Today, I will focus on the remaining part of Chapter 2 of [1].

We begin by recall our moduli problem

1 Statement of Moduli Problem

**Theorem 1.1.** Let $X$ be a decent $p$-divisible group over a perfect field $L$, $W = W(L)$, consider the moduli functor $M : \text{Nilp}_W \to \text{Sets}$, where

\[ M(S) = \{(X, \rho) : X \times_L \bar{S} \to X \times_S \bar{S} \}/\cong \]

Where $(X_1, \rho_1) \cong (X_2, \rho_2)$ if and only if $\rho_1 \circ \rho_2^{-1}$ lifts to an isomorphism $X_2 \to X_1$.

Then $M$ is representable by a formal scheme over $\text{Spf} W$, which is formally of locally of finite type, and each irreducible component of $M_{\text{red}}$ is projective over $L$.

**Remark.**
- Let $J(\mathbb{Q}_p)$ be the group of quasi-isogeny on $X$, acts on the right on $M$, by $(X, \rho) \cdot \gamma = (X, \rho \circ \gamma)$. Hence acts on geometric spaces and their cohomology groups constructed from $M$.
- Drinfeld rigidity lemma implies $\text{Aut}(X, \rho) = \{\text{id}\}$
- The moduli functor $M = M_X$ only depends on isogeny class of $X$.

In order to understand the “discrete part” of $M$, we introduce

**Definition 1.2.** (1) If $f : X \to Y$ is an isogeny of $p$-divisible groups over $S$. Then the order of $\ker f$ is $p^h$, for some $Z_{\geq 0}$ valued, locally constant function $h$. If $h$ is constant, we call it the **height** of $f$.

(2) $f : X \to Y$ be a quasi-isogeny. Assume $p^n f$ is an isogeny, we define the **height** of $f$ by

\[ \text{ht}(f) = \text{ht}(p^n f) - \text{ht}(p^n) \]

For example, the height of multiplication by $p$ is the height of $X$.

**Remark.** For isogenies, one has $\text{ht}(f_1 \circ f_2) = \text{ht}(f_1) + \text{ht}(f_2)$. Thus the height of a quasi-isogeny is well-defined, and the above relation also holds for quasi-isogenies.
Height is a discrete invariant of a quasi isogeny. Define $\mathcal{M}(h)(S) = \{(X, \rho) | \text{ht}(\rho) = h\}$. Then $\mathcal{M}(h)$ is an open and closed functor of $\mathcal{M}(h)$ is an open and closed subfunctor of $\mathcal{M}$. And $\mathcal{M} = \bigsqcup_{h \in \mathbb{Z}} \mathcal{M}(h)$. Thus, it suffices to show each $\mathcal{M}(h)$ is representable. Or one can define $\widetilde{\mathcal{M}} = \bigsqcup_{h = 0}^{h \leq X - 1} \mathcal{M}(h)$, and it suffices to show $\widetilde{\mathcal{M}}$ is representable.

2 Some examples

We focus on $L = \bar{L}$ and $X$ is a height 2, dim 1 $p$-divisible group. We study the corresponding functor.

Up to isogeny, they are classified by associated isocrystal, by Dieudonné-Manin classification, they are classified by Newton polygons from $(0, 0)$ to $(2, 1)$.

Remark. We have the following facts: for a $p$-divisible group $G$ over a perfect field $k$ of characteristic $p$

- $G$ is étale $\iff D(G)_q$ is isoclinic of slope 0
- $G$ is formal (defined later) $\iff D(G)_q$ has no zero slope.

Now we consider associated $\mathcal{M}$ for these $X$.

