Perfectoidization and perfect prismatic complex

\[ \Delta_{X/A, \text{perf}} \]

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STAGE seminar

May 14th, 2021
Outline

1. What is perfection in char \( p \) ..?

2. Construction via prismatic cohomology

3. Universal property, applications
Perfection in char $p$

$k$ perfect field of char $p > 0$. \( \{\text{perfect ring over } k\} \xrightarrow{\text{forget}} \{\text{rings over } k\} \)
has a left adjoint:

The perfection of a $k$-algebra $R$

\[ R_{\text{perf}} := \text{colim}(R \xrightarrow{\phi} R \xrightarrow{\phi} R \xrightarrow{\phi} \ldots), \text{ where } \phi : x \to x^p. \]
Observations (to be generalized later):

- $R_{\text{perf}}$ is a $R$-algebra via first term of colim, or by adjointness.
- $R_{\text{perf}}$ is independent of $k$.
- $R \to R_{\text{perf}}$ is the universal map from $R$ to a perfect $k$-algebra.
- Zariski closed=Strongly Zariski closed: $R_1 \to R$ a surjective map with perfect $R_1$, then $R \to R_{\text{perf}}$ is surjective.
- perfect rings are reduced. Frobenius is zero on higher $\pi_i, i > 0$. 
Reconstruction via derived de Rham cohomology

dR_{-}/k = \text{left Kan extension of de Rham complex } \Omega^{*}_{-}/k \text{ on polynomial } k\text{-algebras.}

φ on R ↷ \phi_k\text{-semilinear endomorphism } φ_R : dR_R/k \to dR_R/k.

The perfection of dR_R/k

dR_{R/k, perf} := \text{colim}(dR_R/k \phi_R \to dR_R/k \phi_R \to dR_R/k \phi_R \to ..).
dR−/k = left Kan extension of de Rham complex Ω∗−/k on polynomial k-algebras.

φ on R → φk-semilinear endomorphism φR : dR/R/k → dR/R/k.

The perfection of dR/R/k

dR/R/k,perf := colim(dR/R/k φR dR/R/k φR dR/R/k φR ..).

The projection dR/R/k → R gives dR/R/k,perf ≅ Rperf: we reduce to the case R is a polynomial algebra, then \( d(x^p) = px^{p-1}dx = 0 \), so colimits of \( \Omega^i_{R/k} \) (i > 0) under \( \phi_R \) is zero.
Reconstruction via derived prismatic cohomology

\((A, I) = (W(k), p)\) the perfect prism corresponding to \(k\). \(R\) a \(A/I = k\) algebra. \(\Delta_{R/A} \in D(A)\), with \(R \to \overline{\Delta}_{R/A}\). \(I = (p), \phi(p) \subseteq (p), \phi\) still acts on \(\overline{\Delta}_{R/A}\).

**Proposition**

The map \(R \to \overline{\Delta}_{R/A}\) gives \(\overline{\Delta}_{R/A,\text{perf}} \cong R_{\text{perf}}\).
Reconstruction via derived prismatic cohomology

$(A, I) = (W(k), p)$ the perfect prism corresponding to $k$. $R$ a $A/I = k$ algebra. $\Delta_{R/A} \in D(A)$, with $R \to \overline{\Delta}_{R/A}$. $I = (p)$, $\phi(p) \subseteq (p)$, $\phi$ still acts on $\overline{\Delta}_{R/A}$.

**Proposition**

The map $R \to \overline{\Delta}_{R/A}$ gives $\overline{\Delta}_{R/A,\text{perf}} \cong R_{\text{perf}}$.

WLOG $R$ is a polynomial algebra. By Hodge-Tate comparison

$gr_{i}^{HT}(\overline{\Delta}_{R/A}) = \Omega^{i}_{R/k}$, only need to check $gr_{i}^{HT}(\phi) = 0, i > 0,$

$gr_{0}^{HT}(\phi) = \phi_{R}, i > 0$. WLOG $R = k[x]$. By crystalline comparison, reduce to de Rham cohomology of $\mathbb{A}^{1}_{W}$ over $W$, where mod $p$ Hodge-Tate filtration = canonical filtration.
Main players

$S$ $p$-complete ring, $X = \text{Spf}(S)$.

