Frobenius on p-adic modular forms and the theta operator

Zhiyu Zhang

STAGE seminar

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Outline

p-adic modular forms

Classical modular forms with coefficients in a ring R_0 (weight k, level n, meromorphic at infinity) can be thought as R-valued functions on triples (E, ω, α_n), where E is an elliptic curve over a R_0 algebra R, ω is a non-vanishing differential on E, and α_n is a level n structure, together satisfying some transformation laws e.g $f(E, \lambda\omega) = \lambda^{-k} f(E, \omega)$ for any $\lambda \in R^\times$. Its value at the Tate curve $\mathrm{Tate}(q)$ with the canonical differential will give the classical q-expansion.

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Note the weight $p-1$ modular form Hasse invariant has q-expansion 1, hence for any lift $A(q)$ of it to char zero (a weight $p-1$ modular form), we know $A^{p^n-1} \to A^{-1}$ $(n \to +\infty)$ p-adically. Therefore, A^{-1} is a p-adic modular form in the sense of Serre. So to geometrize p-adic modular forms, we shall work on some "p-adic space" where a lifting of Hasse invariant is invertible (or at least non-zero), this is why we choose to work over the non-vanishing locus of Hasse invariant i.e the ordinary locus.
To get p-adic modular forms, Katz uses test objects (E, ω, α_n, Y) with growth condition $r \in [0, 1]$, where $Y E_{p-1}(E, \omega) = a$. Here the base is a complete DVR R_0 with characteristic $(0, p)$, $a \in R_0$ is fixed such that $v_p(a) = r, v_p(p) = 1$, and $(n, p) = 1$.

Katz's approach to p-adic modular forms

- Geometrically, p-adic modular forms "=" sections of powers of the Hodge bundle over the ordinary locus of modular curves.
- geometry of modular curves with natural compactification, Tate curve at the cusp $\Rightarrow$ q-expansion principle.

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- overconvergent modular forms are good p-adic modular forms such that the section can be extended to the overconvergent locus $X(r)$ (a rigid space as we're removing disc with small radius r) for some $0 < r \leq 1$.
- growth condition , canonical subgroup $\Rightarrow$ overconvergent modular forms, $U_p : M_k^\dagger(r) \to M_k^\dagger(pr)$ $(0 \leq r < \frac{1}{p+1})$, spectral theory of the compact operator U_p.

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- **differential operator e.g theta operator can also be defined geometrically for p-adic / mod p modular forms**.
- people have been trying to generalize these geometric constructions and constructions of p-adic L-functions to more general setting ...

Gauss-Manin connection – complex picture

If R is the ring of holomorphic functions of τ on upper half plane, and $E \to R$ be the relative elliptic curve "$\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$", whose affine model is

$$y^2 = 4x^3 - \frac{E_4}{12}x + \frac{E_6}{216},\ E_i \in R.$$

Let's denote the complex coordinate on E by z, then the embedding $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau \hookrightarrow \mathbb{P}^2$ is given by $x = \wp_\tau(z), y = \wp'_\tau(z)$.

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The dual $H_1^{dR}(E/R)$ is a free R-module with basis γ_1, γ_2, which can thought as the paths from 0 to τ and from 0 to 1 respectively.
The Gauss-Manin connection on $H^1_{dR}(E/R)$ is defined as the dual connection on $H_1^{dR}(E/R)$ such that

$$\int_{\gamma_i} \nabla_\tau(\xi) = \frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\gamma_i} \xi, \quad \forall \xi \in H^1_{dR}(E/R), i = 1, 2$$

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here $\nabla_\tau = \nabla(\frac{d}{d\tau})$, and $\int_{\gamma_i}\xi \in R$ i.e a holomorphic function on upper half plane so we can take its derivative. Let $\omega = \frac{dx}{y}, \eta = \frac{xdx}{y}$ be the standard basis, taking the integration along γ_i we get periods of E as functions of τ (they can be also regarded as the connection matrix between the basis ω, η and γ_1^*, γ_2^*).

Connection matrix

By definition $\omega = dz, \eta = \wp_\tau(z)dz$, and we compute these integrals using power series expansion. Let $P = E_2 = 1 - 24\sum_{n\geq 1}(\sum_{d\geq 1, d|n} d)q^n$.

