FF curve III

E non arch local field $\mathcal{O}_E \ni \pi$ $\mathbb{F}_q$

$C / \mathbb{F}_q$ complete non arch alg closed field

$\rightsquigarrow$ FF curve

$$X_{C,E} = Y_{C,E} / \phi_C^{\mathbb{Z}} \quad \text{adic space} / E$$

$$X_{C,E} = X_C = X$$

$$Y_{C,E} \overset{\text{open}}{\subseteq} \mathrm{Spa}\, \underbrace{W_{\mathcal{O}_E}(\mathcal{O}_C)}_{\text{flat deformation of } \mathcal{O}_C \text{ to } \mathcal{O}_E}$$

where $\pi \neq 0$ and $[\varpi] \neq 0$ $\quad \varpi \in C$ pseudo unif

$$X_{C,E}^{cl} \hookrightarrow |X_{C,E}|$$

$$X_{C,E}^{cl} = \{\text{untilt } C^\sharp/E \text{ of } C\} / \phi_C^{\mathbb{Z}}$$

Any con affinoid open subset $\mathrm{Spa}\, A \subseteq X_{C,E}$

has A a PID

and $\mathrm{Spm}\, A = X_{C,E}^{cl} \cap |\mathrm{Spa}\, A| \subseteq |X_{C,E}|$

- for any closed pt $y \in Y_{C,E}^{cl}$, $\exists\, t \in C$ $0 < |t| < 1$ s.t. $y = V(\pi - [t])$

it's subtle that t is not unique

Classification Theorem for Vector bundles

Isocrystals Recall An isocrystal is a pair (V, ϕ) V f.d $\breve{E}$-vect space

$\breve{E}/E$ completion of max unramified ext

$\| W_{\mathcal{O}_E}(\overline{\mathbb{F}_q})[\frac{1}{\pi}]$

$+\ \phi_V : V \simeq V$

$\phi_{\breve{E}}$-linear auto

$\rightsquigarrow$ E-linear $\otimes$-category Isoc_E

Examples 0) unit $(\breve{E}, \phi_{\breve{E}})$

1) 1-dim obj $(\breve{E}, b\phi_{\breve{E}})$ for some $b \in \breve{E}$

any such is iso to $(\breve{E}, \pi^n \phi_{\breve{E}})$ for a unique $n \in \mathbb{Z}$

pf: $a^{-1} b\, \phi(a) \sim b$ $\quad a \in \breve{E}^\times$

but $\underline{a^{-1}\phi(a)} \in \mathcal{O}_{\breve{E}}^{\times}$ can be any possible element

$$\mathrm{Hom}((\breve{E}, \pi^n \phi_{\breve{E}}), (\breve{E}, \pi^m \phi_{\breve{E}})) = \begin{cases} E & m = n \\ 0 & m \neq n \end{cases}$$

2) $\lambda = \frac{s}{r} \in \mathbb{Q}$ $(s, r) = 1$ $r > 0$

$$(V_\lambda, \phi_{V_\lambda}) = \left(\breve{E}^r, \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ \pi^s & & & \end{pmatrix} \phi_{\breve{E}} \right)$$

Q: Can we do an integral isocrystal theory ?

Q: what if $(r, s) \neq 1$ what obj do you get ?

A: get a direct sum e.g $E(\frac{2}{2}) = E(1) \oplus E(1)$

<u>Thm</u> (Dieudonne - Manin)

$$\mathrm{Isoc}_E = \bigoplus_{\lambda \in \mathbb{Q}} \mathrm{Isoc}_E^\lambda$$

"isoclinic of slope λ"

$$\mathrm{Isoc}_E^\lambda := (E \text{ f.d v.s}) \otimes V_\lambda$$

$\mathrm{End}(V_\lambda) = D_\lambda$ central div alg of inv $\lambda \in \mathbb{Q}/\mathbb{Z}$

