$^{\mathbb{L}}\mathcal{CM} \cap \mathcal{KR}$

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Short summary (following [1]):

- review the set up with analogues between local and global setting.
- define the derived intersection and the arithmetic intersection paring, and explain why both are necessary and useful.
- analyze the support of $^{\mathbb{L}}\mathcal{CM} \cap \mathcal{KR}$, state the precise relation between local and global intersection numbers.
- introduce the idea of p-adic uniformization with some motivating examples.

1 The set up

Let $F_0 = \mathbb{Q}$, F be an imaginary quadratic field and fix an embedding $\Phi : F \hookrightarrow \mathbb{C}$ (denote it by $a \mapsto a$). Choose a herm space V/F of $\dim_F = n$ s.t $V \otimes_\Phi \mathbb{C} \cong \mathrm{diag}\{1_{n-1}, -1\}$ (as usual, the type at infinity is $U(n-1,1)$), and a herm space V_0/F of $\dim_F = 1$ which is positively definite. Define $G = U(V), G^{\mathbb{Q}} = GU(V), Z^{\mathbb{Q}} = GU(V_0) = Res_{F/F_0}(\mathbb{G}_m), \tilde{G} = Z^{\mathbb{Q}} \times_{\mathbb{G}_m} G^{\mathbb{Q}}$.

Then we have the RSZ Shimura variety $Sh_{\tilde{G}}$ over the reflex field $E = F$ and special cycles on it (0-dim CM and n-2 dim KR in n-1 dim Sh). To do intersection, we need integral models.

To simplify the problem, we introduce Δ as the product of all bad primes (2, p ramified in F, p s.t $V \otimes_{\mathbb{Q}} \mathbb{Q}_p$ non-split), and the base will be $B = \mathrm{Spec}\ O_F[\frac{1}{\Delta}]$.

And we work with **no level structure at good places** but **arbitrary level structure at bad places**. Fix an O_F-lattice Λ in V that is self-dual away from Δ. Choose an open compact subgroup $K_G = \prod_v K_{G,v}$ inside $\{g \in G(\mathbb{A}_f) | g(\Lambda \otimes \widehat{\mathbb{Z}}) = \Lambda \otimes \widehat{\mathbb{Z}}\}$ s.t $K_{G,v} = \mathrm{Stab}(\Lambda_v)$ for $v \nmid \Delta$, and $K^\circ_{Z^{\mathbb{Q}}} = \prod_{v|F} O_{F,v}$ the maximal compact subgroup in $Z^{\mathbb{Q}}(\mathbb{A}_f)$. Define $K_{\tilde{G}} = K^\circ_{Z^{\mathbb{Q}}} \times_{\mathbb{G}_m} K_G$.

We will use the notation $(-)_\Delta := \prod_{p|\Delta}(-)_p$ e.g $V_p = V \otimes_{\mathbb{Q}} \mathbb{Q}_p$, $T(A)_p$ is the p-adic rational Tate module of A and so on.

The set up:

$$\begin{array}{ccc} & & \mathcal{Z}(u) \\ & & \big\downarrow \quad \text{(local)} \\ \mathcal{N}^\gamma & \longrightarrow & \mathcal{N} \end{array}$$

$$\begin{array}{ccc} & & \mathcal{Z}(\xi,\mu) \\ & & \downarrow \quad \text{(global)} \\ \mathcal{CM}_R(g) & \longrightarrow & \mathcal{M} \end{array}$$

Definition 1. The moduli problem for $\mathcal{M} = \mathcal{M}_{K_{\tilde{G}}}(\tilde{G})$ over B associates to each locally noetherian $O_E[\frac{1}{\Delta}]$-scheme S the groupoid of tuples $(A_0, i_0, \lambda_0, A, i, \lambda, \overline{\eta})$, where

- A_0 is an elliptic curve over S, A is a n-dim abelian scheme over S;
- $i_0 : O_F \to \mathrm{End}(A_0)$ is a O_F action on A_0, s.t for all $a \in O_F$
$$\mathrm{char}\,(a|\text{ Lie } A_0) = T - a$$
- $i : O_F[\frac{1}{\Delta}] \to \mathrm{End}(A) \otimes \mathbb{Z}[\frac{1}{\Delta}]$ s.t for all $a \in O_F[\frac{1}{\Delta}]$
$$\mathrm{char}\,(a|\text{ Lie } A) = (T-a)^{n-1}(T-\bar{a})$$
- λ_0 (w.r.t λ) is a principal polarization (w.r.t a prime-to-Δ principal polarization) on A_0 (w.r.t A) such that the induced Rosati involution coincides with the Galois involution on O_F (w.r.t $O_F[\frac{1}{\Delta}]$);
- $\overline{\eta}$ is a $K_{G,\Delta}$-orbit of isometries of hermitian modules (as locally constant sheaves on S) $\eta : \mathrm{Hom}_F(T(A_0), T(A))_\Delta \cong V_\Delta$.