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Proposition

$\omega_1 = \tau, \omega_2 = 1, \eta_1 = \tau P, \eta_2 = -\frac{\pi^2}{3}P$. We have

$$\nabla(\theta)\begin{pmatrix}\omega_{\mathrm{can}}\\ \eta_{\mathrm{can}}\end{pmatrix} = \begin{pmatrix}\frac{-P}{12} & 1\\ \frac{P^2-12\theta P}{144} & \frac{P}{12}\end{pmatrix}\begin{pmatrix}\omega_{\mathrm{can}}\\ \eta_{\mathrm{can}}\end{pmatrix}$$

where we choose a normalization: $\omega_{\mathrm{can}} = 2\pi i w, \omega_{\mathrm{can}} = \frac{1}{2\pi i}\eta$, $\theta = \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}$ $(q = e^{2\pi i\tau})$ (the theta operator).

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If $q \mapsto 0$, we get a nilpotent matrix $\begin{pmatrix} -\frac{1}{12} & 1 \\ -\frac{1}{144} & \frac{1}{12} \end{pmatrix}$, so the connection has non-trivial unipotent monodromy at $q = 0$.

Summary

Conclusion: analytically, theta operator occurs naturally via Gauss-Manin connection on de Rham cohomology of universal elliptic curve.
Today we will see more about this for p-adic modular forms in a geometrical way using canonical subgroup, hence work over the ordinary locus. The overconvergent story is similar.

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There is one and only one way to attach to every $(E/R, Y)$ (R a p-adically complete R_0-algebra) a finite flat rank p subgroup scheme $H \subseteq E$, called the canonical subgroup of E/R such that:

1. The formation of H commutes with arbitrary change of base of p-adically complete R_0-algebras.
2. if $p/a = 0$ in R, then H is the kernel of Frobenius $E \to E^{(p)}$.
3. if E/R is the Tate curve $\mathrm{Tate}(q^n)$ over $R_0/p^N((q))$, then $H = \mu_p$.

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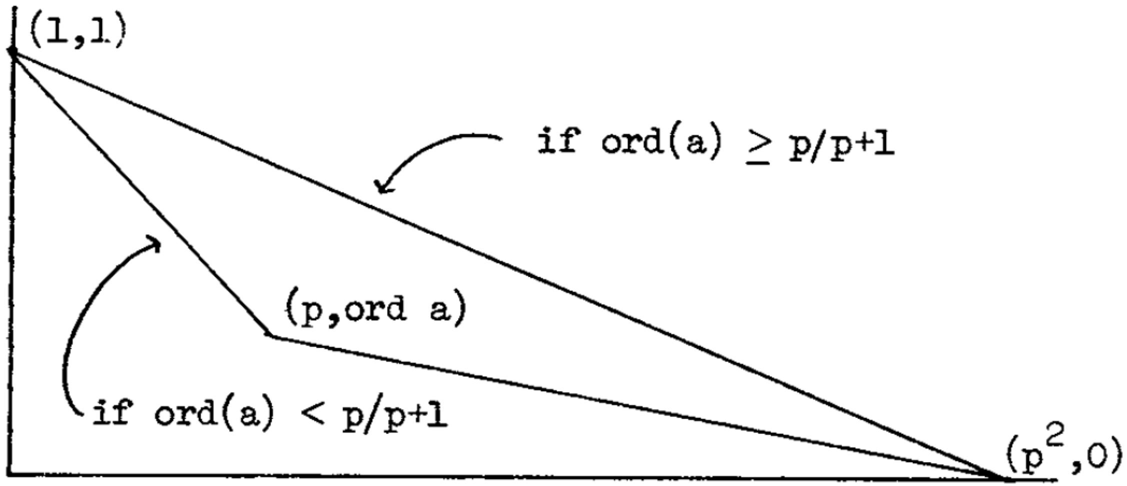
Proof: $E[p] = \{X | [p](X) = 0\}$, the filtration on roots of $[P]$ by valuation gives a subgroup filtration of $E[p]$ ($v(X +_E Y) \geq max\{v(X), v(Y)\}$), Hasse invariant is essentially p-th coefficient of the power series $[p](X)$, if $r < \frac{p}{p+1}$ then looking at Newton polygon we can pick up p roots with larger valuations than others.

