



Higher Signs for Coxeter Groups

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Abstract

We define and study cocycles on a Coxeter group in each degree generalizing the sign function. When the Coxeter group is a Weyl group, we explain how the degree three cocycle arises naturally from geometric representation theory.

Keywords Coxeter groups · Cocycles · Hecke category

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1 Introduction

Throughout these notes, we fix a Coxeter group (W, S) with length function $|\cdot| : W \rightarrow \mathbb{Z}_{\geq 0}$. The group homomorphism $\epsilon : W \rightarrow \{\pm 1\}$ given by $w \mapsto (-1)^{|w|}$ is called the *sign* for W . The sign of W can be viewed as a 1-cocycle of W valued in the group $\{\pm 1\} \cong \mathbb{F}_2$. In these notes, for every positive integer n , we define and study a canonical n -cocycle of W valued in \mathbb{Z} or \mathbb{F}_2 depending on the parity of n .

The main result is the following:

1.1 Theorem *Let (W, S) be a Coxeter group. For any even (resp. odd) integer $n \geq 1$, there is a unique n -cocycle $\epsilon_n^W \in Z^n(W, \mathbb{Z})$ (resp. $\epsilon_n^W \in Z^n(W, \mathbb{F}_2)$) satisfying two conditions*

- (1) *If $n \geq 2$ and $(x_1, x_2, \dots, x_n) \in W^n$ satisfy $|x_i x_{i+1}| = |x_i| + |x_{i+1}|$ for some $1 \leq i \leq n-1$, then $\epsilon_n^W(x_1, \dots, x_n) = 0$.*
- (2) *For any $s \in S$, $\epsilon_n^W(s, s, \dots, s) = 1$.*

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We will construct a cocycle universal among those satisfying the first condition above (called *collapsing*) (see §2.9). We will give formulas for computing these cocycles in §3.

We will study ϵ_3^W in more detail and explain how ϵ_3^W shows up naturally in geometric representation theory when W is a finite Weyl group (see §4). It is natural to ask the following:

1.2 Question Do the higher cocycles ϵ_n^W ($n \geq 4$) show up in geometric representation theory (e.g., Hecke categories)?

1.3 Notation For a set X , $\mathbb{Z}[X]$ denotes the free abelian group with basis $\{[x] \mid x \in X\}$. For an integer n , we denote

$$\epsilon(n) = \begin{cases} +, & \text{if } n \text{ is even;} \\ -, & \text{if } n \text{ is odd.} \end{cases}$$

When W is a Weyl group, we adopt the following notation: Let G be a semisimple group over an algebraically closed field k with a Borel subgroup B and a maximal torus $T \subset B$ such that the corresponding Weyl group is W with simple reflections S given by B .

2 Universal Collapsing Cocycle

2.1 Definition Let A be a $\mathbb{Z}[W]$ -module and $n \geq 2$ be an integer. A cocycle $\zeta \in Z^n(W, A)$ is called *collapsing* if, for any $x_1, \dots, x_n \in W$, whenever $|x_i x_{i+1}| = |x_i| + |x_{i+1}|$ for some $1 \leq i \leq n - 1$, we have $\zeta(x_1, \dots, x_n) = 0$.

2.2 Example (Tits section) Suppose (W, S) is a Weyl group (see §1.3 for notation). Tits defines a section to $\pi : N_G(T) \rightarrow W$ as follows: For any $s \in S$, let $\varphi_s : \text{SL}_2 \rightarrow G$ be the unique homomorphism that restricts to $\alpha^\vee : \mathbb{G}_m \rightarrow T$ on the diagonal torus \mathbb{G}_m of SL_2 . Let $\tau(s) = \varphi_s\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$. For $w \in W$ with reduced expression $w = s_1 \cdots s_m$, Tits proved that $\tau(w) := \tau(s_1) \cdots \tau(s_m)$ is independent of the reduced expression. The map $\tau : w \mapsto \tau(w)$ defines a section to π . The map $\Delta : W^2 \ni (x, y) \mapsto \tau(x)\tau(y)\tau(xy)^{-1} \in T[2]$ defines a collapsing 2-cocycle on W valued in the two-torsion points $T[2]$ of T .

The goal of this section is to describe the universal object among pairs (A, ζ) where A is a $\mathbb{Z}[W]$ -module and ζ is a collapsing n -cocycle of W valued in A .

2.3 Walls, Chambers, etc. We recall the geometric realization of a Coxeter group (W, S) following [2, Ch. V, §4].

Let $E = \mathbb{R}^S$ equipped with basis $\{e_s\}_{s \in S}$ and let E^* be its dual. There is a faithful representation of W on E^* such that $s \in S$ acts by a reflection. Let $H_s \subset E^*$ be the

hyperplane fixed by s (equivalently H_s is the kernel of e_s). A *wall* is a hyperplane in E^* of the form $w \cdot H_s$ for some $w \in W, s \in S$. Let \mathcal{H} be the set of walls; it is in bijection with reflections in W (the W -conjugates of S).

For $H \in \mathcal{H}$, $E^* - H$ has two connected components called *reflecting half-spaces*. Let \mathcal{D} be the set of reflecting half-spaces. Let $C_0 \subset E^*$ be an open cone defined by $C_0 = \{x \in E^* \mid \langle x, e_s \rangle > 0, \forall s \in S\}$. The W -translates of C_0 are called *chambers*. Let \mathcal{C} be the set of chambers.

There is an involution $\sigma : \mathcal{D} \rightarrow \mathcal{D}$ sending D to $E^* - \overline{D}$. For each $H \in \mathcal{H}$, let $\{D_H^+, D_H^-\}$ be the two connected components of $E^* - H$, where D_H^+ contains C_0 . For $H = H_s$, we denote $D_{H_s}^\pm$ by D_s^\pm .

2.4 For each $D \in \mathcal{D}$ and $C \in \mathcal{C}$, let

$$v_D(C) = \begin{cases} 1, & \text{if } C \subset D; \\ 0, & \text{if } C \subset \sigma(D). \end{cases}$$

The free abelian group $\mathbb{Z}[\mathcal{D}]$ carries an action of the involution σ . Let $\mathbb{Z}[\mathcal{D}]^+$ (resp. $\mathbb{Z}[\mathcal{D}]^-$) be the subgroup of $\mathbb{Z}[\mathcal{D}]$ consisting of those elements such that $\sigma(a) = a$ (resp. $\sigma(a) = -a$). Then $\mathbb{Z}[\mathcal{D}]^\pm$ is the free abelian group with basis $[D_H^\pm] \pm [D_H^\mp]$ for $H \in \mathcal{H}$. Both $\mathbb{Z}[\mathcal{D}]^\pm$ are $\mathbb{Z}[W]$ -modules.

2.5 Definition Let $n \in \mathbb{Z}_{\geq 1}$. Let Z_n^W be the function

$$Z_n^W : W^n \rightarrow \mathbb{Z}[\mathcal{D}]$$

$$\underline{x} = (x_1, \dots, x_n) \mapsto \sum_{D \in \mathcal{D}} \left(\prod_{i=0}^{n-1} (v_D(C_i) - v_D(C_{i+1})) \right) [D]. \tag{2.1}$$

Here $C_i = x_1 \cdots x_i C_0$ for $1 \leq i \leq n$.

From the definition, we see that $[D]$ appears in $Z_n^W(\underline{x})$ if and only if the sequence of chambers (C_0, \dots, C_n) is *H-alternating* in the sense that C_i and C_{i+1} lie on different sides of H , for all $0 \leq i \leq n - 1$. The following alternative formula for Z_n^W is immediate from the definition:

2.6 Lemma For $(x_1, \dots, x_n) \in W^n$, let $C_i = x_1 \cdots x_i C_0$. Then

$$Z_n^W(x_1, \dots, x_n) = \sum_{H \in \mathcal{H}; (C_0, \dots, C_n) \text{ is } H\text{-alternating}} (-1)^{\lfloor n/2 \rfloor} [D_H^+] + (-1)^{\lfloor (n+1)/2 \rfloor} [D_H^-]. \tag{2.2}$$

2.7 Proposition The function Z_n^W is an n -cocycle valued in $\mathbb{Z}[\mathcal{D}]^{\epsilon(n)}$, i.e., $Z_n^W \in Z^n(W, \mathbb{Z}[\mathcal{D}]^{\epsilon(n)})$.

Proof By (2.2), $Z_n(x_1, \dots, x_n)$ is a linear combination of $[D_H^+] + [D_H^-]$ if n is even (so that $[n/2] = [(n + 1)/2]$) and otherwise a linear combination of $[D_H^+] - [D_H^-]$. This shows that Z_n^W takes values in $\mathbb{Z}[\mathcal{D}]^{\epsilon(n)}$.