<u>Sketch of proof</u> $\forall$ $V = (V, \phi_V) \in \mathrm{Isoc}_E$

define $\mu(V) = \dfrac{\deg(V)}{\mathrm{rk}(V)}$

$\deg(V) \leftarrow \det V \in \mathrm{Iso}_E$ of rk 1, $n = \deg V$

$\mathrm{rk}(V) \leftarrow \dim_{\breve{E}} V$

$\rightsquigarrow$ "Harder - Narasimhan formalism of slopes"

$\rightsquigarrow$ (semi-) stable objects:

(V, ϕ_V) (semi-) stable if

$\forall \quad 0 \subsetneq (V', \phi_{V'}) \subsetneq (V, \phi_V)$

$$\mu(V') \leq \mu(V)$$

$(<)$ ← stable

like rk 1 case

no map between objs <u>with diff slopes</u>

$\rightsquigarrow$ " HN filtration" decreasing separated exhausted fil ($\mathbb{Q}$- indexed)

$$V^{\geq \lambda} \subseteq V$$

s.t. each $V^{\lambda} := V^{\geq\lambda} \Big/ \bigcup_{\lambda' > \lambda} V^{\geq \lambda'}$

is semi-stable of slope λ

Note $\mu'(V) := -\mu(V)$ also gives HN formalism

$\Rightarrow$ the filtration is canonical split

$$\mathrm{Isoc}_E = \bigoplus_{\lambda \in \mathbb{Q}} \mathrm{Isoc}_E^{\lambda} \leftarrow \text{"semi-stable of slope } \lambda \text{"}$$

$\lambda = 0$: want $\mathrm{Isoc}^{0}_{E} \xleftarrow{\cong}$ f.d E-vs

$(W \otimes_E \check{E}, \mathrm{id} \otimes \phi_{\check{E}}) \longleftarrow\!\shortmid W$

ess surj $\iff$ $\forall$ all $(V, \phi_V) \in \mathrm{Isoc}^{0}_{E}$

it has many ϕ-inv elements $\longleftrightarrow$ Similiar idea for classifing vect on FF curve but more diff and different idea

i.e $W = V^{\phi_V = \mathrm{id}}$, we shall have

$W \otimes_E \check{E} \xrightarrow{\cong} V$

idea show that V has a ϕ_V-stable lattice $L \subseteq V$ $\phi_V(L) = L$ (e.g $\sum_i \phi_V^i(L_0)$)

then $L^{\phi_L = \mathrm{id}}$ is finite free over $\mathcal{O}_E$ with correct rank by Artin-Scheier theory

for general λ, $(V, \phi_V) \in \mathrm{Isoc}^{\lambda}_{E}$

$\underline{\mathrm{Hom}}((V_\lambda, \phi_{V_\lambda}), (V, \phi_V)) \in \mathrm{Isoc}^{0}_{E}$

0

Back to FF curve

Note $\overline{\mathbb{F}}_q \subseteq C$

Q: Do we have good int model of FF curve

$\rightsquigarrow$ $Y_{C,E} \longrightarrow \mathrm{Spa}\, \breve{E}$

$\phi_C \circlearrowleft Y_{C,E}$, $\phi_E \circlearrowleft \mathrm{Spa}\,\breve{E}$

pull back functor $\underset{\sim}{\rightsquigarrow}$ $\mathrm{Isoc}_{\breve{E}} \longrightarrow \{\phi_C\text{-equiv } VB \mid Y_{C,E}\}$