A picture



Motivation

The starting point for p-adic differential operator, is the unit-root splitting of the Hodge filtration over the ordinary locus, which is **an analogue** to the Hodge decomposition over $\mathbb{C}$ (which can be used to define Maass-Shimura type differential operator in complex setting). Then Katz applies such differential operators to Eisenstein series to get p-adic Eisenstein series. Sum of their values at some CM points will give p-adic interpolation for L-values over CM fields in the end.

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From now on, we work universally and $p > 3$. Let $R = M(W(\mathbb{F}_q), 1, n, 0)$ be the ring of p-adic modular functions (growth $a = 1$, weight 0) of level n defined over $W(\mathbb{F}_q)$, i.e R represents (the p-adic completion) of the ordinary locus.
Let $E \to R$ be the universal elliptic curve with Hasse$(E) \in (R/p)^\times$ (as $r = 1$), $H \subseteq E$ the canonical subgroup (unique lifting of the multiplicative type group scheme $\ker(Fr : E \to E^{(p)})$ mod p in our ordinary case).
We have the quotient map $\pi : E \to E' := E/H$.

Motivation for appendix 2 of [Kat73]

Observation: E/H over R still has invertible Hasse, so by universality of E there is a unique ring map $\varphi : R \to R$ so that the base change of $E \to R$ along φ gives $E' = E^{(\varphi)} \to R$. The base change of φ is denoted by $\varphi_E : E' \to E$.
In terms of moduli interpretation, the map $E \mapsto E/C_{can}(E)$ (called Delinge-Tate map) gives a self map of the ordinary locus.

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Therefore $H^1_{dR}(E'/R) = H^1_{dR}(E^{(\varphi)}/R) \overset{\text{flat base change}}{=} H^1_{dR}(E/R)^{(\varphi)}$, and π^* on H^1_{dR} gives a φ-linear endormorphism on $H^1_{dR}(E/R)$, which we denote by $F(\varphi)$.
Question: How to understand this Frobenius action explicitly? How to define theta operator? Can we extend it to the compactification?

Hodge filtration

$\underline{\omega}_{E/R} = H^0(E, \Omega^1)$ is the sheaf of invariant 1-differentials, and we have the Hodge filtration of locally free R-modules

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note $\frac{1}{6} \in R$, and we can (non-canonically) split this sequence at least locally on R:
For (E, ω), we have a unique pair of meromorphic functions with poles only at ∞, of orders 2 and 3 resp., denoted by X, Y so that

$$\omega = \frac{dX}{Y} \quad \text{generates } \underline{\omega} \text{ and } E\colon Y^2 = 4X^3 - \frac{E_4}{12}X - \frac{E_6}{216}, E_i \in R$$

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$H^1_{dR}(E/R) \cong H^0(E, \Omega^1(2\infty))$ has a basis $\omega = \frac{dX}{Y}, \eta := \frac{XdX}{Y}$ (differential of second kind, and η is the Serre dual of ω). If we change by $\omega \mapsto \lambda\omega$, then $X \mapsto \lambda^{-2}X, Y \mapsto \lambda^{-3}Y, \eta \mapsto \lambda^{-1}\eta$.

Hodge filtration

The Frobenius $F(\varphi)$ preserves the Hodge filtration by functoriality.

Lemma A2.1 of [Kat73]

On $H^0(E, \Omega^1)$, $F(\varphi) = p\varphi$; and on $H^1(E, O_E)$, $F(\varphi) = \varphi$. Here the ring homomorphism φ gives natural endomorphism of O_E hence on its cohomology and cohomology of its Serre dual, for which we still denote by φ.

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Proof.

