SURVEY ON GEOMETRIC REPRESENTATION THEORY

ZHIWEI YUN

Abstract. We give an overview of some fundamental results in geometric representation theory, emphasizing aspects related to the Langlands program.

1. Introduction

Representation theory aims to classify different ways a given group (or a ring) can act on mathematical objects. A large part of the existing theory is about linear representations, namely the actions of groups or rings on vector spaces as linear transformations.

In many cases, representations of groups are constructed by geometric methods, typically by first considering manifolds or algebraic varieties related to the group, and then studying sheaves on them and their cohomology. Early successes of the geometric methods include the construction of discrete series representations of real Lie groups, and the construction of general representations of finite groups of Lie type. Since then, geometric representation theory became an active branch of representation theory, and is now intimately connected to many other subjects such as algebraic geometry (derived category of coherent sheaves, moduli spaces, mirror symmetry), number theory (representations of p-adic groups and automorphic representations, Langlands program), topology (knot theory, generalized cohomology theories), combinatorics and mathematical physics.

Due to the limitation of my expertise, I will focus on those topics in geometric representation theory that are directly related to the Langlands program. As a result, there are obvious omissions of important topics such as modular representations, quantum groups, affine and general Kac-Moody Lie algebras, etc. We also try to illustrate fundamental constructions by using SL_2 or GL_2 as examples.

2. Finite-dimensional representations of reductive groups

In this article, k always denotes an algebraically closed field. By reductive groups we shall always mean *connected* reductive groups.

2.1. **Reductive groups.** Reductive groups G over k are classified by quadruples called *root data*

$$(X, \Phi, X^\vee, \Phi^\vee) \tag{2.1}$$

where X is a free abelian groups of finite rank, $X^\vee = \mathrm{Hom}(X, \mathbb{Z})$, $\Phi \subset X$ is a finite subset forming a root system with coroots $\Phi^\vee \subset X^\vee$ in bijection with Φ. See [56, Theorem 9.6.2]. From G, such a root datum is obtained by choosing a maximal torus $T \subset G$, taking $X = \mathrm{Hom}(T, \mathbb{G}_m)$ to be the character lattice of T and Φ (resp. $\Phi^\vee$) to be the roots (resp. coroots) of G with respect to T.

Semisimple algebraic groups G over k correspond to root data where Φ span $X \otimes \mathbb{Q}$ as a rational vector space. Among such, simple algebraic groups correspond to those with irreducible root systems. Up to central isogeny, simple algebraic groups fall into 4 infinite series and 5 exceptional types:

$$\mathrm{SL}_n, \mathrm{SO}_{2n+1}, \mathrm{Sp}_{2n}, \mathrm{SO}_{2n}, E_6, E_7, E_8, F_4, G_2.$$

Chevalley showed that, reductive groups over fields of various characteristics with the same root datum can be glued together to give a group scheme G defined over $\mathbb{Z}$, the *Chevalley group* attached to a root datum. It allows us to obtain a reductive group G_R over any commutative ring R by base change, and talk

2010 *Mathematics Subject Classification.* Primary ; Secondary .
Supported partially by the Simon Foundation and the Packard Foundation.

about the group of R-points $G(R)$. Not all reductive groups over R are obtained from Chevalley groups by base change: we only get *split groups*. In this article we mostly discuss split groups.

Below we shall talk about the representation theory of groups of the kind $G(R)$ starting from a root datum as in (2.1) (or equivalently, a Chevalley group G over $\mathbb{Z}$), where R can be a finite field $\mathbb{F}_q$, a p-adic field $\mathbb{Q}_p$ or $\mathbb{F}_q((t))$, real or complex numbers $\mathbb{R}$ or $\mathbb{C}$, or the ring of adèles $\mathbb{A}_F$ of a global field F.

2.2. **Flag variety.** Flag varieties are the basic geometric avatar for constructing representations in various settings.

First let G be a reductive group over an algebraically closed field k. The flag variety $\mathcal{B}$ of G is the moduli space of Borel subgroups of G. It is a projective variety over k with a transitive action of G. If we choose a Borel subgroup $B \subset G$ defined over k, we have a G-equivariant isomorphism $\mathcal{B} \cong G/B$.

