SCIENCE CHINA
Mathematics

Special Issue on Pure Mathematics
• ARTICLES •

November 2019 Vol. 62 No. 11: 2355–2370
https://doi.org/10.1007/s11425-019-1612-8

Semilinear automorphisms of classical groups and quivers

Dedicated to Professor Lo Yang's 80th Anniversary, with Admiration

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Received April 28, 2019; accepted October 13, 2019; published online October 16, 2019

Abstract For a classical group G over a field F together with a finite-order automorphism θ that acts compatibly on F, we describe the fixed point subgroup of θ on G and the eigenspaces of θ on the Lie algebra $\mathfrak{g}$ in terms of cyclic quivers with involution. More precise classification is given when $\mathfrak{g}$ is a loop Lie algebra, i.e., when $F = \mathbb{C}((t))$.

Keywords semilinear automorphisms, classical groups, quiver

MSC(2010) 11E39, 11E57, 17B40

Citation: Yang J, Yun Z. Semilinear automorphisms of classical groups and quivers. Sci China Math, 2019, 62: 2355–2370, https://doi.org/10.1007/s11425-019-1612-8

1 Introduction

1.1 Cyclically graded Lie algebras

Let $\mathfrak{g}$ be a Lie algebra of a connected simple algebraic group G over a field k. A *cyclic grading* on $\mathfrak{g}$ is a decomposition

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_i,$$

where $m \in \mathbb{N}$ and $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for all $i, j \in \mathbb{Z}/m\mathbb{Z}$. The summand $\mathfrak{g}_0$ is a Lie subalgebra; let G_0 denote the corresponding connected subgroup of G. When m is prime to $\mathrm{char}(k)$ and k contains all m-th roots of unity, such a cyclic grading corresponds to an automorphism θ of $\mathfrak{g}$ of order divisible by m, under which $\mathfrak{g}_i$ is the eigenspace of θ with eigenvalue ζ^i (where ζ is a fixed primitive m-th root of unity).

The invariant theory of the action of G_0 on $\mathfrak{g}_i$ has been much studied by Vinberg and his school. The G_0 action on $\mathfrak{g}_i$ share many nice properties of the adjoint action of G on $\mathfrak{g}$. In [2], Reeder et al. singled out *stable gradings* (when $\mathfrak{g}_i$ has stable vectors under G_0) and connect them with regular elliptic elements in the Weyl group of G. When $\mathfrak{g}$ is a classical Lie algebra, the subgroup G_0 as well as its action on $\mathfrak{g}_i$ can be described in terms of cyclic quivers with involution (see [4, Sections 6–8]).

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From the Lie-theoretic perspective it is natural to consider cyclic gradings on Kac-Moody algebras, starting from the loop Lie algebras. In the case of a loop Lie algebra $\mathfrak{g}\otimes k(\!(t)\!)$, it is interesting to consider not only those cyclic gradings coming from $k(\!(t)\!)$-linear automorphisms of $\mathfrak{g}$, but also those coming from $k(\!(t)\!)$-semilinear automorphisms. For example, let ζ be a root of unity in k, we may consider a finite order automorphism θ of $\mathfrak{g}\otimes k(\!(t)\!)$ such that $\theta(X\otimes a(t))=\theta(X)\otimes a(\zeta t)$ for all $a(t)\in k(\!(t)\!)$ and $X\in\mathfrak{g}$. In this paper we generalize the quiver description in [4] for cyclic gradings on classical Lie algebras to a setting that includes finite-order semilinear automorphisms of loop Lie algebras of classical type.

1.2 Convention

For a field k and $n\in\mathbb{N}$, $\mu_n(k)$ denotes the group of n-th roots of unity in $k^\times$.

Let A be an associative ring, and M,M' two left A-modules. Let ζ be an automorphism of A. A map $f:M\to M'$ is called (A,ζ)-*semilinear* if

$$f(av)=\zeta(a)f(v),\quad \forall\, a\in A,\ v\in M.$$

For a left A-module M, let $\mathrm{End}_A(M)$ denote the set of A-linear endomorphisms of M and $\mathrm{Aut}_A(M)$ denote the group of A-linear automorphisms of M.

If A contains a field k, and M is a left A-module M of finite dimensions over k, let $\mathbf{GL}_{A/k}(M)$ be the algebraic group over k whose R-points (for any commutative k-algebra R) are $R\otimes_k A$-linear automorphisms of $R\otimes_k M$. When $A=k$ we write $\mathbf{GL}_k(M)$ for $\mathbf{GL}_{k/k}(M)$, which is the usual general linear group. If A is commutative, then $\mathbf{GL}_{A/k}(M)=\mathbf{R}_{A/k}\mathbf{GL}_A(M)$ is the Weil restriction of the general linear group $\mathbf{GL}_A(M)$ from A to k.

If, moreover, the A-module M carries a k-bilinear pairing $\langle\cdot,\cdot\rangle:M\times M\to B$ valued in some k-vector space B, we denote by $\mathbf{Aut}_{A/k}(M,\langle\cdot,\cdot\rangle)$ the algebraic subgroup of $\mathbf{GL}_{A/k}(M)$ preserving the pairing.

1.3 The setup and the main result

Throughout the paper, let k be a field. Let F be a finite separable k-algebra together with an automorphism $\zeta\in\mathrm{Aut}(F)$ of order $n\in\mathbb{N}$ such that $k=F^\zeta$. We allow F to be a product of fields.

Let V be a finite type F-module. Let θ be an (F,ζ)-semilinear automorphism of V. Let m be a multiple of n such that m/n is invertible in k. Assume

$$\theta^m=\beta\cdot\mathrm{id}_V,\qquad \text{for some } \beta\in k^\times.$$

Then θ acts on the Weil restriction $\mathbf{GL}_{F/k}(V)$ and on the Lie algebra $\mathrm{End}_F(V)$ by conjugation. As a warm-up, in Section 2 we describe the fixed point subgroup of θ on $\mathbf{GL}_{F/k}(V)$ and the θ-eigenspaces on $\mathrm{End}_F(V)$ in terms of cyclic quivers decorated by division algebras. The more complicated case where $\mathbf{GL}_{F/k}(V)$ is replaced with a classical group G defined using V and a symmetric bilinear form, a symplectic form or a Hermitian form on it is considered in Section 3.

Our main result is Theorem 3.12, which gives a complete description of the fixed subgroup H of θ on G and eigenspaces $\mathfrak{g}(\xi)$ of θ on $\mathfrak{g}=\mathrm{Lie}\ G$ in terms of cyclic quivers with involution decorated by division algebras and pairings. The strategy of the proof is to realize V as a module over a certain semisimple (non-commutative) algebra A_β, and to extract linear-algebraic data from the multiplicity spaces of simple A_β-modules in V.

In Section 4 we specialize to the case of classical loop Lie algebras, and make the description in Theorem 3.12 more precise. The result in this case can be summarized in the following rough form: when G comes from a polarization on V and $\mathrm{Nm}_{F/k}(\xi)$ is a primitive (m/n)-th root of unity, we can associate to the situation a cyclic quiver Q_ξ with m/n or $m/2n$ vertices (i.e., there is a vector space M_i on each vertex i of Q_ξ over k or a quadratic extension of k). The quiver Q_ξ is equipped with an involution $(-)^\diamondsuit$, and the vector spaces on i and $i^\diamondsuit$ are dual to each other. For $i=i^\diamondsuit$, M_i is equipped with a symmetric bilinear, skew-symmetric bilinear or Hermitian form. Then $H=G^{\mathrm{Ad}(\theta)}$ is the automorphism group of the $(M_i)_{i\in I}$ preserving the pairings and forms; $\mathfrak{g}(\xi)$ is the space of representations of the quiver Q_ξ in the vector spaces (M_i) satisfying a certain self-adjointness conditions with respect to the pairings.

1.4 Examples of the setup

(1) $n = 1$ so $k = F$. In this case, V is a finite-dimensional k-vector space with a k-linear operator θ such that θ^m is a scalar.

(2) $m = n$. In this case θ gives a descent datum of V to a k-vector space V', i.e., $V = V' \otimes_k F$.

(3) $k = \mathbb{R}$, $F = \mathbb{C}$ and $n = 2$. In this case V is a complex vector space with a complex *anti-linear* automorphism θ of finite order.

(4) k is a discrete valuation field and F is a tamely ramified Galois extension of k of degree n. This includes the case $k = \mathbb{C}(\!(t^n)\!)$ and $F = \mathbb{C}(\!(t)\!)$ with the action $\zeta(a(t)) = a(\zeta_n t)$ for some primitive n-th root of unity ζ_n, which arises from the loop Lie algebra setting discussed in the beginning.

(5) Let k be a field containing a finite field $\mathbb{F}_q$, and $F = k \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$ with the action of ζ by q-Frobenius on the $\mathbb{F}_{q^n}$-factor. An F-vector space V with an (F, ζ)-semilinear automorphism θ appears in the study of the generic fiber of Shtukas by Drinfeld [1, Section 2] (which Drinfeld calls "F-spaces").

2 Linear case

2.1 The problem

We are in the setup of Subsection 1.3. Let $G = \mathbf{GL}_F(V)$, the general linear group over F. Let $\mathfrak{g} = \mathrm{End}_F(V)$ be the Lie algebra of G. Let $\mathrm{Ad}(\theta)$ (respectively, $\mathrm{ad}(\theta)$) denote the conjugation action of θ on G (respectively, $\mathfrak{g}$): $\mathrm{Ad}(\theta)(g) = \theta g \theta^{-1}$ for $g \in G$ (respectively, $\mathrm{ad}(\theta)(\varphi) = \theta\varphi\theta^{-1}$ for $\varphi \in \mathfrak{g}$). Then $\mathrm{Ad}(\theta)$ is an automorphism of the Weil restriction $\mathbf{R}_{F/k}G = \mathbf{GL}_{F/k}(V)$, and $\mathrm{ad}(\theta)$ is an (F, ζ)-semilinear automorphism of $\mathfrak{g}$. Our goal is to understand the following in terms of quivers:

(1) The fixed point group $H := (\mathbf{R}_{F/k}G)^{\mathrm{Ad}(\theta)}$ as an algebraic group over k. Note that

$$H(k) = \{g \in \mathrm{Aut}_F(V) \mid g\theta = \theta g\}.$$

(2) For $\xi \in F^\times$, the H-module

$$\mathfrak{g}(\xi) := \{\varphi \in \mathrm{End}_F(V) \mid \theta\varphi\theta^{-1} = \xi\varphi\}.$$

with the action of $h \in H$ by $h : \varphi \mapsto h\varphi h^{-1}$.