Now we show Z_n^W is a cocycle. Write Z_n^W simply as Z_n . Let $x_1, \dots, x_n, x_{n+1} \in W$. Then

$$\begin{aligned} \delta Z_n(x_1, \dots, x_{n+1}) &= x_1 Z_n(x_2, \dots, x_{n+1}) \\ &\quad + \sum_{j=1}^n (-1)^j Z_n(x_1, \dots, x_j x_{j+1}, \dots, x_{n+1}) \\ &\quad + (-1)^{n+1} Z_n(x_1, \dots, x_n). \end{aligned}$$

Fix $D \in \mathcal{D}$, let $h_j = v_D(C_j)$. For $1 \leq j \leq n$, the coefficient of $[D]$ in $Z_n(x_1, \dots, x_j x_{j+1}, \dots, x_{n+1})$ is $(h_0 - h_1) \cdots (h_{j-1} - h_{j+1}) \cdots (h_n - h_{n+1})$. The coefficient of $[D]$ in the last term $Z_n(x_1, \dots, x_n)$ is by definition $(h_0 - h_1) \cdots (h_{n-1} - h_n)$. The coefficient of $[D]$ in $x_1 Z_n(x_2, \dots, x_{n+1})$ is the same as the coefficient of $[x_1^{-1} D] =: [D']$ in $Z_n(x_2, \dots, x_{n+1})$, which is the product of $v_{D'}(C_0) - v_{D'}(x_2 C_0)$, $v_{D'}(x_2 C_0) - v_{D'}(x_2 x_3 C_0)$, \dots , $v_{D'}(x_2 \cdots x_n C_0) - v_{D'}(x_2 \cdots x_{n+1} C_0)$. Since $v_{D'}(x_2 \cdots x_i C_0) = v_D(x_1 \cdots x_i C_0) = v_D(C_i) = h_i$, we see that the coefficient of $[D]$ in $x_1 Z_n(x_2, \dots, x_{n+1})$ is $(h_1 - h_2) \cdots (h_n - h_{n+1})$. Therefore, the coefficient of $[D]$ in $\delta Z_n(x_1, \dots, x_{n+1})$ is

$$\begin{aligned} (h_1 - h_2) \cdots (h_n - h_{n+1}) &+ \sum_{j=1}^n (-1)^j (h_0 - h_1) \cdots (h_{j-1} - h_{j+1}) \cdots (h_n - h_{n+1}) \\ &+ (-1)^{n+1} (h_0 - h_1) \cdots (h_{n-1} - h_n). \end{aligned} \tag{2.3}$$

Write $d_j = h_j - h_{j+1}$, then $h_{j-1} - h_{j+1} = d_{j-1} + d_j$. Expanding the RHS into the d_j 's, we get each monomial $d_0 d_1 \cdots \widehat{d_j} \cdots d_n$ appearing twice with opposite signs, for all $0 \leq j \leq n$. Therefore, (2.3) is zero, hence $\delta Z_n = 0$. \square

2.8 Lemma *The cocycle Z_n^W satisfies the following two properties:*

- (1) *If $n \geq 2$ then Z_n^W is collapsing.*
- (2) *For any $s \in S$, $Z_n^W(s, s, \dots, s) = (-1)^{[n/2]} [D_s^+] + (-1)^{[(n+1)/2]} [D_s^-]$.*

Proof

- (1) If $|x_i x_{i+1}| = |x_i| + |x_{i+1}|$, then for any $H \in \mathcal{H}$, the three chambers $C_{i-1} = x_1 \cdots x_{i-1} C_0$, C_i and C_{i+1} either lie on the same side of H , or H separates C_{i-1} with C_i and C_{i+1} , or H separates C_{i-1}, C_i with C_{i+1} . In any case H is not (C_{i-1}, C_i, C_{i+1}) -alternating. Therefore $Z_n^W(x_1, \dots, x_n) = 0$ in this case.
- (2) For $s \in S$, H_s is the only wall separating C_0 and sC_0 , so only $[D_s^\pm]$ appear in $Z_n^W(s, s, \dots, s)$. The coefficients follow from (2.2). \square

The main result of this section is that Z_n^W is the universal collapsing n -cocycle on W .

2.9 Theorem *Let $n \geq 2$ be an integer. Let A be a $\mathbb{Z}[W]$ -module and $\zeta \in Z^n(W, A)$ be a collapsing cocycle. Then there exists a unique W -equivariant homomorphism $a : \mathbb{Z}[D]^{e(n)} \rightarrow A$ such that $\zeta = a(Z_n^W)$.*

To prove the theorem, we need two lemmas:

2.10 Lemma *Suppose $n \geq 2$ and $\zeta \in Z^n(W, A)$ is collapsing. For $s \in S$ let $a_s := \zeta(s, s, \dots, s) \in A$. Let $s, s' \in S$ and let $w \in W$ be such that $ws w^{-1} = s'$, then*

$$wa_s = \begin{cases} a_{s'}, & \text{if } wD_s^+ = D_{s'}^+; \\ (-1)^n a_{s'}, & \text{if } wD_s^+ = D_{s'}^-. \end{cases}$$

Proof We have $ws = s'w$. Consider the case $wD_s^+ = D_{s'}^+$; then wC_0 and C_0 lie on the same side of $H_{s'}$, which implies $|ws| = |s'w| > |w|$.

Since $|ws| = |w| + |s|$, the cocycle condition $(\delta\zeta)(w, s, \dots, s) = 0$ together with the collapsing of ζ implies that in the expansion of $(\delta\zeta)(w, s, \dots, s)$, the only nonzero terms are $w\zeta(s, \dots, s) = wa_s$ and $-\zeta(ws, s, \dots, s)$ (here we use $n \geq 2$). Therefore, $wa_s = \zeta(ws, s, \dots, s) = \zeta(s'w, s, \dots, s)$. Then we apply the cocycle condition $(\delta\zeta)(s', w, s, \dots, s) = 0$ to get $\zeta(s'w, s, \dots, s) = \zeta(s', ws, s, \dots, s) = \zeta(s', s'w, s, \dots, s)$. Then we apply $(\delta\zeta)(s', s', w, s, \dots, s) = 0$, etc. In this way we get

$$wa_s = \zeta(s'w, s, \dots, s) = \zeta(s', s'w, s, \dots, s) = \dots = \zeta(s', \dots, s'w).$$

Finally, we apply the cocycle condition $(\delta\zeta)(s', \dots, s', w) = 0$ to conclude $\zeta(s', \dots, s'w) = \zeta(s', \dots, s') = a_{s'}$. Combined with the above equality we get $wa_s = a_{s'}$.

When $wD_s^+ = D_{s'}^-$, we write $w = w's$; then $w's w^{-1} = s'$ and $w'D_s^+ = wD_s^- = D_{s'}^+$. From the case already proven, we get $w'a_s = a_{s'}$. It remains to show that $sa_s = (-1)^n a_s$ for then $wa_s = w'sa_s = (-1)^n w'a_s = (-1)^n a_{s'}$. For this we apply $(\delta\zeta)(s, s, \dots, s) = 0$ and using collapsing we see $(\delta\zeta)(s, s, \dots, s) = s\zeta(s, \dots, s) + (-1)^{n+1}\zeta(s, \dots, s) = 0$; hence $sa_s = (-1)^n a_s$ holds. □

2.11 Lemma *Let $\xi \in Z^n(W, A)$ be a collapsing n -cocycle if $n \geq 2$ or satisfying $\xi(1) = 0$ if $n = 1$. If $\xi(s, \dots, s) = 0$ for all $s \in S$, then $\xi = 0$.*

Proof We prove $\xi(\underline{x})$ is identically zero by induction on the total length $L(\underline{x}) := \sum_{i=1}^n |x_i|$.

If $L(\underline{x}) < n$, then some $x_i = 1$ and hence $\xi(\underline{x}) = 0$ since it is collapsing. If $L(\underline{x}) = n$, then either some $|x_i| \geq 2$, or all x_i are simple reflections. In the latter case, if all x_i are the same simple reflection, then $\xi(\underline{x}) = 0$ by assumption; otherwise $\xi(\underline{x}) = 0$ since it is collapsing.

Now assume $L(\underline{x}) \geq n + 1$, and assume $\xi(\underline{x}') = 0$ whenever $L(\underline{x}') < L(\underline{x})$. Let $\mu(\underline{x}) = \min\{1 \leq i \leq n; |x_i| \geq 2\}$. We prove $\xi(\underline{x}) = 0$ by downward induction on $\mu(\underline{x})$.

If $\mu(\underline{x}) = n$, then $x_1, \dots, x_{n-1} \in S \cup \{1\}$. Since ξ is collapsing, $\xi(\underline{x}) = 0$ unless all x_i are equal to the same $s \in S$ for $1 \leq i \leq n - 1$. In the case $x_1 = \dots = x_{n-1} = s$, if $|sx_n| > |x_n|$, we still have $\xi(\underline{x}) = 0$ by collapsing. If $|sx_n| < |x_n|$, we write $x_n = sx'_n$, then the cocycle condition $(\delta\xi)(s, s, \dots, s, s, x'_n) = 0$ together with the collapsing condition implies $\xi(\underline{x}) = \xi(s, \dots, s) = 0$ by assumption.

Now assume $\xi(\underline{x}') = 0$ for those \underline{x}' such that either $L(\underline{x}') < L(\underline{x})$ or $L(\underline{x}') = L(\underline{x})$ and $\mu(\underline{x}') > \mu(\underline{x})$. Let $\mu = \mu(\underline{x})$. By the same argument as in the previous paragraph, $\xi(\underline{x}) = 0$ unless $x_1 = \dots = x_{\mu-1} = s \in S$, and $x_\mu = sx'_\mu$ where $|x_\mu| > |x'_\mu|$. Now apply the cocycle condition $(\delta\xi)(s, s, \dots, s, x'_\mu, x_{\mu+1}, \dots, x_n) = 0$ (the first μ entries are s). Expanding, we get the terms

$$s\xi(s, s, \dots, s, x'_\mu, x_{\mu+1}, \dots, x_n) \quad (\text{first } \mu - 1 \text{ entries are } s); \tag{2.4}$$

$$\xi(s, \dots, 1, \dots, s, x'_\mu, x_{\mu+1}, \dots, x_n) \quad (\text{there are } \mu - 1 \text{ such terms}); \tag{2.5}$$

$$\xi(s, \dots, s, sx'_\mu, \dots, x_n) = \xi(\underline{x}); \tag{2.6}$$

$$\xi(s, \dots, s, x'_\mu x_{\mu+1}, x_{\mu+2}, \dots, x_n) \quad (\text{first } \mu \text{ entries are } s); \tag{2.7}$$

$$\xi(s, \dots, s, x'_\mu, \dots, x_i x_{i+1}, \dots, x_n) \quad (\text{first } \mu \text{ entries are } s, \mu < i < n), \tag{2.8}$$

$$\xi(s, \dots, s, x'_\mu, x_{\mu+1}, \dots, x_{n-1}). \tag{2.9}$$

Now (2.4) has total length less than $L(\underline{x})$ and hence vanishes by inductive hypothesis; (2.5) is zero by collapsing, (2.7), (2.8) and (2.9) have total length $\leq L(\underline{x})$ and the μ -invariant strictly larger than $\mu(\underline{x})$, hence they also vanish by inductive hypothesis. Since the sum of all terms is zero, the only remaining term $\xi(\underline{x})$ must also be zero. This finishes the induction step. \square

2.12 Proof of Theorem in §2.9 For $s \in S$ let $a_s := \zeta(s, s, \dots, s) \in A$. Define a linear map $a : \mathbb{Z}[\mathcal{D}]^{\epsilon(n)} \rightarrow A$ by sending $\mathbf{D}_H := (-1)^{\lfloor n/2 \rfloor} [D_H^+] + (-1)^{\lfloor (n+1)/2 \rfloor} [D_H^-]$ to wa_s if $w \in W$ and $s \in S$ are such that $wD_s^+ = D_H^+$ (such (w, s) always exists). To show a is well defined we need to show that if (w', s') also satisfies $w'D_{s'}^+ = D_H^+$, then $wa_s = w'a_{s'}$. Now $v = (w')^{-1}w$ satisfies $vD_{s'}^+ = D_{s'}^+$. By Lemma in §2.10, $va_{s'} = a_{s'}$; hence $wa_s = w'a_{s'}$. Therefore, a is well defined.

