -\sum_{j=1}^r l_j(C) \).

It suffices to find for each \( 1 \leq i_0 \leq r \) a very special cycle \( C'_i \) such that
\[
(C'_i,X^*_i)_M \neq 0.
\]
Then \( C' = \sum_{i=1}^r l_i(C)/l'_i(C')C'_i \) is what we want.

To find such \( C'_i \), apply basic uniformization we only need to only need to find special 1-cycle \( C'_i \) in a given Bruhat-Tits stratum \( BT(L^\circ) \cong \mathbb{P}^{n-1} \). Here \( L^\circ \) is a vertex lattice in \( V^{(v)} \) of type \( t_0 = 1 = 0 \). Choose an orthogonal decomposition of vertex lattices
\[
L^\circ_v = (L^\circ_v)^{\oplus} \oplus M_v
\]
where \((L^\circ_v)^{\oplus}\) is a self-dual lattice of rank 2. It gives an decomposition \( L_v = L^\circ_v \oplus M_v \).

Then we can choose
\[
C'_i = \mathcal{Z}(M) = BT(L^\circ)_v \cong \mathbb{P}^{1} \subseteq \mathcal{N}_L^v.
\]
The case \( t_0 = n - 1 \) follows from duality.

13.7. The case \( 1 < t_0 < n - 1 \). If \( 1 < t_0 < n - 1 \), the irreducible components \((N^*)_{\text{red}}^{(v)} \) and \((N^*)_{\text{red}}^{(v)} \) are of dimension less than \( n - 1 \) by Theorem 8.11. So the Balloon \( M_{k_v} \) and Ground strata \( \mathcal{M}^*_{k_v} \) don’t lie in the basic locus of \( \mathcal{M}_{k_v} \).

Remark 13.18. The different behavior of geometry of Balloon–Ground stratum can already be seen locally. Using Bruhat–Tits stratification in Section 5.2 we see \( \mathcal{N}_{\text{red}}^{(v)} \) is always connected. But \( \mathcal{N}^{(v)}_{\text{red}} \) and \( \mathcal{N}^*_{\text{red}} \) are connected and non-empty if and only if \( 2 < t_0 < n - 2 \).

Proposition 13.19. Any geometric connected component of \( M_{k_v} \) contains a basic point \( x \in M_{k_v} \).

Proof. This follows from [13, Prop. 3.5.4], using the compatibility of the basic uniformization and the complex uniformization of geometric connected components (13.1). Firstly, we can work on any fiber \( M_{O_{E_v,0}} \) of the projection \( M_{O_{E_v}} \rightarrow Z^{(v)}(\mathcal{Q})/(Z^{(v)}(\mathcal{A}_{0,f})/K_G) \).

As \( M_0 \) is flat and proper over \( O_{E_v} \), we have an isomorphism \( \pi_0(\mathcal{M}_{k_v,0}) \cong \pi_0(\mathcal{M}_{k_v,0}) \). Recall by Proposition 13.1 we have
\[
\pi_0(\mathcal{M}_{k_v,0}) = G^{ab}(F_0)\backslash G^{ab}(K_f)/\det(K_G).
\]
The basic uniformization
\[
\Theta_0 : \mathcal{M}_{E_v,0}^{(v)} \cong G^{(v)}(F_0)\backslash [\mathcal{N}_{O_{E_v}}^{(v)} \times G^{(v)}(\mathcal{A}_{0,f})/K^{(v)}_G]
\]
is compatible with the projection
\[
\pi_0(\mathcal{M}_{k_v,0}^{(v)}) \rightarrow \pi_0(\mathcal{M}_{k_v,0}) \cong \pi_0(\mathcal{M}_{E_v,0}^{(v)}) = G^{ab}(F_0)\backslash G^{ab}(K_f)/\det(K_G).
\]
We are reduced to show the projection
\[
G^{(v)}(F_0)\backslash [\mathcal{N}_{O_{E_v}}^{(v)} \times G^{(v)}(\mathcal{A}_{0,f})/K^{(v)}_G] \rightarrow G^{ab}(F_0)\backslash G^{ab}(K_f)/\det(K_G)
\]
is surjective, which follows from \([G^{(v)}]^{ab} = G^{ab} = T^1 \) in our cases.

Conjecture 13.20. (1) Any geometric connected component of the Balloon (resp.) Ground stratum contains a basic point.
In any geometric connected component of $\mathcal{M}_{k_0}$, the Balloon and Ground stratum are geometrically irreducible.

Remark 13.21. These conjectures make sense for mod $p$ fibers of general Shimura varieties (with suitable integral models). Conjecture (1) is replaced by that any geometric irreducible component of the mod $p$ fiber contains a basic point. It is closely related to the Hecke orbit conjecture for PEL type Shimura varieties [27, Prop. 4.2 (2)]. Conjecture (2) is replaced by irreducibility of non-basic Kottwitz–Rapoport strata, see [13, Thm. 3.7.1, Prop. 3.7.3] for a proof under certain unramified assumptions.

Proposition 13.22. Conjecture (13.20(2)) holds i.e., in any geometric connected component of $\mathcal{M}_{k_0}$, the Balloon and Ground stratum are geometrically irreducible.

Proof. The proof in [13, Section 3.5-3.7] works for our Shimura varieties: the arguments for Proposition 3.6.1, 3.6.2, 3.7.4 and 3.7.3 in loc.cit. works on the Iwahori level. The reason is that we can consider $\leq 1$ dimensional minimal Kottwitz–Rapoport strata at Iwahori level, and our Shimura variety is proper.

Return to our parahoric level for $L_v$. Using the definition of $\mathcal{M}^\circ$ (resp. $\mathcal{M}^\bullet$) as vanishing locus of Lie $\lambda$ (resp. Lie $\lambda^\vee$) in Proposition 11.6 at the parahoric level for $L_v$ we may identify Balloon–Ground strata with (closures) of Kottwitz–Rapoport strata on $\mathcal{M}_{k_0}$.

We have the Axiom (4c) of Rapoport–He [17] by basic uniformization theorem for RSZ Shimura varieties at our level, see Section 12.6 and [13, Appendix]. The Axiom (4c) implies that for every EKOR stratum at our parahoric level, there is a Kottwitz–Rapoport stratum at Iwahori level surjecting onto it, hence we know the irreducibility of non-basic EKOR strata at our level. And our Balloon (resp. Ground) stratum is the closure of a maximal dimensional Kottwitz–Rapoport stratum hence the closure of a non-basic EKOR stratum, thus it’s irreducible. □

Proposition 13.23. If $1 < t_0 < n - 1$, then for any 1-cycle $C$ in $\mathcal{M}_{k_0}$, there exists a very special 1-cycle $C'$ in $\mathcal{M}_{k_0}^\circ$ such that

$$(C', X_i)_{\mathcal{M}} = (C, X_i)_{\mathcal{M}}$$

for any irreducible component $X_i$ of $\mathcal{M}_{k_0}$.

Proof. We can work in a given connected component $X$ of $\mathcal{M}_{k_0}$. By previous proposition on Conjecture (13.20(2)), there are only 2 irreducible components in this component i.e., the Balloon stratum $X^\circ$ and the Ground stratum $X^\bullet$. Hence we only need to show there exists a very special 1-cycle $C'$ such that

$$(C', X^\bullet) = - (C, X^\circ)$$

is non-zero. By basic uniformization, any two nearby minimal dimensional Bruhat–Tits strata $\mathbb{P}^{n-t}$ and $\mathbb{P}^{n-t'}$ intersect at a point. Choose any $\mathbb{P}^1 \subseteq \mathbb{P}^{n-t}$ passing exactly only one such point will do the job. □

14. Global modularity via modification

In this section, we introduce the modification method towards modularity. We establish the modularity of arithmetic theta series at maximal parahoric levels when intersecting with 1-cycles on $\mathcal{M}$, in particular with modified derived Hecke CM cycles.

Let $\mathcal{X} \to \text{Spec } \mathcal{O}_E[\Delta^{-1}]$ be a pure dimensional, regular, flat and proper scheme with smooth generic fiber $X$. Set $R_\Delta = \mathbb{R}/\text{span}_\mathbb{Q}\{\log p|\exists v \in \Delta, v|p\}$.

14.1. Truncated arithmetic intersection pairing. We recall basics of the Gillet–Soulé arithmetic intersection theory, see [9] for a reference. We consider arithmetic divisors to relate local and global intersection numbers, and recall the $\Delta$-truncated version arithmetic intersection theory.