Since $\mathrm{ad}(\theta)$ is not F-linear, $\mathfrak{g}(\xi)$ is not an eigenspace of $\mathrm{ad}(\theta)$ in the traditional sense. In particular, for different ξ, the subspaces $\mathfrak{g}(\xi)$ are not necessarily linearly independent.

Definition 2.1. (1) Let $F\langle\boldsymbol{\theta}\rangle$ be the non-commutative polynomial ring over F in one variable $\boldsymbol{\theta}$ with the relation $\boldsymbol{\theta} a = \zeta(a)\boldsymbol{\theta}$ for all $a \in F$.

(2) Let A_β be the quotient of $F\langle\boldsymbol{\theta}\rangle$ by the ideal generated by the central element $\boldsymbol{\theta}^m - \beta$.

By construction, A_β is an associative F-algebra. An A_β-module is an F-vector space U together with an (F, ζ)-semilinear automorphism

$$T : U \to U$$

satisfying

$$T^m = \beta \cdot \mathrm{id}_U.$$

In particular, V is an A_β-module with $\boldsymbol{\theta}$ acting by θ.

2.2 Twisting by $\boldsymbol{\xi}$

Let

$$\Xi_{m/n} = \{\xi \in F^\times \mid \mathrm{Nm}_{F/k}(\xi) \in \mu_{m/n}(k)\}.$$

For $\xi \in \Xi_{m/n}$, let μ_ξ be the F-linear automorphism of A_β sending $\boldsymbol{\theta}$ to $\xi\boldsymbol{\theta}$. This defines an action of $\Xi_{m/n}$ on A_β. For an A_β-module V, let V^ξ be the same F-vector space V with the action of A_β twisted by μ_ξ, i.e., the new action of $\boldsymbol{\theta}$ on $v \in V^\xi$ is $\boldsymbol{\theta} \cdot v = \xi\theta(v)$.

2.3 Reformulation of the problem

We may rewrite H and $\mathfrak{g}(\xi)$ in terms of the A_β-module structure on V:

(1) $H = \mathbf{GL}_{A_\beta/k}(V)$ as an algebraic group over k (see Subsection 1.2 for convention).

(2) For $\xi \in \Xi_{m/n}$, we have $\mathfrak{g}(\xi) = \mathrm{Hom}_{A_\beta}(V^\xi, V)$. Note that if $\xi \notin \Xi_{m/n}$, then $\mathfrak{g}(\xi) = 0$.

2.4 Classification of A_β-modules

Let $L_\beta = k[\boldsymbol{\theta}^n] \subset A_\beta$; this is the center of A_β. Let $b = \boldsymbol{\theta}^n \in L_\beta$. Then $L_\beta \cong k[\mathbf{b}]/(\mathbf{b}^{m/n} - \beta)$ (the image of $\mathbf{b}$ in L_β is b) is a separable k-algebra (since m/n is prime to $\mathrm{char}(k)$ by assumption). Let

$$L_\beta = \prod_{i\in I} L_i$$

be the decomposition of L_β into a product of fields, with the index set I in natural bijection with the underlying set of $\mathrm{Spec}\, L_\beta$. Let $b_i \in L_i$ be the image of b. Then

$$A_\beta = \prod_{i\in I} A_i, \quad \text{with } A_i = (L_i \otimes_k F\langle\boldsymbol{\theta}\rangle)/(\boldsymbol{\theta}^n - b_i).$$

Lemma 2.2. *The algebra A_i is a central simple algebra over L_i.*

Proof. The presentation of A_i is the standard one for a cyclic algebra of degree n^2 over L_i. In particular, A_i is a central simple algebra over L_i. □

By the above lemma, for each $i \in I$, there is up to isomorphism a unique simple A_i-module. *We fix a simple A_i-module S_i for each $i \in I$.* Let $D_i = \mathrm{End}_{A_i}(S_i)^{\mathrm{opp}}$. Then D_i is a central division algebra over L_i. Let $n_i = \dim_{D_i^{\mathrm{opp}}}(S_i)$, then

$$A_i = \mathrm{End}_{D_i^{\mathrm{opp}}}(S_i) \cong M_{n_i}(D_i), \quad \text{and} \quad \dim_{L_i}(D_i) = (n/n_i)^2.$$

We view S_i as a right D_i-module with the right D_i-action given by the left $D_i^{\mathrm{opp}} = \mathrm{End}_{A_i}(S_i)$-action on S_i.

Corollary 2.3. *The algebra A_β is a semisimple k-algebra with the set of simple modules up to isomorphism given by $\{S_i\}_{i\in I}$. Any A_β-module V is canonically isomorphic to a direct sum*

$$V \cong \bigoplus_{i\in I} S_i \otimes_{D_i} M_i, \tag{2.1}$$

where $M_i = \mathrm{Hom}_{A_i}(S_i, V)$ viewed as a left D_i-module using the right D_i-action on S_i.

2.5 The group H

Now we are ready to describe the group H using the canonical decomposition (2.1) for the A_β-module V. We have an isomorphism of algebraic groups over k,

$$H = \mathbf{GL}_{A_\beta/k}(V) \cong \prod_{i\in I} \mathbf{GL}_{D_i/k}(M_i). \tag{2.2}$$

Under the above isomorphism, if $g \in H$ corresponds to $(g_i)_{i\in I}$ on the right-hand side, then

$$g(u \otimes x) = u \otimes g_i(x), \quad \forall\, i \in I,\ u \in S_i,\ x \in M_i. \tag{2.3}$$

2.6 The quiver Q_ξ

The action of $\xi \in \Xi_{m/n}$ on A_β induces an action on its center L_β by

$$\mu_\xi : b \mapsto \mathrm{Nm}_{F/k}(\xi) \cdot b,$$

hence a permutation on $I = \mathrm{Spec}\ L_\beta$. We denote this permutation by $i \mapsto \overline{\xi}(i)$. Let Q_ξ be the directed graph with vertex set I and an arrow $i \to \overline{\xi}(i)$ for each $i \in I$. Let E be the set of arrows of Q_ξ. Each vertex $i \in I$ is decorated by the division algebra D_i.

In general, Q_ξ is a disjoint union of cycles of not necessarily the same size. In the special case where k contains all (m/n)-th roots of unity, L_β is Galois over k, and Q_ξ is a disjoint union of cycles of equal size.

2.7 The H-module $\mathfrak{g}(\xi)$

Let $e : i \to \overline{\xi}(i)$ be an arrow in Q_ξ. The automorphism μ_ξ of A_β restricts to an isomorphism $A_i \xrightarrow{\sim} A_{\overline{\xi}(i)}$, hence a non-canonical isomorphism $\eta_e : (S_i)^\xi \cong S_{\overline{\xi}(i)}$. Once we fix a choice of η_e, we get an isomorphism $\eta_e^\flat : D_i \cong D_{\overline{\xi}(i)}$ by applying $\mathrm{End}_{A_\beta}(-)^{\mathrm{opp}}$ to the source and target of η_e. Note that even when e is a self loop at $i = \overline{\xi}(i)$, the automorphism $\eta_e^\flat$ of D_i and even its restriction to the center L_i may not be the identity.

We have a decomposition of V^ξ as an A-module using the maps η_e,

$$V^\xi = \bigoplus_{i\in I} (S_i)^\xi \otimes_{D_i} M_i \cong \bigoplus_{i\in I} S_{\overline{\xi}(i)} \otimes_{D_i} M_i.$$

Here the action of D_i on $S_{\overline{\xi}(i)}$ is via the isomorphism $\eta_e^\flat$ for the arrow $e : i \to \overline{\xi}(i)$. Hence

$$\mathfrak{g}(\xi) = \mathrm{Hom}_{A_\beta}(V^\xi, V) = \bigoplus_{e : i \to \overline{\xi}(i)} \mathrm{Hom}_{D_i}(M_i, M_{\overline{\xi}(i)}), \tag{2.4}$$

where the sum runs over all arrows e of Q_ξ. Here $M_{\overline{\xi}(i)}$ is viewed as a D_i-module via the isomorphism $\eta_e^\flat$.

Under the isomorphism (2.4), if $\varphi \in \mathrm{Hom}_{A_\beta}(V^\xi, V)$ corresponds to $(\varphi_e)_{e\in E}$ on the right-hand side, then

$$\varphi(u \otimes x) = \eta_e(u) \otimes \varphi_e(x), \quad \forall\, e : i \to \overline{\xi}(i),\ u \in S_i,\ x \in M_i. \tag{2.5}$$

To summarize, $\mathfrak{g}(\xi)$ is the space of representations of the quiver Q_ξ (decorated by division algebras D_i) with a fixed dimension vector $\dim_{D_i}(M_i)$ at vertex i.

Example 2.4. Consider the case $F = \mathbb{C}(\!(t)\!)$ and ζ acts on F by change of variables $t \mapsto \zeta_n t$ for some primitive n-th root of unity. Then $k = \mathbb{C}(\!(\tau)\!)$ where $\tau = t^n$. Without loss of generality, we may assume $\beta = t^{nr} = \tau^r$ for some $r \in \mathbb{Z}$. Then $L = \mathbb{C}(\!(\mathbf{b})\!)/(\mathbf{b}^{m/n} - \tau^r)$. Let $\ell = \gcd(m/n, r)$. Then I can be identified with μ_ℓ, with $L_\epsilon \cong k[\mathbf{b}]/(\mathbf{b}^{\frac{m}{n\ell}} - \epsilon\tau^{\frac{r}{\ell}})$ for $\epsilon \in \mu_\ell$. We have $D_\epsilon = L_\epsilon$ since there are no nontrivial division algebras over L_ϵ.