If G is of classical type, $\mathcal{B}$ can be described in linear algebra terms. For example, if $G = \mathrm{Sp}(V)$ is the symplectic group attached to a symplectic vector space V over k of dimension $2n$, then $\mathcal{B}(k)$ is the set of complete isotropic flags in V

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V_n^{\perp} \subset V_{n-1}^{\perp} \subset \cdots \subset V_1^{\perp} \subset V$$

where $\dim_k V_i = i$ for all i.

One can define the flag scheme for a Chevalley group as well, which will be a smooth projective scheme over $\mathbb{Z}$, whose base change to an algebraically closed field k recovers the flag variety over k discussed above.

2.3. **Borel-Weil-Bott.** Algebraic representations of a reductive group G over k are finite-dimensional k-vector spaces V together with a morphism of algebraic groups $\rho : G \to \mathrm{GL}(V)$. Let $\mathrm{Rep}(G)$ denote the category of algebraic representations of G.

In this subsection, all algebraic groups are defined over $k = \mathbb{C}$.

Let T be a maximal torus in G, so that we have a root datum as in (2.1), with $X = \mathrm{Hom}(T, \mathbb{G}_m)$, etc. Let B be a Borel subgroup containing T, which defines the subset of positive roots $\Phi^+ \subset \Phi$ and positive coroots $\Phi^{\vee,+} \subset \Phi^\vee$. An element $\lambda \in X$ is called *dominant* if $\langle \lambda, \alpha^\vee \rangle \geq 0$ for any $\alpha^\vee \in \Phi^{\vee,+}$. Let X^+ denote the set of dominant weights.

By Cartan-Weyl theory, isomorphism classes of irreducible algebraic representations of G are classified by their highest weights $\lambda \in X^+$: for $\lambda \in X^+$, the corresponding irreducible representation V_λ is characterized by having a unique eigenline $\ell_\lambda \subset V_\lambda$ under B, on which B acts through the character $B \to T \xrightarrow{\lambda} \mathbb{G}_m$.

The Borel-Weil construction [54] gives a geometric realization of irreducible algebraic representations of G as (holomorphic, or equivalently algebraic) sections of line bundles over the flag variety $\mathcal{B}$. For each $\lambda \in X$, consider the line bundle $\mathcal{L}_\lambda$ on $\mathcal{B}$ whose total space is the quotient of $G \times \mathbb{A}^1$ by the right action of B by

$$(g, a) \cdot b = (gb, \lambda(\bar{b}^{-1})a)$$

for $g \in G, a \in \mathbb{A}^1$, $b \in B$, and $\bar{b} \in T$ denotes the image of b under the projection $B \to T$. For $\lambda \in X^+$, consider the line bundle $\mathcal{L}_{-\lambda}$ on $\mathcal{B}$ constructed above. Then its global sections carries a natural action of G, and is isomorphic to the irreducible algebraic representation dual to V_λ:

$$\mathrm{H}^0(\mathcal{B}, \mathcal{L}_{-\lambda}) \cong V_\lambda^*.$$

2.3.1. **Example.** For $G = \mathrm{SL}_2$. The flag variety in this case is $\mathbb{P}^1$ with homogeneous coordinates $[x, y]$, and the action of $g \in \mathrm{SL}_2$ on $[x, y]$ is by matrix multiplication $\binom{x}{y} \mapsto g\binom{x}{y}$. It is well-known that line bundles on $\mathbb{P}^1_{\mathbb{C}}$ are of the form $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$ for integers $n \in \mathbb{Z}$. Here $\mathcal{O}(1)$ is the universal quotient bundle whose fiber at $[x, y]$ is the quotient line $\mathbb{C}^2/\mathbb{C}\binom{x}{y}$.

When $n \geq 0$, $V_n := \mathrm{H}^0(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}(n))$ can be identified with homogeneous degree n polynomials in two variables x and y, on which $\mathrm{SL}_2(\mathbb{C})$ acts by linear change of variables. This is the familiar realization of the $(n+1)$-dimensional irreducible representation of SL_2, and as n varies in $\mathbb{Z}_{\geq 0}$, they exhaust all irreducible algebraic representations of SL_2.