Choose a perfect prism $(A, I)$, with $A/I \to S$.

$\Delta_{X/A} \in D(A)$, $\Delta_{X/A,\text{perf}} := (\text{colim}_\phi \Delta_{X/A})^\wedge \in D_{(p,I)-\text{comp}}(A)$.

The (derived) “perfectoidization” of $S$

$S_{\text{perfd}} := \Delta_{X/A,\text{perf}} \otimes_A^L A/I \in D_{p-\text{comp}}(S)$. 
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$S_{\text{perfd}} := \Delta_{X/A,\text{perf}} \otimes^L_A A/I \in D_{p-\text{comp}}(S)$.

Why the name? Is $S_{\text{perfd}}$ (derived) perfectoid / independent of $(A, I)$ / the classical perfection in char $p$ case? Is $S_{\text{perfd}}$ universal?
Why is $S_{\text{perfd}}$ nice?

- HT comparison + goodness of $L_{X/(A/I)}$ + derived Nakayama $\sim$ control $\Delta_{X/A}$ hence $S_{\text{perfd}}$, get descent and base change.
- Study general $S$ via reduction to nice $A/I \to S$. For nice $S$ (e.g. quasiregular semi-perfectoids, in particular perfectoids), show $S_{\text{perfd}}$ is the classical universal perfectoid over $S$. 
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- Universality of $S_{\text{perfd}}$ comes from derived “functorial” universality of $\Delta_{X/A}$, at least if $\Delta_{X/A} \otimes^L_A A/I$ is discrete.

$\Delta_{X/A}$ has a derived $\delta$-ring (even “derived prism”) structure (we need simplicial tools to see this). Therefore $\Delta_{X/A,\text{perf}}$ is a derived perfect $\delta$-ring, and $S_{\text{perfd}} := \Delta_{X/A,\text{perf}} \otimes^L_A A/I$ is “derived perfectoid”.

3 levels of universality in category theory

For $X \in \mathcal{C}$, weakly initial $<$ “functorial” initial $<$ initial (uniqueness).

“Functorial”: for any $Y \in \mathcal{C}$, There is a map $X \rightarrow Y$, functorial on $Y$.

Initial object $\times$ something can be ’functorial’ initial.

- By design, cohomology of a site $R\Gamma(-, \mathcal{O}) = R\lim(*)$ e.g $\Delta_{X/A}$, has good “functorial” universality (provided it’s discrete and in the site). But the true universality i.e uniqueness is subtle: the conjecture in [BS] Lemma 7.7 seems open in general.
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- But via computations on simple examples and descent, still get universality of $S \to S_{\text{perfd}}$ in good cases, and prove things needed for comparison theorems.
Applications

Why is perfectoidization powerful? Many applications related to descent:

- Zariski closed $\Rightarrow$ Strongly Zariski closed: If $S$ is semiperfectoid, then $S \to S_{\text{perfd}}$ is surjective and universal among $S \to \text{perfds}$.

- $\Delta_{X/A,\text{perf}} \cong R\Gamma((X/A)^{\text{perf}}_{\Delta}, O_{\Delta})$, and arc descent for $S \to S_{\text{perfd}}$ (next time).

- The étale comparison $R\Gamma_{\text{et}}(X_\eta, \mathbb{Z}/p^n) \cong (\Delta_{X/A}[1/d]/p^n)^{\phi=1}$ for perfect prisms, can be proved via descent to the perfectoidization (next time).

Now we do some recollections.
Prismatic cohomology

**Conceptual leap:** we don’t need a Frobenius on $R$, only Frobenius on the test objects.

$S$ $p$-complete ring, $X = \text{Spf}(S)$. Choose a perfect prism $(A, I)$, with $A/I \to S$.

The prismatic site of $X$ over $A/I$

$(X/A)_\Delta$ is the opposite of the category of prisms $(B, J)$ with a map $(A, I) \to (B, J)$ and a map $\text{Sp}(B/J) \to X$ over $\text{Sp}(A/I)$.

$\mathcal{O}_\Delta : (B, J) \mapsto B$. $\overline{\mathcal{O}}_\Delta : (B, J) \mapsto B/J = B/IB$.