The morphism between tuples is defined in a obvious way.

Remark 1. A herm structure on $\mathrm{Hom}_F(T(A_0), T(A))_\Delta$ is implicitly used in the definition. This is a feature of RSZ shimura variety: we naturally get a hermitian lattice $\mathrm{Hom}_{O_F}(A_0, A)$ (KR lattice) using polarization i.e

$$\langle x, y\rangle = \lambda_0^{-1} \circ y^\vee \circ \lambda \circ x \in \mathrm{End}_{O_F}(A) \otimes \mathbb{Q} = F$$

More importantly, this allows us to define KR divisors.

The KR divisor encodes the homomorphism between A_0 and A with given size (like the coefficient of theta series encodes the number of vector with given norm).

The input : $\xi \in F_{0,+}$, $\mu \in V_\Delta / K_{G,\Delta}$.

Definition 2. The KR cycle $\mathcal{Z}(\xi,\mu)$ is the moduli problem associated to the groupoid of tuples $(A_0, i_0, \lambda_0, A, i, \lambda, \overline{\eta}, u)$ with $(A_0, i_0, \lambda_0, A, i, \lambda, \overline{\eta}) \in \mathcal{M}$. And $u \in \mathrm{Hom}_{O_F}(A_0, A) \otimes \mathbb{Z}[\frac{1}{\Delta}]$ such that $\langle u, u\rangle = \xi$ and $\overline{\eta}(u) = \mu$ in the quotient $V_\Delta / K_{G,\Delta}$.

The CM cycle is a variant of the “big CM cycle” of Bruinier-Kudla–Yang and Howard. The ”big“ means it’s attached to a large field extension of F. Roughly it parameterizes objects with action by a large CM order.

The input: Let F_0' be a totally real extension of F_0 of degree n, $F' = F_0' \otimes_{F_0} F$. Consider a 1-dim F'/F_0' herm space W such that $Res_{F'/F} W \cong V$, so $F'^1 = U(W) \hookrightarrow G = U(V)$. We choose $g_0 \in F'^1$ such that $R = O_F[\frac{1}{\Delta}][g_0]$ is an order in F' i.e $R \otimes \mathbb{Q} = F'$. Then $\mathrm{char}_F(g_0) \in O_F[\frac{1}{\Delta}][T]$ is irreducible of deg n. Finally, choose any $g \in G(F_\Delta) = \prod_{v|\Delta} G(F_v)$.

Definition 3. The CM cycle $\mathcal{CM}_R(g)$ is the moduli problem associated to the groupoid of tuples $(A_0, i_0, \lambda_0, A, i, \lambda, \overline{\eta}, \varphi)$ with $(A_0, i_0, \lambda_0, A, i, \lambda, \overline{\eta}) \in \mathcal{M}$ and $\varphi \in End_{O_F}(A) \otimes \mathbb{Z}[\frac{1}{\Delta}]$ such that

- compatiblilty: $\varphi^* \lambda = \lambda$.
- under any $\eta : \mathrm{Hom}_F(T(A_0), T(A))_\Delta \cong V_\Delta$ s.t $\eta \in \overline{\eta}$, φ transforms to an element in $G(F_\Delta)$ which is only well-defined **up to $K_{G,\Delta}$-conjugacy**. We require $\eta(\varphi) \in K_{G,\Delta} g K_{G,\Delta}$.
- (key) $\mathrm{char}_F(g_0)$ annihilates the endomorphism φ (i.e $\mathrm{char}_F(\varphi) = \mathrm{char}_F(g_0)$ as the latter is irreducible).

The last condition roughly says R acts on A by $g_0 \mapsto \varphi$.