$\| \text{ descent}$

$VB(X_{C,E})$

simplified proof, but use deep thm of period map Lubin-Tate tower, Drinfeld

$V \longmapsto \mathcal{E}(V)$

Let $\mathcal{O}_{X_{C,E}}(\lambda) := \mathcal{E}(V_{-\lambda})$

difficult computation

Thm (Fargues-Fontaine '10, $\breve{E}$ p-adic; Hartl-Pink '04, $E = \mathbb{F}_q((t))$; Kedlaya '04, E p-adic

in the language of rigid varieties, Y_C is, but X_C is not

only ϕ-equiv VB on Y_C

Any VB $\mathcal{E}$ on $X_{C,E}$ is isom to a direct sum of $\mathcal{O}(\lambda)$

i.e $\mathrm{Isoc}_E \longrightarrow VB(X_{C,E})$

is ess surjective

1) any $\mathcal{E}$ also has a HN filtration

$$\mathcal{E}^{\geq\lambda} \subseteq \mathcal{E}$$

2) $\mathrm{Isoc}^{\lambda}(E) \simeq VB^{\lambda}(X_{C,E})$

3) The HN filtration split (but not uniquely)

1) like for all smooth proj curves

2) Similar to $\mathbb{P}^1$, but not other curves

3) Similar to $g=0,1$, but not higher genus

By our convention ($\mathcal{O}(\lambda) = \mathcal{E}(Y_{-\lambda})$)

with odd structure

$$H^0(X_{C,E}, \mathcal{O}(n)) = \begin{cases} \text{inf-dim } E\text{-vs}, & n>0 \\ E, & n=0 \\ 0, & n<0 \end{cases}$$

$$H^1(X_{C,E}, \mathcal{O}(n)) = \begin{cases} 0 & n>0 \\ 0 & n=0 \\ \text{inf-dim } E\text{-vs} & n<0 \end{cases}$$

Q: Serre duality ? ← BC spaces

Q: de Rham coh ? not defined but have étale coh

Let $C^\#/E$ some untilt of C

$$\operatorname{Spa} C^\# \xhookrightarrow{i} X_{C,E}$$

$$0 \longrightarrow I \longrightarrow \mathcal{O}_{X_{C,E}} \longrightarrow i_* C^\# \longrightarrow 0$$

$I \cong \mathcal{O}(-1)$

Q: $C^\#/E$ independent of $C^\#$?

$\rightsquigarrow H^1(X_{C,E}, \mathcal{O}(-1)) = C^\#/E$

"Banach–Colmez Spaces": built from f.d

$C^\#$-vs + f.d E-vs

Similarly (twist by $O(1)$)

$$0 \to \mathcal{O} \to \mathcal{O}(1) \to i_* C^\# \to 0$$

$$\rightsquigarrow \quad 0 \to E \to H^0(X, \mathcal{O}(1)) \to C^\# \to 0$$

Next Goal: proof of the classification thm

(a new proof!)

no hard computation

using heavily perfectoid spaces, diamonds, v-descent

Some reductions (Same in all known proofs)

1) line bundles

Firstly, prove that "$\mathcal{O}(1)$ is ample"

Thm (Kedlaya-Liu '15)

For any vector bundle $\mathcal{E}$ on $X_{C,E}$

$\forall\ n \gg 0$, $\mathcal{E}(n) = \mathcal{E} \otimes_{\mathcal{O}} \mathcal{O}(n)$

is globally generated, and $H^1(X, \mathcal{E}(n)) = 0$

(Note: $H^i(X_{C,E}, \mathcal{E}) = 0 \quad \forall\ i \geq 2$)

because $|X_{C,E}|$ is a curve

pf: not hard, just need to do right estimates

Cor (GAGA)

$$P = \bigoplus_{n \geq 0} H^0(X_{C,E}, \mathcal{O}(n))$$

$$X^{alg}_{C,E} := \mathrm{Proj}(P)$$

No, $|X^{alg}_{C,E}| = X^{cl}_{C,E} \cup \{\eta\}$

all other pts of $|X_{C,E}|$ go to η

$\rightsquigarrow$ natural map of locally ringed spaces

$$X_{C,E} \xrightarrow{f} X^{alg}_{C,E}$$

Q: is the top space the same

s.t. $f^* : VB(X^{alg}_{C,E}) \longrightarrow VB(X_{C,E})$ eq

— preserves cohomology

$(X^{alg}_{C,E}$ regular, notherian, Krull dim 1, locally spectrum of a PID)