Let $f = f(E, \omega)\omega$ be a section of $\underline{\omega} = e^*\Omega^1_{E/R}$ (e the zero section). By definition, $\varphi(f) = f(E/H, \pi^{\vee,*}(\omega))\omega$. H is of multiplicative type, $H^\vee$ is étale. So $\pi^\vee$ is étale, $\pi^{\vee,*}(\omega) = \lambda\omega^{(\varphi)}$ for some $\lambda \in R^\times$. So $\varphi(f) = f(E^{(\varphi)}, \lambda\omega^{(\varphi)}) = \lambda^{-1}\varphi(f(E, \omega))\omega$. By definition $F(\varphi)(f) = \pi^*((f(E, \omega)\omega)^{(\varphi)}) = \varphi(f(E, \omega))\pi^*(\omega^{(\varphi)}) = p\varphi(f)$ (note $\pi^*(\lambda\omega^{(\varphi)}) = [p]^*(\omega) = p\omega$). The dual case is similar. $\square$

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So if we work modulo p i.e over R/p, we see $F(\varphi)=0$ on $\underline{\omega}$, this is essentially due to $d(x^p)=0$.

Canonical splitting of Hodge filtration

Therefore, locally under the basis ω, η, $\mathrm{F}(\varphi)\begin{pmatrix} \omega \\ \eta \end{pmatrix} = \begin{pmatrix} \mathrm{p}/\lambda & 0 \\ \mathrm{c} & \lambda \end{pmatrix}\begin{pmatrix} \omega \\ \eta \end{pmatrix}$ for some $\lambda \in R^{\times}, c \in R$.

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There is a unique $f \in R$ s.t $F(\varphi)(f\omega + \eta) \in R(f\omega + \eta)$.

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$F(\varphi)(f\omega + \eta) = \varphi(f)\frac{p}{\lambda}\omega + c\omega + \lambda\eta$, so we want $\varphi(f)\frac{p}{\lambda} + c = \lambda f$. Namely, we need to show $T : R \to R$ by $T(f) = \frac{c}{\lambda} + \frac{p}{\lambda^2}\varphi(f)$ has a unique fixed point. As T is a contraction and R is p-adically complete, this follows from the contraction mapping theorem. $\square$

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This characterization is independent of the local basis ω, η.
Conclusion: Hodge filtration has a φ-stable splitting $a\eta \to af_\eta$ over the entire ordinary locus using Frobenius, and in fact it is the largest open subset for this to hold, see [Kat77]. In other words, $H^1_{\mathrm{dR}}(E/F) := \underline{\omega} \oplus U$, $U = H^1_{\mathrm{dR}}(E/F)^{\mathrm{Frob}}$ is generated by fixed (up to a unit λ) vectors of Frobenius $F(\varphi)$, so we get a retraction $r : H^1_{dR}(E/R) \to \underline{\omega} = H^0(E, \Omega^1)$.

Theta operator

Over the ordinary locus X^{ord}, we have Gauss-Manin connection and Kodaira-Spencer isomorphism $KS : \underline{\omega}^{\otimes 2} \cong \Omega^1_{X^{ord}}$. Here KS is defined by

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and it can be thought as the dual of the tangent mapping of the classifying map from the base scheme of the elliptic curve (here it is the ordinary locus) to the modular curve M_n, so it shall be an isomorphism (we will see the computation at infinity later). We define **the theta operator** over the ordinary locus by

$$\underline{\omega}^{\otimes k} \hookrightarrow Sym^k(H^1_{dR}) \xrightarrow{\nabla_{GM}} Sym^k(H^1_{dR}) \otimes \Omega^1 \xrightarrow{\mathrm{KS}^{-1}} Sym^k(H^1_{dR}) \otimes \underline{\omega}^{\otimes 2} \xrightarrow{\mathrm{proj}} \underline{\omega}^{\otimes k+2}$$

Note the projection map uses the splitting $r : H^1_{dR}(E/R) \to H^0(E, \Omega^1)$, and the Gauss-Manin connection on $Sym^k(H^1_{dR})$ is defined by $\nabla(a_1 \otimes .. \otimes a_k) = \sum_{i=1}^k a_1 \otimes ... \otimes \nabla(a_i) \otimes ... \otimes a_k$.

Extension to the whole modular curve

A calculation (we will see the computation at infinity later) shows that θ has at most simple poles at the supersingular points, and Hasse invariant has simple zeros at the supersingular points. In other words, we can extend $A\theta$ (A is a lifting of Hasse invariant) to the whole modular curve. In literature, sometimes this is denoted by θ (mapping weight k one to weight $k+p+1$ one), and the original operator is denote by θ_0.