Let ξ be a primitive m-th root of unity in $\mathbb{C}$. So we have $\xi \in \Xi_{m/n}$. The action of $\overline{\xi}$ on $I = \mu_\ell$ is via multiplication by $\xi^{m/\ell} \in \mu_\ell$. In particular, Q_ξ is a single cycle of length ℓ, with the vertices decorated by $L_\epsilon \cong \mathbb{C}(\!(\tau^{\frac{nr}{m}})\!)$. In this case, we may rename the M_i ($i \in I = \mu_\ell$) by $M_0, M_1, \ldots, M_{\ell-1}$ so that $\mathfrak{g}(\xi)$ is the space of representation of the following cyclic quiver over $\mathbb{C}(\!(\tau^{\frac{nr}{m}})\!)$:

$$\begin{tikzcd}
 & M_1 \arrow[r] & \cdots \arrow[dr] & \\
M_0 \arrow[ur] & & & \cdots . \arrow[dl] \\
 & M_{\ell-1} \arrow[ul] & \cdots \arrow[l] &
\end{tikzcd}$$

3 Polarized case

In this section we extend the results of the previous section from $G = \mathbf{GL}_F(V)$ to other classical groups. We remark that even in the case $G = \mathbf{GL}_F(V)$, we have not covered all finite order automorphisms of G in the previous section; only inner ones are considered. The outer ones will be covered as a special case of the polarized setting in this section (see Example 3.1).

We continue with the setup in Subsection 1.3. For the rest of the paper we assume $\mathrm{char}(k) \neq 2$.

3.1 Involution

Let $\sigma : F \to F$ be an involution that commutes with ζ (σ maybe trivial). In particular, σ restricts to an involution on k. For example, when n is even, we may take $\sigma = \zeta^{n/2}$, in which case $\sigma|_k$ is trivial.

3.2 Polarization

Let $\epsilon \in \{\pm 1\}$. Let

$$\langle \cdot, \cdot \rangle : V \times V \to F$$

be a non-degenerate pairing such that

(1) $\langle \cdot, \cdot \rangle$ is F-linear in the first variable.

(2) $\langle x, y \rangle = \epsilon\sigma(\langle y, x \rangle)$. (This implies that $\langle \cdot, \cdot \rangle$ is (F, σ)-semilinear in the second variable.)

(3) $\langle \theta x, \theta y \rangle = c\zeta(\langle x, y \rangle)$, for some $c \in (F^\times)^\sigma$ such that

$$\mathrm{Nm}_{F/k}(c)^{m/n} = \beta\sigma(\beta). \tag{3.1}$$

Let $G = \mathbf{Aut}_{F/F^\sigma}(V, \langle \cdot, \cdot \rangle) \subset \mathbf{GL}_{F/F^\sigma}(V)$; this is an algebraic group over F^σ. Let $\mathfrak{g} = \mathrm{Lie}\, G$ be the Lie algebra over F^σ. When σ is trivial and $\epsilon = 1$ (respectively, $\epsilon = -1$), G is the full orthogonal group (respectively, symplectic group) attached to $(V, \langle \cdot, \cdot \rangle)$. When σ is nontrivial, we may rescale $\langle \cdot, \cdot \rangle$ to reduce to the case $\epsilon = 1$, in which case G is the unitary group attached to the Hermitian space $(V, \langle \cdot, \cdot \rangle)$. By the property (3) of the pairing $\mathrm{Ad}(\theta)$ preserves the subgroup G of $\mathbf{GL}_{F/F^\sigma}(V)$. Similarly, $\mathrm{ad}(\theta)$ acts on the Lie algebra $\mathfrak{g}$.

Example 3.1. Take $F = k \times k$, and let $\zeta = \sigma$ be the swapping of two factors. In this case, we write $V = V_0 \oplus V_1$ for some k-vector spaces V_0 and V_1 using the idempotents in F. The pairing $\langle \cdot, \cdot \rangle$ identifies V_1 as the k-linear dual V_0^* of V_0. The automorphism θ of V sends V_0 to $V_1 = V_0^*$ and $V_1 = V_0^*$ to V_0. We have $G = \mathbf{GL}_k(V_0)$, and $\mathrm{Ad}(\theta)$ is an *outer* automorphism of G.

3.3 The problem

In the situation above, we try to understand the following in terms of quivers "with polarizations":

(1) $H := (\mathbf{R}_{F^\sigma/k^\sigma} G)^{\mathrm{Ad}(\theta)}$ as an algebraic group over k^σ. Note that $H(k^\sigma) = \{g \in \mathrm{Aut}_F(V) | \langle gx, gy \rangle = \langle x, y \rangle, \forall x, y \in V; g\theta = \theta g\}$.

(2) Let $\xi \in \Xi_{m/n} \cap F^\sigma$. Consider the k^σ-vector space with H-action

$$\mathfrak{g}(\xi) := \{\varphi \in \mathrm{End}_F(V) \mid \theta\varphi\theta^{-1} = \xi\varphi,\ \langle \varphi x, y \rangle + \langle x, \varphi y \rangle = 0,\ \forall\, x, y \in V\}.$$

If $\xi \notin \Xi_{m/n} \cap F^\sigma$, then the similarly defined $\mathfrak{g}(\xi)$ is zero.

As in Subsection 2.3 we may describe H and $\mathfrak{g}(\xi)$ using the A_β-module structure on V:

(1) $H = \mathbf{Aut}_{A_\beta/k^\sigma}(V, \langle \cdot, \cdot \rangle)$.

(2) For $\xi \in \Xi_{m/n} \cap F^\sigma$,

$$\mathfrak{g}(\xi) = \{\varphi \in \mathrm{Hom}_{A_\beta}(V^\xi, V) \mid \langle \varphi x, y \rangle + \langle x, \varphi y \rangle = 0,\ \forall\, x, y \in V\}.$$

Let $n' = [F^\sigma : k^\sigma]$. Note that n' is either n or $n/2$ (the latter happens if and only if $\sigma = \zeta^{n/2}$). When $n' = n$, θ^n is an F-linear automorphism of V satisfying

$$\langle \theta^n x, \theta^n y \rangle = \mathrm{Nm}_{F/k}(c) \langle x, y \rangle, \quad \forall\, x, y \in V.$$

If $n' = n/2$, then $\theta^{n'}$ is an (F, σ)-semilinear automorphism of V satisfying

$$\langle \theta^{n'} x, \theta^{n'} y\rangle = \mathrm{Nm}_{F^\sigma/k}(c)\sigma\langle x, y\rangle, \quad \forall\, x, y \in V.$$

In any case, $\mathrm{Ad}(\theta^{n'})$ gives an automorphism of G (over F^σ), and $\mathrm{ad}(\theta^{n'})$ gives an automorphism of $\mathfrak{g}$.

The following proposition describes the pair $(H, \mathfrak{g}(\xi))$ after base change to F^σ in terms of the F^σ-linear action of $\theta^{n'}$ on G and $\mathfrak{g}$.

Proposition 3.2. *We have canonical isomorphisms*

$$H_{F^\sigma} \cong G^{\mathrm{Ad}(\theta^{n'})},$$
$$\mathfrak{g}(\xi) \otimes_{k^\sigma} F^\sigma \cong \{\varphi \in \mathfrak{g} \mid \mathrm{ad}(\theta^{n'})\varphi = \xi\varphi\}$$

compatible with the natural actions of the first row on the second row.

Proof. We have an isomorphism $F^\sigma \otimes_{k^\sigma} V \cong V \oplus V \oplus \cdots \oplus V$ (n' factors) sending $x \otimes v$ to $(\zeta^i(x)v)_{0 \leqslant i \leqslant n'-1}$. Under this isomorphism, $\mathrm{id}_{F^\sigma} \otimes \theta$ acts cyclically on the n' factors, so that $\mathrm{id}_{F^\sigma} \otimes \theta^{n'}$ acts on each. The pairing $\langle \cdot, \cdot \rangle$ defines a pairing on the first factor of V, and determines the pairings on the rest by property (3) of the pairing. An element $g \in H_{F^\sigma}$ (respectively, $\varphi \in \mathfrak{g}(\xi) \otimes_{k^\sigma} F^\sigma$) is uniquely determined by its action on the first factor of V, on which it has to commute with $\theta^{n'}$. $\square$

3.4 Duality for A_β-modules

Let U be an A_β-module which is finite-dimensional over F. Let $U^* = \mathrm{Hom}_F(U, F)$ be the F-linear dual of U. Let $U^\diamondsuit$ be U^* with the action of F twisted by σ, i.e., for $a \in F$, $u^* \in U^\diamondsuit = U^*$ and $u \in U$, $(a \cdot u^*, u) = \sigma(a)(u^*, u)$, where (u^*, u) denotes the canonical pairing between U^* and U. We define an A_β-module structure on $U^\diamondsuit$ by requiring the action of $\boldsymbol{\theta}$ to be (F, ζ)-semilinear and satisfy

$$(\boldsymbol{\theta} u^*, u) = c\zeta(u^*, \boldsymbol{\theta}^{-1}u), \quad \forall\, u \in U,\ u^* \in U^\diamondsuit.$$

One readily checks that $\boldsymbol{\theta}^m u^* = \mathrm{Nm}_{F/k}(c)^{m/n}\beta^{-1}u^* = \sigma(\beta)u^*$ under the (old) F-action on U^*, and hence $\boldsymbol{\theta}^m u^* = \beta \cdot u^*$ under the (new) F-action on $U^\diamondsuit$.

The assignment $U \mapsto U^\diamondsuit$ gives a contravariant auto-equivalence on the category of finite-dimensional A_β-modules. On morphisms, it sends $f : U \to W$ to the transpose $f^\vee : W^\diamondsuit = W^* \to U^* = U^\diamondsuit$.

We have a canonical isomorphism of A_β-modules $U \cong (U^\diamondsuit)^\diamondsuit$ given by sending $u \in U$ to the (F, σ)-semilinear function $u^* \mapsto \sigma(u^*, u)$ on $u^* \in U^*$ (which is the same as an F-linear function on $U^\diamondsuit$).

For $\xi \in \Xi_{m/n}$ (so that the twisting functor $(-)^\xi$ on A_β-modules is defined as in Subsection 2.2), we have a canonical isomorphism

$$(U^\xi)^\diamondsuit \cong (U^\diamondsuit)^{\sigma(\xi)^{-1}}, \tag{3.2}$$

which is the identity on the underlying F-vector spaces.