How about line bundles $\mathcal{L}_\lambda$ for general $\lambda \in X$? Bott [13] proved that the cohomology groups $\mathrm{H}^*(\mathcal{B}, \mathcal{L}_\lambda)$ is non-vanishing in at most one degree, which can be determined by the combinatorial data of the W-action on X. In that non-vanishing degree, the cohomology group of $\mathcal{L}_\lambda$ again gives an irreducible representation of G. In the SL_2-example, it is easily seen by Serre duality that when $n \le -2$, $\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(n))$ is dual to $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(-n-2)) \cong V_{-n-2}$.

2.4. **Geometric Satake equivalence.** A totally different way of realizing algebraic representations of G is by considering geometric objects attached to another group $G^\vee$, the Langlands dual group of G. Let us work over $k = \mathbb{C}$. The root datum of $G^\vee$ is obtained from that of G by swapping roots and coroots: if (2.1) is the root datum of G, then the root datum of $G^\vee$ is

$$(X^\vee, \Phi^\vee, X, \Phi).$$

For example, if $G = \mathrm{GL}_n$, then $G^\vee$ is also isomorphic to GL_n; if $G = \mathrm{Sp}_{2n}$, then $G^\vee \cong \mathrm{SO}_{2n+1}$.

Let $G^\vee(\!(t)\!)$ be the loop group of $G^\vee$: its $\mathbb{C}$-points are $G^\vee(\mathbb{C}(\!(t)\!))$; similarly, let $G^\vee[\![t]\!]$ be the arc group whose $\mathbb{C}$-points are $G^\vee(\mathbb{C}[\![t]\!])$.

The affine Grassmannian $\mathrm{Gr}_{G^\vee}$ of $G^\vee$ is an infinite dimensional version of the (partial) flag variety of $G^\vee$. Its $\mathbb{C}$-points is the coset space $G(\mathbb{C}(\!(t)\!))/G(\mathbb{C}[\![t]\!])$. For $G^\vee = \mathrm{GL}_n$, we can identify $\mathrm{Gr}_{G^\vee}(\mathbb{C})$ with the space of $\mathbb{C}[\![t]\!]$-lattices in $\mathbb{C}(\!(t)\!)^n$ (rank n $\mathbb{C}[\![t]\!]$-submodules of $\mathbb{C}(\!(t)\!)^n$). One can write $\mathrm{Gr}_{G^\vee}$ as an increasing union of projective schemes, each inside a Grassmannian variety of growing size. The loop group $G^\vee(\!(t)\!)$ acts transitively on $\mathrm{Gr}_{G^\vee}$ with stabilize at the base point equal to $G[\![t]\!]$.

Let $T^\vee$ be a maximal torus of $G^\vee$ whose cocharacter lattice $\mathrm{Hom}(\mathbb{G}_m, T^\vee)$ is identified with the character lattice X of T. For $\lambda \in X$, let $t^\lambda \in T^\vee(\mathbb{C}(\!(t)\!))$ denote the image of $t \in \mathbb{C}(\!(t)\!)^\times = \mathbb{G}_m(\mathbb{C}(\!(t)\!))$ under the cocharacter $\lambda : \mathbb{G}_m \to T^\vee$. We may view t^λ as a $\mathbb{C}$-point of $\mathrm{Gr}_{G^\vee}$. For $\lambda \in X^+$, let $\mathrm{Gr}_\lambda \subset \mathrm{Gr}_{G^\vee}$ be the orbit of t^λ under $G^\vee[\![t]\!]$. Then $\{\mathrm{Gr}_\lambda\}_{\lambda \in X^+}$ are exactly the $G^\vee[\![t]\!]$-orbits on $\mathrm{Gr}_{G^\vee}$. Let $\mathrm{Gr}_{\le\lambda}$ be the closure of Gr_λ in $\mathrm{Gr}_{G^\vee}$, which is a projective variety, often singular. The varieties $\mathrm{Gr}_{\le\lambda}$ should be thought of as analogs of the Schubert varieties (in the usual Grassmannian) for the loop group $G^\vee(\!(t)\!)$.