$\Delta_{X/A} = R\Gamma((X/A)_\Delta, \mathcal{O}_\Delta) \in D(A)$, with a natural map $S \to \overline{\Delta}_{X/A}$.
The Hodge-Tate filtration

Proposition

\[ \overline{\Delta}_{X/A} := \Delta_{X/A} \otimes^L_A A/I \] admits a natural increasing \( \mathbb{N} \)-indexed filtration, with \( i \)-th graded piece given by the derived \( p \)-completion of
\[ \wedge^i L_{S/(A/I)} \{-i\}[-i]. \]

Idea: Universal property of the de Rham complex \( \rightsquigarrow \) the comparison map from de Rham to Hodge-Tate cohomology.
Reminder of Hodge-Tate in char $p$

$k$ perfect field char $p > 0$. $X$ over $k$ smooth, relative Frobenius $F : X \to X^{(p)}$. Two filtration on de Rham complex $\Omega^*_{X/k}$:

- Hodge filtration = stupid filtration $\sim$ Hodge spectral sequence $E_1^{pq} = H^q(X, \Omega^p_{X/k})$.

- Conjugate filtration = canonical filtration $\sim$ Hodge-Tate spectral sequence $E_2^{pq} = H^p(X, H^q(\Omega^*_{X/k}))$.

Hodge-Tate filtration is a generalization of conjugate filtration, via Cartier isomorphism.
Perfection in mixed characteristic

\[ \Delta_{X/A, \text{perf}} := (\text{colim}_\phi \Delta_{X/A})^\wedge \in D_{(p,I) - \text{comp}}(A). \]

The (derived) "perfectoidization" of \( S \)

\[ S_{\text{perfd}} := \Delta_{X/A, \text{perf}} \otimes_A^L A/I \in D_{p - \text{comp}}(S) \] is a commutative algebra object.

The universal classical perfectoid of \( S \)

\[ S_{\text{perfd'}} := \text{the universal classical perfectoid over } S \text{ (if it exists)}, \text{ i.e} \]
\[ S_{\text{perfd'}} \text{ is perfectoid, and for any } S \to R \text{ with } R \text{ perfectoid, there is a unique map } S_{\text{perfd'}} \to R \text{ extending it.} \]
Independence of base

Proposition

Let \((A, I) \to (B, J)\) be a map of perfect prisms, and let \(S\) be a \(p\)-complete \(B/J\)-algebra. Then the natural map gives an isomorphism \(\Delta_{S/A} \cong \Delta_{S/B}\). In particular, \(\Delta_{S/A,\text{perf}} \cong \Delta_{S/B,\text{perf}}\), \(S_{\text{perfd}}\) is independent of \((A, I)\).

Proof.

Use HT comparison, we’re reduced to show \(L^\wedge_{(B/J)/(A/I)} = 0\), which follows from that \(A/I\) and \(B/J\) are both perfectoid.
The formation of $\Delta_{X/A}$ commutes with base change in the sense that for any map of bounded prisms $(A, I) \to (B, J)$, $\Delta_{X_B/B} = B \otimes_A^L \Delta_{X/A}$. We can check directly that $S \to S'_{\text{perfd}}$ also commutes with base change of the perfect prism.
$S_{\text{perfd}} = S = S'_{\text{perfd}}$ for perfectoid $S$

So if $S$ perfectoid, $(S/A)_\Delta$ has an object $(A_{inf}(S), \text{Ker}\theta_S)$, hence a map $\Delta_{S/A} \to A_{inf}(S)$, it’s an isomorphism: apply derived Nakayama and $HT$, done by $L^\wedge_{S/(A/I)} = 0$. We see $S_{\text{perfd}} = S$, in particular discrete. Note $(A_{inf}(S), \text{Ker}\theta_S)$ is also the initial object in $(S/A)_\Delta$:

**Proposition**

Let $(A, I)$ be a perfect prism corresponding to a perfectoid ring $R = A/I$. Then for any prism $(B, J)$, any map $A/I \to B/J$ of commutative rings lifts uniquely to a map $(A, I) \to (B, J)$ of prisms.

Proof: use the relation between deformation theory and cotangent complex, we’re done by $L^\wedge_{A/Z_p} = 0$. 