An useful observation (which relates the local and global definition) is that CM cycle is a union of connected components of the fixed point locus of a Hecke correspondence, i.e we have the diagram:

$$\begin{array}{ccccc} \mathcal{CM}_R(g) & \hookrightarrow & \mathcal{M}_{[K_G g K_G]} & \longrightarrow & Hk_{[K_G g K_G]} \\ & & \downarrow & \ulcorner & \downarrow \\ & & \mathcal{M} & \xhookrightarrow{\Delta} & \mathcal{M} \times_{O_E[\frac{1}{\Delta}]} \mathcal{M} \end{array}$$

Here the Hecke stack $Hk_{[K_G g K_G]}$ parameterizes (A_1, A_2, φ) where $A_i \in \mathcal{M}$ and $\varphi : A_1 \to A_2$ is compatible with polarization and lies in $K_G g K_G$ under transformation by η_1 and η_2. The map $\mathcal{M}_{[K_G g K_G]} \to F[T]_{deg=n}$ sending (A, φ) to $\mathrm{char}_F(\varphi)$ is analogous to Hitchin map in some sense, and the fiber of $\mathrm{char}(g_0)$ is precisely $\mathcal{CM}_R(g)$ (by definition).

Theorem 1. $\mathcal{Z}(\xi, \mu) \to \mathcal{M}$ is étale locally a Cartier divisor, it's flat over B. $\mathcal{CM}_R(g)$ is **proper over** B, $\mathcal{CM}_R(g)|_{B-\mathrm{Ram}(R)}$ is finite étale over $B - \mathrm{Ram}(R)$.

Remark 2. KR and CM cycles are not literally closed substack of $\mathcal{M}$ in general, but the forgetful maps are finite and unramified. So étale locally they are disjoint union of closed immersions, see Stack project Tag 04HJ.

Remark 3. The properness of CM cycles is due to the fact that toric part of a semi-abelian scheme will have too small dimension to have an action of R. Because of properness, when considering the intersection of CM and KR cycles we can avoid discussing compactification of the Shimura variety.

Example 1. When $n = 1$, roughly $\mathcal{M}$ is (moduli of CM elliptic curves by O_F) $\times_B$ (moduli of CM elliptic curves by O_F), CM cycle is disjoint union of copys of $\mathcal{M}$, KR divisor has empty generic fiber due to Kottwitz condition and is supported in supersingular locus of special fiber.

2 Intersection of KR divisors and CM cycles

A computation of expected dimension of $\mathcal{CM}_R(g)$:

$$\dim Hk_{[K_G g K_G]} + \dim \mathcal{M} - \dim(\mathcal{M} \times_{O_E[\frac{1}{\Delta}]} \mathcal{M}) = n + n - (2n-1) = 1$$

Problem: the CM cycle is not really a 1-cycle and not flat over B in general, it's "fat" at p ramified in R i.e can have very large dimension (which is the difficult part of AFL).

We really want a 1-cycle, then the intersection of CM and KR divisor is expect to be zero dimensional, and one can count lengths to define the intersection number.

The derived intersection will solve this problem.

Definition 4. For a closed scheme $Y \hookrightarrow X$, $K'_{0,Y}(X)$ is the $\mathbb{Q}$-coefficient Grothendieck group of coherent O_X-modules on X with support in Y. Define

$$\mathrm{Fil}^i K'_{0,Y}(X) = \bigcup_{Z \hookrightarrow Y, \mathrm{codim}_X Z \geq i} \mathrm{Im}(K'_0(Z) \to K'_{0,Y}(X))$$

and similarly Fil_i the filtration by dimension.

Remark 4. Why $\mathbb{Q}$ coefficient? This is due to the use of Adams operations in the proof of some theorems about K-groups, and they are not always true over $\mathbb{Z}$. A related analogue is that formal groups $\mathbb{G}_m$ and $\mathbb{G}_a$ are isomorphic over $\mathbb{Q}$, but not over $\mathbb{Z}$.

Example 2. $K'_0(\mathbb{Z}) \overset{rk}{\cong} \mathbb{Q}$, $K'_{0,p}(\mathbb{Z}) \overset{length}{\cong} \mathbb{Q}$.

Definition 5. If X is regular, for any two coherent O_X modules $\mathcal{F}_\infty, \mathcal{F}_\in$, we define

$$\mathcal{F}_1 \otimes^{\mathbb{L}}_{O_X} \mathcal{F}_2 = \sum_i (-1)^i [Tor_i^{O_X}(\mathcal{F}_1, \mathcal{F}_2)] \in K'_{0,Supp(\mathcal{F}_1)\cap Supp(\mathcal{F}_2)}(X)$$

(by regularness, higher enough terms vanish).