An arithmetic divisor on $X$ is a $\mathbb{Q}$-linear combinations of tuples $(Z, g_Z)$, where $Z$ is a divisor on $X$, and $g_Z = (g_{Z,v})_v$ ($v$ runs over all infinite places of $E$) is a tuple of Green functions on $X_v(\mathbb{C}) \backslash Z_v(\mathbb{C})$ with

$$dd^c g_{Z,v} + \delta_{Z_v(\mathbb{C})} = [\omega_v], \quad \omega_v \text{ smooth } (1,1)-\text{form on } Z_v(\mathbb{C}).$$
Fix a Kähler metric on $X_p(\mathbb{C})$. The Green function $g_{Z,\nu}$ is called admissible if $\omega_\nu$ is harmonic with respect to the metric. Then the arithmetic Chow group $\widehat{\text{Ch}}^1(X)$ (with $\mathbb{Q}$-coefficients) is generated by arithmetic divisors, modulo the relation given by the $\mathbb{Q}$-span of principal arithmetic divisors i.e., tuples $\text{div}(f) := (\text{div}(f), (-\log |f|^2_\nu))_{\nu|\infty}$ associated to rational functions $f \in E(X)^\nu$:

$$\widehat{\text{Ch}}^1(X)_\mathbb{Q} := \{\text{Arithmetic divisors}\}/\{(\text{div}(f), (-\log |f|^2_\nu))_{\nu|\infty} \}_{f \in E(X)^\nu}.$$  

An admissible Green function for a given divisor $Z$ exists, and is unique up to adding locally constant functions on $X_p(\mathbb{C})$. Consider the subgroup $\widehat{\text{Ch}}^{1,\text{adm}}(X)$ of the arithmetic Chow group generated by tuples $(Z, (g_{Z,\nu})_\nu)$ with admissible Green functions $g_{Z,\nu}$. There is a natural map

$$\widetilde{\text{Ch}}^{1,\text{adm}}(X) \to \text{Ch}^1(X), \quad (Z, g_Z) \mapsto Z_E.$$  

For any place $\nu$ of $E$, define $\widetilde{\text{Ch}}^1_{\nu}(X) \subseteq \widetilde{\text{Ch}}^{1,\text{adm}}(X)$ as the subgroup generated by

- $(0, c_\nu)$ where $c_\nu$ is a locally constant function on $X_p(\mathbb{C})$, if $\nu|\infty$.
- $(X_{k_\nu}, 0)$ where and $X_{k_\nu}$ is an irreducible component of the special fiber $X_{k_\nu}$ at $\nu$, if $\nu$ is a finite place with residue field $k_\nu$.

Let $\widetilde{\text{Ch}}^1_{\text{Vert}}(X) \subseteq \widetilde{\text{Ch}}^{1,\text{adm}}(X)$ be the subgroup generated by $\widetilde{\text{Ch}}^1_\nu(X)$ for all places $\nu$ of $E$.

**Lemma 14.1.** The projection $\widetilde{\text{Ch}}^{1,\text{adm}}(X) \to \text{Ch}^1(X)$, $(Z, g_Z) \mapsto Z_E$ induces an isomorphism

$$\widetilde{\text{Ch}}^{1,\text{adm}}(X)/\widetilde{\text{Ch}}^1_{\text{Vert}}(X) \cong \text{Ch}^1(X).$$  

For any finite place $\nu$ such that $X_\nu := X \otimes O_{E,\nu}$ is smooth over $O_{E,\nu}$, we have $\widetilde{\text{Ch}}^1_{\nu}(X) \subseteq \sum_{\nu|\infty} \widetilde{\text{Ch}}^1_{\nu,\infty}(X)$.

**Proof.** The first part follows from the definition and dimension reasons. For the second part, assume that $X_\nu$ is smooth over $O_{E,\nu}$. Then any $X_{k_\nu, i}$ is smooth and is the reduction of a connect component $X_{k_\nu}$ of $X_\nu$. Choose an element $a$ in $E$ which is a unit away from $\nu$. As the arithmetic divisor $\text{div}(a|_{X_{k_\nu, i}})$ is principal, we see that $(X_{k_\nu, i}, 0) \in \sum_{\nu|\infty} \widetilde{\text{Ch}}^1_{\nu,\infty}(X)$. \hfill $\square$

Let $C_1(X)$ be the group of 1-cycles on $X$ (with $\mathbb{Q}$-coefficient). Consider the standard arithmetic intersection pairing $\mathbb{Z}$ between $\mathbb{Q}$-vector spaces:

$$\langle \cdot, \cdot \rangle : \text{Ch}^1(X) \times C_1(X) \to \mathbb{R}$$  

and its $\Delta$-truncated version

$$\langle \cdot, \cdot \rangle : \text{Ch}^1(X) \times C_1(X) \to \mathbb{R}_\Delta.$$  

Consider the subgroup $C_1(X)_{\text{deg}=0}$ consisting of 1-cycles with zero degree on each connected component of $X$ over $E$, which is the orthogonal complement of $\sum_{\nu|\infty} \text{Ch}^1_{\nu,\infty}(X)$. Let $C_1(X)^\perp \subseteq C_1(X)$ be the orthogonal complement of $\text{Ch}^1_{\text{Vert}}(X)$. From Lemma 14.1, the $\Delta$-truncated arithmetic intersection pairing induces a pairing

$$\langle \cdot, \cdot \rangle_{\text{adm}} : \text{Ch}^1(X) \times C_1(X)^{\perp} \to \mathbb{R}_\Delta.$$  

14.2. Green functions and arithmetic special divisors. Let $K$ be a neat compact open subgroup of $G(A_0, f)$, and fix an embedding $\nu : E \hookrightarrow \mathbb{C}$.

Choose a Schwartz function $\phi \in \mathcal{S}(V(A_0, f))^K$. Choose any compact open subgroup $K_0 \subseteq \text{SL}_2(A_0, f)$ such that $\phi$ is invariant under $K_0$ by the Weil representation.

For $\xi \in F_0$, we have two kinds of Green functions $\mathbb{Z}$ for the Kudla–Rapoport divisor $Z(\xi, \phi) := Z(\xi, \phi) \otimes_{E,\nu} \mathbb{C}$ on $M_{G, K} \otimes_{E,\nu} \mathbb{C}$:

- The *Kudla Green function* $G^K(\xi, h_\infty, \phi)$ with a variable $h_\infty \in \text{SL}_2(F_0, \infty)$, which occurs in the analytic side naturally via computations of archimedean orbit integrals,
- The *automorphic Green function* $G^B(\xi, h_\infty, \phi) = G^K(\xi, \phi)$ for $\xi \in F_0, +$ which is admissible.
Consider the difference $g^{K-B}(\xi, h_\infty, \phi) = \begin{cases} g^K(\xi, h_\infty, \phi) - g^B(\xi, \phi), & \text{for } \xi \in F_0^+, \\ g^K(\xi, h_\infty, \phi), & \text{else.} \end{cases}$

We form the generating series of Green functions as the archimedean part of the generating series of arithmetic special divisors ($h \in SL_2(\mathbb{A}_0)$):

$$g^\gamma(h, \phi) := \sum_{\xi \in F_0} g^\gamma(\xi, h_\infty, \omega(h)\phi) W^{(n)}(h_\infty), \quad ? = K, B, K - B. \quad (14.6)$$

**Theorem 14.2.** The generating function $g^{K-B}(h, \phi)$ when evaluating at a degree-zero cycle on $\mathcal{M}_{\tilde{G}, \tilde{K}}(\mathbb{C})$, is a smooth function on $SL_2(\mathbb{A}_0)$, invariant under left $SL_2(F_0)$-action and right $K_0$-action, and of parallel weight $n$.

**Proof.** See [38, Thm. 3.13].

Let $\tilde{\omega}$ be an extension of the automorphic line bundle $\omega$ to $\mathcal{M}_{\tilde{G}, \tilde{K}}$ endowed with the natural Petersson metric. Define constant terms

$$\mathcal{Z}^{B}(0, \phi) = -\phi(0)c_1(\tilde{\omega}) \in \hat{CH}^1(\mathcal{M}_{\tilde{G}, \tilde{K}}), \quad (14.7)$$

$$\mathcal{Z}^{K}(0, h_\infty, \phi) = -\phi(0)c_1(\tilde{\omega}) + (0, (g^K_\nu(\xi, h_\nu, \phi))_\nu) \in \hat{CH}^1(\mathcal{M}_{\tilde{G}, \tilde{K}}). \quad (14.8)$$

Set $g^K(\xi, \phi) := g^K(\xi, 1, \phi)$. For $\xi \in F_0$, define arithmetic Kudla–Rapoport divisors ($? = K, B, K - B$)

$$\mathcal{Z}^\gamma(\xi, h_\infty, \phi) = (Z(\xi, \phi, (g^K_\nu(\xi, h_\nu, \phi))_\nu) \in \hat{CH}^1(\mathcal{M}_{\tilde{G}, \tilde{K}}), \quad (14.9)$$

$$\mathcal{Z}^\gamma(\xi, \phi) = (Z(\xi, \phi, (g^K_\nu(\xi, \phi))_\nu) \in \hat{CH}^1(\mathcal{M}_{\tilde{G}, \tilde{K}}). \quad (14.10)$$

We have that $\mathcal{Z}^B(\xi, \phi) \in \hat{CH}^{1, \text{adm}}(\mathcal{M}_{\tilde{G}, \tilde{K}})$.