Let $K \subset G^\vee(\mathbb{C})$ be a maximal compact subgroup. As a topological space, $\mathrm{Gr}_{G^\vee}(\mathbb{C})$ is homeomorphic to the space of polynomial loops $f : S^1 \to K$ such that $f(1) = 1$ (polynomial means that any matrix coefficient of K pulls back to a finite Fourier series on S^1).

Now consider the intersection homology $\mathrm{IH}^*(\mathrm{Gr}_{\le\lambda})$ of $\mathrm{Gr}_{\le\lambda}$ in $\mathbb{C}$-coefficients. A deep theorem of Lusztig [39], Drinfeld, Ginzburg [26], and Mirković-Vilonen [48] says that there is a canonical action of G (not $G^\vee$) on the total intersection homology $\mathrm{IH}^*(\mathrm{Gr}_{\le\lambda})$, realizing an isomorphism of G-representations

$$\mathrm{IH}^*(\mathrm{Gr}_{\le\lambda}) \cong V_\lambda. \tag{2.2}$$

2.4.1. **Example.** Consider the case $G = \mathrm{PGL}_2$, hence $G^\vee = \mathrm{SL}_2$. In this case, $\mathrm{Gr}_{G^\vee}$ has a complex cell decomposition with one cell in each complex dimension $0, 1, 2, \cdots$. It is homotopy equivalent to the based loop space $\Omega\mathrm{SU}_2 \cong \Omega(S^3)$. Identify X^+ with $\mathbb{Z}_{\ge 0}$ in an obvious way, then $\mathrm{Gr}_{\le n}$ is the union of complex cells of dimension $0, 1, 2, \cdots, 2n$. In this case, $\mathrm{Gr}_{\le n}$ happens to be rationally smooth, so that its intersection homology coincides with the usual cohomology $\mathrm{H}^*(\mathrm{Gr}_{\le n})$, which has total dimension $2n + 1$.

There is a canonical action of $G = \mathrm{PGL}_2$ on $\mathrm{H}^*(\mathrm{Gr}_{\le n})$: the diagonal torus acts on $\mathrm{H}^{2i}(\mathrm{Gr}_{\le n})$ with weight $-2n + 2i$; the action of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathrm{Lie}\, G$ is by cup product with the Chern class of an ample line bundle on $\mathrm{Gr}_{G^\vee}$, the determinant line bundle.

The full statement of the geometric Satake equivalence also includes geometric realizations of tensor product of algebraic representations of G, as well as all linear algebraic relations among such tensor products. For example, the tensor product $V_\lambda \otimes V_\mu$ of two irreducible representations with highest weights λ and μ is realized as the intersection homology of the convolution Schubert variety $\widetilde{\mathrm{Gr}}_{\le\lambda,\le\mu}$ inside $(\mathrm{Gr}_{G^\vee})^2$. The G-isomorphisms $V \otimes W \cong W \otimes V$ for $V, W \in \mathrm{Rep}(G)$ also has a geometric interpretation via the "fusion" construction [48]. All these structures are packaged into an equivalence of abelian tensor (symmetric monoidal) categories

$$P_{G^\vee[\![t]\!]}(\mathrm{Gr}_{G^\vee}) \xrightarrow{\sim} \mathrm{Rep}(G) \tag{2.3}$$

where the left side is the category of $G^\vee[\![t]\!]$-equivariant perverse sheaves on $\mathrm{Gr}_{G^\vee}$ (supported on finitely many $G^\vee[\![t]\!]$-orbits), equipped with the tensor structure given by convolution. The equivalence (2.3) is called the *geometric Satake equivalence*. A comprehensive reference is [65].

Another feature of the equivalence (2.3) is that it works for general coefficients. Namely for any commutative ring R, if one considers perverse sheaves with R-coefficients on the left side of (2.3), then the resulting R-linear tensor category is canonically equivalent to $\mathrm{Rep}(G_R)$, the tensor category of algebraic representations of the split group G over R whose root datum is the same as that of G. This equivalence has been used to study modular representations of reductive groups.

2.5. **Quiver varieties.** Let $\mathfrak{g}$ be a simple, simply-laced Lie algebra over $\mathbb{C}$. Nakajima [49] gave a construction of finite-dimensional simple modules of $\mathfrak{g}$ in terms of certain moduli space of quiver representations.