The key point is that **the derived intersection has the correct dimension** (in the Grothendieck group):

Theorem 2. Let X be a regular scheme, and consider two closed subschemes $Y, Z \hookrightarrow X$. Then we have

$$Y \cap^{\mathbb{L}} Z := O_Y \otimes^{\mathbb{L}}_{O_X} O_Z \in \mathrm{Fil}^{\mathrm{codim}\, Y + \mathrm{codim}\, Z} K'_{0,Y\cap Z}(X)$$

Proof. See [5] Prop 5.5. The idea is to use Adams operations to separate each graded part. One can show Fil^i equals the sum of eigenspaces of ψ^k with eigenvalue $\geq k^i$, and ψ^k preserves the derived tensor product. □

Example 3. $\mathbb{Z}/p^n \otimes_{\mathbb{Z}} \mathbb{Z}/p^n$ is obviously not zero, but $\mathbb{Z}/p^n \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n \in \mathrm{Fil}^{1+1} K_{0,p}(\mathbb{Z}) = 0$.

Example 4. For two lines in $\mathbb{CP}^1$ (they can be identical), the derived intersection number is always 1. So we can avoid moving lemma and handle things uniformly.

Upshot: we can endow fixed point locus a derived structure, and get a derived CM cycle

$$^{\mathbb{L}}\mathcal{CM}_R(g) := \left[\mathcal{O}_{\mathrm{Hk}_{[K_G g K_G]}} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{M}}\right]\Big|_{\mathcal{CM}_R(g)} \in \mathrm{Fil}_1 K'_{0,\mathcal{CM}_R(g)}(\mathcal{M})$$

.

Definition 6. $\mathcal{Z}_{1,c}(\mathcal{M})$ is the group generated by proper (over the base B) 1-cycles on $\mathcal{M}$ (with $\mathbb{Q}$-coefficient) quotient by 1-cycles that are supported on proper (over the base B) substacks Y of the special fibers and are rationally equivalent to zero on Y.

With $\mathbb{Q}$ coefficients, we have a natural map

$$\mathrm{Fil}_1 K'_{0,\mathcal{CM}_R(g)}(\mathcal{M}) \longrightarrow \mathrm{Gr}_1 K'_{0,\mathcal{CM}_R(g)}(\mathcal{M}) \xrightarrow{\cong} \mathrm{Ch}_{1,\mathcal{CM}_R(g)}(\mathcal{M}) \longrightarrow \mathcal{Z}_{1,c}(\mathcal{M})$$

where the middle isomorphism is a variant with supports of the theorem $K'_0(X) \otimes \mathbb{Q} \cong \oplus_i CH^i(X) \otimes \mathbb{Q}$ proved by Grothendieck using Adams operations.

The image of the derived CM cycle is an element in the $\mathcal{Z}_{1,c}(\mathcal{M})$, which we still denote by $^{\mathbb{L}}\mathcal{CM}_R(g)$.

Then we define the intersection pairing. Consider any pure dimensional flat (not necessarily proper) morphism of regular schemes $\mathcal{M}_0 \to B_0 = \mathrm{Spec} O_E$ with smooth generic fiber, so now the base is B_0. The arithmetic Picard group $\widehat{Pic}(\mathcal{M}_0)$ is defined as isomorphism classes of hermitian line bundles on X (line bundle $\mathcal{L}$ with a hermitian metric on $\mathcal{L} \otimes_{\mathbb{Z}} C$).

Fact: $\widehat{Pic}(\mathcal{M}_0) \cong \widehat{Ch}^1(\mathcal{M}_0)$ by $\widehat{\mathcal{L}} = (\mathcal{L}, ||-||) \mapsto (\mathrm{div}(s), -\log ||s||^2)$ where s is any non-zero rational section of $\mathcal{L}$.

Definition 7. Any element $[\mathcal{Z}] \in \widehat{Ch}^1(\mathcal{M}_0)$ is represented by a hermitian line bundle $\widehat{\mathcal{L}}$. For any $[C] \in \mathcal{Z}_{1,c}(\mathcal{M})$, define $([\mathcal{Z}], [C])$ as the arithmetic degree of $\widehat{\mathcal{L}}|_C$. This gives an arithmetic intersection pairing ($\mathbb{Q}$-coefficient)

$$(\cdot, \cdot) : \widehat{Ch}^1(\mathcal{M}_0) \times \mathcal{Z}_{1,c}(\mathcal{M}_0) \to \mathbb{R}$$

Remark 5. Here the arithmetic degree on C is defined as follows. The general case is combination of two cases: (vertical) C is a projective curve over a closed point $\mathrm{Spec}\mathbb{F}_q$ of B_0. Then $\widehat{deg}\widehat{\mathcal{L}}$ is the usual degree of line bundle on algebraic curve times $logq$, which is well-defined as C is projective. (horizontal) $C = \mathrm{Spec} O_K$ the integral ring of a number field K over E, then the arithmetic degree is (choose any nonzero rational section s)

$$\widehat{deg}\widehat{\mathcal{L}} := \log \#\mathcal{L}/O_K s - \sum_{\sigma: K \hookrightarrow \mathbb{C}} \log ||s||_\sigma.$$