### 14.3. Modularity of arithmetic theta series at maximal parahoric levels

Return to the set up in Section [11] we have the integral model

$$\mathcal{M} = \mathcal{M}_{\tilde{G}, \tilde{K}} \rightarrow \text{Spec } O_E[\Delta^{-1}].$$

The level $\tilde{K}$ is related to a chosen lattice $L$ such that $L_{v_0}$ is a vertex lattice of type $0 \leq t_0 \leq n$.

**Theorem 14.3.** Let $\phi \in S(V(\mathbb{A}(f)))^K$ be a Schwartz function such that $\phi$ is invariant under a compact open subgroup $K_0 \subseteq SL_2(\mathbb{A}(f))$ by the Weil representation. Then for any $C^\infty \in C_c(\mathbb{A})^+$, the generating function $h \in SL_2(\mathbb{A}_0) \mapsto (Z(h, \phi, C^\infty))_{\text{adm}}$ lies in $\mathcal{A}_{\text{hol}}(SL_2(\mathbb{A}_0), K_0, n)_{\mathbb{Q}} \otimes \mathbb{R} _\Delta \mathbb{R}$.

**Proof.** This follows from Lemma [14.1] and Proposition [12.2].

Let $\phi \in S(V(\mathbb{A}(f)))^K$ be of the form $\phi_i$ ($i = 1, 2$) introduced in [12.6]. Then $\phi$ is invariant under a compact open $K_0 \subseteq SL_2(\mathbb{A}(f))$ such that for $v \not\in \Delta$ we have $K_{0,v} = SL_2(O_{v})$ if $L_v$ is self dual; $K_{0,v} = \Gamma_0(v)$ is a upper congruence subgroup if $v \neq v_0$ and $L_v$ is not self dual; $K_{0,v} = \Gamma_0(v_0)$ if $i = 1$ and $K_{0,v} = \Gamma_0(v_0)$ is a lower triangular congruence subgroup if $i = 2$. Consider the generating series

$$\mathcal{Z}^B(h_\infty, \phi) = \sum_{\xi \in F_0} \mathcal{Z}^B(\xi, \phi) W^{(n)}(h_\infty).$$

**Theorem 14.4.** Assume $E = F$ (e.g. if $F$ is Galois over $\mathbb{Q}$). Then any 1-cycle $C \rightarrow \mathcal{M}$ over $O_E[\Delta^{-1}]$, the function

$$(\mathcal{Z}^B(h_\infty, \phi), C)$$

is a holomorphic modular form on $SL_2(F_0)$ of parallel weight $n$ of level $K_0$.

**Remark 14.5.** In particular, in the case $n = 2$ we answer the question in [51, Remark 4.13].

In the rest of this section, we prove this theorem. From Theorem [14.3] and the modifications over $\mathbb{C}$ (Proposition [13.12] which uses $E = F$) and $\mathbb{F}_q$ (Proposition [13.17] and Proposition [13.23]), it is sufficient to show the theorem for 1-cycles $C$ of the following two forms:
(1) $C$ is the normal integral model of a Hecke CM cycle $C = CM(\alpha, \mu_G, \Delta)$, where $\alpha$ is maximal order at any finite place $v$ of $F_0$ outside $\Delta$ and is unramified at $v_0$. In the set up \[15.2\] we choose the function $\Phi'$ to match the function $\Phi = \Phi_\Delta \otimes \prod_{i \in \Delta}(1_U(L_i) \otimes 1_{L_i})$ with $\Phi_\Delta := \phi_\Delta \otimes 1_{\mu_G, \Delta}$ in $S((S(V_0) \times V')(\mathbb{A}_0))$. Then the intersection number $(\hat{Z}(h_{\infty}, \phi), CM(\alpha, \mu_G, \Delta))$ is expected to match the automorphic generating function $\partial_{|\alpha|}(h_{\infty}, \Phi')$ in Section \[15.2\].

The expected matching (hence the modularity) is reduced to local computations by \[15.15\] and Proposition \[15.18\]. We are done by ATCs for unramified maximal orders in \[10\] AFL for maximal orders in \[37\] Cor. 9.1 and archimedean computations in \[59\] Lem. 14.4.

(2) $C$ is a very special 1-cycle in the basic locus of $M_{k,n}$ where $n$ is a place of $E$ over an inert place $v_0$ of $F_0$. This is done in the next section 14.4.

14.4. Modularity for very special 1-cycles in basic locus. Assume $C$ is a very special 1-cycle in the basic locus of $M_{k,n}$ where $n$ is a place of $E$ over an inert place $v_0$ of $F_0$.

To show $(\hat{Z}(h_{\infty}, \phi), C)$ is modular, we use basic uniformization and local modularity results (Section \[9\]. In the end, we will find $(\hat{Z}(h_{\infty}, \phi), C)$ can be adelized into a theta series for suitable Schwartz function $\phi' \in S(V^{(v_0)}(\mathbb{A}_{0,f}))$ in the nearby hermitian space of $V$ at $\aleph_0$. Without loss of generality, we can assume $C$ is of the form

$$C = C_0 \otimes 1_{\bar{\nu}_{\mathfrak{m}} \to M_{k,n,0}}$$

for some 1-cycle $C_0 \subseteq N^{\text{red}}$ and $g_0 \in G(\mathbb{A}_{0,f}/ \mathbb{A}_{f,v_0})$. Note the functions $(Z(u), C_0)$ and $(Z(u), C_0)$ $(u \in V^{(v_0)}(F_0, v_0) \setminus 0)$ extend to smooth functions on $V^{(v_0)}(F_0, v_0)$ by Theorem 9.1 under the assumption $C$ is a very special 1-cycle. So we have a Schwartz function $\phi'_C \in S(V^{(v_0)}(\mathbb{A}_{0,f}))$ defined by

$$\phi'_{C,v}(u) := \begin{cases} \phi_v(g_v^{-1}u) & \text{if } v \neq v_0, \\ (Z(u), C_0) & \text{if } v = v_0, i = 1, \\ (Y(u), C_0) & \text{if } v = v_0, i = 2. \end{cases}$$

Theorem 14.6. We have an equality

$$(\hat{Z}(h_{\infty}, \phi), C) = \phi'_C(0) + \sum_{\xi \in F_0^+} \sum_{u \in V^{(v_0)}(F_0)} \phi'_{C}(u)W^{(v)}_{\xi}(h_{\infty}).$$

Hence $(\hat{Z}(\xi, \phi), C)$ is the $\xi$-th Fourier coefficient of the theta series $\Theta(h, \phi'_C)$ for $\phi'_C$ on $V^{(v_0)}$. 

Proof. The non-constant term follows from basic uniformization in Proposition 12.17 directly. The constant term is done below. \[\Box\]

Remark 14.7. For $n = 2, t = 1$, basic uniformization is used in \[25\] Section 4.3 to show the modularity of their arithmetic theta series on orthogonal Shimura curves when intersecting with vertical projective lines $\mathbb{P}^1$. We will use the local duality between $Z$ and $Y$ cycles in Theorem 9.1 later to relate $(\hat{Z}(h_{\infty}, \phi_1), C)$ and $(\hat{Y}(h_{\infty}, \phi_2), C)$.

14.5. Constant terms. Recall that $\hat{Z}(0, \phi) = -\phi(0)c_1(\mathbb{Z})$, we need to show

$$-\phi(0) \deg(w|\mathcal{C}) = \phi'_C(0).$$

Identifying the prime-to-$v_0$ part, this reduces to the following proposition.

Proposition 14.8. For a very special 1-cycle $C$, we have

$$-\deg(w|\mathcal{C}) = \lim_{u \to 0} (Z(u), C_0)$$

and

$$q \deg(w|\mathcal{C}) = \lim_{u \to 0} (Y(u), C_0).$$

Proof. As $C$ is very special, via pullback we only need to do the case $n = 2$ and $t_v = 1$ i.e., the unitary Shimura curve case with Drinfeld uniformization at $v_0$.

We now use computations in Theorem 9.3. We firstly do the case $C = \mathbb{P}^1_{\mathcal{C}} = \mathbb{P}(L^2/\pi L^2)$ for a type 0 lattice $L^2$. Then by Theorem 9.6 the right hand side is 1 for $Z$-cycles and $-q$ for $Y$-cycles. So we are reduced to show

$$\deg(w|\mathcal{C}) = -1.$$
This is true by relative Dieudonné theory: by equation 8.20 a special pair \((A, B)\) (here \(A = L^\circ\)) determines a point \(x = \varpi B/\varpi A^\circ\) in \(P(L^\circ/\varpi(L^\circ)\vee)\). The line bundle \(\omega = (\text{Lie } X)_{1}\) at \(x\) gives the line \(M_1/V_2M_0 \cong B^\perp/\varpi A^\circ\) in \(P(L^\circ/\varpi(L^\circ)\vee)\). So \(\omega\) is the tautological line bundle \(O(-1)\) on \(P^1_L\), hence has degree \(-1\).