3.5 Involution on the quiver

We continue to use the notation $L_\beta,\ I,\ Q_\xi,\ \overline{\xi}$ introduced in Subsections 2.4 and 2.6.

Let $\sigma_c : L_\beta \to L_\beta$ be the involution that is σ on k and $\sigma_c(b) = \mathrm{Nm}_{F/k}(c)b^{-1}$. The relation (3.1) implies that σ_c is a well-defined ring automorphism. It induces an involution on the set $I = \mathrm{Spec}\ L_\beta$ which we denote by $i \mapsto i^\diamondsuit$. In other words σ_c restricts to an isomorphism $L_i \cong L_{i^\diamondsuit}$. For $\xi \in \Xi_{m/n} \cap F^\sigma$, direct calculation shows that $\sigma_c \circ \mu_\xi \circ \sigma_c = \mu_{\xi^{-1}}$ as automorphisms of L_β. Therefore the involution $(-)^\diamondsuit$ on I reverses the arrows of the quiver Q_ξ.

3.6 Pairing between simple A_β-modules

The involution $U \mapsto U^\diamondsuit$ on A_β-modules induces an involution on the set of isomorphism classes of simple A_β-modules. In particular, for each $i \in I$, $S_i^\diamondsuit$ is a simple A_β-module isomorphic to $S_{i^\diamondsuit}$ by comparing the actions of L_β. For each i, an isomorphism of A_β-modules $\alpha_i : S_i^{\diamondsuit} \cong S_{i^\diamond}$ is the same data as a perfect pairing

$$\langle\cdot,\cdot\rangle_i : S_i \times S_{i^\diamond} \to F \tag{3.3}$$

satisfying

(1) $\langle\cdot,\cdot\rangle_i$ is F-linear in the first variable and (F,σ)-semilinear in the second variable.

(2) $\langle\boldsymbol{\theta}u,\boldsymbol{\theta}v\rangle_i = c\zeta(\langle u,v\rangle_i)$.

Indeed, α_i determines the pairing $\langle\cdot,\cdot\rangle_i$ characterized by $\langle u,\alpha_i(u^*)\rangle_i = (u^*,u)$, for $u \in S_i, u^* \in S_i^{\diamondsuit} = S_i^*$. Conversely, any pairing $\langle\cdot,\cdot\rangle_i$ as above induces a map $S_{i^\diamond} \to S_i^* = S_i^{\diamondsuit}$ which is F-linear by first property and intertwines the $\boldsymbol{\theta}$-action by the second. In particular, any nonzero pairing $\langle\cdot,\cdot\rangle_i$ satisfying (1) and (2) above must be a perfect pairing. We call a pairing as in (3.3) satisfying (1) and (2) above *admissible*.

Let $\langle\cdot,\cdot\rangle_i$ be a perfect admissible pairing $S_i \times S_{i^\diamond} \to F$. For each $d \in D_i$, there is a unique $\delta_i(d) \in D_{i^\diamond}$ such that

$$\langle ud,v\rangle_i = \langle u,v\delta_i(d)\rangle, \quad \forall\, u \in S_i,\ v \in S_{i^\diamond}. \tag{3.4}$$

Here we write the action of $D_i = \mathrm{End}_{A_\beta}(S_i)^{\mathrm{opp}}$ on S_i as right multiplication. The assignment $d \mapsto \delta_i(d)$ defines a (k,σ)-semilinear isomorphism of algebras

$$\delta_i : D_i \to D_{i^\diamond}^{\mathrm{opp}},$$

which restricts to $\sigma_c : L_i \cong L_{i^\diamond}$ on the centers. Note that δ_i depends on the choice of the admissible pairing $\langle\cdot,\cdot\rangle_i$.

When $i = i^{\diamondsuit}$, an admissible pairing $\langle\cdot,\cdot\rangle_i : S_i \times S_i \to F$ is called *Hermitian* if it satisfies $\langle v,u\rangle_i = \sigma(\langle u,v\rangle_i)$ for all $u,v \in S_i$. It is called *skew-Hermitian* if it satisfies $\langle v,u\rangle_i = -\sigma(\langle u,v\rangle_i)$ for all $u,v \in S_i$.

Lemma 3.3. *Suppose $i = i^{\diamondsuit}$. Then one of the following happens*:

(1) *There exists a perfect Hermitian admissible pairing on S_i.*

(2) *All admissible pairings on S_i are skew-Hermitian. This can only happen when $D_i = L_i$, $\sigma_c|_{L_i} = \mathrm{id}$ and $\sigma \neq \mathrm{id}_F$ (in particular, $\sigma = \zeta^{n/2}$ and $b_i^2 = \mathrm{Nm}_{F/k}(c)$).*

Proof. Start with any nonzero admissible pairing $(u,v) \mapsto \langle\langle u,v\rangle\rangle$ on S_i. Then $(u,v) \mapsto \sigma\langle\langle v,u\rangle\rangle$ is another admissible pairing. Therefore, if $\langle\langle u,v\rangle\rangle + \sigma\langle\langle v,u\rangle\rangle$ is not identically zero, it gives a perfect Hermitian admissible pairing.

Now suppose a perfect Hermitian admissible pairing on S_i does not exist. This means $\langle\langle u,v\rangle\rangle + \sigma\langle\langle v,u\rangle\rangle = 0$ for any $u,v \in S_i$ and any admissible pairing $\langle\langle\cdot,\cdot\rangle\rangle$ on S_i. In other words, all admissible pairings on S_i are skew-Hermitian. Pick any perfect skew-Hermitian admissible pairing $\langle\cdot,\cdot\rangle_i$ on S_i. Let $\delta_i : D_i \xrightarrow{\sim} D_i^{\mathrm{opp}}$ be the corresponding isomorphism characterized by (3.4). For $d \in D_i$, the pairing $\langle u,v\rangle_d := \langle ud,v\rangle_i$ is also admissible, hence also skew-Hermitian. Then we have $\sigma\langle ud,v\rangle_i = \sigma\langle u,v\rangle_d = -\langle v,u\rangle_d = -\langle vd,u\rangle_i = -\langle v,u\delta_i(d)\rangle_i = \sigma\langle u\delta_i(d),v\rangle_i$ for all $u,v \in S_i$, hence $\delta_i(d) = d$ for all $d \in D_i$. In this case, D_i must be commutative since δ_i is an anti-automorphism of D_i. Hence $D_i = L_i$. Since δ_i restricts to σ_c on L_i, we must have $\sigma_c|_{L_i} = \mathrm{id}$.

It remains to show that $\sigma \neq \mathrm{id}_F$ when the above situation happens. Suppose in contrary that $\sigma = \mathrm{id}_F$, then $\langle\cdot,\cdot\rangle_i$ is skew-symmetric, hence $\langle u,u\rangle_i = 0$ for any $u \in S_i$. Moreover, $\langle(a\otimes\ell)u,u\rangle_i = a\ell\langle u,u\rangle = 0$ for any $a \in F, \ell \in L_i$. Since $D_i = L_i$, we have $A_i \cong M_n(L_i)$ and $F \otimes_k L_i$ is maximal abelian subalgebra in A_i. Hence S_i is a rank one free $F \otimes_k L_i$-module. If we choose $u \in S_i$ generating S_i as an $F \otimes_k L_i$-module, then $\langle S_i,u\rangle_i = 0$, contradicting the fact that $\langle\cdot,\cdot\rangle_i$ is a perfect pairing. This finishes the argument. $\square$

3.7 Choice of admissible pairings

For the rest of the section, for each $i = i^{\diamondsuit}$ we fix a perfect Hermitian admissible pairing $\langle\cdot,\cdot\rangle_i$ on S_i if there exists one; otherwise we fix a perfect skew-Hermitian admissible pairing $\langle\cdot,\cdot\rangle_i$ on S_i. Moreover, for $i \neq i^{\diamondsuit}$, we choose perfect admissible pairings $\langle\cdot,\cdot\rangle_i$ and $\langle\cdot,\cdot\rangle_{i^\diamond}$ such that

$$\langle v,u\rangle_{i^\diamond} = \sigma(\langle u,v\rangle_i), \quad \forall\, u \in S_i,\ v \in S_{i^\diamond}. \tag{3.5}$$

By our choice, for each $i \in I$, there is a sign $\epsilon_i \in \{\pm 1\}$ such that

$$\langle v, u\rangle_{i^\diamond} = \epsilon_i \sigma(\langle u, v\rangle_i), \quad \forall\, u \in S_i,\ v \in S_{i^\diamond}. \tag{3.6}$$

Moreover, the case $\epsilon_i = -1$ can happen only in the situation (2) of Lemma 3.3.

Define $\delta_i : D_i \xrightarrow{\sim} D_{i^\diamond}^{\mathrm{opp}}$ using the chosen $\langle\cdot,\cdot\rangle_i$ as in Subsection 3.6. The property (3.6) implies

$$\delta_{i^\diamond} \circ \delta_i = \mathrm{id}_{D_i}.$$

In particular, for $i = i^\diamond$, δ_i is an anti-involution on D_i.

Lemma 3.4. *There is a unique pairing*

$$\{\cdot,\cdot\}'_i : M_i \times M_{i^\diamond} \to D_i$$

characterized by the following property:

$$\langle u \otimes x, v \otimes y\rangle = \langle u\{x, y\}'_i, v\rangle_i, \quad \forall\, u \in S_i,\ v \in S_{i^\diamond},\ x \in M_i,\ y \in M_{i^\diamond}. \tag{3.7}$$

Moreover, the pairing $\{\cdot,\cdot\}'_i$ satisfies the following identities for all $x \in M_i, y \in M_{i^\diamond}$ and $d \in D_i$:

$$\{dx, y\}'_i = d\{x, y\}'_i, \tag{3.8}$$

$$\{x, \delta_i(d)y\}'_i = \{x, y\}'_i d, \tag{3.9}$$

$$\{y, x\}'_{i^\diamond} = \epsilon\epsilon_i \delta_i(\{x, y\}'_i) \tag{3.10}$$

(*indeed* (3.9) *follows from* (3.8) *and* (3.10)).