Remark 6. In fact, as we ignore the bad primes (let S be a finite set of places of F_0 containing all those bad places), we shall work in reduced arithmetic chow group $\widehat{Ch}^1_\circ(\mathcal{M}) := \widehat{Ch}^1(\mathcal{M})/Ch^1_{|S|}(\mathcal{M})$, and $\mathbb{R}_S := \mathbb{R}/\{\log p | S_p \neq \varnothing\}$. Similar definition gives

$$(\cdot, \cdot) : \widehat{Ch}^1_\circ(\mathcal{M}) \times \mathcal{Z}_{1,c}(\mathcal{M}) \to \mathbb{R}_S$$

Remark 7. we only need the case when the generic fibers do not intersect. Then it's the sum of local intersection pairings at each place of E. For a non-archimedean place v, it's defined as $\log q_v$ times the Euler-Poincare characteristic of the derived tensor product on $\mathcal{M} \otimes_{O_E} O_{E,(v)}$ (so it's really a computation of the intersection). For an archimedean place, the local intersection number is the value of the Green's function at the complex point of the CM cycle.

In fact, the intersection of KR cycles and CM cycles lies in a very special closed substack namely the basic locus (it's supersingular locus in our case).

Fact: supersingular abelian varieties over algebraically closed field k form a single isogeny class.

Theorem 3. (9.2 in [1]) For the support of the intersection of the special divisor $Z(\xi, \mu)$ and the CM cycle ${}^{\mathbb{L}}CM_R(g)$ on $\mathcal{M}$, we have

1. The support does not meet the generic fiber.
2. If $v|v_0|p$ in $E|F_0|\mathbb{Q}$ s.t v_0 is split in F, then the support does not meet special fiber $\mathcal{M} \otimes_{O_E} k_v$.
3. If $v|v_0|p$ in $E|F_0|\mathbb{Q}$ s.t v_0 is inert in F, then the support meets special fiber $\mathcal{M} \otimes_{O_E} k_v$ only in the basic locus.

Proof. A k-point (k an algebraically closed field of char=p, p can be ∞) in the intersection gives

$$\begin{array}{ccc} & & A \\ & & \downarrow \varphi \quad \text{(object in } \mathcal{CM} \cap \mathcal{KR}) \\ A_0 & \xrightarrow{u} & A \end{array}$$

(Precisely speaking, we only have $\varphi \in \mathrm{End}_{O_F}(A) \otimes \mathbb{Z}[\frac{1}{\Delta}]$, so we shall work in the category of abelian varieties up to isogeny.) Define a F-linear map $\phi : A_0^n \to A$ by $\phi = (\varphi^i u)_{i=0,\dots,n-1}$.

We claim it's an isogeny. As the dimension of A_0^n and A are the same, if it's not an isogeny, the O_F-stable kernel must have positive dimension. So there is a non-zero $a = (a_i) \in \mathrm{Hom}_{O_F}(A_0, A_0^n) \cong O_F^n$ s.t $\phi \circ a = 0$. In other words, $f = \sum_{i=0}^{n-1} a_i \varphi^i : A \to A$ has a positive dimension kernel. But as $\mathrm{char}_F(\varphi)$ is irreducible, $F[\varphi]$ is a field so f is invertible in $F[\varphi]$ hence an isogeny, we get a contradiction. For general F_0, we can use herm structure on KR lattice to show that $\varphi^i u$ are linear independent as the action of g_0 by φ is regular semi-simple on V.

(Another proof for $F_0 = \mathbb{Q}$: Recall that for any abelian variety A' we have $2 \dim A' \geq [\mathrm{End}^0(A') : \mathbb{Q}]_{red}$, see [8] Prop 1.3 and 3.1. The image of ϕ is an abelian subvariety A' of A which is φ-stable and has positive dimension as $\langle u, u \rangle > 0$. Now $F' \hookrightarrow End^0(A')$ we get $2 \dim A' \geq [F' : \mathbb{Q}] = 2n$, hence $\dim A' = \dim A$, $A' = A$, ϕ is surjective.)