To see a is W -equivariant, we need to show that for any $H \in \mathcal{H}$ and $w \in W$, $a(w\mathbf{D}_H) = wa(\mathbf{D}_H)$. Let $(w', s) \in W \times S$ be such that $w'D_s^+ = D_H^+$. Then $a(\mathbf{D}_H) = w'a_s$ by definition. Now wD_H^+ is either D_{wH}^+ or D_{wH}^- . If $wD_H^+ = D_{wH}^+$, then $w\mathbf{D}_H = \mathbf{D}_{wH}$; hence $a(w\mathbf{D}_H) = a(\mathbf{D}_{wH})$. Since $D_{wH}^+ = ww'D_s^+$, we have $a(\mathbf{D}_{wH}) = ww'a_s$ by definition. Therefore, $a(w\mathbf{D}_H) = a(\mathbf{D}_{wH}) = ww'a_s = wa(\mathbf{D}_H)$ holds. If $wD_H^+ = D_{wH}^-$, then $w\mathbf{D}_H = (-1)^n \mathbf{D}_{wH}$; hence $a(w\mathbf{D}_H) = (-1)^n a(\mathbf{D}_{wH})$. Since $D_{wH}^- = wD_H^+ = ww'D_s^+$, we have $D_{wH}^+ = ww'sD_s^+$; hence $a(\mathbf{D}_{wH}) = ww'sa_s$ by definition. By Lemma in

§2.10, $sa_s = (-1)^n a_s$; hence $a(\mathbf{D}_{wH}) = (-1)^n ww' a_s$. Combining these facts we get $a(w\mathbf{D}_H) = (-1)^n a(\mathbf{D}_{wH}) = ww' a_s = wa(\mathbf{D}_H)$. This verifies that a is W -equivariant.

Now let $\xi = \zeta - a \circ Z_n^W \in Z^n(W, A)$, which is again collapsing. From the construction of a and Lemma in §2.8, we have

$$\xi(s, s, \dots, s) = a_s - a(\mathbf{D}_s) = 0, \quad \forall s \in S. \tag{2.10}$$

By Lemma in §2.11, ξ is identically zero; hence $\zeta = a \circ Z_n^W$.

Finally, we prove the uniqueness of a . Suppose another W -equivariant map $a' : \mathbb{Z}[\mathcal{D}]^{\epsilon(n)} \rightarrow A$ satisfies $\zeta = a' \circ Z_n^W$; then $b = a - a'$ satisfies $b \circ Z_n^W = 0$. Now $b \circ Z_n^W(s, s, \dots, s) = b(\mathbf{D}_s) = 0$ for all $s \in S$. By W -equivariance, $b(\mathbf{D}_H) = 0$ for all $H \in \mathcal{H}$; hence $b = 0$. The proof is complete.

2.13 Remark The analogue of the collapsing condition for a 1-cocycle $\xi \in Z^1(W, A)$ is the normalization $\xi(1) = 0$. However, Z_1^W is not in general the universal normalized 1-cocycle for W .

2.14 Example When $n = 2$, the universal collapsing 2-cocycle Z_2^W defines an extension

$$1 \rightarrow \mathbb{Z}[\mathcal{H}] \rightarrow W^\# \rightarrow W \rightarrow 1. \tag{2.11}$$

Here $W^\# = \mathbb{Z}[\mathcal{H}] \times W$ as a set, with multiplication $(a, x)(b, y) = (a + xb + Z_2^W(x, y), xy)$, where $a, b \in \mathbb{Z}[\mathcal{H}] = \mathbb{Z}[\mathcal{D}]^+$, $x, y \in W$.

When W is a Weyl group, let $P_W \subset B_W$ be the pure braid group and the braid group associated with W . Let $\mathfrak{a} = \mathbb{X}_*(T)_\mathbb{C}$ and $\mathfrak{a}^{\text{rs}} \subset \mathfrak{a}$ be the complement of the root hyperplanes. Then $P_W \cong \pi_1(\mathfrak{a}^{\text{rs}}, *)$. Each root α gives a map $\mathfrak{a}^{\text{rs}} \rightarrow \mathbb{C}^\times$; hence a homomorphism $v_\alpha : P_W \rightarrow \pi_1(\mathbb{C}^\times) = \mathbb{Z}$. The homomorphism v_α only depends on the root hyperplane H_α ; hence they together define a homomorphism $v : P_W \rightarrow \mathbb{Z}[\mathcal{H}]$. It is easy to see using the characterization of Z_2^W that the extension (2.11) is the pushout of the extension $1 \rightarrow P_W \rightarrow B_W \rightarrow W \rightarrow 1$ via the homomorphism v .

2.15 Cup Product Equip $\mathbb{Z}[\mathcal{D}]$ with the (not necessarily unital) ring structure such that $[D] \cdot [D] = [D]$ for all $D \in \mathcal{D}$ and $[D] \cdot [D'] = 0$ if $D \neq D'$ (i.e., the basis elements $[D]$ are orthogonal idempotents). Then $\mathbb{Z}[\mathcal{D}]^+ \oplus \mathbb{Z}[\mathcal{D}]^-$ is a $\mathbb{Z}/2\mathbb{Z}$ -graded subalgebra of $\mathbb{Z}[\mathcal{D}]$. In particular, we have the multiplication map $\mu_n : (\mathbb{Z}[\mathcal{D}]^-)^{\otimes n} \rightarrow \mathbb{Z}[\mathcal{D}]^{\epsilon(n)}$.

From the definition of Z_n^W , the following proposition is easily verified:

2.16 Proposition *The cocycle Z_n^W is the image of the n -th cup power of Z_1^W under μ_n :*

$$Z^1(W, \mathbb{Z}[\mathcal{D}]^-)^{\otimes n} \xrightarrow{\cup} Z^n(W, (\mathbb{Z}[\mathcal{D}]^-)^{\otimes n}) \xrightarrow{\mu_n} Z^n(W, \mathbb{Z}[\mathcal{D}]^{\epsilon(n)}).$$

Here \cup means the cup product on the level of cochains, see [1, §7].

Proof Let $Z'_n = \mu_n((Z_1^W)^{\cup n})$. Then we have $Z'_n(x_1, \dots, x_n) = Z_1^W(x_1) \cdot x_1 Z_1^W(x_2) \cdots x_1 \cdots x_{n-1} Z_1^W(x_n)$ (where \cdot is the multiplication in $\mathbb{Z}[D]$). For $H \in \mathcal{H}$, $[D_H^\pm]$ appear in $\mu_n((Z_1^W)^{\cup n})$ if and only if $[D_H^\pm]$ appear in each term $x_1 \cdots x_{i-1} Z_1^W(x_i)$, $1 \leq i \leq n$, which happens if and only if (C_0, C_1, \dots, C_n) is H -alternating. Moreover, $[D_H^+]$ appears in $x_1 \cdots x_{i-1} Z_1^W(x_i)$ with coefficient $(-1)^{i-1}$; therefore it appears in Z'_n with coefficient $(-1)^{\lfloor n/2 \rfloor}$. Comparing with (2.2) we conclude that $Z'_n = Z_n^W$. \square

2.17 We have short exact sequences of $\mathbb{Z}[W]$ -modules

$$0 \rightarrow \mathbb{Z}[D]^+ \xrightarrow{i^+} \mathbb{Z}[D] \xrightarrow{p^-} \mathbb{Z}[D]^- \rightarrow 0 \tag{2.12}$$

$$0 \rightarrow \mathbb{Z}[D]^- \xrightarrow{i^-} \mathbb{Z}[D] \xrightarrow{p^+} \mathbb{Z}[D]^+ \rightarrow 0 \tag{2.13}$$

where i^\pm are the natural inclusions, and $p^\pm([D]) = [D] \pm [\sigma(D)]$ for all $D \in \mathcal{D}$.

2.18 Proposition For $n \geq 1$, the cohomology class $[-Z_{n+1}^W] \in H^{n+1}(W, \mathbb{Z}[D]^{\epsilon(n+1)})$ is the image of $[Z_n^W]$ under the connecting homomorphism

$$H^n(W, \mathbb{Z}[D]^{\epsilon(n)}) \rightarrow H^{n+1}(W, \mathbb{Z}[D]^{\epsilon(n+1)})$$

attached to (2.12) when n is odd and to (2.13) when n is even.