Proof. For fixed $u \in S_i,\ x \in M_i$ and $y \in M_{i^\diamond}$, the assignment $v \mapsto \langle u \otimes x, v \otimes y\rangle$ is an (F, σ)-semilinear function $S_{i^\diamond} \to F$, and therefore can be written as $v \mapsto \langle u', v\rangle_i$ for a unique $u' \in S_i$. The assignment $u \mapsto u'$ gives an F-linear endomorphism of S_i. We claim that $u \mapsto u'$ is moreover A_β-linear. Indeed, it is enough to check $\langle \boldsymbol{\theta} u \otimes x, v \otimes y\rangle = \langle \boldsymbol{\theta} u', v\rangle_i$, which follows by comparing the property (3) of $\langle\cdot,\cdot\rangle$ and the property (2) of $\langle\cdot,\cdot\rangle_i$. Since $u \mapsto u'$ is A_β-linear, there is a unique $d \in D_i$ such that $u' = ud$ for all $u \in S_i$. We then define $\{x, y\}'_i = d \in D_i$. The properties (3.8) and (3.9) are easy to verify by using (3.4). The property (3.10) is verified by using the property (2) of the pairing $\langle\cdot,\cdot\rangle$ on V and the property (3.6) of the pairings $\langle\cdot,\cdot\rangle_i$. □

Definition 3.5. Let $\{\cdot,\cdot\}_i$ be the L_i-valued pairing

$$\begin{aligned} \{\cdot,\cdot\}_i : M_i \times M_{i^\diamond} &\to L_i \\ (x, y) &\mapsto \mathrm{Trd}_{D_i/L_i}\{x, y\}'_i, \end{aligned}$$

where $\mathrm{Trd}_{D_i/L_i} : D_i \to L_i$ is the reduced trace.

Remark 3.6. The D_i-valued pairing $\{\cdot,\cdot\}'_i$ satisfying (3.8) can be recovered from the L_i-valued pairing $\{\cdot,\cdot\}_i$. Indeed, $\{x, y\}'_i$ is the unique element $z \in D_i$ such that $\mathrm{Trd}_{D_i/L_i}(dz) = \{dx, y\}_i$ for all $d \in D_i$.

Corollary 3.7 (The corollary of Lemma 3.4)**.** *The pairing $\{\cdot,\cdot\}_i$ is L_i-linear in the first variable and (L_i, σ_c)-semilinear in the second variable, and*

$$\{y, x\}_{i^\diamond} = \epsilon\epsilon_i \sigma_c(\{x, y\}_i), \quad \forall\, x \in M_i,\ y \in M_{i^\diamond}.$$

Lemma 3.8. *Let $g \in \mathrm{Aut}_{A_\beta}(V)$ correspond to a family of automorphisms $g_i \in \mathrm{Aut}_{D_i}(M_i)$ under* (2.2). *Then g preserves the form $\langle\cdot,\cdot\rangle$ on V if and only if for all $i \in I$,*

$$\{x, y\}_i = \{g_i(x), g_{i^\diamond}(y)\}_i, \quad \forall\, x \in M_i,\ y \in M_{i^\diamond}. \tag{3.11}$$

Proof. By (2.3), for $u \in S_i,\ v \in S_{i^\diamond},\ x \in M_i$ and $y \in M_{i^\diamond}$, we have

$$\langle g(u \otimes x), g(v \otimes y)\rangle = \langle u \otimes g_i(x), v \otimes g_{i^\diamond}(y)\rangle = \langle u\{g_i(x), g_{i^\diamond}(y)\}'_i, v\rangle_i.$$

Comparing with (3.7) we get $\{x, y\}'_i = \{g_i(x), g_{i^\diamond}(y)\}'_i$. By Remark 3.6, this is equivalent to (3.11). □

After fixing the admissible pairings between the simple A_β-modules, we are going to choose a family of A_β-module isomorphisms $\eta_e : S_i^\xi \to S_{\overline{\xi}(i)}$ for each arrow $e : i \to \overline{\xi}(i)$ of Q_ξ.

Let $e : i \to \overline{\xi}(i)$ be an arrow in Q_ξ such that $e = e^\diamondsuit$, i.e., $\overline{\xi}(i) = i^\diamondsuit$ (the case $i = i^\diamondsuit$ is allowed). Note that in this case $\sigma_{c\xi^{-1}} : \mathbf{b} \mapsto \mathrm{Nm}_{F/k}(c\xi^{-1})\mathbf{b}^{-1}$ is an automorphism of L_i.

An isomorphism of A_β-modules $\eta_e : S_i^\xi \xrightarrow{\sim} S_{\overline{\xi}(i)} = S_{i^\diamond}$ is called *self-adjoint* if $\langle u, \eta_e(v)\rangle_i = \sigma\langle v, \eta_e(u)\rangle_i$ for all $u, v \in S_i = S_i^\xi$; η_e is called *skew-self-adjoint* if $\langle u, \eta_e(v)\rangle_i = -\sigma\langle v, \eta_e(u)\rangle_i$ for all $u, v \in S_i = S_i^\xi$.

Lemma 3.9. *Let $e : i \to \overline{\xi}(i)$ be an arrow in Q_ξ such that $e = e^\diamondsuit$. Then one of the following happens*:

(1) *There exists a self-adjoint A_β-linear isomorphism $\eta_e : S_i^\xi \xrightarrow{\sim} S_{\overline{\xi}(i)} = S_{i^\diamond}$.*

(2) *All elements in $\mathrm{Hom}_{A_\beta}(S_i^\xi, S_{i^\diamond})$ are skew-self-adjoint. This can only happen when $D_i = L_i$, $\sigma_{c\xi^{-1}}|_{L_i} = \mathrm{id}$ and $\sigma \neq \mathrm{id}_F$ (in particular, $\sigma = \zeta^{n/2}$ and $b_i^2 = \mathrm{Nm}_{F/k}(c\xi^{-1})$).*

Proof. The argument is similar to that of Lemma 3.3. For any A_β-linear map $\eta : S_i^\xi \to S_{\overline{\xi}(i)}$, define $\eta^* : S_i^\xi \to S_{\overline{\xi}(i)}$ by requiring $\langle u, \eta(v)\rangle_i = \sigma\langle v, \eta^*(u)\rangle_i$ for all $u, v \in S_i$. Then η^* is also A_β-linear, and $\eta^{**} = \eta$. If $\eta + \eta^*$ is nonzero, it gives a self-adjoint isomorphism.

Now suppose a self-adjoint η does not exist. This implies $\eta + \eta^* = 0$ for all $\eta \in \mathrm{Hom}_{A_\beta}(S_i^\xi, S_{i^\diamond})$, i.e., all η are skew-self-adjoint. Fix a skew-self-adjoint isomorphism $\eta_e : S_i^\xi \xrightarrow{\sim} S_{i^\diamond}$. For any $d \in D_i$, $u \mapsto \eta_e(ud)$ again belongs to $\mathrm{Hom}_{A_\beta}(S_i^\xi, S_{i^\diamond})$, hence it is also skew-self-adjoint. Therefore, for $u, v \in S_i$, $\langle u, \eta_e(v)\eta_e^\flat(d)\rangle_i = \langle u, \eta_e(vd)\rangle_i = -\sigma\langle v, \eta_e(ud)\rangle_i = \langle ud, \eta_e(v)\rangle_i = \langle u, \eta_e(v)\delta_i(d)\rangle_i$. Hence $\eta_e^\flat = \delta_i$ as maps $D_i \to D_{i^\diamond}$. Since $\eta_e^\flat$ is an algebra isomorphism while δ_i is an anti-isomorphism, we conclude that D_i is commutative hence $D_i = L_i$. Moreover, $\mu_\xi|_{L_i} = \eta_e^\flat|_{L_i} = \delta_i|_{L_i} = \sigma_c|_{L_i}$, which implies $\sigma_{c\xi^{-1}}|_{L_i} = \mathrm{id}$. Finally, to rule out the case $\sigma = \mathrm{id}_F$, we use the same argument as in Lemma 3.3. By skew-self-adjointness we have $\langle au\ell, \eta_e(u)\rangle_i = 0$ for all $a \in F, \ell \in L_i$ and $u \in S_i$; choosing u to be a generator of the rank one $F \otimes_k L_i$-module S_i we get $\langle S_i, \eta_e(u)\rangle_i = 0$ which is a contradiction. □

3.8 Choice of the isomorphisms η_e

Next, for each arrow $e : i \to i^\diamondsuit$ in Q_ξ, we fix an isomorphisms of A_β-modules

$$\eta_e : S_i^\xi \xrightarrow{\sim} S_{\overline{\xi}(i)}$$

as follows. If $e \neq e^\diamondsuit$ (say $e : i \to \overline{\xi}(i)$, $e^\diamondsuit : \overline{\xi}(i)^\diamondsuit \to i^\diamondsuit$), then we choose η_e and $\eta_{e^\diamond}$ so that

$$\sigma\langle v, \eta_e(u)\rangle_{\overline{\xi}(i)^\diamond} = \langle u, \eta_{e^\diamond}(v)\rangle_i, \quad \forall\, u \in S_i,\ v \in S_{\overline{\xi}(i)^\diamond}.$$

If $e = e^\diamondsuit$, we choose η_e to be self-adjoint if there exists one; otherwise we choose η_e to be skew-self-adjoint. By our choice, for each arrow e there is a sign $\epsilon_e \in \{\pm 1\}$ such that

$$\sigma\langle v, \eta_e(u)\rangle_{\overline{\xi}(i)^\diamond} = \epsilon_e\langle u, \eta_{e^\diamond}(v)\rangle_i, \quad \forall\, u \in S_i,\ v \in S_{\overline{\xi}(i)^\diamond}. \tag{3.12}$$

Moreover, $\epsilon_e = -1$ can only happen in the situation (2) of Lemma 3.9.