If $p = \infty$, then any isogeny is separable so it induces a $F \otimes k$-isomorphism $\mathrm{Lie}(A_0)^n \cong \mathrm{Lie}A$), but this contradicts different Kottwitz conditions $(n, 0)$ and $(n-1, 1)$ on two sides. If $p = v_0$ is split in F, then $O_F \otimes \mathbb{Z}_p \cong \mathbb{Z}_p \times \mathbb{Z}_p$ (the first projection corresponds to Φ) acts on corresponding p-divisible groups and gives splitting

$$A_0^n[p^\infty] = X_0^{(1)} \times X_0^{(2)}, A[p^\infty] = X^{(1)} \times X^{(2)}.$$

Kottwitz condition forces $\dim X_0^{(2)} = 0$, $\dim X^{(2)} = 1$ (for general F_0 the dimension is true mod n), but the F-linear isogeny ϕ gives an isogeny between $X_0^{(2)}$ and $X^{(2)}$, a contradiction! Finally, as A_0 is a CM elliptic curve with CM by O_F, if $p = v_0$ is inert in F then A_0 is supersingular (this is classical Deuring's criterion, otherwise $\mathrm{End}^0(A) = F$ but the p-Frob Fr in it satisfies $\mathrm{Norm}(Fr) = p$, which shows p is split in F, contradiction). A is isogenous to A_0^n hence also supersingular. $\square$

Remark 8. We only know $\mathrm{char}_F(g_0)$ is irreducible in $F[T]$, it can be reducible in $F_v[T]$ for every finite place v (e.g choose any non-solvable Galois CM extension of F and take the irreducible polynomial of a generator; Toy model: $x^4 + 1 \in \mathbb{Q}[x]$ is reducible over $\mathbb{Q}_p$ for all odd primes p). So we avoid the use of Δ-Tate modules.

3 Local-global intersection numbers

Now we introduce weight version of two cycles. Let $\Phi = \varphi_0 \otimes \varphi \in \mathcal{S}((G \times V)(\mathbb{A}_{0,f}))$, here

- $\varphi_0 = 1_{K_G^\Delta} \otimes \varphi_{0,\Delta}$ with $\varphi_{0,\Delta} \in \mathcal{S}(\prod_{v|\Delta} G(F_v), K_{G,\Delta})$.
- $\varphi = 1_{\Lambda^\Delta} \otimes \varphi_\Delta$ with $\varphi_\Delta \in \mathcal{S}(V_\Delta)^{K_{G,\Delta}}$.

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Definition 8.

$$\mathcal{Z}(\xi, \varphi) := \sum_{\mu \in V\left(F_{0,\Delta}\right)_\xi / K_{G,\Delta}} \varphi(\mu)\mathcal{Z}(\xi, \mu) \quad (7.2)$$

here $V(F_{0,\Delta})_\xi = \{\mu \in V(F_{0,\Delta}) | \langle \mu, \mu \rangle = \xi\}$ and we update $\mathcal{Z}(\xi, \varphi)$ to an element in $\widehat{Ch}^1_\circ(\mathcal{M})$ using Bruinier's Green function;

$${}^{\mathbb{L}}\mathcal{CM}_R(\varphi_0) = \sum_{g \in K_G \backslash G\left(\mathbb{A}_{0,f}\right)/K_G} \varphi_0(g) {}^{\mathbb{L}}\mathcal{CM}_R(g)$$

$$\mathrm{Int}(\xi, \Phi) := \frac{1}{[E : F]\tau(Z^{\mathbb{Q}})} (\widehat{\mathcal{Z}}(\xi, \varphi), {}^{\mathbb{L}}\mathcal{CM}_R(\varphi_0))$$

here $\tau(Z^{\mathbb{Q}}) := \# Z^{\mathbb{Q}}(\mathbb{Q}) \backslash Z^{\mathbb{Q}}(\mathbb{A}_f)/K^\circ_{Z^{\mathbb{Q}}}$ is the class number of F;

$$\mathrm{Int}(\tau, \Phi) := \sum_{\xi \in F_0, \xi \geq 0} \mathrm{Int}(\xi, \Phi) q^\xi$$

We know for CM and KR cycles, the intersection number $\mathrm{Int} = \sum_{v_0 \text{ inert}} \mathrm{Int}_{v_0} + \mathrm{Int}_\infty$. For finite places, we have