The case \(C = P^1_L\) for a type 2 lattice \(L^*\) is similar. \(\square\)

**Remark 14.9.** See [29, Section 11.1] for similar computations for orthogonal Shimura curves.

**Part 4. The proof of ATC for \(p > 2\)**

For \(p > 2\) and any \(p\)-adic local field \(F_0\) unramified over \(\mathbb{Q}_p\), we now prove arithmetic transfer conjectures 6.8 and 6.3. By equivalences in Theorem 7.5 we only need to prove the semi-Lie version ATC in part (1) of Conjecture 6.3 for any vertex lattice \(L\) in \(F/F_0\)-hermitian spaces.

The strategy is local-global: we globalize the data and produce a pair of automorphic forms both on geometric side (using modularity) and analytic side. Their Fourier coefficients away from \(v\) are matched.

By double induction we get global identities for all Fourier coefficients. We deduce some local identities at \(v\) and finally the desired local equalities by subtracting local modification terms.

**15. The global analytic and geometric side**

In this section, we construct the global analytic side and geometric side in the semi-Lie setting at maximal parahoric levels, following [59, Section 12, 14, 15]. Recall the set up in Section 11.

- \(F/F_0\) is a totally imaginary quadratic extension of a totally real number field. Choose a CM type \(\Phi\) of \(F\) and a distinguished element \(\varphi_0 \in \Phi\).
- \(V\) is a \(F/F_0\)-hermitian space of dimension \(n \geq 1\) that has signature \(\{(n-1,1)_{\varphi_0}, (n,0)_{\varphi \in \Phi - \{\varphi_0\}}\}\).
- Fix a distinguished finite place \(v_0\) of \(F_0\) that is inert in \(F\), with residue characteristic \(p > 2\). Fix an integer \(0 \leq t_0 \leq n\).
- \(\Delta\) is a finite collection of places of \(F_0\) and \(L\) is a hermitian lattice satisfying assumptions in 11.2 Assume \(v_0 \notin \Delta\) and \(L_{v_0}\) is a vertex lattice of type \(t_0\).

We have the integral model 11.2

\[ M = M_{G,\bar{K}} \rightarrow \text{Spec } O_E[\Delta^{-1}] \]

of the RSZ Shimura variety with level \(\bar{K}\) for \(L\) and \(\Delta\).

**15.1. Analytic generating functions and derivatives at \(s = 0\)**

We firstly globalize the analytic side in Section 2. Fix an orthogonal \(F\)-basis of \(V\) to endow it with a \(F_0\)-rational structure \(V_0\) and a \(F/F_0\) semi-linear involution \((-)\) with fixed subspace \(V_0\).

Consider the symmetric space

\[ S(V_0) = \{ \gamma \in \text{GL}(V) | \gamma \gamma = id \} \]  \hspace{1cm} (15.1)

and the natural \((F_0 \times F_0)/F_0\)-hermitian space

\[ V' = V_0 \times (V_0)^* \]  \hspace{1cm} (15.2)

Consider orbit integrals for the action of \(h \in \text{GL}(V_0)\) on the product \((\gamma, u_1, u_2) \in S(V_0) \times V'\) diagonally by

\[ h. (\gamma, u_1, u_2) = (h^{-1} \gamma h, h^{-1} u_1, u_2 h) \]  \hspace{1cm} (15.3)

Let \(\alpha \in F[t]\) be a degree \(n\) irreducible conjugate self-reciprocal monic polynomial. Then \(F' := F[t]/(\alpha(t))\) is a field with its subfield \(F_0'\) fixed by the involution.

Consider the subscheme \(S(V_0)(\alpha) \subseteq S(V_0)\) of elements with characteristic polynomial \(\alpha\). Then \(S(V_0)(\alpha)(F_0)\) consists of exactly one \(\text{GL}(V_0)(F_0)\)-orbit. Consider a decomposable function

\[ \Phi^\prime = \otimes_v \Phi^\prime_v \in S((S(V_0) \times V')(A_0)) \]
such that for every infinite place \( v \mid \infty \) of \( F_0 \), \( \Phi'_v \) is the (partial to \( \alpha \)) Gaussian test function introduced in \[59\] Section 12.4.

For \((\gamma, u') \in (S(V_0)(\alpha) \times V')(F_0)\) that is regular semisimple (equivalently \( u' \neq 0 \) as \( \alpha \) is irreducible), define the global orbit integral \((v \text{ runs over all places of } F_0)\):

\[
\text{Orb}((\gamma, u'), \Phi', s) = \prod_v \text{Orb}((\gamma, u'), \Phi'_v, s)
\] (15.4)

Set the adele quotient \([\text{GL}(V_0)] = \text{GL}(V_0)/(F_0)\backslash \text{GL}(V_0)(A_0)\). For \( h \in \text{SL}_2(A_0), s \in \mathbb{C} \), consider the regularized integral in \[59\] Section 12.4:

\[
\mathbb{J}_\alpha(h, \Phi', s) = \int_{g \in [\text{GL}(V_0)]} \left( \sum_{(\gamma, u') \in (S(V_0)(\alpha) \times V')(F_0)} \omega(h)\Phi'(g^{-1}.(\gamma, u')) \right) |g|^s \eta(g) dg.
\] (15.5)

Denote by \([(S(V_0)(\alpha) \times V')(F_0)] \) (resp. \([(S(V_0)(\alpha) \times V')(F_0)]_{rs} \) ) the set of (resp. regular semisimple) \( \text{GL}(V_0)(F_0) \)-orbits in \((S(V_0)(\alpha) \times V')(F_0)\).

By \[59\] Thm. 12.14, we see \( \mathbb{J}_\alpha(h, \Phi', s) \) is a smooth function of \((h, s)\), entire in \( s \in \mathbb{C} \) and is left invariant under \( h \in \text{SL}_2(F_0) \) by Poisson summation formula. We have a decomposition:

\[
\mathbb{J}_\alpha(h, \Phi', s) = \mathbb{J}^1_\alpha(h, \Phi', s)_0 + \sum_{(\gamma, u') \in [(S(V_0)(\alpha) \times V')(F_0)]_{rs}} \text{Orb}((\gamma, u'), \omega(h)\Phi', s),
\] (15.6)

where \( \mathbb{J}^1_\alpha(h, \Phi', s)_0 \) is the term over the two regular nilpotent orbits as in \[59\] Section 12.6.

For \( \xi \in F_0^\times \), the \( \xi \)-th Fourier coefficient \((1.2)\) of \( \mathbb{J}_\alpha(\cdot, \Phi', s) \) is equal to

\[
\sum_{(\gamma, u') \in [(S(V_0)(\alpha) \times V')(F_0)]_{rs}} \text{Orb}((\gamma, u'), \omega(h)\Phi', s).
\] (15.7)

Here \( V'_\xi \) is the \( F_0 \)-subscheme of \( V' \) defined by \( \{(u_1, u_2) \in V'|u_2(u_1) = \xi\} \). Then we introduce

\[
\partial \mathbb{J}_\alpha(h, \Phi') := \left. \frac{d}{ds} \right|_{s=0} \mathbb{J}_\alpha(h, \Phi', s).
\] (15.8)

\[
\partial \text{Orb}((\gamma, u'), \Phi'_v) := \left. \frac{d}{ds} \right|_{s=0} \text{Orb}((\gamma, u'), \Phi'_v, s).
\] (15.9)

By Leibniz’s rule, we have the standard decomposition:

\[
\partial \mathbb{J}_\alpha(h, \Phi') = \partial \mathbb{J}^1_\alpha(h, \Phi')_0 + \sum_v \partial \mathbb{J}^1_{\alpha,v}(h, \Phi'_v),
\] (15.10)

where \( v \) runs over all places of \( F_0 \) and

\[
\partial \mathbb{J}^1_{\alpha,v}(h, \Phi'_v) := \sum_{(\gamma, u') \in [(S(V_0)(\alpha) \times V')(F_0)]_{rs}} \partial \text{Orb}((\gamma, u'), \omega(h)\Phi'_v) \cdot \text{Orb}((\gamma, u'), \omega(h)\Phi'^{\nu} v).
\] (15.11)