Lemma 3.10. *Let $\varphi \in \mathrm{Hom}_{A_\beta}(V^\xi, V)$ correspond to a family of maps $\varphi_e \in \mathrm{Hom}_{D_i}(M_i, M_{\overline{\xi}(i)})$ for each arrow $e : i \to \overline{\xi}(i)$ in Q_ξ (see (2.4)). Then $\varphi \in \mathfrak{g}(\xi)$ if and only if for each arrow $e : i \to \overline{\xi}(i)$,*

$$\{y, \varphi_e(x)\}_{\overline{\xi}(i)^\diamond} + \epsilon\epsilon_e\sigma_{c\xi^{-1}}(\{x, \varphi_{e^\diamond}(y)\}_i) = 0, \quad \forall\, x \in M_i,\ y \in M_{\overline{\xi}(i)^\diamond}. \tag{3.13}$$

Proof. The map φ lies in $\mathfrak{g}(\xi)$ if and only if

$$\langle\varphi(u \otimes x), v \otimes y\rangle + \langle u \otimes x, \varphi(v \otimes y)\rangle = 0, \quad \forall\, i \in I,\ u \in S_i,\ x \in M_i,\ v \in S_{\overline{\xi}(i)^\diamond},\ y \in M_{\overline{\xi}(i)^\diamond}. \tag{3.14}$$

Let $e : i \to \overline{\xi}(i)$ so that $e^\diamondsuit : \overline{\xi}(i)^\diamondsuit \to i^\diamondsuit$. We have by (2.5) and (3.7),

$$\langle\varphi(u \otimes x), v \otimes y\rangle = \langle\eta_e(u) \otimes \varphi_e(x), v \otimes y\rangle_{\overline{\xi}(i)} = \langle\eta_e(u)\{\varphi_e(x), y\}'_{\overline{\xi}(i)}, v\rangle_{\overline{\xi}(i)}. \tag{3.15}$$

By (3.6) the above is equal to $\epsilon_{\overline{\xi}(i)}\langle v, \eta_e(u)\{\varphi_e(x), y\}'_{\overline{\xi}(i)}\rangle_{\overline{\xi}(i)^\diamond}$. Hence

$$\langle \varphi(u \otimes x), v \otimes y\rangle = \epsilon_{\overline{\xi}(i)}\langle v, \eta_e(u)\{\varphi_e(x), y\}'_{\overline{\xi}(i)}\rangle_{\overline{\xi}(i)^\diamond}. \tag{3.16}$$

On the other hand, by (2.5), and (3.7),

$$\langle u \otimes x, \varphi(v \otimes y)\rangle = \langle u \otimes x, \eta_{e^\diamond}(v) \otimes \varphi_{e^\diamond}(y)\rangle_i = \langle u\{x, \varphi_{e^\diamond}(y)\}'_i, \eta_{e^\diamond}(v)\rangle_i.$$

By (3.12) and the definition of $\eta_e^\flat$, we have

$$\langle u\{x, \varphi_{e^\diamond}(y)\}'_i, \eta_{e^\diamond}(v)\rangle_i = \epsilon_e \sigma\langle v, \eta_e(u\{x, \varphi_{e^\diamond}(y)\}'_i)\rangle_{\overline{\xi}(i)^\diamond} = \epsilon_e \sigma \langle v, \eta_e(u)\eta_e^\flat(\{x, \varphi_{e^\diamond}(y)\}'_i)\rangle_{\overline{\xi}(i)^\diamond}.$$

Therefore

$$\langle u \otimes x, \varphi(v \otimes y)\rangle = \epsilon_e \langle v, \eta_e(u)\eta_e^\flat(\{x, \varphi_{e^\diamond}(y)\}'_i)\rangle_{\overline{\xi}(i)^\diamond}. \tag{3.17}$$

Plugging (3.16) and (3.17) into (3.14), we get

$$\epsilon_{\overline{\xi}(i)}\langle v, \eta_e(u)\{\varphi_e(x), y\}'_{\overline{\xi}(i)}\rangle_{\overline{\xi}(i)^\diamond} + \epsilon_e\langle v, \eta_e(u)\eta_e^\flat(\{x, \varphi_{e^\diamond}(y)\}'_i)\rangle_{\overline{\xi}(i)^\diamond} = 0$$

for all $u \in S_i,\ v \in S_{\overline{\xi}(i)^\diamond}$, which is equivalent to

$$\epsilon_{\overline{\xi}(i)}\{\varphi_e(x), y\}'_{\overline{\xi}(i)} + \epsilon_e \eta_e^\flat(\{x, \varphi_{e^\diamond}(y)\}'_i) = 0, \quad \forall\, x \in M_i,\ y \in M_{\overline{\xi}(i)^\diamond}. \tag{3.18}$$

By (3.10) we have

$$\epsilon_{\overline{\xi}(i)}\{\varphi_e(x), y\}'_{\overline{\xi}(i)} = \epsilon\delta_{\overline{\xi}(i)}(\{y, \varphi_e(x)\}'_{\overline{\xi}(i)^\diamond}), \tag{3.19}$$

hence (3.18) is equivalent to

$$\epsilon\delta_{\overline{\xi}(i)}(\{y, \varphi_e(x)\}'_{\overline{\xi}(i)^\diamond}) + \epsilon_e\eta_e^\flat(\{x, \varphi_{e^\diamond}(y)\}'_i) = 0. \tag{3.20}$$

Taking reduced trace and using $\sigma_c \circ \mu_\xi = \sigma_{c\xi^{-1}} : L_i \xrightarrow{\sim} L_{\overline{\xi}(i)^\diamond}$ we get (3.13), which is equivalent to (3.20) by Remark 3.6. □

The above lemma motivates the following definition.

Definition 3.11. For an arrow $e : i \to i^\diamondsuit$ fixed by $(-)^\diamondsuit$, and a sign $\epsilon' \in \{\pm 1\}$, we define $\mathfrak{h}^{\epsilon'}(M_i, M_{i^\diamond})$ to be the set of maps $\varphi_i : M_i \to M_{i^\diamond}$ such that

$$\varphi(dx) = \eta_e^\flat(d)\varphi(x), \quad \{y, \varphi(x)\}_i = \epsilon'\sigma_{c\xi^{-1}}(\{x, \varphi(y)\}_i), \quad \forall\, d \in D_i,\ x \in M_i,\ y \in M_{i^\diamond}.$$

3.9 Shape of Q_ξ with involution

Each connected component of Q_ξ is a directed cycle. Let Π be the set of connected components of Q_ξ. The involution $(-)^\diamondsuit$ induces an involution on Π. Let $\underline{\Pi}$ be the set of orbits of Π under $(-)^\diamondsuit$. For $\alpha \in \underline{\Pi}$, let Q_ξ^α be the union of the components contained in α. Note that $\#\alpha$ equals 1 or 2. We have a decomposition

$$Q_\xi = \coprod_{\alpha \in \underline{J}} Q_\xi^\alpha.$$

Corresponding to this decomposition, we have

$$H = \prod_{\alpha \in \underline{\Pi}} H^\alpha; \quad \mathfrak{g}(\xi) = \bigoplus_{\alpha \in \underline{\Pi}} \mathfrak{g}(\xi)^\alpha$$

such that H^α acts on $\mathfrak{g}(\xi)^\alpha$.

The directed graph Q_ξ^α ($\alpha \in \underline{J}$) with involution $(-)^\diamondsuit$ takes one of the follow shapes:

(CC-ℓ) Q_ξ^α is a disjoint union of two direct cycles with $(-)^\diamondsuit$ mapping one to the other. We label the vertices by $1,\dots,\ell,1^\diamondsuit,\dots,\ell^\diamondsuit$ as follows ($\ell \geqslant 1$):

$$\begin{tikzcd}
1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[d] & & \ell^\diamondsuit \arrow[r] & (\ell-1)^\diamondsuit \arrow[r] & \cdots \arrow[d] \\
\ell \arrow[u] & (\ell-1) \arrow[l] & \cdots \arrow[l] & & 1^\diamondsuit \arrow[u] & 2^\diamondsuit \arrow[l] & \cdots \arrow[l]
\end{tikzcd}.$$

(VV-ℓ) Q_ξ^α is a directed cycle with two distinct vertices and no arrow fixed by $(-)^\diamondsuit$. We label the vertices as follows so that 0 and ℓ are fixed by $(-)^\diamondsuit$ ($\ell = 0$ is allowed):

$$\begin{tikzcd}
& 1 \arrow[r] & \cdots \arrow[r] & (\ell-1) \arrow[dr] & \\
0 = 0^\diamondsuit \arrow[ur] & & & & \ell = \ell^\diamondsuit \arrow[dl] \\
& 1^\diamondsuit \arrow[ul] & \cdots \arrow[l] & (\ell-1)^\diamondsuit \arrow[l] &
\end{tikzcd}.$$

(VE-ℓ) Q_ξ^α is a directed cycle with exactly one vertex and one arrow fixed by $(-)^\diamondsuit$. We label the vertices as follows so that $0 = 0^\diamondsuit$ and $e: \ell \to \ell^\diamondsuit$ is fixed by $(-)^\diamondsuit$ ($\ell = 0$ is allowed):

$$\begin{tikzcd}
& 1 \arrow[r] & \cdots \arrow[r] & \ell \arrow[dd, "e"] \\
0 = 0^\diamondsuit \arrow[ur] & & & \\
& 1^\diamondsuit \arrow[ul] & \cdots \arrow[l] & \ell^\diamondsuit \arrow[l]
\end{tikzcd}.$$

(EE-ℓ) Q_ξ^α is a directed cycle with no vertex and exactly two arrows fixed by $(-)^\diamondsuit$. We label the vertices as follows so that $e: 1^\diamondsuit \to 1$ and $e': \ell \to \ell^\diamondsuit$ are fixed by $(-)^\diamondsuit$ ($\ell = 1$ is allowed):

$$\begin{tikzcd}
1 \arrow[r] & \cdots \arrow[r] & \ell \arrow[d, "e'"] \\
1^\diamondsuit \arrow[u, "e"] & \cdots \arrow[l] & \ell^\diamondsuit \arrow[l]
\end{tikzcd}.$$

Our convention is such that in the cases (CC-ℓ), (VV-ℓ), and (EE-ℓ) the graph Q_ξ^α has 2ℓ vertices, while in the case (VE-ℓ) it has $2\ell+1$ vertices.

3.10 The contragredient action

For $i \in I$ and $g \in \mathrm{Aut}_{D_i}(M_i)$, we define $g^* \in \mathrm{Aut}_{D_{i^\diamondsuit}}(M_{i^\diamondsuit})$ so that

$$\{gx, y\}_i = \{x, g^* y\}_i, \quad \forall\, x \in M_i,\ y \in M_{i^\diamondsuit}.$$

The assignment $g \mapsto g^{*,-1}$ defines an isomorphism of algebraic groups

$$\mathbf{GL}_{D_i/k^\sigma}(M_i) \cong \mathbf{GL}_{D_{i^\diamondsuit}/k^\sigma}(M_{i^\diamondsuit}).$$

For each $\alpha \in \underline{\Pi}$, Lemmas 3.8 and 3.10 give a description of H^α and $\mathfrak{g}(\xi)^\alpha$ in each case classified in Subsection 3.9. We summarize our results so far in the following theorem.