Proof Let \tilde{Z}_n be the lifting of Z_n^W to a function $W^n \rightarrow \mathbb{Z}[D]$ defined by $\tilde{Z}_n(x_1, \dots, x_n) = \sum_H (-1)^{\lfloor n/2 \rfloor} [D_H^+]$ where H runs over those walls such that (C_0, C_1, \dots, C_n) is H -alternating (as usual $C_i = x_1 \cdots x_i C_0$). To prove the proposition, it suffices to check that

$$\delta \tilde{Z}_n = -Z_{n+1}^W.$$

Expanding $\delta \tilde{Z}_n(x_1, \dots, x_{n+1})$ as we did for δZ_n^W in the proof of Proposition in §2.7. Note that all the terms except for the first one $x_1 \tilde{Z}_n(x_2, \dots, x_{n+1})$ only involve $[D_H^+]$ for various $H \in \mathcal{H}$. Let $h_i = v_{D_H^+}(C_i)$. The coefficient of $[D_H^+]$ in $\delta Z_n^W(x_1, \dots, x_{n+1}) - x_1 \tilde{Z}_n(x_2, \dots, x_{n+1})$ is the sum of all but the first term in (2.3). Since the sum in (2.3) is zero, the coefficient of $[D_H^+]$ in $\delta Z_n^W(x_1, \dots, x_{n+1}) - x_1 \tilde{Z}_n(x_2, \dots, x_{n+1})$ is $-(h_1 - h_2) \cdots (h_n - h_{n+1})$, which is nonzero if and only if (C_1, \dots, C_{n+1}) is H -alternating, in which case the coefficient is ± 1 . Since we know $\delta Z_n = 0$, $\delta \tilde{Z}_n$ must take values in $\mathbb{Z}[D]^{\epsilon(n+1)}$; therefore, if (C_1, \dots, C_{n+1}) is H -alternating, and that $[D_H^\pm]$ both appear with nonzero coefficients in $\delta \tilde{Z}_n(x_1, \dots, x_{n+1})$, $[D_H^-]$ must appear with nonzero coefficient in $x_1 \tilde{Z}_n(x_2, \dots, x_{n+1})$. This means $x_1^{-1} D_H^- = D_{H'}^+$ for some $H' = x_1^{-1} H$; hence $C_0 \subset x_1^{-1} D_{H'}^-$, or $C_1 \subset D_H^-$, i.e., $(C_0, C_1, \dots, C_{n+1})$ is H -alternating. In this case, the coefficient of $[D_H^+]$ in $\delta \tilde{Z}_n(x_1, \dots, x_{n+1})$ is $-(h_1 - h_2) \cdots (h_n - h_{n+1}) = -\prod_{i=0}^n (h_i - h_{i+1})$, which is the negative of the coefficient of $[D_H^+]$ in Z_{n+1}^W . This completes the proof. \square

3 Higher Signs

3.1 Definition

- (1) For $n \geq 2$ even, let $\chi_n : \mathbb{Z}[\mathcal{D}]^+ = \mathbb{Z}[\mathcal{H}] \rightarrow \mathbb{Z}$ be the linear map sending each basis element $[D_H^+] + [D_H^-]$ to $(-1)^{n/2}$ (where $H \in \mathcal{H}$).
- (2) For $n \geq 1$ odd, let $\chi_n : \mathbb{Z}[\mathcal{D}]^- \rightarrow \mathbb{F}_2$ be the linear map sending each $[D_H^+] - [D_H^-]$ to $1 \in \mathbb{F}_2$. (Of course all the χ_n are the same for n odd.)
- (3) For $n \geq 1$, define $\epsilon_n^W \in Z^n(W, \mathbb{Z})$ for n even and $\epsilon_n^W \in Z^n(W, \mathbb{F}_2)$ for n odd to be the image of Z_n^W under the W -invariant map χ_n .

Concretely, when n is even and $\underline{x} = (x_1, \dots, x_n) \in W^n$, $\epsilon_n(\underline{x})$ is the number of walls $H \in \mathcal{H}$ such that (C_0, C_1, \dots, C_n) is alternating with respect to H . Here $C_i = x_1 \cdots x_i C_0$ for $1 \leq i \leq n$. When n is odd, $\epsilon_n(\underline{x}) \in \mathbb{F}_2$ is the parity of the number of such walls.

3.2 Proof of Theorem in §1.1 Since ϵ_n^W is the homomorphic image of a collapsing cocycle, it is also collapsing. By Lemma (2) in §2.8 and the definition of χ_n , we have $\epsilon_n^W(s, \dots, s) = 1$. By Lemma in §2.11, these two properties (collapsing and values at (s, \dots, s)) also characterize ϵ_n^W as \mathbb{Z} -valued or \mathbb{F}_2 -valued cocycles.

3.3 Remark (Lusztig) One can define ϵ_n^W without using the notion of chambers and walls. Indeed, for $w \in W$ there is a well-defined set of reflections \mathbf{T}_w in W as in [2, Ch. IV, §1.4, Lemmas 1 and 2] (and $\#\mathbf{T}_w = |w|$). For $(x_1, \dots, x_n) \in W^n$, let $\mathbf{T}(x_1, \dots, x_n)$ be the intersection

$$\mathbf{T}(x_1, \dots, x_n) := \bigcap_{i=1}^n x_1 \cdots x_{i-1} \cdot \mathbf{T}_{x_i},$$

where the $i = 1$ term is \mathbf{T}_{x_1} . Then

$$\epsilon_n^W(x_1, \dots, x_n) = \begin{cases} \#\mathbf{T}(x_1, \dots, x_n), & \text{if } n \text{ is even;} \\ \#\mathbf{T}(x_1, \dots, x_n) \pmod 2, & \text{if } n \text{ is odd.} \end{cases}$$

To see this, we only need to observe that two chambers xC_0 and xyC_0 lie on different sides of the wall H_r given by some reflection $r \in W$ if and only if $r \in x\mathbf{T}_y$.

3.4 Remark One can refine the cocycles ϵ_n^W slightly as follows: Let \underline{S} be the set of W -orbits on \mathcal{H} ; the map $S \rightarrow \underline{S}$ sending s to the W -orbit of H_s is surjective. For n even, we have the W -invariant map $\tilde{\chi}_n : \mathbb{Z}[\mathcal{D}]^+ \rightarrow \mathbb{Z}[\underline{S}]$ sending $[D_H^+] + [D_H^-]$ to $(-1)^{n/2}[\underline{H}]$, where \underline{H} is the W -orbit of $H \in \mathcal{H}$. For n odd, we also have the W -invariant map $\tilde{\chi}_n : \mathbb{Z}[\mathcal{D}]^- \rightarrow \mathbb{F}_2[\underline{S}]$ sending $[D_H^+] - [D_H^-]$ to $[\underline{H}]$. The image of Z_n^W under $\tilde{\chi}_n$ defines an n -cocycle $\tilde{\epsilon}_n^W \in Z^n(W, \mathbb{Z}[\underline{S}])$ if n is even and $\tilde{\epsilon}_n^W \in Z^n(W, \mathbb{F}_2[\underline{S}])$ if n is odd.

3.5 Proposition *If $n \geq 1$ is odd, then the cohomology class $[-\epsilon_{n+1}^W] \in H^{n+1}(W, \mathbb{Z})$ is the image of $[\epsilon_n^W]$ under the Bockstein homomorphism*

$$\beta : H^n(W, \mathbb{F}_2) \rightarrow H^{n+1}(W, \mathbb{Z}).$$

Proof Let $\theta_{n+1} : \mathbb{Z}[D] \rightarrow \mathbb{Z}$ be the linear map sending all $[D]$ to $(-1)^{(n+1)/2}$. We have a commutative diagram of $\mathbb{Z}[W]$ -modules where the rows are exact sequences

$$\begin{CD} 0 @>>> \mathbb{Z}[D]^+ @>{i^+}>> \mathbb{Z}[D] @>{p^-}>> \mathbb{Z}[D]^- @>>> 0 \\ @. @VV\chi_{n+1}V @VV\theta_{n+1}V @VV\chi_nV @. \\ 0 @>>> \mathbb{Z} @>{2}>> \mathbb{Z} @>>> \mathbb{Z}/2\mathbb{Z} @>>> 0 \end{CD}$$

This gives a commutative diagram of connecting homomorphisms

$$\begin{CD} H^n(W, \mathbb{Z}[D]^-) @>>> H^{n+1}(W, \mathbb{Z}[D]^+) \\ @VV\chi_nV @VV\chi_{n+1}V \\ H^n(W, \mathbb{F}_2) @>{\beta}>> H^{n+1}(W, \mathbb{Z}) \end{CD}$$

The equality $[-\epsilon_{n+1}^W] = \beta[\epsilon_n^W]$ then follows from Proposition in §2.18. □

The cocycles ϵ_1^W and ϵ_2^W are easy to compute.

3.6 Proposition

- (1) *The cocycle ϵ_1^W is the sign homomorphism $W \ni x \mapsto |x| \pmod 2$.*
- (2) *For $x, y \in W$, $\epsilon_2^W(x, y) = \frac{1}{2}(|x| + |y| - |xy|) \in \mathbb{Z}$.*

The following properties of ϵ_n^W follow directly from the uniqueness in Theorem in §1.1:

3.7 Lemma

- (1) *ϵ_n^W is invariant under any automorphism of (W, S) .*
- (2) *For any $x_1, \dots, x_n \in W$, we have $\epsilon_n^W(x_1, \dots, x_n) = \epsilon_n^W(x_n^{-1}, \dots, x_1^{-1})$.*
- (3) *Suppose $(W', S') \subset (W, S)$ is a standard parabolic subgroup, then $\epsilon_n^W|_{W'} = \epsilon_n^{W'}$.*
- (4) *Let (W_1, S_1) and (W_2, S_2) be Coxeter groups and $(W, S) = (W_1 \times W_2, S_1 \amalg S_2)$.*

Let $p_i : W \rightarrow W_i$ be the projection for $i = 1, 2$. Then $\epsilon_n^W = p_1^ \epsilon_n^{W_1} + p_2^* \epsilon_n^{W_2}$.*

3.8 Lemma

- (1) *Suppose $W = \mathbb{Z}/2\mathbb{Z}$, then the cohomology class $[\epsilon_n^W]$ is the nonzero element in $H^n(W, \mathbb{Z}) = \mathbb{F}_2$ if n is even or in $H^n(W, \mathbb{F}_2) = \mathbb{F}_2$ if n is odd.*
- (2) *If $w \in W$ is conjugate to a product of m commuting simple reflections (so w is itself an involution), then the class $[\epsilon_n^W|_{\langle w \rangle}] = m \pmod 2$ as an element in $H^n(\langle w \rangle, \mathbb{Z}) = \mathbb{F}_2$ if n is even or $H^n(W, \mathbb{F}_2) = \mathbb{F}_2$ if n is odd.*

3.9 Remark The maps χ_n are not compatible with the ring structure on $\mathbb{Z}[D]$ in general (besides the case $W = \mathbb{Z}/2\mathbb{Z}$). Therefore, in general there is no reason to expect that ϵ_n^W be related to cup powers of the sign ϵ_1^W in any obvious way.