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The case $S$ is semiperfectoid

**Proposition**

If $S$ is semiperfectoid i.e there is a surjection $R \to S$ with $R$ perfectoid, then $S_{\text{perfd}}'$ exists.

**Proof.**

We can cut out the perfect prism for $S_{\text{perfd}}'$ inside the perfect prism $(A_{inf}(R), d)$ for $R$ (we know $R_{\text{perfd}}' = R = R_{\text{perfd}}$), using the kernel of $A_{inf}(R) \to S$. We need to do transfinite induction, to make it both $d$-torsion free and derived complete.
Derived “functorial” universality of $\Delta_{X/A}$

**Proposition**

Assume $\Delta_{X/A}$ is concentrated in degree zero. Then the pair $(\Delta_{X/A}, I\Delta_{X/A})$ gives a prism over $(A, I)$, with a map $R \to \Delta_{X/A}$. For any prism $(B, J)$ over $(A, I)$ equipped with a map $R \to B/J$, there is a map $(\Delta_{X/A}, I\Delta_{X/A}) \to (B, J)$, functorial on $(B, J)$.
Proposition
Assume $\overline{\Delta}_{X/A}$ is concentrated in degree zero. Then the pair $(\Delta_{X/A}, I\Delta_{X/A})$ gives a prism over $(A, I)$, with a map $R \to \overline{\Delta}_{X/A}$. For any prism $(B, J)$ over $(A, I)$ equipped with a map $R \to B/J$, there is a map $(\Delta_{X/A}, I\Delta_{X/A}) \to (B, J)$, functorial on $(B, J)$.

Proof.
$\Delta_{X/A} = \Gamma((X/A)_\Delta, \mathcal{O}_\Delta) = \lim_{(B, J)} (B, J)$. Use Cech-Alexander complexes, and the canonical simplicial resolution of $X$, we see the existence of a derived $\delta$-structure i.e a section of $W_2(-) \to (-)$ on $\Delta_{X/A}$. $\Delta_{X/A}$ is discrete by assumption, so $(\Delta_{X/A}, I\Delta_{X/A})$ gives a prism over $(A, I)$. The universality is clear by definition of $R\Gamma$. 

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Discreteness of perfection when $S$ is semiperfectoid

For any $X$, $\Delta_{X/A,\text{perf}}$ always lies in $D^{\geq 0}$ i.e $H^i(\Delta_{X/A,\text{perf}}) = 0, i < 0$, because Frobenius is zero on higher homotopy groups $\pi_i, i > 0$.

If $S$ is semiperfectoid, then $\Omega^1_{S/(A/I)} = 0$. By HT comparison, this implies $L_{X/A[-1]} \cdot \Delta_{X/A,\text{perf}} \in D^{\leq 0}$. So $\Delta_{X/A,\text{perf}}$ is discrete. By previous proposition, it’s a classical perfect $\delta$-ring and $d$-torsion free. Hence $S_{\text{perfd}}$ is discrete and perfectoid. By equivalence of perfectoid rings and perfect prisms, we see

**Proposition**

If $S$ is semiperfectoid, then $S \to S_{\text{perfd}}$ satisfies “functorial” universality among $S \to \text{perfds}$, in particular there is a section $S_{\text{perfd}} \to S_{\text{perfd}}'$ to the map $S_{\text{perfd}}' \to S_{\text{perfd}}$. 

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Andre’s Flatness lemma

Proposition

Let $R$ be a perfectoid ring. For any set $\{f_s \in R\}_{s \in I}$ of elements of $R$, there exists a $p$-completely faithfully flat map $R \to R_\infty$ of perfectoid rings such that each $f_s$ admits a compatible system of $p$-power roots in $R_\infty$. In other words, the map $\#: R^b \to R$ is surjective locally for the $p$-completely flat topology.
Andre’s Flatness lemma

WLOG $\#I = 1$. Let $S$ be the $p$-adic completion of $R[x^{1/p^\infty}]/(x - f)$, so $R \to S$ is $p$-completely faithfully flat. We reduce the problem to $S$, but $S$ is not perfectoid in general. We only know $S$ is semiperfectoid. Let $(A, I)$ be the perfect prism corresponding to $R$. We shall show that $S_{\text{perfd}}$ solves the problem. We know discreteness and perfectoidness, and only need to check $S_{\text{perfd}}$ is $p$-completely faithfully flat over $R$. 
Andre’s Flatness lemma