The nilpotent term \( \partial \mathbb{J}^1_\alpha(h, \Phi'^{\nu} v) \) is part of the 0-th Fourier coefficient of \( \partial \mathbb{J}_\alpha(h, \Phi') \) and we do not need its precise formula in this paper. Consider the Fourier expansion

\[
\partial \mathbb{J}^1_{\alpha,v}(h, \Phi'_v) = \sum_{\xi \in F_0} \partial \mathbb{J}^1_{\alpha,v}(\xi, h, \Phi'_v),
\] (15.12)

where

\[
\partial \mathbb{J}^1_{\alpha,v}(\xi, h, \Phi'_v) = \sum_{(\gamma, u') \in [(S(V_0)(\alpha) \times V'_v)(F_0)]_{rs}} \partial \text{Orb}((\gamma, u'), \omega(h)\Phi'_v) \cdot \text{Orb}((\gamma, u'), \omega(h)\Phi'^{\nu} v).
\] (15.13)

It follows from similar properties of \( \mathbb{J}_\alpha(h, \Phi', s) \) that \( \partial \mathbb{J}^1_\alpha(h, \Phi') \) is a smooth function on \( h \in \text{SL}_2(A_0) \), left invariant under \( \text{SL}_2(F_0) \), and of parallel weight \( n \). If \( \Phi' \) is \( K_0 \)-invariant under the Weil representation, then \( \partial \mathbb{J}^1_\alpha(h, \Phi') \) is right \( K_0 \)-invariant.
15.2. Comparison to intersection numbers. Following [38 Section 8.5], consider a decomposable function \( \Phi = \bigotimes_{v<\infty} \Phi_v \in \mathcal{S}( (U(V) \times V)(K_{0,v}) ) \).

Let \( \Phi' = \bigotimes_{v<\infty} \Phi'_v \in \mathcal{S}( (S(V_0) \times V')(K_{0,v}) ) \) be a decomposable function as above such that

- for every finite place \( v \) of \( F_0 \), \( \Phi'_v \) partially (relative to \( \alpha \)) transfers to \( \Phi_v \).
- for every \( v|\infty \), \( \Phi'_v \) is the chosen (partial to \( \alpha \)) Gaussian test function.

If \( v \) is a finite place of \( F_0 \) that splits in \( F \), then \( \partial \mathcal{J}_{\alpha,v}(h, \Phi') = 0 \) by [38 Prop. 3.6].

If \( v \) is non-split (including the case \( v|\infty \)), consider the \( v \)-nearby positive definite hermitian space \( V^{(v)} \) of \( V \). By the definition of local transfers, the \( (\gamma, u') \)-term in the summation (15.11) of \( \partial \mathcal{J}_{\alpha,v}(h, \Phi') \) is 0, unless \( (\gamma, u') \) matches an orbit \( (g, u) \in [(U(V^{(v)}) \times V^{(v)})(F_0)]_\alpha \).

Let \( v \not\in \Delta \) be a finite non-split place. As \( \Phi'_v (v|\infty) \) are (partial) Gaussian test functions, we have

\[
\partial \mathcal{J}_{\alpha,v}(h, \Phi') = \sum_{\xi \in F_0, \xi \geq 0} \partial \mathcal{J}_{\alpha,v}(\xi, 1, \omega(h_f)\Phi') \frac{W_{\xi,1}(h_\infty)}{W_{\xi,1}(1)}.
\]

(15.14)

For simplicity, we normalize the value at \( h = 1 \) by

\[
\partial \mathcal{J}_{\alpha,v}(\xi, \Phi') := \frac{1}{W_{\xi,1}(1)} \partial \mathcal{J}_{\alpha,v}(\xi, 1, \Phi').
\]

By the computation of archimedean orbit integrals for Gaussian test functions [59 Section 12], we have

\[
\partial \mathcal{J}_{\alpha,v}(\xi, \Phi') = \sum_{(\gamma, u') \in ( V(V_{0}(\alpha)) \times V'(F_0))_{\alpha}} \partial \text{Orb}((\gamma, u'), \Phi'_v) \cdot \text{Orb}((\gamma, u'), \Phi^{u',v}_{\infty}).
\]

(15.15)

Recall \( K \) is the level associated to \( L \) and \( \Delta \). Choose \( \Phi \) to be of the form \( \Phi_i = \phi_{CM} \otimes \phi_i \in \mathcal{S}( (U(V) \times V)(K_{0,v})) \) for \( i = 1, 2 \), where

1. For \( v \not\in \Delta \) and \( v \neq v_0 \), we have \( \Phi_{1,v} = 1_{U(L_v)} \otimes 1_{L_v} \).
2. At \( v_0 \), we have \( \Phi_{1,v_0} = 1_{U(L_{v_0})} \otimes 1_{L_{v_0}} \) and \( \Phi_{2,v_0} = 1_{U(L_{v_0})} \otimes 1_{L_{v_0}} \).

Recall the volume factor \( \tau(\mathbb{Z}^2) = \# Z^2(\mathbb{Q}) \setminus Z^2(L_f) \). For \( \xi \in F_0 \), we define the \text{global arithmetic intersection number} as

\[
\text{Int}((\xi, \Phi_i) := \frac{1}{\tau(\mathbb{Z}^2)(E : F)} (\hat{Z}^B(\xi, \phi_i) \cdot \hat{\mathcal{C}} M(\alpha, \phi_{CM})) \in \mathbb{R}.
\]

(15.16)

Recall these cycles are introduced in [12.6] and [12.15].

**Theorem 15.1.** Assume that \( \xi \neq 0 \), consider a place \( v \) of \( E \) above a finite place \( v \) of \( F_0 \). Consider the support of \( Z(\xi, \phi_i) \cap \mathcal{C} M(\alpha, \phi_{CM}) \).

(i) The support does not meet the generic fiber \( M \otimes \mathcal{O}_E E \).

(ii) If \( v \) is split in \( F \), then the support does not meet the special fiber \( M \otimes \mathcal{O}_E \kappa_v \).

(iii) If \( v \) is inert in \( F \), then the support meets the special fiber \( M \otimes \mathcal{O}_E \kappa_v \) only in its basic locus.

**Proof.** The proof of [59 Theore 9.2] can be applied directly to our levels. \( \square \)

Hence \( \text{Int}((\xi, \Phi_i) \Phi = \Phi_i \) are well-defined, and can be decomposed into a sum of archimedean contributions and non-archimedean contributions away from \( \Delta \) in \( \mathbb{R}_{\Delta} \):

\[
\text{Int}((\xi, \Phi) = \sum_{v \in \Sigma_{F_0} \setminus \Delta} \text{Int}_v((\xi, \Phi)).
\]

**Proposition 15.2.** Assume \( \xi \neq 0 \) and \( v \not\in \Delta \).

1. If \( v \) is inert in \( F \) and \( v \neq v_0 \), then

\[
\text{Int}_v((\xi, \Phi_i) = 2 \log q_v \sum_{(g, u) \in (U(V^{(v)}) \times V^{(v)})(F_0)]} \text{Int}_v((g, u) \cdot \text{Orb}((g, u), \Phi^{u,v}_{\infty})).
\]

(15.17)

Here \( \text{Int}_v((g, u) \) is the local quantity defined in the AFL (i.e., Conjecture 6.3) for rank \( n \) self-dual lattices) with respect to the quadratic extension \( F_v / F_0^v \).
(2) If \( v = v_0 \), then
\[
\text{Int}_{v_0}(\xi, \Phi_i) = 2 \log q_{v_0} \sum_{(g, u) \in [(U(V^{v_0}))(\alpha) \times V^{v_0}](F_0)} \text{Int}^i_{v_0}(g, u) \cdot \text{Orb}((g, u), \Phi_i^{v_0}).
\] (15.18)

Here \( \text{Int}^i_{v_0}(g, u) \) (resp. \( \text{Int}^2_{v_0}(g, u) \)) is the local quantity defined in Part (1) (resp. (2)) of Conjecture [12.4] for rank \( n \) and type \( t_0 \) vertex lattice with respect to the quadratic extension \( F_{v_0}/F_{0, v_0} \).

(3) If \( v|\infty \), then
\[
\text{Int}^i_v(\xi, \Phi_i) = \sum_{(g, u) \in [(U(V^{v}))(\alpha) \times V^{v}](F_0)} \text{Int}_v(g, u) \cdot \text{Orb}((g, u), \Phi_i^v). \quad (15.19)
\]

Here \( \text{Int}_v(g, u) \) is defined as the special value of the function
\[
\text{Int}_v(g, u) = \mathcal{G}^K(u, h_\infty)(z_g). \quad (15.20)
\]

**Proof.** This follows the proof of [59, Thm. 9.4, Thm. 10.1], replacing basic uniformization results in loc. cit. by basic uniformization results for Kudla–Rapoport cycles and Hecke CM cycles proved in Section [12.6].