In the rest of the section we collect some useful formulas for computing ϵ_n^W and more generally collapsing cocycles.

3.10 Lemma *Let $x_1, \dots, x_n \in W$ and let $\ell \geq 1$ be odd. Suppose there is a partial product of ℓ consecutive entries $x_j \cdots x_{j+\ell-1} = 1$, then $\zeta(x_1, \dots, x_n) = 0$ for any collapsing n -cocycle $\zeta \in Z^n(W, A)$.*

Proof By Theorem in §2.9 it suffices to treat the case $\zeta = Z_n^W$. If $x_j \cdots x_{j+\ell-1} = 1$, consider the sequence of $\ell + 1$ chambers $(C_{j-1}, C_j, \dots, C_{j+\ell-1})$. The condition $x_j \cdots x_{j+\ell-1} = 1$ implies $C_{j+\ell-1} = C_{j-1}$. For any $H \in \mathcal{H}$, this even number of chambers cannot be H -alternating if the first one is the same as the last one. Therefore $Z_n^W(x_1, \dots, x_n) = 0$ by (2.2). □

3.11 Lemma *Let $x_1, \dots, x_n \in W$ and let $\ell \geq 2$ be even. If $x_1 \cdots x_\ell = 1$, then*

$$\epsilon_n^W(x_1, x_2, x_3, \dots, x_n) \equiv \epsilon_{n-1}^W(x_2, x_3, \dots, x_n) \pmod{2}. \tag{3.1}$$

Similarly, if $x_{n-\ell+1} \cdots x_n = 1$, then

$$\epsilon_n^W(x_1, x_2, \dots, x_{n-1}, x_n) \equiv \epsilon_{n-1}^W(x_1, x_2, \dots, x_{n-1}) \pmod{2}. \tag{3.2}$$

Proof We prove (3.1). Since $x_1 \cdots x_\ell = 1$, the corresponding chain of chambers (C_0, C_1, \dots, C_n) satisfies $C_\ell = C_0$. For any $H \in \mathcal{H}$, (C_0, C_1, \dots, C_n) is H -alternating if and only if (C_1, \dots, C_n) is H -alternating, if and only if $(C_0, x_2 C_0, \dots, x_2 \cdots x_n C_0)$ is $x_2 H$ -alternating. This implies (3.1). The proof of (3.2) is similar. □

3.12 Lemma *Let $x_1, \dots, x_n \in W$ and $1 \leq i \leq n$. Let $x_i = s_1 \cdots s_m$ be a reduced expression. Then for any collapsing n -cocycle $\zeta \in Z^n(W, A)$, we have*

$$\zeta(x_1, \dots, x_n) = \sum_{j=1}^m \zeta(x_1, \dots, x_{i-1} s_1 \cdots s_{j-1}, s_j, s_{j+1} \cdots s_m x_{i+1}, \dots, x_n).$$

Here, if $i = 1$, the first term should be s_j ; if $i = n$, the last term should be s_j .

Proof We prove this by induction on m . For $m = 0$ the statement is vacuous. Now assume the statement is proved for $m - 1$. Applying the cocycle condition $(\delta\zeta)(x_1, \dots, x_{i-1}, s_1, s_2 \cdots s_m, x_{i+1}, \dots, x_n) = 0$ we get a sum of terms most of which contain two consecutive entries s_1 and $s_2 \cdots s_m$, which vanishes since ζ is collapsing. The remaining terms are

$$\begin{aligned} &\zeta(x_1, \dots, x_{i-1}, s_1, s_2 \cdots s_m, x_{i+1}, \dots, x_n) \quad (\text{if } i = 1, \text{ the first entry is } s_2 \cdots s_m); \\ &\zeta(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n); \\ &\zeta(x_1, \dots, x_{i-1}, s_1, s_2 \cdots s_m, x_{i+1}, x_{i+2}, \dots, x_n) \quad (\text{if } i = n, \text{ the last entry is } s_1). \end{aligned}$$

The sum of the three terms above is zero. We then apply inductive hypothesis to the first term above to conclude. □

4 Geometric Origin of e_3^W

When W is a Weyl group, the cocycle e_3^W naturally shows up in the study of convolutions in certain Hecke categories. In fact this was our motivation to study collapsing cocycles for general Coxeter groups. Below we recall the geometric context.

4.1 Monodromic Hecke Category In [3] we studied monodromic Hecke categories for reductive groups. Let G be a connected reductive group over an algebraically closed field k , B a Borel subgroup of G with unipotent radical U and $T \subset B$ a maximal torus. Let \mathcal{L} and \mathcal{L}' be two rank one character sheaves on T . In [3] we studied the derived category ${}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$ of complexes on $U \backslash G / U$ equivariant under the left and right translation by T with respect to \mathcal{L} and \mathcal{L}' , respectively. These categories appear naturally in the study of representations of finite groups of Lie type.

Now fix a W -orbit \mathfrak{o} of rank one character sheaves on T . The direct sum $\mathcal{D}_{\mathfrak{o}} := \bigoplus_{\mathcal{L}, \mathcal{L}' \in \mathfrak{o}} ({}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}})$ carries a monoidal structure under convolution

$$\star : {}_{\mathcal{L}''}\mathcal{D}_{\mathcal{L}'} \times {}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}} \rightarrow {}_{\mathcal{L}''}\mathcal{D}_{\mathcal{L}}. \tag{4.1}$$

In [3, §4], we gave a decomposition of $\mathcal{D}_{\mathfrak{o}}$ into blocks. For $\mathcal{L}, \mathcal{L}' \in \mathfrak{o}$, let ${}_{\mathcal{L}'}W_{\mathcal{L}} = \{w \in W \mid w\mathcal{L} = \mathcal{L}'\}$. Let $W_{\mathcal{L}}^{\circ}$ be the subgroup of W generated by reflections with respect to the roots α such that $\alpha^{\vee, *}\mathcal{L}$ is the trivial local system on \mathbb{G}_m . Then the blocks in ${}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$ are indexed by the set ${}_{\mathcal{L}'}\underline{W}_{\mathcal{L}} = {}_{\mathcal{L}'}W_{\mathcal{L}} / W_{\mathcal{L}}^{\circ} = W_{\mathcal{L}'}^{\circ} \backslash {}_{\mathcal{L}'}W_{\mathcal{L}}$ (see [3, §4.1]). Moreover, each coset $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$ contains a unique element $w^{\beta} \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$ of minimal length ([3, Lemma 4.2]).

In [3, §4.4] we introduced a groupoid Ξ whose objects set is \mathfrak{o} and its morphism set $\text{Hom}_{\Xi}(\mathcal{L}, \mathcal{L}')$ is ${}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$, the set of blocks in ${}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$. The multiplication on the groupoid Ξ is compatible with convolution on $\mathcal{D}_{\mathfrak{o}}$. In [3, §5.8] we have defined a 3-cocycle σ on the groupoid Ξ that captures certain signs that appear in the associativity constraint of the convolution, which we now recall.

Let $\mathcal{L}, \mathcal{L}', \mathcal{L}'' \in \mathfrak{o}$, and let $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$ and $\gamma \in {}_{\mathcal{L}''}\underline{W}_{\mathcal{L}'}$ be blocks. Let $\dot{w}^{\beta}, \dot{w}^{\gamma}$ be liftings of w^{β}, w^{γ} to $N_G(T)$. Let $\Delta(\dot{w}^{\beta})_{\mathcal{L}}$ be the standard perverse sheaf obtained from a shifted local system on $B\dot{w}^{\beta}B$ by extension by zero (equipped with a trivialization of its stalk at \dot{w}^{β} , see [3, §2.11]). By [3, §5.5], there is a *canonical* isomorphism in ${}_{\mathcal{L}''}\mathcal{D}_{\mathcal{L}}$

$$\text{can}_{\dot{w}^\gamma, \dot{w}^\beta} : \Delta(\dot{w}^\gamma)_{\mathcal{L}'} \star \Delta(\dot{w}^\beta)_{\mathcal{L}} \cong \Delta(\dot{w}^\gamma \dot{w}^\beta)_{\mathcal{L}}.$$

For another $\mathcal{L}''' \in \mathfrak{o}$, and block $\delta \in \mathcal{L}''' \underline{W}_{\mathcal{L}''}$ consider the following diagram of isomorphisms:

$$\begin{array}{ccc}
 & \Delta(\dot{w}^\delta \dot{w}^\gamma)_{\mathcal{L}'} \star \Delta(\dot{w}^\beta)_{\mathcal{L}} & \\
 \text{can}_{\dot{w}^\delta, \dot{w}^\gamma} \star \text{id} \nearrow & & \searrow \text{can}_{\dot{w}^\delta \dot{w}^\gamma, \dot{w}^\beta} \\
 \Delta(\dot{w}^\delta)_{\mathcal{L}''} \star \Delta(\dot{w}^\gamma)_{\mathcal{L}'} \star \Delta(\dot{w}^\beta)_{\mathcal{L}} & & \Delta(\dot{w}^\delta \dot{w}^\gamma \dot{w}^\beta)_{\mathcal{L}} \quad (4.2) \\
 \text{id} \star \text{can}_{\dot{w}^\gamma, \dot{w}^\beta} \searrow & & \nearrow \text{can}_{\dot{w}^\delta, \dot{w}^\gamma \dot{w}^\beta} \\
 & \Delta(\dot{w}^\delta)_{\mathcal{L}''} \star \Delta(\dot{w}^\gamma \dot{w}^\beta)_{\mathcal{L}} &
 \end{array}$$

However, this diagram is not necessarily commutative. Let $\sigma(\dot{w}^\delta, \dot{w}^\gamma, \dot{w}^\beta)_{\mathcal{L}} \in \overline{\mathbb{Q}}_{\mathcal{L}}^{\times}$ be the ratio of the upper composition over the lower. It is easy to see that $\sigma(\dot{w}^\delta, \dot{w}^\gamma, \dot{w}^\beta)_{\mathcal{L}}$ is independent of the choices of liftings of w^β, w^γ and w^δ ; therefore, we may denote it by $\sigma(\delta, \gamma, \beta)_{\mathcal{L}}$. From the pentagon axiom for the associativity of the convolution functor, we get that σ is a 3-cocycle on the groupoid Ξ , i.e., $\sigma \in Z^3(\Xi, \overline{\mathbb{Q}}_{\mathcal{L}}^{\times})$.