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Finally, we need to check θ recovers $\theta = q\frac{d}{dq}$ on q-expansion. Here we only consider the original operator (as A has q-expansion 1 mod p, the results are the same at least for mod p modular forms).

Extension to the compactification

The Hodge bundle $\underline{\omega}$ extends to X and characterised by global sections of its k-th power being the same as weight k modular forms with coefficient in the base field. The Hodge filtration also extends to short exact sequence of locally free sheaves. The Gauss-Manin connection extends to a connection with logarithmic poles

$$\nabla : H^1_{dR} \to H^1_{dR} \otimes \omega^1(\log D),$$

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In paritcualr at infinity, we consider the Tate curve Tate(q) over $\mathbb{Z}[1/N]((q))$ with its canonical invariant differential form $\omega_{can} = \frac{dq}{q}$ and canonical $\Gamma_1(N)$ level structure $\alpha_{N,can}$. The Gauss-Manin connection on $H^1_{dR}(\mathrm{Tate}(q)/\mathbb{Z}[1/N]((q)))$ can be defined similarly as in the complex setting.
The canonical subgroup is μ_p, complex analytically it's $\frac{1}{p}\mathbb{Z}/\mathbb{Z} + \mathbb{Z}\tau \hookrightarrow \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ with quotient Tate(q^p). The Frobenius map $\varphi : \mathbb{Z}[1/N]((q)) \to \mathbb{Z}[1/N]((q))$ sends q to q^p.

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Denote $u_{can} := \nabla\left(q\frac{d}{dq}\right)(\omega_{can})$. u_{can} is a horizontal section i.e $\nabla u_{can} = 0$ as the periods of ω_{can} are $1, \tau$, and they are killed by $(\frac{d}{d\tau})^2$.

Tate curves

At infinity, we have

Proposition

1. u_{can} is fixed by $F(\varphi)$, and spans U as a rank 1 locally free R module. Hence, U can also be characterized as the rank 1 φ-stable sub-module of horizontal sections.
2. $u_{can} = \frac{-P(q)}{12}\omega_{can} + \eta_{can}$, so $\langle \omega_{can}, u_{can} \rangle_{dR} = 1$, ω_{can}, u_{can} form a basis of H^1_{dR}. Under this basis, $F(\varphi) = \mathrm{diag}\{p, 1\}\varphi$. Moreover, $KS(\omega_{can}^2) := \langle \omega_{can}, \nabla\omega_{can} \rangle_{dR} = \frac{dq}{q}$.

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Sketch of proof: The second part over $\mathbb{Z}[1/N]$ follows from previous computation over $\mathbb{C}$ and analytically by appropriate integral normalization. For the first part, note $F(\varphi)$ commutes with ∇ by functoriality of Gauss-Manin connection, but u_{can} is the unique horizontal section up to scalar (this is for monodromy reason or by direct computation), hence $F(\varphi)u_{can} = au_{can}$ for some $a \in \mathbb{Z}$. The Frobenius action on H^2_{dR} is multiplication by $\deg(\pi) = p$ twisted by φ, and $\omega_{can} \wedge u_{can}$ is identified with $\langle \omega_{can}, u_{can}\rangle_{dR} = 1$. As $\pi^*(\omega_{can}^{(\varphi)}) = p\omega_{can}$, we know $a = 1$.

Computation

for any section f of $\underline{\omega}^k$ over the ordinary locus,

$$\begin{aligned}\nabla_{GM}\left(f(q)\cdot\omega_{\mathrm{can}}^{\otimes \mathrm{k}}\right) &= \nabla\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)\left(f(q)\cdot\omega_{\mathrm{can}}^{\otimes \mathrm{k}}\right)\cdot\frac{\mathrm{d}q}{q} \\ &\overset{KS(\omega_{can}^2)=\frac{dq}{q}}{\mapsto} \nabla\left(q\frac{\mathrm{d}}{\mathrm{dq}}\right)\left(f(q)\cdot\omega_{\mathrm{can}}^{\otimes \mathrm{k}}\right)\cdot\omega_{\mathrm{can}}^{\otimes 2} \\ &= q\frac{\mathrm{d}}{\mathrm{d}q}\left(f(q)\right)\cdot\omega_{\mathrm{can}}^{\otimes \mathrm{k}+2} + \mathrm{k}\cdot f(q)\cdot\omega_{\mathrm{can}}^{\otimes \mathrm{k}+1}\cdot\nabla\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)\left(\omega_{\mathrm{can}}\right)\end{aligned}.$$