<u>Cor</u> Any line bundle $\mathcal{L} \in \mathrm{Pic}(X_{C,E})$ is isomorphic to $\mathcal{O}(D)$ for some divisor $D \in \bigoplus_{x \in X^{cl}_{C,E}} \mathbb{Z}$

<u>prop</u> $\forall\ x \in X^{cl}_{C,E}$

$$\mathcal{O}(x) \simeq \mathcal{I}_x^{-1} \cong \mathcal{O}_{X_{C,E}}(1)$$

(next time: use Lubin-Tate theory)

<u>Cor</u> $\mathcal{O}(D) \cong \mathcal{O}(\deg D)$, so

$$\mathrm{Pic}(X_{C,E}) \cong \mathbb{Z}$$

$$\mathcal{O}(1) \longleftrightarrow 1$$

(like $\mathbb{P}^1$)

(check $\mathcal{O}(n) \not\cong \mathcal{O}(m)$)

Now, define $\deg \mathcal{E} :=$ image of $\det \mathcal{E}$
$= \Lambda^{rk} \mathcal{E}$

$$\mu(\mathcal{E}) = \frac{\deg \mathcal{E}}{rk\, \mathcal{E}} \quad (\mathcal{E} \neq 0)$$

$\rightsquigarrow$ HN filtration on the curve

as classical set up

$\rightsquigarrow$ 1)

<u>Remaining Step</u>: classify semi-stable VB of slope

$$\text{f.d } E\text{-vs} \xrightarrow{\cong} VB(X_{C,E})^{0}$$

$$W \longmapsto W \otimes_E \mathcal{O}_{X_{C,E}}$$

<u>Key</u> If $\mathcal{E} / X_{C,E}$ semi-stable of slope 0

then $\underline{H^0(X_{C,E}, \mathcal{E}) \neq 0}$ i.e $\mathcal{O}_X \xrightarrow{\neq 0} \mathcal{E}$

so induction ✓

Idea

enlarge C

consider the functor on $\{C'/C$

$$C'/C \longmapsto H^0(X_{C',E}, \mathcal{E}|_{X_{C',E}})$$

shall be a geometric obj (like a space)

whose C'-valued pts are $H^0(X_{C',E}, \mathcal{E}|\ )$

need more test objects to define a space

perfectoid C-algs $\rightsquigarrow$ a sheaf on aff perfd space$/C$

Ex Fix $C^\#/E$ untilt of C

$$R \text{ perfd } C\text{-alg} \xmapsto[\text{tilting equi }/C]{} \exists! \text{ untilt } R^\#/C^\#$$

$\rightsquigarrow$

$$\begin{tikzcd} \mathrm{Spa}\, R^\# \arrow[r, hook, "i_{R^\#}"] \arrow[d] & X_{R,E} \arrow[d] \\ \mathrm{Spa}\, C^\# \arrow[r, hook, "i_{C^\#}"] & X_{C,E} \end{tikzcd}$$

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to i_{R^\# *}\mathcal{O}_{R^\#} \to 0$$

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to i_{C^\# *}\mathcal{O}_{C^\#} \to 0$$

$$\Rightarrow H^1(X_{R,E}, \mathcal{O}(-1)) = R^\#/E$$

so get (in equal char, $R^\# = R$)

$$H^1(\mathcal{O}(-1)) = \mathbb{A}^1_C / E \qquad E \subset \mathbb{A}^1_C$$

locally profinite gp

closed subset

quotient of $\mathbb{A}^1_C$ by pro-étale equiv relation

General picture

$H^0(-,\mathcal{E})$, $H^1(-,\mathcal{E})$ are

"quotients of perfd space (char p) by pro-étale relations" diamonds ← they're motivations

like alg spaces

$\supseteq$ {BC spaces} = "quotients of schemes by étale relations"

geometry

Next time:

pf of classification (Lubin-Tate $\leftrightarrow$ line bundle) diamonds

Q: equal char $H^0(X, \mathcal{O}(n)) = ?$

$= m_C^{*}$