It suffices to show \( A \to \Delta_{S/A, \text{perf}} \) is \((p, I)\)-completely faithfully flat, which is implied by that \( A \to \Delta_{S/A} \) is \((p, I)\)-completely faithfully flat. So we only need to check \( \overline{\Delta}_{S/A} \) is \( p \)-completely faithfully flat over \( R \).

Use HT filtration, we see it’s faithful as the zero graded piece \( gr_0 = R \). We only need to check \( L_{S/R}[-1] \) (noting \( \wedge^i L_{S/R}[-i] = \wedge^i (L_{S/R}[-1]) \)) is \( p \)-complete flat over \( R \).
Consider $R \rightarrow R[x^{1/p^\infty}] = R' \rightarrow S$. $L_{R'/R}^\wedge = 0$ by perfectoidness. We only need to show $\wedge^i L_{S/R'}[-i]$ is $p$-completely faithfully flat over $R$. But $S$ is the quotient of $R'$ by non-zero divisor $x - f$, so $L_{S/R'}[-1]$ is simply isomorphic to $S$, hence $p$-completely flat over $R$. We’re done.
Zariski closed=Strongly Zariski closed

**Proposition**

Let \( R \) be a perfectoid ring, and let \( S = R/J \) be a \( p \)-complete quotient (so \( S \) is semiperfectoid). Then there is a universal map \( S \to S' \) with \( S' \) being a perfectoid ring. Moreover, this map is surjective.

We just need to check \( S \to S_{\text{perfd}} \) is surjective.
Assume first that the kernel $J \subseteq R$ of $R \to S$ is the $p$-completion of an ideal generated by a set $\{x_i\}$ of elements that lie in the image of the map $\# : R^b \to R$.

In this case, if $J_\infty$ denotes the $p$-completion of the ideal generated by $x_1/p^n \#$, then check directly that the $R/J_\infty$ is perfectoid, and $S \cong R/J \to R/J_\infty$ is the universal map from $S$ to a perfectoid ring.
This proves the assertion in this case. In general, as the surjectivity of a map of $p$-complete $R$-modules can be detected after $p$-completely faithfully flat base change (and $S_{\text{perf}}$ commutes with base change), we reduce to previous case by Andre’s flatness lemma.
A complex $M$ of $A$-modules is $I$-completely flat if for any $I$-torsion $A$-module $N$, the derived tensor product $M \otimes^L AN$ is concentrated in degree 0. This implies in particular that $M \otimes^L_A A/I$ is concentrated in degree 0, and is a flat $A/I$-module.

**Proposition**

Let $(A, I)$ be a bounded prism. For any $(p, I)$-completely flat $A$-complex $M \in D(A)$. Then $M$ is discrete and classically $(p, I)$-complete.
In fact, one can deduce the following thing from section 7 – 8 of [BS]

Proposition
Consider any $p$-complete ring $S$ over a perfectoid ring $A/I$ where $(A, I)$ is a perfect prism.

- If $S_{\text{perfd}'}$ exists, then $S_{\text{perfd}}$ is discrete and agrees with $S_{\text{perfd}'}$
- If $S_{\text{perfd}}$ is discrete, then $S_{\text{perfd}'}$ exists and agrees with $S_{\text{perfd}}$. 
The uniqueness part in the universality can be also deduced from the compatibility between \((-\)\)_{\text{perfd}} and derived tensor product \(- \otimes^L -\): assume \(S_{\text{perfd}}\) is discrete and \(R\) is a perfectoid ring. Let \(f_1, f_2 : S_{\text{perfd}} \to R\) be two maps over \(S\), then they induce a map \((f_1, f_2) : S_{\text{perfd}} \otimes_S^L S_{\text{perfd}} \to R\), which factors through \((S_{\text{perfd}} \otimes_S^L S_{\text{perfd}})_{\text{perfd}} = S_{\text{perfd}}\).
References

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Thank you!