Let Q be an oriented graph (or quiver) whose underlying graph is the Dynkin diagram of $\mathfrak{g}$. In particular, the vertex set I of Q is in bijection with simple roots $\{\alpha_i\}_{i\in I}$ of $\mathfrak{g}$. The orientation of Q is arbitrary. Let $Q^\square$ be the oriented graph obtained from Q by adding one vertex $i^\square$ for each $i\in I$, with an arrow $i\to i^\square$ for each $i\in I$. The added vertices $\{i^\square\}_{i\in I}$ are called framing vertices. Let $\overleftrightarrow{Q^\square}$ be the quiver obtained from $Q^\square$ by adding an opposite arrow to each existing arrow.

Let $\mathbf{v}=(v_i)_{i\in I}$ and $\mathbf{w}=(w_i)_{i\in I}\in\mathbb{Z}^I_{\ge 0}$. Let $R\binom{\mathbf{w}}{\mathbf{v}}$ be the moduli space of representations of the doubled quiver $\overleftrightarrow{Q^\square}$ where the vector space at i is $\mathbb{C}^{v_i}$, and the vector space at $i^\square$ is $\mathbb{C}^{w_i}$, for all $i\in I$.

The group $\mathrm{GL}(\mathbf{v})=\prod_{i\in I}\mathrm{GL}(v_i)$ acts on $R\binom{\mathbf{w}}{\mathbf{v}}$ by acting on the vector spaces $\mathbb{C}^{v_i}$. Viewing $R\binom{\mathbf{w}}{\mathbf{v}}$ as the cotangent bundle of the representation space of $Q^\square$ with the same dimension vectors, we have a moment map $\mu: R\binom{\mathbf{w}}{\mathbf{v}}\to\mathfrak{gl}(\mathbf{v})^*$. Let $M_0\binom{\mathbf{w}}{\mathbf{v}}$ be the Hamiltonian reduction $\mu^{-1}(0)/\!\!/\mathrm{GL}(\mathbf{v})$; it is an affine scheme. On the other hand, for a character $\theta:\mathrm{GL}(\mathbf{v})\to\mathbb{G}_m$, one can form the GIT version of the Hamiltonian reduction $M_\theta\binom{\mathbf{w}}{\mathbf{v}}=(\mu^{-1}(0)\cap R^{\theta-\mathrm{st}}\binom{\mathbf{w}}{\mathbf{v}})/\mathrm{GL}(\mathbf{v})$ using the θ-stable locus. Under a genericity condition on θ, $M_\theta\binom{\mathbf{w}}{\mathbf{v}}$ is a smooth symplectic variety equipped with a projective morphism $\pi: M_\theta\binom{\mathbf{w}}{\mathbf{v}}\to M_0\binom{\mathbf{w}}{\mathbf{v}}$ which sometimes turns out to be a symplectic resolution. The reduced zero fiber $\Lambda\binom{\mathbf{w}}{\mathbf{v}}=\pi^{-1}(0)_{\mathrm{red}}$ is a projective Lagrangian subscheme of $M_\theta\binom{\mathbf{w}}{\mathbf{v}}$.

Nakajima constructed an action of $\mathfrak{g}$ on the following vector space

$$L_{\mathbf{w}}=\bigoplus_{\mathbf{v}\in\mathbb{Z}^I_{\ge 0}}\mathrm{H}_{top}\left(\Lambda\binom{\mathbf{w}}{\mathbf{v}}\right). \tag{2.4}$$

Here H_{top} denotes top degree homology. The sum turns out to be finite in this case. The action of $\mathfrak{g}$ is constructed by specifying the actions of the raising and lowering operators e_i, f_i for each simple root α_i. The action of e_i is defined using a certain Lagrangian correspondence between $M_\theta\binom{\mathbf{w}}{\mathbf{v}}$ and $M_\theta\binom{\mathbf{w}}{\mathbf{v}+\epsilon_i}$, where ϵ_i is the dimension vector whose i-coordinate is 1 and zero elsewhere. Nakajima proves that $L_{\mathbf{w}}$ is a simple $\mathfrak{g}$-module with highest weight $\lambda=\sum_{i\in I}w_i\varpi_i$, where $\{\varpi_i\}$ are the fundamental weights dual to the simple coroots $\{\alpha_i^\vee\}$. The summand in (2.4) indexed by $\mathbf{v}$ is the weight space with weight $\lambda-\sum_{i\in I}v_i\alpha_i$.