Now we form the generating series (\( h_\infty \in SL_2(F_0, \infty) \))
\[
\text{Int}(h_\infty, \Phi) = \text{Int}(0, \Phi) + \sum_{\xi \in F_{0, +}} \text{Int}(\xi, \Phi)W_\xi(h_\infty). \quad (15.21)
\]

16. **The end of proof**

16.1. **The double induction method.** We introduce a double induction lemma which generalizes the vanishing lemma [59, Lem. 13.6] to Iwahori levels. Recall \( \psi \) is a fixed non-trivial additive character of \( A_0 \), whose the level at \( v \) is denoted by \( c_v \). For any finite place \( v \) of \( F_0 \), let \( \varpi_v \) be a uniformizer of \( F_{0, v} \). By definition, \( \psi_v \) is trivial on \( \varpi_v^{-c_v}O_{F_{0, v}} \).

For a left \( \mathcal{N}^+(F_0) \)-invariant continuous function \( f : SL_2(A_0) \rightarrow \mathbb{C} \) and \( \xi \in F_0 \), consider its \( \xi \)-th Fourier coefficient \([12.3] \):
\[
W_{\xi, f}(h) = \int_{F_0 \backslash A_0} f \left( \begin{array}{cc} 1 & b \\ 0 & 1 \end{array} \right) \psi_{F_0}(-\xi b) db.
\]

Here we use left multiplication. There is a Fourier expansion (by an absolute convergent sum) for \( h \in SL_2(A_0) \):
\[
f(h) = \sum_{\xi \in F_0} W_{\xi, f}(h). \quad (16.1)
\]

**Proposition 16.1.** Let \( B \) be a finite collection of finite places of \( F_0 \). Let \((f_1, f_2)\) be a pair of continuous functions on \( SL_2(F_0) \backslash SL_2(A_0) \) right invariant under some compact open subgroup \( K_0 \subseteq SL_2(A_0) \). Assume

(1) For all \( v \in B \), we have \( f_1 \left( h \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \right) = f_2(h), \quad \forall h \in SL_2(A_0) \).

(2) There exists \( k_{1v}, k_{2v} \in \mathbb{Z} \) such that \( f_1 \) is right \( \left( \begin{array}{cc} \varpi_v^{k_v}O_v \\ 0 \end{array} \right) \)-invariant. We require \( k_{1v} + k_{2v} \leq 0 \) for all but possibly one place \( v_0 \in B \). If such \( v_0 \) exists, we assume \( k_{1v_0} + k_{2v_0} \leq 1 \).

(3) For all \( \xi \in F_0 \) with \( v(\xi) = -c_v - k_v \) for all \( v \in B \) and any element \( h_\infty \in SL_2(F_{0, \infty}) \), we have
\[
W_{\xi, f_1}(h_\infty) = 0.
\]

Then \( f_1, f_2 \) are both constant.

**Proof.** If \( B \) is empty, then \( f_i(h_\infty) \) are left invariant under \( \left( \begin{array}{cc} 1 & F_{0, \infty} \\ 0 & 1 \end{array} \right) \) and right invariant under \( K_{0, \infty} \), hence are constant for \( h_\infty \in SL_2(F_{0, \infty}) \). By strong approximation of \( SL_2, F_0 \) at \( \infty \), we see that \( f_i(h) \) are both constant. In general, we do induction on \( \#B \). Choose any \( v \in B \). We may assume \( v = v_0 \). Fix any element \( b_v \in F_{0, v_0} \) such that \( v_0(b_v) \geq \max\{k_{1v_0}, k_{2v_0}\} - 1 \).
Set $\tilde{B} := B - \{v_0\}$ and $\tilde{k}_iv := k_{iv}$ for all $v \in \tilde{B}$. Consider the pair of functions on $SL_2(A_0)$:

$$\tilde{f}_i(h) := f_i(h \begin{pmatrix} 1 & b_{iv} \\ 0 & 1 \end{pmatrix}_{v_0}) - f_i(h).$$

We hope to apply induction to $(\tilde{B}, \tilde{f}_i)$. For any $v \in \tilde{B}$, the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{v}$ commutes with $\begin{pmatrix} 1 & b_{iv} \\ 0 & 1 \end{pmatrix}_{v_0}$ so conditions (1) and (2) still hold. Let $\xi \in F_0$ such that $v(\xi) = -c_v - k_{iv}$ for all $v \in \tilde{B}$ (with no control of $v_0(\xi)$). By condition (2), $f_i$ is right $\begin{pmatrix} 1 & \varpi_{v_0}^{-1}O_{v_0} \\ 0 & 1 \end{pmatrix}$-invariant. If $v_0(\xi) \leq -k_{iv} - c_v - 1$ then $\psi_{-\xi}$ is non-trivial on $\varpi_{v_0}^{-1}O_{v_0}$, which implies that $W_{\xi, f_i}(h_{\infty}) = 0$.

By condition (3), if $v_0(\xi) = -k_{iv} - c_v$ we have that $W_{\xi, f_i}(h_{\infty}) = 0$ hence $W_{\xi, f_i}(h_{\infty}) = 0$.

Moreover, we have

$$W_{\xi, f_i}(h_{\infty}) = (\psi(\xi b_{v_0}) - 1)W_{\xi, f_i}(h_{\infty}).$$

Hence if $v_0(\xi) \geq -v_0(b_{v_0}) - c_v$, then $W_{\xi, f_i}(h_{\infty}) = 0$. As $v_0(b_{v_0}) \geq \max\{k_{1v_0}, k_{2v_0}\} - 1$, we see condition (3) i.e., $W_{\xi, f_i}(h_{\infty}) = 0$ always holds.

By induction applied to $(\tilde{f}_i, \tilde{B})$, we see that $f_i$ is right $\begin{pmatrix} 1 & \varpi_{v_0}^{-1}O_{v_0} \\ 0 & 1 \end{pmatrix}$-invariant for $i = 1, 2$.

By the condition $f_1(h \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{v_0}) = f_2(h)$, we see $f_1$ is invariant under the group generated by $\begin{pmatrix} \varpi_{v_0}^{-1}O_{v_0} \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & \varpi_{v_0}^{-1}O_{v_0} \\ 0 & 1 \end{pmatrix}$, as $k_{1v_0} + k_{2v_0} \leq 1$. As $k_{1v_0} + k_{2v_0} \leq 1$, this group is equal to $SL_2(F_{0, v_0})$. So $f_1(h)$ is right invariant under $SL_2(F_{0, v_0})$ and therefore it must be a constant by strong approximation at $v_0$. Hence $f_2(h) = f_1(h \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{v_0})$ is also a constant. The result follows.

**Remark 16.2.** In classical languages, consider a cusp form $f \in S_k(\Gamma_0(p))$ with Iwahori level at $p$. If $f$ has Fourier coefficients $a_n(f) = 0$ for all $(n, p) = 1$ and $a_{n/p}(w_p f) = 0$ for all $(n, p) = 1$, then $f = 0$.

**Remark 16.3.** By Proposition 5.19, we have $Z(u) \cong N_{n-1}^{[t_0]}$ if $v_F((u, u)_{\gamma}) = 0$ and $Y(u) \cong N_{n-1}^{[t_0]}$ if $v_F((u, u)_{\gamma}) = -1$. This matches the above proposition for $k_{1v_0} = 0$ and $k_{2v_0} = 1$.

### 16.2. Globalization

Let $F_{v_0}/F_{0, v_0}$ be an unramified quadratic extension of $p$-adic local fields with residue fields $F_q^*/F_q$. Let $L_{v_0}$ be a vertex lattice of type $t_0$ in a $n$-dimensional $F_{v_0}/F_{0, v_0}$ hermitian space $V_{v_0}$. Denote by $V_{v_0}$ the nearby hermitian space of $V_{v_0}$.

We formulated the semi-Lie ATC Conjecture 6.3 for $L_{v_0}$ and any regular semisimple pair $(g_0, u_0) \in (U(V_{v_0}^{\psi_{\infty}}) \times V_{v_0}^{\psi_{\infty}})(F_{0, v_0})$.