Example 2.1. When $X = E[p^\infty]$, where $E$ supersingular elliptic curve over $L$. We will show that as a formal scheme

$$\mathcal{M} = \bigsqcup_{h \in \mathbb{Z}} \text{Spf}(W[[x]])$$

Remark. $\mathcal{M}_{\text{red}} = \mathcal{M}(L)$ is disjoint union of points

We will show a more general result:

Proposition 2.2. If $X$ comes from a formal group of dim 1, height $n$ over $L = \bar{L}$, then

$$\mathcal{M}_X = \bigsqcup_{h \in \mathbb{Z}} \text{Spf}(W(L)[[x_1, \cdots, x_n]])$$

Some background:

Definition 2.3. An $n$-dimensional commutative formal group law over ring $A$ is a power series $F \in A[[X_1, \cdots, X_n, Y_1, \cdots, Y_n]]$ which satisfies some formal group axioms.

For example, we have $\overline{G}_a = X + Y, \overline{G}_m = X + Y + XY$, or completion of abelian scheme over $A$ along zero section, or Lubin-Tate formal group appeared in local class field theory.

Definition 2.4. We say $F$ is $p$-divisible if $[p] : A[[X]] \to A[[X]]$ is finite locally free. The rank of $[p]$ is $p^h$ for some $h$. $h$ is called height.

One has the following result of Tate and Messing

Theorem 2.5 (Tate-Messing). The category of $p$-divisible formal group fully faithfully embeds in to category of $p$-divisible groups, which preserve height and dimension.
The essential image above is called formal $p$-divisible groups.

We will mainly focus on 1-dim formal group law. When $F$ is a 1-dim formal group law over a field of characteristic $p$, the notions above can be characterized in an easier way:

**Proposition 2.6.** For 1-dim formal group law $F$ over a field of characteristic $p$, then $F$ is $p$-divisible if and only if $[p] \neq 0$. In this case, $[p] = g(X^{p^h})$ for some $g$ with $g'(0) \neq 0$. Such $h$ is the height of $F$.

We $[p] = 0$, we also say $F$ has height $\infty$.

**Theorem 2.7.** If $L = \bar{L}$, then for each height $h \in \{1, 2, \cdots, \infty\}$, there exists a unique (up to isomorphism) 1-dim formal group $F_0$ of height $h$ over $L$.

**Fact:** $\text{End}(F) = \mathbb{Z}_p[\Pi], \Pi^h = p, \Pi a = \sigma(a)\Pi$.

We can consider the following Lubin-Tate deformation functor: Fix $F$ above. Define $\mathcal{C}_L$, the category of local Artin rings with a fixed surjection $A \to L$

Consider the following deformation functor:

$$\mathcal{D} : \mathcal{C}_L \to \text{Sets}$$

which sends $A \to L$ to isomorphism classes of $\{F, \iota\}$, where $F$ is a formal group law over $A$ and $\iota$ is an isomorphism $F \otimes_A L \cong F_0$.

One has the following theorem of Lubin and Tate

**Theorem 2.8.** $\mathcal{D}$ is representable by $\text{Spf}(W(k)[[x_1, \cdots, x_{n-1}]]$.

Now we find its relationship with R-Z moduli problem

**Lemma 2.9.** For 1-dim $p$ divisible formal group, quasi-isogeny of height 0 is equivalent to an isomorphism

**Proof.** Self quasi-isogeny is $D^x$, and both morphisms above corresponds to $O_D^x$. \qed

Now if $R$ is artinian local ring, $\text{Spec} R \in \text{Nil}_W$.

**Proposition 2.10.** $\mathcal{D}(R) \cong \mathcal{M}(0)(R)$ canonically.

**Proof.** $\mathcal{M}(0)(R)$ consists of quasi-isogeny of height 0 on $R/p$, which is equivalent to quasi-isogeny of height 0 on $L$, by Drinfeld rigidity, thus by the lemma, which is equivalent to an isomorphism. And by a result of Tate, $X$ is $p$-divisible over $R \in \mathcal{C}_L$, then $X$ is connected if and only if $X$ is formal. Thus, $\mathcal{M}(0)(R)$ is equivalent to $\mathcal{D}(R)$ \qed

**Remark.**

- We didn’t prove $\mathcal{M}(0)$ is representable by $\sqcup \text{Spf} W(k)[[x_1, \cdots, x_{n-1}]]$ above. (Since we need to compare to other $S \in \text{Nil}_W \setminus \mathcal{C}_L$. But once we know $\mathcal{M}$ representable and is formally of locally of finite type, and $\mathcal{M}$ is a point, then we can conclude $\mathcal{M}(0) = \mathcal{D}$.