4.2 Our next goal is to relate the cocycle $\sigma \in Z^3(\Xi, \overline{\mathbb{Q}}_{\mathcal{L}}^{\times})$ to the cocycle $e_3^W \in Z^3(W, \mathbb{F}_2)$. By definition, there is a map of groupoids

$$\pi : \Xi \rightarrow [\text{pt}/W] \tag{4.3}$$

sending a morphism $\beta \in \text{Hom}_{\Xi}(\mathcal{L}, \mathcal{L}') = \mathcal{L}' \underline{W}_{\mathcal{L}}$ to $w^\beta \in W$.

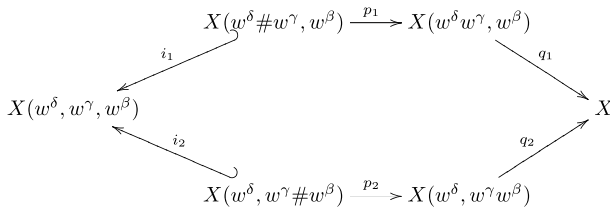
We generalize the notion of a collapsing cocycle to the groupoid Ξ . Let $n \geq 2$ and A be an abelian group. A cocycle $\zeta \in Z^n(\Xi, A)$ is collapsing if, whenever β_1, \dots, β_n are composable morphisms in Ξ (so that $\beta_1 \cdots \beta_n$ is defined), and $|w^{\beta_i} w^{\beta_{i+1}}| = |w^{\beta_i}| + |w^{\beta_{i+1}}|$ for some $1 \leq i \leq n - 1$, we have $\zeta(\beta_1, \dots, \beta_n) = 0$. We have the following generalization of Lemma in §2.11:

4.3 Lemma *Let $n \geq 2$ and $\zeta \in Z^n(\Xi, A)$ be a collapsing cocycle. Suppose for any morphism β in Ξ containing a simple reflection $s \in S$, we have $\zeta(\beta^{(-1)^{n-1}}, \dots, \beta^{-1}, \beta) = 0$. Then $\zeta = 0$.*

Proof Sketch The proof is almost identical as that of Lemma in §2.11. For composable morphisms $\beta = (\beta_1, \dots, \beta_n)$ in Ξ , we prove the vanishing of $\zeta(\beta_1, \dots, \beta_n)$ by induction on $L(\overline{\beta}) := \sum_{i=1}^n |w^{\beta_i}|$. The only nontrivial point is to make sense of reduced word decomposition of a morphism in Ξ . Let $\beta \in \text{Hom}_{\Xi}(\mathcal{L}, \mathcal{L}')$, and $w^\beta = s_{i_N} s_{i_{N-1}} \cdots s_{i_1}$ a reduced expression in W . Let $\mathcal{L}_j = s_{i_j} \cdots s_{i_1} \mathcal{L}$. Then for any $1 \leq j < j' \leq N$, the partial product $s_{i_{j'}} \cdots s_{i_j}$ is the minimal length element in its coset $\beta_{[j, j']} \in \mathcal{L}_{j'} \underline{W}_{\mathcal{L}_j}$, i.e., $w^{\beta_{[j, j']}} = s_{i_{j'}} \cdots s_{i_j}$. This follows from [3, Lemma 4.6 (3)(4)]. In particular, we may write $\beta = \beta_{[N, N]} \circ \beta_{[1, N-1]}$ and $w^\beta = s_{i_N} w^{\beta_{[1, N-1]}}$ so that $|w^{\beta_N}| = N = 1 + |w^{\beta_{[1, N-1]}}|$. This decomposition is used in the induction step. \square

4.4 Theorem *The cocycle σ takes values in $\{\pm 1\}$. If we identify $\{\pm 1\}$ with \mathbb{F}_2 , we have $\sigma = \pi^* \epsilon_3^W$.*

Proof We first check that σ is collapsing. For this we look more closely into the geometry behind the cocycle σ . Let X be the flag variety of G , and $\mathbf{O}(w) \subset X \times X$ be the G -orbit containing $(B, \text{Ad}(w)B)$. For any sequence of elements $w_n, \dots, w_1 \in W$, let $X(w_n, \dots, w_1) \subset X^n$ be the variety classifying sequences of Borel subgroups (B_1, \dots, B_n) such that $(B, B_1) \in \mathbf{O}(w_1)$, and $(B_i, B_{i+1}) \in \mathbf{O}(w_{i+1})$ for $1 \leq i \leq n - 1$. For composable morphisms δ, γ and β in Ξ , let $X(w^\delta \# w^\gamma, w^\beta) \subset X(w^\delta, w^\gamma, w^\beta)$ be the closed subvariety where $(B_1, B_3) \in \mathbf{O}(w^\delta w^\gamma)$; let $X(w^\delta, w^\gamma \# w^\beta) \subset X(w^\delta, w^\gamma, w^\beta)$ be the closed subvariety where $(B, B_2) \in \mathbf{O}(w^\gamma w^\beta)$. We have a diagram



The maps are the obvious inclusion and forgetful maps. We take the fiber of all spaces in the above diagram over the point $B_3 = \text{Ad}(w^\delta w^\gamma w^\beta)B \in X$, and name them $Y(?)$ instead of $X(?)$. Then the stalk of $\Delta(\dot{w}^\delta)_{\mathcal{L}''} \star \Delta(\dot{w}^\gamma)_{\mathcal{L}'} \star \Delta(\dot{w}^\beta)_{\mathcal{L}}$ at $\dot{w}^\delta \dot{w}^\gamma \dot{w}^\beta$ is $H_c^*(Y(w^\delta, w^\gamma, w^\beta), \mathcal{F}[|w^\delta| + |w^\gamma| + |w^\beta|])$ for some rank one local system \mathcal{F} constructed from $\mathcal{L}'', \mathcal{L}'$ and \mathcal{L} . The restriction of \mathcal{F} to $Y(w^\delta \# w^\gamma, w^\beta)$ and $Y(w^\delta, w^\gamma \# w^\beta)$ are canonically isomorphic to $\overline{\mathbb{Q}}_\ell$. By [3, §5.3], the maps $q_1 p_1$ and $q_2 p_2$ are affine space bundles of dimension $d = \frac{1}{2}(|w^\delta| + |w^\gamma| + |w^\beta| - |w^\delta w^\gamma w^\beta|)$ when restricted to the point $\text{Ad}(w^\delta w^\gamma w^\beta)B$. Therefore, $Y(w^\delta \# w^\gamma, w^\beta)$ and $Y(w^\delta, w^\gamma \# w^\beta)$ are isomorphic to \mathbb{A}^d . The fact that $\text{can}_{\dot{w}^\delta, \dot{w}^\gamma}$ and $\text{can}_{\dot{w}^\gamma, \dot{w}^\beta}$ are isomorphisms implies that the restriction maps

$$\begin{aligned}
 i_1^* &: H_c^{-|w^\delta w^\gamma w^\beta|}(Y(w^\delta, w^\gamma, w^\beta), \mathcal{F}[|w^\delta| + |w^\gamma| + |w^\beta|]) \rightarrow H_c^{2d}(Y(w^\delta \# w^\gamma, w^\beta), \overline{\mathbb{Q}}_\ell) \cong \overline{\mathbb{Q}}_\ell, \\
 i_2^* &: H_c^{-|w^\delta w^\gamma w^\beta|}(Y(w^\delta, w^\gamma, w^\beta), \mathcal{F}[|w^\delta| + |w^\gamma| + |w^\beta|]) \rightarrow H_c^{2d}(Y(w^\delta, w^\gamma \# w^\beta), \overline{\mathbb{Q}}_\ell) \cong \overline{\mathbb{Q}}_\ell
 \end{aligned}$$

are isomorphisms. The number $\sigma(\delta, \gamma, \beta)$ is the ratio of these two isomorphisms.

Suppose $|w^\delta| + |w^\gamma| = |w^\delta w^\gamma|$; then $Y(w^\delta \# w^\gamma, w^\beta) = Y(w^\delta, w^\gamma, w^\beta)$; therefore $Y(w^\delta, w^\gamma \# w^\beta) = Y(w^\delta, w^\gamma, w^\beta)$ as well for dimension reasons. In this case, both i_1^* and i_2^* are the identity maps; hence $\sigma(\delta, \gamma, \beta) = 1$. The case $|w^\gamma| + |w^\beta| = |w^\gamma w^\beta|$ is proved similarly. This proves that σ is collapsing.