Assume that $F_{0, v_0}$ is unramified over $\mathbb{Q}_p$. We can find a CM quadratic extension $F/F_0$ of a totally real field $F_0$ with an distinguished embedding $\varphi_0 : F_0 \rightarrow \mathbb{R}$, a $n$-dimensional $F/F_0$-hermitian space $V$, a CM type $\Phi$ of $F$ such that

- $\Phi$ is unramified at $v_0$.
- Either the reflex field $E = F$ i.e., $F^k \subseteq F$ so the assumption in Theorem 14.3 holds, or we assume that Conjecture 13.10 holds.
- There exists a place $v_0|p$ of $F_0$ inert in $F$ such that the completion of $F/F_0$ at $v_0$ is exactly $F_{v_0}/F_{0, v_0}$.
- All places $v \neq v_0$ of $F_0$ above $p$ are split in $F$.
- $V$ is of signature $(n - 1, 0)$ at $\varphi_0$, and $(n, 0)$ at all other real places of $F_0$.
- The localization $V \otimes_F F_{v_0}$ is isomorphic to the hermitian space $V_{v_0}$ we start with. And the localization $V_p = \bigoplus_{v \mid p} V_v$ contains a lattice $L_p = \prod_{v \mid p} L_v$ such that $L_v$ is self-dual for $v \neq v_0$ above $p$, and that $L_{v_0}$ a vertex lattice of type $t_0$ at $v_0|p$.
Let $V^{(v_0)}$ be the nearby $F/F_0$ hermitian space of $V$ at $v_0$ so the localization of $V^{(v_0)}$ at $v_0$ is isomorphic to $V^{(v_0)}_{rs}$. As in \cite{33} Section 10.1, by locally constancy of intersection numbers \cite{39} (whose argument is quite general and applies to our Rapoport–Zink space $\mathcal{N}_{L_{v_0}}$), we can find a pair $(g, u) \in (U(V^{(v_0)}) \times V^{(v_0)})(F_0)_{rs}$ such that

- $(g, u)$ is $v_0$-closely enough to $(g_0, u_0)$ such that two sides of Conjecture \ref{not_def} are the same for $(g, u)$ and $(g_0, u_0)$.
- The characteristic polynomial $\alpha$ of $g$ is irreducible over $F$.
- The norm $\xi_0 = \langle u, u \rangle \in F_0$ is non-zero.
- For all places $v \neq v_0$ of $F_0$ above $p$, the orbit integral for the $U(V_v)$-orbit of $(g, u)$

\[
\operatorname{Orb}(g(u), 1_{U(L_v)} \times 1_{L_v}) \neq 0.
\]

Choose a finite collection $\Delta$ of places of $F_0$ with $v_0 \notin \Delta$. We can enlarge $\Delta$ to assume that $\Delta$ and $L$ are in the set up of Section \ref{not_def}. Thus we obtain the integral model

\[
\mathcal{M}_{G, K} \rightarrow \operatorname{Spec} O_E[\Delta^{-1}]
\]

for the RSZ shimura variety associated to $V$ with level for $\Delta$ and $L$.

By \cite{59} Prop. 13.8, we can enlarge $\Delta$ and shrink $K_{G, \Delta}$ to find $\phi_\Delta \in \mathcal{S}(V(F_0, \Delta))$ and $\phi_{CM, \Delta} \in \mathcal{S}(K_{G, \Delta}\backslash U(V)(F_0, \Delta)/K_{G, \Delta})$ such that the following holds:

- $L$ is self-dual away from $\Delta$ and $v_0$.
- For the function

\[
\Phi^{\nu} = (\phi_{CM, \Delta} \otimes 1_{K_{G, v_0}^\Delta}) \otimes (\phi_\Delta \otimes 1_{L^{\nu}_{v_0}}) \in \mathcal{S}(U(V(\mathbb{A}_{0, f}^{v_0})) \times V(\mathbb{A}_{0, f}^{v_0})),
\]

its orbit integral $\operatorname{Orb}((g_1, u_1), \Phi^{\nu})$ at any $(g_1, u_1) \in (U(V^{(v_0)}) \times V^{(v_0)})(F_0)$ is zero, unless $(g_1, u_1) = (g, u)$ in $[(U(V^{(v_0)}) \times V^{(v_0)})(F_0)]_{rs}$.

Now set

\[
\Phi_1 := \Phi^{\nu} \otimes 1_{U(L_{v_0})} \otimes 1_{L_{v_0}^\nu}, \quad \Phi_2 := \Phi^{\nu} \otimes 1_{U(L_{v_0})} \otimes 1_{L_{v_0}^\nu}.
\]

\[\Phi_1 := \Phi^{\nu} \otimes 1_{S(L_{v_0}, L_{v_0}^\nu)} \otimes 1_{L_{v_0}^\nu} \otimes 1_{L_{v_0}^\nu}. \quad \Phi_2 := 1_{S(L_{v_0}, L_{v_0}^\nu)} \otimes 1_{L_{v_0}^\nu} \otimes 1_{L_{v_0}^\nu}.
\]

By the proven explicit transfer conjectures at $v_0$ in Part \ref{not_def}, $\Phi_1'$ partially (relative to $\alpha$) transfers to $\Phi_1$. By \ref{not_def} and \ref{not_def} in Section \ref{not_def}, we see that

**Proposition 16.4.** For $i = 1, 2$, part (i) of the Conjecture \ref{not_def} for $(g_0, u_0)$ follows from the semi-global identities:

\[
2\partial J_{\alpha, v_0}(\xi_0, \Phi_i) = - \operatorname{Int}_{v_0}(\xi_0, \Phi_i) \in \mathbb{R}. \tag{16.4}
\]

**Remark 16.5.** The proof in loc.cit. uses the assumption that all places $v \neq v_0$ of $F_0$ above $p$ are split in $F$. We can thus prove AFL for $L_v$ with $p > 2$ via such globalization (without assuming $E = F$). For out set up of ATCs, we may relax the assumption by allowing other places $v \neq v_0$ of $F_0$ above $p$ to be also inert in $F$. This makes sure that we can choose $F/\mathbb{Q}$ to be Galois, e.g. the cyclotomic extension $F/F_0 = \mathbb{Q}/\mathbb{Q}(\zeta_n) = \mathbb{Q}(\zeta_{n+1})$ with $n = q + 1$. Then we could deduce the ATC for the vertex lattice $\mathcal{L}_{v_0}$ as in the Proposition \ref{not_def} by Leibniz’s rule, using the known AFL for the self-dual lattice $L_v$ $(\forall v \neq v_0)$ and the known Jacquet–Rallis transfers proved in Part \ref{not_def} for vertex lattices $L_v$ $(\forall v \neq v_0)$.

By archimedean computations in \cite{59} Lem 14.4 and $\mathbb{Q}$-linear independence of logarithms of primes (see also \cite{33} Cor. 10.3)), we see Conjecture \ref{not_def} for $(g_0, u_0)$ follows from the semi-global identities:

\[
2\partial J_{\alpha}(\xi_0, \Phi_i) + \operatorname{Int}^{K-B}(\xi, \Phi_i) = - \operatorname{Int}(\xi_0, \Phi_i) \in \mathbb{R}_\Delta. \tag{16.5}
\]
16.3. **Modification.** Possibly enlarging \( \Delta \) (keeping \( v_0 \not\in \Delta \)), we do the following modifications:

1. We choose another irreducible conjugate self-reciprocal degree \( n \) monic polynomial \( \alpha_m \in O_F[1/|\Delta|][t] \) such that \( \alpha_m \) is an maximal order away from \( \Delta \) and unramified at \( v_0 \). Choose \( \mu_0 \in K_{G.G}/G(F_0,\Delta)/K_{G,\Delta} \) such that \( \mathcal{CM}(\alpha_m,\mu_0)(C) \) is non-empty. From Section 13.1 there exists a rational number \( \lambda_0 \in \mathbb{Q} \), such that

\[
\mathcal{C}_1 = \mathcal{CM}(\alpha,\phi_{CM}) - \lambda_0 \mathcal{C}_{\alpha_m}
\]

is of degree 0 over \( \mathbb{C} \). On the analytic side, the corresponding contribution is

\[
2\lambda_0 \partial \mathcal{C}_{\alpha_m}(\xi,\Phi'_{cm})
\]

where \( \Phi'_{cm} \) are (relative to \( \alpha_m \)) Gaussian transfers for \( \Phi_{tm} := (\Phi)_{\Delta} \otimes (1_{\mu_0} \otimes \phi_{\Delta}) \). We can enlarge \( \Delta \) such that \( \Phi_{cm} \) are also standard away from \( \Delta \). By the known ATC (and AFL) for unramified maximal orders (Section 10), the local intersection number at \( v_0 \) for \( \mathcal{C}_{\alpha_m} \) can be computed directly to match the \( v_0 \)-part of \( 2\lambda_0 \partial \mathcal{C}_{\alpha_m}(\xi,\Phi'_{cm}) \) on the analytic side.

2. By Section 13.5 for any place \( v_0 \) of \( E \) over \( v_0 \) we can find a very special 1-cycle \( \mathcal{C}_{v_0} \) in the basic locus of \( \mathcal{M}_{k_0} \) such that

\[
(X_j,\mathcal{C}_i) = (X_j,\mathcal{C}_{v_0}) \in \mathbb{Q} \log p
\]

for all irreducible components \( X_j \) of \( \mathcal{M}_{k_0} \). On the analytic side, the corresponding contribution is \( \tilde{\mathbb{Z}}^j(\xi,\phi_1,\mathcal{C}_{v_0}) \) which is \( \xi \)-th Fourier coefficient of a theta series \( \Theta(h,\phi_1^\tau) \) on \( V^{(v_0)} \) hence modular by Theorem 13.6.