Theorem 3.12. *The isomorphism type of the directed graph Q_ξ together with the involution $(-)^\diamondsuit$ on it depends only on $(k, \sigma|_k, \beta, \mathrm{Nm}_{F/k}(c), \mathrm{Nm}_{F/k}(\xi))$.*

For each $\alpha \in \underline{\Pi}$, the pair $(H^\alpha, \mathfrak{g}(\xi)^\alpha)$ is described as follows according to the shape of Q_ξ^α:

(1) *If Q_ξ^α is of shape* (CC-ℓ), *then*

$$H^\alpha \cong \prod_{i=1}^{\ell} \mathbf{GL}_{D_i/k^\sigma}(M_i),$$

$$\mathfrak{g}(\xi)^\alpha \cong \bigoplus_{i=1}^{\ell} \mathrm{Hom}_{D_i}(M_i, M_{i+1}).$$

Here $M_{\ell+1} = M_1$, and M_{i+1} is viewed as a D_i-module by $\eta_e^\flat : D_i \cong D_{i+1}$ (where e is the arrow $i \to i+1$). The factors $\mathbf{GL}_{D_i/k^\sigma}(M_i)$ and $\mathbf{GL}_{D_{i+1}/k^\sigma}(M_{i+1})$ act on $\mathrm{Hom}_{D_i}(M_i, M_{i+1})$ by $(g_i, g_{i+1})\cdot\varphi = g_{i+1}\circ\varphi\circ g_i^{-1}$.

(2) *If Q_ξ^α is of shape* (VV-ℓ), *then*

$$H^\alpha \cong \mathbf{Aut}_{D_0/k^\sigma}(M_0, \{\cdot,\cdot\}_0) \times \prod_{i=1}^{\ell-1} \mathbf{GL}_{D_i/k^\sigma}(M_i) \times \mathbf{Aut}_{D_\ell/k^\sigma}(M_\ell, \{\cdot,\cdot\}_\ell),$$

$$\mathfrak{g}(\xi)^\alpha \cong \bigoplus_{i=0}^{\ell-1} \mathrm{Hom}_{D_i}(M_i, M_{i+1}).$$

The action of H^α on $\mathfrak{g}(\xi)^\alpha$ is as explained in the case (CC-ℓ), *by viewing H^α as a subgroup of $\prod_{i=0}^{\ell} \mathbf{GL}_{D_i/k^\sigma}(M_i)$.*

(3) *If Q_ξ^α is of shape* (VE-ℓ), *then*

$$H^\alpha \cong \mathbf{Aut}_{D_0/k^\sigma}(M_0, \{\cdot,\cdot\}_0) \times \prod_{i=1}^{\ell} \mathbf{GL}_{D_i/k^\sigma}(M_i),$$

$$\mathfrak{g}(\xi)^\alpha \cong \left(\bigoplus_{i=0}^{\ell-1} \mathrm{Hom}_{D_i}(M_i, M_{i+1})\right) \oplus \mathfrak{h}^{-\epsilon\epsilon_e}(M_\ell, M_{\ell^\diamond}).$$

The action of H^α on $\mathrm{Hom}_{D_i}(M_i, M_{i+1})$ is as explained in the case (CC-ℓ), *viewing $\mathbf{Aut}_{D_0/k^\sigma}(M_0, \{\cdot,\cdot\}_0)$ as a subgroup of $\mathbf{GL}_{D_0/k^\sigma}(M_0)$. The action of $\mathbf{GL}_{D_\ell/k^\sigma}(M_\ell)$ on $\mathfrak{h}^{-\epsilon\epsilon_e}(M_\ell, M_{\ell^\diamond})$ is induced from its natural action on M_ℓ and the contragredient action on $M_{\ell^\diamond}$ given by $g \mapsto g^{*,-1}$ (see Subsection* 3.10).

(4) *If Q_ξ^α is of shape* (EE-ℓ), *then*

$$H^\alpha \cong \prod_{i=1}^{\ell} \mathbf{GL}_{D_i/k^\sigma}(M_i),$$

$$\mathfrak{g}(\xi)^\alpha \cong \mathfrak{h}^{-\epsilon\epsilon_e}(M_{1^\diamond}, M_1) \oplus \left(\bigoplus_{i=1}^{\ell-1} \mathrm{Hom}_{D_i}(M_i, M_{i+1})\right) \oplus \mathfrak{h}^{-\epsilon\epsilon_{e'}}(M_\ell, M_{\ell^\diamond}).$$

The action of H^α on $\mathrm{Hom}_{D_i}(M_i, M_{i+1})$ is as explained in the case (CC-ℓ). *The action of $\mathbf{GL}_{D_\ell/k^\sigma}(M_1)$ on $\mathfrak{h}^{-\epsilon\epsilon_e}(M_{1^\diamond}, M_1)$ and the action of $\mathbf{GL}_{D_\ell/k^\sigma}(M_\ell)$ on $\mathfrak{h}^{-\epsilon\epsilon_{e'}}(M_\ell, M_{\ell^\diamond})$ are as explained in the* (VE-ℓ) *case.*

Here is a more precise description of the factors $\mathbf{Aut}_{D_i/k^\sigma}(M_i, \{\cdot,\cdot\}_i)$ that appear in H in the above theorem. The statement follows immediately from Corollary 3.7.

Proposition 3.13. *Let $i = i^\diamondsuit$ be a vertex in Q_ξ. Then*

(1) *If $\sigma = \mathrm{id}_F$ and $\sigma_c|_{L_i} = \mathrm{id}$, then $\mathbf{Aut}_{D_i/k^\sigma}(M_i, \{\cdot,\cdot\}_i)$ is an orthogonal group (respectively, symplectic group) when $\epsilon = 1$ (respectively, $\epsilon = -1$).*

(2) *If $\sigma \neq \mathrm{id}_F$ and $\sigma_c|_{L_i} = \mathrm{id}$ (in particular, $\sigma|_k = \mathrm{id}$, hence $\sigma = \zeta^{n/2}$), then $\mathbf{Aut}_{D_i/k^\sigma}(M_i, \{\cdot,\cdot\}_i)$ is either an orthogonal or a symplectic group.*

(3) *If $\sigma_c|_{L_i} \neq \mathrm{id}$, then $\mathbf{Aut}_{D_i/k^\sigma}(M_i, \{\cdot,\cdot\}_i)$ is a unitary group.*

Proof. The map $g \mapsto g^*$ defined in Subsection 3.10 is an anti-involution on $\mathrm{End}_{D_i}(M_i)$. When $\sigma_c|_{L_i} = \mathrm{id}$, it is an involution of the first kind; when $\sigma_c|_{L_i} \neq \mathrm{id}$, it is an involution of the second kind. Therefore in the former case the corresponding isometry group is an orthogonal or symplectic group, while in the latter case it is a unitary group. This proves (2) and (3).

It remains to show in the case (1), the type of $\mathbf{Aut}_{D_i/k^\sigma}(M_i, \{\cdot,\cdot\}_i)$ is the same as that of G. We already know that $\mathbf{Aut}_{D_i/k^\sigma}(M_i, \{\cdot,\cdot\}_i)$ is either an orthogonal or a symplectic group. By Proposition 3.2, H_F is the fixed point subgroup of G (orthogonal or symplectic) under $\mathrm{Ad}(\theta^n)$. Hence the simple factors of $H_F = G^{\mathrm{Ad}(\theta^n)}$ are either of type A or of the same type as G. Therefore $\mathbf{Aut}_{D_i/k^\sigma}(M_i, \{\cdot,\cdot\}_i)$ has the same type as G. □

4 Loop Lie algebras of classical type

In this section we continue with the setup in Section 3. We specialize to the case $k = \mathbb{C}((\tau))$. Then F is a finite separable k-algebra with $\mathrm{Aut}_k(F) \cong \mathbb{Z}/n\mathbb{Z}$ but we do not require F to be a field. We write $\gamma = \mathrm{Nm}_{F/k}(c) \in k^\times$. Let $\mathrm{val}_\tau : k^\times \to \mathbb{Z}$ be the valuation such that $\mathrm{val}_\tau(\tau) = 1$.

The isomorphism type of $(Q_\xi, (-)^\diamondsuit)$ depends only on $(k, \sigma|_k, \beta, \gamma, \mathrm{Nm}_{F/k}(\xi))$ according to Theorem 3.12. In the following we *assume* $\mathrm{Nm}_{F/k}(\xi) \in \mu_{m/n}(\mathbb{C})$ *to be primitive*. We describe in more details the shape of $(Q_\xi, (-)^\diamondsuit)$ as well as the factors in H and $\mathfrak{g}(\xi)$. The situation simplifies because there are no nontrivial division algebras over L_i in this case, therefore $D_i = L_i$ for all $i \in I$.

4.1 The case $\sigma|_k = \mathrm{id}$

In this case $\gamma^{m/n} = \beta^2$. We distinguish two cases according to the parity of m/n.

4.1.1 *m/n is odd*

In this case, $\mathrm{val}_\tau(\beta)$ is divisible by m/n hence $b^{m/n} = \beta$ has m/n distinct solutions in k, i.e., L splits into m/n-factors of k (all $L_i = k$). The graph Q_ξ is a single cycle of length m/n. Since m/n is odd, it must be of type (VE).

The unique vertex $i = i^\diamondsuit$ corresponds to the unique $b_i \in k$ such that $b_i^2 = \gamma$ and $b_i^{m/n} = \beta$. In particular, $\sigma_c|_{L_i} = \mathrm{id}$. The factor $\mathbf{Aut}_k(M_i, \{\cdot,\cdot\}_i)$ in H is either an orthogonal group or a symplectic group over k. When $\sigma|_F = \mathrm{id}$, we have $\epsilon_i = 1$ by Lemma 3.3, hence $\mathbf{Aut}_k(M_i, \{\cdot,\cdot\}_i)$ is an orthogonal group if $\epsilon = 1$ and a symplectic group if $\epsilon = -1$.