In [3, Example 5.7], we have shown for $G = \text{SL}_2$, and a block β containing the nontrivial element in the Weyl group $W = S_2$, we have $\sigma(\beta, \beta^{-1}, \beta) = -1$. This implies for general G , and any block β containing a simple reflection $s \in S$, we have $\sigma(\beta, \beta^{-1}, \beta) = -1$ since the calculation in this case reduces to the SL_2 example by the homomorphism $\text{SL}_2 \rightarrow G$ corresponding to s . Now we identify $\{\pm 1\}$ with \mathbb{F}_2 . We

have seen that $\zeta = \sigma - \pi^* \epsilon_3^W$ is collapsing in $Z^3(\Xi, \mathbb{F}_2)$ and satisfies $\zeta(\beta, \beta^{-1}, \beta) = 0$ for any morphism β in Ξ containing a simple reflection. By Lemma in §4.3, ζ vanishes identically; hence $\sigma = \pi^* \epsilon_3^W$. \square

4.5 Remark A similar sign is observed by Bezrukavnikov and Riche¹ in the usual Hecke category. Let $\Delta(s)$ and $\nabla(s)$ be the $!$ and $*$ extensions of $\overline{\mathbb{Q}}_\ell[1]$ from the Schubert cell $X(s) \subset X = G/B$. Let δ be the skyscraper sheaf at the point stratum $X(1) = \{B\}$. There are canonical isomorphisms $\Delta(s) \star \nabla(s) \cong \delta \cong \nabla(s) \star \Delta(s)$. However, the resulting two isomorphisms

$$(\Delta(s) \star \nabla(s)) \star \Delta(s) \cong \delta \star \Delta(s) = \Delta(s), \tag{4.4}$$

$$\Delta(s) \star (\nabla(s) \star \Delta(s)) \cong \Delta(s) \star \delta = \Delta(s) \tag{4.5}$$

differ by a sign. Indeed, the stalk of $\Delta(s) \star \nabla(s) \star \Delta(s)$ along $X(s)$ can be identified with

$$H_c^*(\mathbb{A}^2, j_* \overline{\mathbb{Q}}_\ell[3]),$$

where $j : \mathbb{A}^2 - C \hookrightarrow \mathbb{A}^2$, and $C = \{(x, y) \in \mathbb{A}^2 \mid xy = 1\}$. The two isomorphisms (4.4) and (4.5) correspond to the two restrictions which are isomorphisms

$$\text{res}_x : H_c^*(\mathbb{A}^2, j_* \overline{\mathbb{Q}}_\ell[3]) \xrightarrow{\sim} H_c^*(\mathbb{A}_x^1, \overline{\mathbb{Q}}_\ell[3]) \cong \overline{\mathbb{Q}}_\ell[1](-1),$$

$$\text{res}_y : H_c^*(\mathbb{A}^2, j_* \overline{\mathbb{Q}}_\ell[3]) \xrightarrow{\sim} H_c^*(\mathbb{A}_y^1, \overline{\mathbb{Q}}_\ell[3]) \cong \overline{\mathbb{Q}}_\ell[1](-1).$$

Here \mathbb{A}_x^1 and \mathbb{A}_y^1 are the coordinate lines $y = 0$ and $x = 0$. The involution $\sigma : (x, y) \mapsto (y, x)$ then interchanges res_x and res_y . To see the sign, it suffices to show that σ^* acts on $H_c^2(\mathbb{A}^2, j_* \overline{\mathbb{Q}}_\ell)$ by -1 . Using the triangle $\overline{\mathbb{Q}}_\ell \rightarrow j_* \overline{\mathbb{Q}}_\ell \rightarrow i_* \overline{\mathbb{Q}}_\ell-1$, where $i : C \hookrightarrow \mathbb{A}^2$, we get $H_c^2(\mathbb{A}^2, j_* \overline{\mathbb{Q}}_\ell) \cong H_c^1(C, \overline{\mathbb{Q}}_\ell)(-1)$. Now $C \cong \mathbb{G}_m$ and σ restricts to the inversion map on C , which acts by -1 on $H^1(C)$ and $H^1(C)$. This is the source of the sign.

5 Restriction of ϵ_3^W to Ω

In this section, for W an irreducible Weyl group, we calculate the restriction of ϵ_3^W to certain abelian subgroups. Such restriction naturally shows up in [3] as the twisting data for the monoidal structure of monodromic Hecke categories, and our calculation here shows that the twisting is often nontrivial.

Below we write ϵ_3^W simply as ϵ_3 . We first collect some useful properties of ϵ_3 .

¹ Private communication.

5.1 Proposition *Let (W, S) be any Coxeter group and $x, y, z \in W$.*

(1) *If $y = s \in S$, then*

$$\epsilon_3(x, s, z) = \begin{cases} 1, & \text{if } |xs| < |x| \text{ and } |sz| < |z|; \\ 0, & \text{otherwise.} \end{cases}$$

(2) *Let $y = s_1 \cdots s_m$ be a reduced expression; then*

$$\epsilon_3(x, y, z) = \sum_{i=1}^m \epsilon_3(xs_1 \cdots s_{i-1}, s_i, s_{i+1} \cdots s_m z).$$

(3) *We have $\epsilon_3(x, y, z) = 0$ if $xyz = 1$.*

(4) *We have $\epsilon_3(y^{-1}, y, z) = \frac{1}{2}(|y| + |z| - |yz|) \pmod 2$ and $\epsilon_3(x, y, y^{-1}) = \frac{1}{2}(|x| + |y| - |xy|) \pmod 2$.*

(5) *We have $\epsilon_3(x, x^{-1}, x) = |x| \pmod 2$.*

(6) *We have $\epsilon_3(x, y, z) = \epsilon_3(y, z, (xyz)^{-1}) = \epsilon_3(z, (xyz)^{-1}, x) = \epsilon_3((xyz)^{-1}, x, y)$.*

Proof (1) follows from definition. (2) is a special case of Lemma in §3.12.

(3) follows from Lemma in §3.10. (4) follows from Lemma in §3.11 and (5) is a special case of (4).

(6) Apply the cocycle condition $(\delta\epsilon_3)(x, y, z, (xyz)^{-1}) = 0$, and using (3) to get the first equality; the others are obtained by iterating the first one. □

5.2 The Group Ω For the rest of the notes, let W be a Weyl group. Let G, T, B be as in §1.3, with the extra assumption that G is simple of adjoint type. Let $\tilde{W} = \Lambda \rtimes W$ be the extended affine Weyl group attached to G , where $\Lambda = \mathbb{X}_*(T)$. Let A be the corresponding fundamental alcove in $\mathbb{X}_*(T)_{\mathbb{R}}$, and let $\Omega \subset \tilde{W}$ be the stabilizer of A . The map $i : \Omega \subset \tilde{W} \rightarrow W$ is injective, and the conjugacy class of i is independent of the choice of B (it will depend on G and not just W , as we shall see by comparing type B and type C).

The finite abelian group $\Omega_{\mathcal{L}} = W_{\mathcal{L}}/W_{\mathcal{L}}^{\circ}$ in [3, §2.4] is a subgroup of Ω . We would like to compute the cohomology classes of the pullback of ϵ_3 to Ω as well as to subgroups of Ω , which play a role in the statement of [3, Theorem 10.12]. This question is non-vacuous only when $\#\Omega$ is even, which happens for types A_n (n odd), B_n, C_n, D_n and E_7 .

5.3 Method To compute $[\epsilon_3|_{\Omega}]$ we need to compute i . Let D be the Dynkin diagram of G , and \tilde{D} be the extended Dynkin diagram of G . It is well-known that $\Omega \cong \text{Aut}(\tilde{D})/\text{Aut}(D)$. Also $\text{Aut}(D) = \text{Aut}(\tilde{D}, \alpha_0)$. We shall compute the projection $\tilde{i} : \text{Aut}(\tilde{D}) \rightarrow W$. Let $\tau \in \text{Aut}(\tilde{D})$; then it is a permutation of affine simple roots. It is then easy to write down the matrix for the action of τ on $V = \mathbb{X}^*(T)_{\mathbb{Q}} = \text{Span}_{\mathbb{Q}}\{\alpha_s \mid s \in S\}$. If $\tau(\alpha_s) = \alpha_0$, then its action on V should read $\tau(\alpha_s) = -\theta = -\sum_s n_s \alpha_s$.

Another general principle: if $X \subset W$, let $\Phi(X) = \{\alpha \in \Phi(G, T) \mid \alpha \perp V^X\}$, and let $W(X) \subset W$ be the subgroup generated by reflections attached to $\alpha \in \Phi(X)$. Then $X \subset W(X)$ and $W(X)$ is a parabolic subgroup of W . This allows us to reduce the calculation to smaller Coxeter groups.

In the next subsections we compute $[\epsilon_3|_\Omega]$ case by case.

5.4 Type A_{n-1} , n Even We have $\Omega \cong \mathbb{Z}/n\mathbb{Z} \subset S_n$ generated by the cyclic permutation $C = s_1 s_2 \cdots s_{n-1}$. We claim that $[\epsilon_3|_\Omega]$ is the nontrivial class in $H^3(\Omega, \mathbb{F}_2) = \mathbb{F}_2$. Let $n = 2m$.

Suppose in the contrary that $\epsilon_3|_\Omega$ is a coboundary, i.e., $\epsilon_3 = \delta\lambda$ for some function $\lambda : \Omega \times \Omega \rightarrow \mathbb{F}_2$; then the fact that ϵ_3 is normalized shows that $\lambda(x, 1) = \lambda(1, 1) = \lambda(1, x)$ for all $x \in \Omega$. We may assume $\lambda(1, 1) = 0$; for otherwise we can change λ to $\lambda + 1$ and its coboundary is still ϵ_3 . Therefore, we assume λ is also normalized, i.e., $\lambda(1, x) = \lambda(x, 1) = 0$ for all $x \in \Omega$.

We have $\epsilon_3(C^m, C^m, C) = \epsilon_3(C, C^m, C^m)$ by Proposition (4) in §5.1. Since $\epsilon_3 = \delta\lambda$, we get

$$\begin{aligned} \epsilon_3(C^m, C^m, C) &= \lambda(C^m, C) + \lambda(C^m, C^{m+1}) + \lambda(C^m, C^m), \\ \epsilon_3(C, C^m, C^m) &= \lambda(C, C^m) + \lambda(C^{m+1}, C^m) + \lambda(C^m, C^m). \end{aligned}$$

The equality of the LHS of the above two equations implies

$$\lambda(C^m, C) + \lambda(C, C^m) + \lambda(C^m, C^{m+1}) + \lambda(C^{m+1}, C^m) = 0. \tag{5.1}$$

However, the above is equal to $\epsilon_3(C^m, C, C^m)$, which we now show is nonzero.