Theorem 4. (9.4 in [1]) For any v_0 not dividing Δ and inert at F and $\Phi_{v_0} = 1_{K^\circ_{G,v}} \otimes 1_{\Lambda^\circ_v}$, we have

$$\mathrm{Int}_{v_0}(\xi, \Phi) = 2 \log q_{v_0} \sum_{(g,u)} \mathrm{Int}_{v_0}(g, u) \cdot \mathrm{Orb}\left((g, u), \Phi^{v_0}\right)$$

where the sum runs over the $G'(\mathbb{Q})$-orbits (g,u) in the product

$$G'(g_0)(\mathbb{Q}) \times V'(F_0)_\xi$$

Here

- $\mathrm{Int}_{v_0}(g,u)$ is the quantity defined in the AFL conjecture (semi-Lie algebra version) for the unramified quadratic extension $F_{v_0}/F_{0,v_0}$.
- V' is **the "nearby" F/F_0-hermitian space** over F, which is positive definite at all archimedean places, and isomorphic to V locally at all non-archimedean places except at v_0. (In particular V' is non-split at v_0.)
- $G' = U(V')$, and $G'(g_0)$ is the subvariety of G' defined by $\mathrm{char}(g) = \mathrm{char}(g_0)$. The orbital integral is the product of the local orbital integral defined by

  $$\mathrm{Orb}((g,u), \Phi_w) := \int_{\mathrm{U}(V)(F_{0,w})} \Phi_w(h \cdot (g,u))dh$$

  with Haar measures on $G(F_{0,v})$ such that $\mathrm{Vol}(K_{G,v}) = 1$.

Remark 9. For arichmedean connection, the Green function is replaced by Kudla's Green function, and local a rchimedean intersection number is defined to be special value of the Green function $G_K(u, h_\infty)$ at the fixed point of g on the locally symmetric domain. Using complex uniformization, the connection between local-global archimedean intersection number is proved and stated in a simliar way as in the p-adic setting. The difference between the two Green functions does not matter in the final proof of AFL, as the difference is a nearly holomorphic modular form.

The key idea for the proof: one only needs to consider the intersection around the basic locus over an inert prime. To count intersection at each point, one can do it on a small tubular neighborhood (algebraically, it means the formal completion along the basic locus). Then one uses p-adic uniformization to relate local and global cycles. The functions used for weight sums contribute to orbit integrals.

4 p-adic uniformization

Over $\mathbb{C}$, we know smooth projective complex curves are (analytic) quotients of $\mathbb{CP}^1, \mathbb{C}, \mathbb{H}$. This is called complex uniformization, which is quite useful e.g you can easily understand the structure of endormophism ring or torsion groups of elliptic curves.

The analog in p-adic world is called p-adic uniformization, which starts with Mumford's work, which is motivated by Tate's work on the uniformization of elliptic curve over p-adic field with split multiplication reduction $E_q(\mathbb{Q}_p^{alg}) \cong \mathbb{Q}_p^{alg,\times}/q_E^{\mathbb{Z}}$.

It is **more powerful** as we can not only parametrize the generic fiber, but also parametrize the integral model and the special fiber (at least for the supersingular locus), and the uniformization is often Galois equivariant. For us, it's a tool to relate local and global intersection numbers and do computations.

First of all, let us recall that the definition of a Shimura variety, is actually an archimedean uniformization of the complex algebraic variety

$$Sh_G(\mathbb{C}) = \coprod_i \Gamma_i \backslash X$$

In p-adic world we expect

$$Sh_G^{ss,\wedge} = \coprod_i \Gamma_i \backslash \mathcal{N}$$

where J is a natural inner form of G that acts on RZ spaces$\mathcal{N}$ and Γ_i are discrete subgroups of J.

Example 5. The complex fiber of modular curve has a uniformization by the upper half-plane. Similarly, the formal completion of the integral model of a Shimura curve over $\breve{\mathbb{Z}}_p$ along the special fiber has an uniformization by the Drinfeld upper half-plane Ω^2 (the generic fiber over $\mathbb{C}_p$ is $\mathbb{P}^1_{\mathbb{C}_p} - \mathbb{P}^1(\mathbb{Q}_p)$) if p is ramified in the quaterion algebra, or Lubin-Tate spaces (if p is split). See Čerednik-Drinfeld's work for more details.

Example 6. Fix a supersingular elliptic curve E_0 over $k = \overline{\mathbb{F}_p}$. Then $R := \mathrm{End}(C)$ which is a maximal order in the quaternion algebra $D = D_{p,\infty}$. All supersingular elliptic curves are isogenous, and there is a bijection between supersingular elliptic curves over k and rank one projective right R modules (both up to isomorphism) given by $C \mapsto \mathrm{Hom}(C, C_0)$. So we see the supersingular locus is related to class number of $R = O_D$.

This example explains the appearance of the twist $G' = U(V')$ of $G = U(V)$: only the unitary group of the non-split V'_p can act on the Rapoport-Zink space.