The unique arrow $e : j \to j^\diamondsuit$ fixed by $(-)^\diamondsuit$ corresponds to the unique $b_j \in k$ such that $b_j^2 = \gamma \mathrm{Nm}_{F/k}(\xi^{-1})$ and $b_j^{m/n} = \beta$. The factor $\mathfrak{h}^{-\epsilon\epsilon_e}(M_j, M_{j^\diamond})$ in $\mathfrak{g}(\xi)$ is isomorphic to either $\wedge^2(M_{j^\diamond})$ or $\mathrm{Sym}^2(M_{j^\diamond})$. When $\sigma|_F = \mathrm{id}$, we have $\epsilon_e = 1$ by Lemma 3.9, hence $\mathfrak{h}^{-\epsilon\epsilon_e}(M_j, M_{j^\diamond})$ is $\wedge^2(M_{j^\diamond})$ if $\epsilon = 1$ and $\mathrm{Sym}^2(M_{j^\diamond})$ if $\epsilon = -1$.

4.1.2 *m/n is even*

In this case we have $\beta = \pm\gamma^{m/2n}$. Whether or not $b^{m/n} = \beta$ has a solution in k depends on the parity of $\mathrm{val}_\tau(\gamma)$.

• When $\mathrm{val}_\tau(\gamma)$ is even, L splits into m/n factors of $L_i = k$. The graph Q_ξ is a single cycle of length m/n. We have two subcases:

(1) When $\beta = \gamma^{m/2n}$, then Q_ξ is of type (VV). Let $i, i' \in I$ be the two vertices fixed by $(-)^\diamondsuit$. Since $\sigma_c|_{L_i} = \mathrm{id}$ and $\sigma_c|_{L_{i'}} = \mathrm{id}$, the factor of H corresponding to i or i' is either an orthogonal group or a symplectic groups (when $\sigma = \mathrm{id}_F$ it is the former if $\epsilon = 1$ and the latter if $\epsilon = -1$).

(2) When $\beta = -\gamma^{m/2n}$, then Q_ξ is of type (EE). The factor in $\mathfrak{g}(\xi)$ corresponding to any arrow $e : i \to i^\diamondsuit$ fixed by $(-)^\diamondsuit$ is isomorphic to either $\wedge^2(M_{i^\diamond})$ or $\mathrm{Sym}^2(M_{i^\diamond})$ (when $\sigma = \mathrm{id}_F$ it is the former if $\epsilon = 1$ and the latter if $\epsilon = -1$).

• When $\mathrm{val}_\tau(\gamma)$ is odd. In this case L splits into a product of fields L_i where each L_i is isomorphic to the unique quadratic extension of k. The graph Q_ξ is a single cycle of length $m/2n$. We have four subcases:

(1) When $m/2n$ is odd and $\beta = \gamma^{m/2n}$, then Q_ξ is of type (VE). The vertex $i = i^\diamondsuit$ corresponds to $b_i^2 = \gamma$, and $\sigma_c|_{L_i} = \mathrm{id}$. The factor $\mathbf{Aut}_{L_i/k}(M_i, \{\cdot,\cdot\}_i)$ is the Weil restriction of an orthogonal group or symplectic group over L_i (when $\sigma = \mathrm{id}_F$ it is the former if $\epsilon = 1$ and the latter if $\epsilon = -1$). The edge $e : j \to j^\diamondsuit$ fixed by $(-)^\diamondsuit$ corresponds to $b_j^2 = -\gamma \mathrm{Nm}_{F/k}(\xi^{-1})$, hence $\sigma_{c\xi^{-1}}|_{L_j} \neq \mathrm{id}$. The corresponding factor $\mathfrak{h}^{-\epsilon\epsilon_e}(M_j, M_{j^\diamondsuit})$ is isomorphic to the space of L_j/k-Hermitian forms on M_j.

(2) When $m/2n$ is odd and $\beta = -\gamma^{m/2n}$, then Q_ξ is of type (VE). The vertex $i = i^\diamondsuit$ corresponds to $b_i^2 = -\gamma$, and $\sigma_c|_{L_i} \neq \mathrm{id}$. The factor $\mathbf{Aut}_{L_i/k}(M_i, \{\cdot,\cdot\}_i)$ is a unitary group over k. The edge $e : j \to j^\diamondsuit$ fixed by $(-)^\diamondsuit$ corresponds to $b_j^2 = \gamma \mathrm{Nm}_{F/k}(\xi^{-1})$, hence $\sigma_{c\xi^{-1}}|_{L_j} = \mathrm{id}$. The corresponding factor $\mathfrak{h}^{-\epsilon\epsilon_e}(M_j, M_{j^\diamondsuit})$ is isomorphic to either $\wedge^2_{L_{j^\diamondsuit}}(M_{j^\diamondsuit})$ or $\mathrm{Sym}^2_{L_{j^\diamondsuit}}(M_{j^\diamondsuit})$ (when $\sigma = \mathrm{id}_F$ it is the former if $\epsilon = 1$ and the latter if $\epsilon = -1$).

(3) When $m/2n$ is even and $\beta = \gamma^{m/2n}$, then Q_ξ is of type (VV). One vertex $i = i^\diamondsuit$ corresponds to $b_i^2 = \gamma$, and $\sigma_c|_{L_i} = \mathrm{id}$. The factor $\mathbf{Aut}_{L_i/k}(M_i, \{\cdot,\cdot\}_i)$ is the Weil restriction of an orthogonal group or symplectic group over L_i (when $\sigma = \mathrm{id}_F$ it is the former if $\epsilon = 1$ and the latter if $\epsilon = -1$). Another vertex $i' = i'^\diamondsuit$ corresponds to $b_{i'}^2 = -\gamma$, and $\sigma_c|_{L_{i'}} \neq \mathrm{id}$. The factor $\mathbf{Aut}_{L_{i'}/k}(M_{i'}, \{\cdot,\cdot\}_{i'})$ is a unitary group over k.

(4) When $m/2n$ is even and $\beta = -\gamma^{m/2n}$, then Q_ξ is of type (EE). One arrow $e = e^\diamondsuit : i \to i^\diamondsuit$ corresponds to $b_i^2 = \gamma \mathrm{Nm}_{F/k}(\xi^{-1})$, and $\sigma_{c\xi^{-1}}|_{L_i} = \mathrm{id}$. The factor $\mathfrak{h}^{-\epsilon\epsilon_e}(M_i, M_{i^\diamondsuit})$ is isomorphic to $\wedge^2_{L_{i^\diamondsuit}}(M_{i^\diamondsuit})$ or $\mathrm{Sym}^2_{L_{i^\diamondsuit}}(M_{i^\diamondsuit})$ (when $\sigma = \mathrm{id}_F$ it is the former if $\epsilon = 1$ and the latter if $\epsilon = -1$). Another arrow $e' = e'^\diamondsuit : i' \to i'^\diamondsuit$ corresponds to $b_i^2 = -\gamma \mathrm{Nm}_{F/k}(\xi^{-1})$, and $\sigma_{c\xi^{-1}}|_{L_i} \neq \mathrm{id}$. The factor $\mathfrak{h}^{-\epsilon\epsilon_e}(M_{i'}, M_{i'^\diamondsuit})$ is isomorphic to the space of L_i/k-Hermitian forms on $M_{i'}$.

4.2 The case $\sigma|_k \neq \mathrm{id}$

We have $k^\sigma = \mathbb{C}(\!(\tau^2)\!)$. Since $c \in F^\sigma$ hence $\gamma \in k^\sigma$, $\mathrm{val}_\tau(\gamma)$ is always even. We have $\gamma^{m/n} = \beta\sigma(\beta)$, which implies that $\mathrm{val}_\tau(\beta)$ is divisible by m/n. Hence L splits into m/n factors of k. The graph Q_ξ is a single cycle of length m/n. We distinguish two cases according to the parity of m/n.

4.2.1 *m/n is odd*

In this case Q_ξ is of type (VE).

The unique vertex $i = i^\diamondsuit$ corresponds to the unique $b_i \in k$ such that $b_i^2 = \gamma$ and $b_i^{m/n} = \beta$. Since $\sigma_c|_{L_i} \neq \mathrm{id}$, the factor $\mathbf{Aut}_k(M_i, \{\cdot,\cdot\}_i)$ in H is either an orthogonal group or a symplectic group over k.

The unique arrow $e : j \to j^\diamondsuit$ fixed by $(-)^\diamondsuit$ corresponds to the unique $b_j \in k$ such that $b_j^2 = \gamma \mathrm{Nm}_{F/k}(\xi^{-1})$ and $b_j^{m/n} = \beta$. Since $\sigma_{c\xi^{-1}}|_{L_j} \neq \mathrm{id}$, the factor $\mathfrak{h}^{-\epsilon\epsilon_e}(M_j, M_{j^\diamondsuit})$ in $\mathfrak{g}(\xi)$ is isomorphic to the space of k/k^σ-Hermitian forms on M_j.

4.2.2 *m/n is even*

In this case Q_ξ is of types (VV) or (EE) according to whether the equations

$$b\sigma(b) = \gamma, \quad b^{m/n} = \beta \tag{4.1}$$

have a common solution in $k^\times$.

• If the equations (4.1) have a common solution in $k^\times$, then Q_ξ is of type (VV). In this case, the two vertices i, i' fixed by $(-)^\diamondsuit$ correspond to two solutions $b_i, b_{i'} = -b_i$ to (4.1). The corresponding factors in H are unitary groups over k^σ.

• If the equations (4.1) do not have a common solution in $k^\times$, then Q_ξ is of type (EE). In this case, the two arrows $e : i \to i^\diamondsuit, e' : i' \to i'^\diamondsuit$ fixed by $(-)^\diamondsuit$ correspond to two solutions $b_i, b_{i'} = -b_i$ to the system of equations

$$b\sigma(b) = \gamma \mathrm{Nm}_{F/k}(\xi^{-1}), \quad b^{m/n} = \beta.$$

The corresponding factors in $\mathfrak{g}(\xi)$ are isomorphic to the space of k/k^{σ}-Hermitian forms on M_i and $M_{i'}$.

Acknowledgements The second author was supported by the Packard Foundation.

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