- Since $\Pi$ is an isogeny of height 1, thus for each height, we have isogeny of height 1, thus $\mathcal{M}$ is disjoint union of $\text{Spf} W(k)[[x_1, \cdots, x_{n-1}]]$, parametrized by height.

**Example 2.11.** Let’s briefly talk about another example, where $X = E[p^\infty]$, for an ordinary elliptic curve $E$. In this case, $\mathcal{M} = \bigsqcup_{m \geq 0} \mathcal{O}_m$, which is a restatement of theory of Serre-Tate coordinates.
3 Proof strategy

Recall that, in the previous week, we have reduced to the case where $L$ is a finite field, and $X$ lifts to $\overline{X}$ on $\text{Spf} \ W(L)$.

The proof strategy is

1. Approximate $\mathcal{M}$ by “controllable” subfunctors $\mathcal{M}_n$ of $\mathcal{M}$

2. Show that each $\mathcal{M}_n$ is representable by a formal scheme, and underlying reduced schemes are eventually the same.

3. Take $\mathcal{M}$ as same space + “limit of sheaves”, then show it is a formal scheme, which really represents $\mathcal{M}$.

Here are some definitions related to “controllable” ones,

**Definition 3.1.**

1. Define $\mathcal{M}^n(S) = \{(X, \rho) \in \mathcal{M}(S) | p^n \rho \text{ is an isogeny}\}$.

2. For quasi-isogeny $\alpha : X \rightarrow Y$, define $q(\alpha) = \text{ht}(p^n \alpha)$, where $n$ is smallest integer such that $p^n \alpha$ is an isogeny,

3. Define $d(\alpha) = q(\alpha) + q(\alpha^{-1})$.

4. Define $\mathcal{M}_c(S) = \{(X, \rho) \in \mathcal{M}(S) | d(\rho_s) \leq c \text{ for any } s \in S\}$.

$d$ is like kind of a metric:

**Lemma 3.2.** $d(\alpha) + d(\beta) \geq d(\alpha + \beta)$

**Proof.** Easy, reduced to additivity of height function.

Now we can state our strategy more clearly:

1. Each $\mathcal{M}_n$ is representable.

2. $\mathcal{M}_c^n$ is representable.

3. Underlying reduced scheme of $\mathcal{M}_c^n$ are eventually the same

4. $\mathcal{M}_c$ is representable.

5. $\mathcal{M}$ is representable.

4 Some details of the proof

**Proposition 4.1.** $\mathcal{M}^n$ is representable

**Proof.** Define $\mathcal{M}^{n,m} = \{(X, \rho) | \text{isogeny of height } m\}$, which is an open and closed subfunctor of $\mathcal{M}$. Suffices to show each $\mathcal{M}^{n,m}$ is representable.

But give such $\rho$ is equivalent to give an isogeny of height $m$, which corresponds to locally free subgroup of $\overline{X}[m]$ of order $p^m$. Thus it is representable by a closed subscheme of Grassmanian (consists of Hopf ideals, which is defined by a polynomial relation). And, we can also take $p$-adic completion of this scheme, since $p$ is locally nilpotent, thus after taking completion, it represents the same functor.
Proposition 4.2. $\mathcal{M}_c^n$ is representable.

Lemma 4.3. For $\alpha : X \to Y$ an isogeny, $\{s \in S | d(\alpha_s) \leq c\}$ is closed.

Proof. Not hard, it follows from the subsets of $S$ where a given quasi-isogeny is an isogeny, which is a proposition proved last time.

Proof of 4.2 Consider the universal $p$-divisible group on $\mathcal{M}_c^n$, since the sets of points where $d(\alpha_s) \leq c$ is closed, taking completion of it suffices.

The last three steps are more complicated, which is based on the key lemma Prop 2.17 in [1].

References