Because the second term is in U (u_{can} is fixed by $F(\varphi)$), hence is zero under projection, we see $\theta = q\frac{d}{dq}$.

Properties of theta operator

In [Kat77], Katz studies properties of theta operator on mod p modular forms. Here we fix an algebraically closed field K of characteristic $p > 0$, an integer $N > 3$ prime to p. Consider the graded ring R_N^* of (meromorphic at cusps) level N modular forms over K, then we have

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Theorem 1 in [Kat77]

- There exists a derivation $A\theta : R_N^* \to R_N^{*+p+1}$ and acts by $q\frac{d}{dq}$ at each cusp, where A is the Hasse invariant.
- If $f \in R_N^k$ has exact filtration k (i.e not of the form Ag for some $g \in R_N^{k-p-1}$) and $p \nmid k$, then $A\theta f \in R_N^k$ has exact filtration $k+p+1$ in particular non-zero.
- If $f \in R_N^{pk}$ and $A\theta f = 0$, then $f = g^p$ for a unique $g \in R_N^k$.

E_2 as a weight 2 p-adic modular form

$$P = E_2 = 1 - 24 \sum_{n \geq 1} (\sum_{d|n} d) q^n$$

is not a classical modular form in the complex setting but close to be a modular form. However, it's in fact q-expansion of a weight two p-adic modular form.

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Definition

For any ordinary elliptic curves $(E/R, \omega)$ where p is nilpotent in R. Let $U \subseteq H^1_{dR}(E/R)$ be the inverse image of the canonical U as above. $H^1_{dR}(E/R) = R\omega \oplus U$, so for any base u of U (at least exists if we work locally), then $\langle u, \omega \rangle \in R^\times$ because the Poincare duality over R is a perfect pairing. We define $\tilde{P}(E, \omega) = 12 \frac{\langle \eta, u \rangle}{\langle \omega, u \rangle}$.

E_2 as a weight 2 p-adic modular form

As $(\omega, \eta) \mapsto (\lambda\omega, \lambda^{-1}\eta)$, we see $\tilde{P}$ is a p-adic modular form of weight 2 and level 1. As we can choose $u = \nabla(\theta)(\omega_{can}) = -\frac{P(q)}{12}\omega_{can} + \eta_{can}$ at infinity, in fact $\tilde{P}(q) = P(q)$.

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Final remarks:

- If $F(\varphi)^n \left(x\frac{dx}{y}\right) = a_n\frac{dx}{y} + b_n\frac{xdx}{y}$ then $P(E, \omega) \equiv \frac{-12a_n}{b_n} \pmod{p^n}$, this is useful in practice to compute special value of E_2, see the note *computational tools for quadratic Chabauty.*
- However, E_2 is not overconvergent (see Coleman's paper). This reflects the feature that theta operator may not preserve overconvergence, which is used in the proof of some classicality results
- Locally under the basis ω, η, $\mathrm{F}(\varphi)\begin{pmatrix} \omega \\ \eta \end{pmatrix} = \begin{pmatrix} \mathrm{p}/\lambda & 0 \\ \mathrm{c} & \lambda \end{pmatrix}\begin{pmatrix} \omega \\ \eta \end{pmatrix}$ for some $\lambda \in R^\times, c \in R$. By modulo p, we see $p|c$, and λ is a local lifting of Hasse invariant.
- One can avoid compactification by regarding Tate(q) as an elliptic curve over $\mathbb{Z}((q))$ and pull everything back along the classifying map.

References

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- Appendix 3 by M. Emerton of F.Gouvea's 1999 lecture at the IAS/Park City Mathematics Institute.

Thank you!