For computation purpose, we need a precise adelic form (as the class number has an adelic expression).

Example 7. (Precise) p prime, $N \geq 5$, $(N, p) = 1$. For the modular curve $X_0(N)$ over $\mathbb{Z}_p$, the uniformization for the integral model along supersingular locus is

$$X_0(N)^{ss,\wedge}_{\breve{\mathbb{Z}}_p} = D^\times \backslash (GL_2(\mathbb{A}_f^p)/K_0(N)) \times RZ_{E_0[p^\infty]}$$

here $K_0(N)$ is the $\Gamma_0(N)$-congruence subgroup of $GL_2(\mathbb{A}_f^p) \cong D^\times(\mathbb{A}_f^p)$, and the Rapoport-Zink space $RZ_{E_0[p^\infty]}$ is isomorphic to countably many disjoint union of Lubin-Tate spaces $M_1 \cong \mathrm{Spf}\ \breve{\mathbb{Z}}_p[[t]]$. Taking points on each side, we recover the previous example

$$X_0(N)^{ss}(\overline{\mathbb{F}_p}) = D^\times \backslash (D \otimes \mathbb{A}_f)^\times / K_0(N) O_{D,p}^\times$$

$(RZ_{E_0[p^\infty]}(\overline{\mathbb{F}_p}) \cong \mathbb{Z} \cong (D \otimes \mathbb{Q}_p)^\times / O_{D,p}^\times)$

Idea: We have a map from (the deform space of quasi-isogeny to E_0) to $X_0(N)^{ss}_{\overline{\mathbb{F}_p}}$

$$(\text{quasi-isogeny } E \to E_0) \mapsto E$$

this is the uniformization map. We can understand the deform space piece by piece: {quasi-isogenies of Abelian varieties $A \to B$ } = {$\widehat{\mathbb{Z}^p}$-lattices Λ in prime-to-p rational Tate module

+ a quasi-isogeny $A[p^\infty] \to B[p^\infty]$ }. And one knows the fiber of uniformization map is essentially $\mathrm{End}^0(E_0) \cong D^\times$.

The general idea are similar. The important thing is to define the **uniformization map** from the defromation space, and then prove it's bijective on points using e.g variants of Dieudonne-Cartier theory and Honda-Tate theorem, and use theorems of p-divisible groups e.g Serre-Tate and Grothendieck-Messing to prove it's étale, then conclude it's an isomorphism.

This partly explains the following theorem:

Theorem 5. Let $\mathcal{M} = \mathcal{M}_{K_{\tilde{G}}}(\tilde{G})$ be the RSZ Shimura variety over O_{E_v} (E is the reflex field, v is good **inert** place of E, $v|v_0|p$ in $E|F_0|\mathbb{Q}$), then the completion of $\mathcal{M}$ along the basic locus in the supersingular locus over $O_{\breve{E}_v}$ has a p-adic uniformization

$$\mathcal{M}^{ss,\wedge}_{O_{\breve{E}_v}} = \tilde{G}'(\mathbb{Q})\backslash[\mathcal{N}_{O_{\breve{E}_v}} \times \tilde{G}(\mathbb{A}_f^{v_0})/K_{\tilde{G}}^{v_0}]$$

here the group $\tilde{G}'$ is defined similarly as $\tilde{G}$ with V replaced by the "nearby" hermitian space V', so $\tilde{G}(\mathbb{A}_f^{v_0}) \cong \tilde{G}'(\mathbb{A}_f^{v_0})$.

By the almost product structure of $\tilde{G}$, there is a projecton to a finite set (finiteness of class number)

$$\mathcal{M}^{ss,\wedge}_{O_{\breve{E}_v}} \to Z^{\mathbb{Q}}(\mathbb{Q})\backslash Z^{\mathbb{Q}}(\mathbb{A}_f)/K^\circ_{Z^{\mathbb{Q}}}$$

with each fiber isomorphic to

$$\mathcal{M}^{ss,\wedge}_{O_{\breve{E}_v},0} = G'(\mathbb{Q})\backslash[\mathcal{N}_{O_{\breve{E}_v}} \times G(\mathbb{A}_f^{v_0})/K_{\tilde{G}}^{v_0}].$$

Return to the relation of local-global intersection numbers, we can compute the intersection number at each fiber. And at each fiber, we also have uniformization results for KR cycles and CM cycles which relates between local and global cycles (the local KR divisor is defined similarly, the local CM cycle is the derived fixed locus ${}^{\mathbb{L}}\mathcal{N}^\gamma$). This explains the strategy of the proof.

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