For \( i = 1, 2 \), consider the 1-cycle (independent of \( i \))

\[
\mathcal{C}^\perp := \mathcal{CM}(\alpha,\phi_{CM}) - \lambda_0 \mathcal{C}_{\alpha_m} - \sum_{v_0 | v_0} \mathcal{C}_{v_0} \in \mathcal{C}_1(\mathcal{M}). \tag{16.6}
\]

Consider the modified generating function on \( h \in \text{SL}_2(A_0) \):

\[
\text{Int}(h,\Phi_1)^\perp := \frac{1}{\tau(Z(\mathbb{Q})[E:F])}(Z(h,\phi_1),\mathcal{C}^\perp)^{\text{adm}}, \tag{16.7}
\]

and the modified archimedean intersection number on \( h \in \text{SL}_2(A_0) \):

\[
\text{Int}^{K-B}(h,\Phi_1)^\perp := \frac{1}{\tau(Z(\mathbb{Q})[E:F])}\sum_{v_0 | \infty} \text{Int}^{K-B}(h,\phi_1)(\mathcal{C}^\perp). \tag{16.8}
\]

And we define the difference functions on the analytic side:

\[
\partial \mathcal{C}^{h,1}(h) := 2\partial \mathcal{C}(h,\Phi_1') - 2\lambda_0 \partial \mathcal{C}_{\alpha_m}(\xi,\Phi'_{cm}) + \sum_{v_0 | v_0} \Theta(h,\phi_1',\mathcal{C}_{v_0}) + \text{Int}^{K-B}(h,\Phi_1^\perp). \tag{16.9}
\]

These functions are holomorphic automorphic forms. Recall \( c_v \) is the level of the fixed non-trivial character \( \psi \) on \( k_0 \) at \( v \). For a compact open subgroup \( K_{0,\Delta} \leq \text{SL}_2(F_0,\Delta) \), consider the compact open subgroup \( K_0 \leq \text{SL}_2(A_0,f) \) given by

\[
K_0 = \Gamma(v_0) \times K_0^{\Delta,v_0,0} \times K_0,\Delta \leq \text{SL}_2(A_0,f)
\]

where \( K_0^{\Delta,v_0,0} := \prod_{v \not\in \Delta} \text{diag} \{ e_v^{-1}, 1 \} \text{SL}_2(O_{F_0,v}) \text{diag} \{ e_v, 1 \} \) and \( \Gamma(v_0) := \text{Ker} \text{SL}_2(O_{F_0,v_0}) \rightarrow \text{SL}_2(k_{v_0}) \). Choose \( K_{0,\Delta} \) to be sufficiently small so that the pair \( (\phi_1,\phi_2) \) are invariant under the Weil representation by \( K_0 \).

**Corollary 16.6.** For \( i = 1, 2 \), the generating function \( \text{Int}(h,\Phi_1)^\perp \) lies in \( A_{h_0}(\text{SL}_2(A_0),K_0,n)_{\mathbb{R} \otimes \mathbb{R}_\Delta,\mathbb{R}} \). Moreover, the dual relation holds:

\[
\text{Int}(h \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{v_0},\Phi_1)^\perp = \gamma_{v_0} \text{vol}(L_{v_0}) \text{Int}(h,\Phi_2)^\perp.
\]

**Proof.** See Theorem 13.3. The dual relation follows from \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). \( \square \)
Theorem 16.8. For \( i = 1, 2 \), the Conjecture 6.3 for \((g_0, u_0)\) follows from the identity of Fourier coefficients
\[
\partial \mathbb{H}_{\text{hol}, i}(\xi_0) = -\text{Int}(\xi_0, \Phi_1)^\perp.
\] (16.11)

Proof. This follows from equation (16.5) ATCs for unramified maximal orders proved in Section 10 and the computation of \( \nu_0\)-term of the theta series \( \Theta(h, \phi_i, \nu_0) \) in Section 14.4.

We now do inductions and assume the semi-Lie version Conjecture 6.3 holds for all maximal parahoric levels for all dimensions (of the hermitian space) smaller than \( n \). Recall we have enlarged \( \Delta \) to assume that \( \alpha_v(t) \) are maximal orders for any finite place \( v \nmid \Delta \) in particular for \( v = v_0 \).

Theorem 16.9. For \( i = 1, 2 \), we have the identity
\[
\partial \mathbb{H}_{\text{hol}, i}(h) = -\text{Int}(h, \Phi_i)^\perp.
\] in the function space \( \mathbb{A}_{\text{hol}}(\text{SL}_2(A_0), K_0, n)^\perp \otimes \mathbb{R}_{\Delta, \overline{\Delta}} \).

Proof. By Corollary 16.6 and Proposition 16.7 both sides are holomorphic modular forms.

For \( i = 1, 2 \), denote the differences by \( \partial_i(h) := \partial \mathbb{H}_{\text{hol}, i}(h) + \text{Int}(h, \Phi_i)^\perp \). Note by the assumption \( F_{0, v_0} \) is unramified over \( \mathbb{Q}_p \), we have \( c_{v_0} = 0 \). We now apply the double induction in Lemma 16.1 for the pair \((f_1, (\gamma_{v_0}, \text{vol}(L_{v_0})))f_2\) and \( B = \{v_0\} \) with \( k_{v_0} = 0, k_{2v_0} = 1 \).

We check the assumptions in Lemma 16.1 the dual relation holds by Corollary 16.9. The support condition follows from the valuation of \( L_{v_0} \) and \( L_{2v_0} \) are 0 and \(-1\) respectively. So we only need to show equalities of Fourier coefficients
\[
\partial \mathbb{H}_{\text{hol}, i}(\xi, h_\infty) = -\text{Int}(\xi, h_\infty, \Phi_i)^\perp.
\]
for all \( \xi \in F_0^\perp \) such that \( v_0(\xi) = 0 \) (resp. \( v_0(\xi) = -1 \)) if \( i = 0 \) (resp. \( v_0(\xi) = -1 \)). Since both sides are holomorphic of the same weight \( n \), we can assume \( h_\infty = 1 \). It is sufficient to prove the following two claims:

Claim \( Z \). For \( \xi \) with \( v_0(\xi) = 0 \), we have
\[
\partial \mathbb{H}_{\text{hol}, 1}(\xi) = -\text{Int}(\xi, \Phi_1)^\perp \in \mathbb{R}_\Delta.
\]

Claim \( Y \). For \( \xi \) with \( v_0(\xi) = -1 \), we have
\[
\partial \mathbb{H}_{\text{hol}, 2}(\xi) = -\text{Int}(\xi, \Phi_2)^\perp \in \mathbb{R}_\Delta.
\]

Adding these local error terms at \( \xi \) again, the Claim \( Z \) follows if we can show the identity
\[
2\partial \mathbb{H}_0(\xi, \Phi_1') + \text{Int}K_{B}(\xi, \Phi_1) = -\text{Int}(\xi, \Phi_1) \in \mathbb{R}_\Delta.
\]

By the factorization and vanishing of archimedean terms, it is sufficient to show
\[
2\partial \mathbb{H}_0(\xi, \Phi_1') + \text{Int}K_{B}(\xi, \Phi_1) = -\text{Int}_v(\xi, g, u) \log q_v.
\]
holds for any finite place \( v \nmid \Delta \) of \( F_0 \). By local-global decompositions on both sides (see Proposition 15.2 and equation 15.15), it is sufficient to prove the local identity
\[
\partial \text{Orb}(\gamma, u') = -\text{Int}_v(g, u) \log q_v.
\]

By our assumption, one of the following two cases holds:

(1) If \( v = v_0 \), then \( v_0(\xi) = 0 \), hence the ATC identities \( \partial \text{Orb}(\gamma, u'), \Phi_{\nu_0}^\perp = -\text{Int}_{v_0}^\perp(g, u) \) holds by Theorem 7.3 and induction.
If \( v \neq v_0 \), then due to our enlargement of \( \Delta \) we know that \( O_F[t]/(\alpha(t)) \) is a maximal order at \( v \). In this case the above identity \( \partial \text{Orb}(U, u') = -\text{Int}_v(g, u) \log q_v \) is also known by the known maximal order AFLs in [31] Cor. 9.1 (which is reprovod for unramified maximal orders in Section 10).

This finishes the proof of Claim \( Z \). The Claim \( Y \) is proved in the same way by applying the dual isomorphism. Hence the theorem follows.

Therefore, we subtract local error terms added on two sides as in Theorem 16.8 and finish the proof of arithmetic transfer conjecture 6.3 for the vertex lattice \( L_{v_0} \), assuming that \( F_0 \) is unramified over \( \mathbb{Q}_p \) if \( 0 < t_0 < n \).

References