The element C is the cyclic permutation: $1 \mapsto 2 \mapsto \cdots \mapsto n \mapsto 1$. The involution C^m interchanges i and $i + m$ for $1 \leq i \leq m$, and hence $|C^m| = m^2$. The permutation $C^m s_1 \cdots s_{m-1}$ is $j \mapsto m + j + 1$ for $1 \leq j \leq m - 1$, $m \mapsto m + 1$, $m + j \mapsto j$ for $0 < j \leq m$. We see that $|C^m s_1 \cdots s_{m-1}| = m^2 + m - 1 = |C^m| + m - 1$. Similarly, $|s_{m+1} \cdots s_{n-1} C^m| = m - 1 + |C^m|$. By Proposition (1)(2) in §5.1, we see

$$\epsilon_3(C^m, C, C^m) = \sum_{j=1}^{n-1} \epsilon_3(C^m s_1 \cdots s_{j-1}, s_j, s_{j+1} \cdots s_{n-1} C^m). \tag{5.2}$$

The automorphism of (W, S) sending s_i to s_{n-i} sends the triple $(C^m s_1 \cdots s_{j-1}, s_j, s_{j+1} \cdots s_{n-1} C^m)$ to $(C^m s_1 \cdots s_{n-j-1}, s_{n-j}, s_{n-j+1} \cdots s_{n-1} C^m)$, which have the same value under ϵ_3 by Lemma (1) in §3.7. Therefore, the RHS of (5.2) reduces to one term $\epsilon_3(C^m s_1 \cdots s_{m-1}, s_m, s_{m+1} \cdots s_{n-1} C^m)$. Now $|C^m s_1 \cdots s_m| = m^2 + m - 2 = |C^m s_1 \cdots s_{m-1}| - 1$, and similarly $|s_m \cdots s_{n-1} C^m| = |s_{m+1} \cdots s_{n-1} C^m| - 1$. By Proposition (1) in §5.1, $\epsilon_3(C^m s_1 \cdots s_{m-1}, s_m, s_{m+1} \cdots s_{n-1} C^m) = 1$; therefore, $\epsilon_3(C^m, C, C^m) \neq 0$, which contradicts (5.1). Therefore, $[\epsilon_3|_\Omega]$ is the nontrivial element in $H^3(\Omega, \mathbb{F}_2)$.

For any subgroup $\Omega' \subset \Omega$, the restriction $[\epsilon_3|_{\Omega'}] \in H^3(\Omega', \mathbb{F}_2)$ is nontrivial if and only if $[\Omega : \Omega']$ is odd.

5.5 Type B_n In this case, $\Omega = \mathbb{Z}/2\mathbb{Z}$, with nontrivial element conjugate to the simple reflection $s_n : (x_1, x_2, \dots, x_n) \mapsto (x_1, x_2, \dots, -x_n)$. Therefore, $[\epsilon_3|_\Omega] \in H^3(\Omega, \mathbb{F}_2) = \mathbb{F}_2$ is always the nontrivial element.

5.6 Type C_n In this case, $\Omega = \mathbb{Z}/2\mathbb{Z}$, the nontrivial element is the involution $\tau : (x_1, \dots, x_n) \mapsto (-x_n, \dots, -x_1)$. We have $\Phi(\tau) = \{\epsilon_1 + \epsilon_n, \epsilon_2 + \epsilon_{n-1}, \dots\}$ (if n is odd, $2\epsilon_{(n+1)/2} \in \Phi(\tau)$) consisting of orthogonal roots. Therefore, $W(\tau)$ is of type $A_1^{[(n+1)/2]}$, and τ is the product of nontrivial elements from each factor. By Lemma (2) in §3.8, $[\epsilon_3|_\Omega] = [\frac{n+1}{2}] \pmod 2 \in H^3(\Omega, \mathbb{F}_2) = \mathbb{F}_2$. Therefore,

$$[\epsilon_3|_\Omega] \begin{cases} = 0, & n \equiv 0, 3 \pmod 4; \\ \neq 0, & n \equiv 1, 2 \pmod 4. \end{cases}$$

5.7 Type D_n, n Odd In this case $\Omega \cong \mathbb{Z}/4\mathbb{Z}$. The three nontrivial elements in $i(\Omega) \subset W$ are $\omega, \omega^2, \omega^3$:

$$\begin{aligned} \omega(x_1, \dots, x_n) &= (x_n, -x_{n-1}, \dots, -x_2, -x_1), \\ \omega^2(x_1, \dots, x_n) &= (-x_1, x_2, \dots, x_{n-1}, -x_n), \\ \omega^3(x_1, \dots, x_n) &= (-x_n, -x_{n-1}, \dots, -x_2, x_1). \end{aligned}$$

Write $n = 2m + 1$. We see that $W(\Omega)$ is of type $D_3 \times A_1^{m-1} \cong A_3 \times A_1^{m-1}$, and ω is (C, s, \dots, s) where $C \in S_4$ is a Coxeter element, and $s \in S_2$ is the nontrivial element. Let $p : \Omega \rightarrow \mathbb{Z}/2\mathbb{Z}$ be the projection; then by Lemma in §3.7,

$$\epsilon_3|_\Omega \sim (m - 1)p^* \epsilon_3^{S_2} + \epsilon_3^{S_4}|_\Omega.$$

Note that p^* induces zero on cohomology in degrees > 1 , therefore,

$$\epsilon_3|_\Omega \sim \epsilon_3^{S_4}|_\Omega$$

which we know is nontrivial by §5.4.

The unique nontrivial subgroup of Ω is $\langle \omega^2 \rangle$. Since the restriction map $H^3(\mathbb{Z}/4\mathbb{Z}, \mathbb{F}_2) \rightarrow H^3(\mathbb{Z}/2\mathbb{Z}, \mathbb{F}_2)$ is trivial, $\epsilon_3|_{\langle \omega^2 \rangle}$ is cohomologically trivial.

5.8 Type D_n, n Even In this case $\Omega \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The three nontrivial elements in $i(\Omega) \subset W$ are as follows:

$$\begin{aligned} \omega_1(x_1, \dots, x_n) &= (x_n, -x_{n-1}, \dots, -x_2, x_1), \\ \omega_2(x_1, \dots, x_n) &= (-x_1, x_2, \dots, x_{n-1}, -x_n) \\ \omega_3(x_1, \dots, x_n) &= (-x_n, -x_{n-1}, \dots, -x_2, -x_1). \end{aligned}$$

We identify Ω with $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ by $\omega_1 \mapsto (1, 0), \omega_2 \mapsto (1, 1)$ and $\omega_3 \mapsto (0, 1)$. Let $p_1, p_2 : \Omega \rightarrow \mathbb{Z}/2\mathbb{Z}$ be the projection to the first and second factors and let $p_3 = p_1 + p_2 : \Omega \rightarrow \mathbb{Z}/2\mathbb{Z}$.

Write $n = 2m$. We see that $W(\Omega)$ is of type $D_2 \times A_1^{m-1} \cong A_1^{m+1}$. We write D_2 for the first two factors of A_1 . Then

$$\omega_1 = (s, 1, s, \dots); \quad \omega_2 = (s, s, 1, \dots); \quad \omega_3 = (1, s, s, \dots),$$

where s denotes the nontrivial element in each A_1 -factor of $W(\Omega)$. Each factor of A_1 corresponds to a projection $\pi_i : \Omega \rightarrow S_2 = \mathbb{Z}/2\mathbb{Z}$, $i = 1, \dots, m + 1$; then $e_3|_\Omega$ is cohomologous to $\sum_{i=1}^{m+1} \pi_i^* e_3^{S_2}$. Since $\pi_1 = p_1, \pi_2 = p_2, \pi_3 = \dots = \pi_{m+1} = p_3$, we thus get

$$e_3|_\Omega \sim p_1^*(e_3^{S_2}) + p_2^*(e_3^{S_2}) + (m - 1)p_3^*(e_3^{S_2}). \tag{5.3}$$

Let $\eta \in H^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{F}_2)$ be the nontrivial element; then $\{\eta^a \otimes \eta^{3-a} \mid 0 \leq a \leq 3\}$ gives a basis for $H^3(\Omega, \mathbb{F}_2)$. Then (5.3) implies

$$[e_3|_\Omega] = m(\eta^3 \otimes 1) + (m - 1)(\eta^2 \otimes \eta) + (m - 1)(\eta \otimes \eta^2) + m(1 \otimes \eta^3) \in H^3(\Omega, \mathbb{F}_2).$$

In particular, $[e_3|_\Omega]$ is always nontrivial.

We have three nontrivial subgroups in Ω , generated by $\omega_i, i = 1, 2, 3$. We see that

$$[e_3|_{\langle \omega_1 \rangle}] = m\eta^3, \quad [e_3|_{\langle \omega_2 \rangle}] = 0, \quad [e_3|_{\langle \omega_3 \rangle}] = m\eta^3.$$

5.9 Type E_7 In this case, $\Omega = \mathbb{Z}/2\mathbb{Z}$, and the nontrivial element $\tau \in \Omega$ has $\dim V^\tau = 4$. Therefore, $W(\tau)$ is a rank 3 parabolic subgroup of W and τ is the longest element in $W(\tau)$ that acts by -1 on the 3-dimensional reflection representation of $W(\tau)$. Now $W(\tau)$ is simply-laced since W is, therefore $W(\tau)$ is of type A_1^3 . This implies that τ is conjugate to a product of three commuting simple reflections in W . By Lemma (2) in §3.8, $[e_3|_\Omega] \in H^3(\Omega, \mathbb{F}_2) = \mathbb{F}_2$ is the nontrivial element.

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References

1. Atiyah, M.F., Wall, C.T.C.: Cohomology of groups. In: Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965), pp. 94–115, Thompson, Washington, D.C. (1967)
2. Bourbaki, N.: Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV–VI. Actualités Scientifiques et Industrielles, No. 1337, Hermann, Paris, p. 288 (1968)
3. Lusztig, G., Yun, Z.: Endoscopy for Hecke categories, character sheaves and representations. Forum Math. Pi **8**, e12 (2020)