2.5.1. **Example.** In type A, Nakajima's construction essentially recovers the earlier construction by Ginzburg [25]. Consider the case $\mathfrak{g}=\mathfrak{sl}_2$, where the quiver Q reduces to a single vertex without edge. The doubled and framed quiver $Q^\square$ has shape

$$\circ \underset{i}{\overset{j}{\rightleftarrows}} \square$$

where $\square$ denotes the framing vertex. Fix any $0\le m\le n$, the corresponding moduli space $M\binom{n}{m}$ is the cotangent bundle of the Grassmannain $\mathrm{Gr}\binom{n}{m}$ of m-planes in $\mathbb{C}^n$, and $\Lambda\binom{n}{m}=\mathrm{Gr}\binom{n}{m}$ is the zero section. Therefore, the simple $\mathfrak{sl}_2$-module with highest weight n is realized on the $(n+1)$-dimensional space

$$V_n=\bigoplus_{0\le m\le n}\mathrm{H}_{top}\left(\mathrm{Gr}\binom{n}{m}\right).$$

The summands above are weight spaces of the Cartan subalgebra of $\mathfrak{sl}_2$. The raising operator $e=\begin{pmatrix}0&1\\0&0\end{pmatrix}\in\mathfrak{sl}_2$ acts by a map $e_{m,m+1}:\mathrm{H}_{top}\left(\mathrm{Gr}\binom{n}{m}\right)\to\mathrm{H}_{top}\left(\mathrm{Gr}\binom{n}{m+1}\right)$ for each m. Consider the

2-step partial flag variety $\mathrm{Gr}\binom{n}{m,m+1} \subset \mathrm{Gr}\binom{n}{m} \times \mathrm{Gr}\binom{n}{m+1}$ classifying linear subspaces $F_m \subset F_{m+1} \subset \mathbb{C}^n$. Let $Z_{m,m+1} \subset T^*\mathrm{Gr}\binom{n}{m} \times T^*\mathrm{Gr}\binom{n}{m+1}$ be the conormal bundle of $\mathrm{Gr}\binom{n}{m,m+1}$, viewed as a Lagrangian correspondence

$$\begin{array}{ccc} & Z_{m,m+1} & \\ \swarrow & & \searrow \\ T^*\mathrm{Gr}\binom{n}{m} & & T^*\mathrm{Gr}\binom{n}{m+1} \end{array}$$

Then $e_{m,m+1}$ is the pull-push operation on homology given by the correspondence $Z(m, m+1, n)$. The construction of the action of the lowering operator uses the same corresponce $Z_{m,m+1}$ but read from right to left.

3. Real and complex groups

In this section we are concerned with infinite-dimensional representations of Lie groups.

3.1. **Category $\mathcal{O}$, Kazhdan-Lusztig conjecture and the Beilinson-Bernstein localization.** Let G be a complex reductive group, and $\mathfrak{g}$ be its Lie algebra. The *category* $\mathcal{O}$ for $\mathfrak{g}$, introduced by Bernstein, I.Gelfand and S.Gelfand [7], can be viewed as a model for studying infinite-dimensional representations of the complex Lie group $G(\mathbb{C})$. For simplicity, we restrict our attention to the principal block $\mathcal{O}_0$. Fix a maximal torus T and a Borel subgroup $B \subset G$ containing T with Lie algebra $\mathfrak{b}$. Let $W = W(G,T)$ be the Weyl group. A finitely generated $\mathfrak{g}$-modules M is in the category $\mathcal{O}_0$ if the following conditions are satisfied:

- The center $Z(U(\mathfrak{g}))$ of the universal enveloping algebra $U(\mathfrak{g})$ acts on M via the same character as its action on the trivial $\mathfrak{g}$-module $\mathbb{C}$;
- The action of $\mathfrak{b}$ on M is locally finite.

For $w \in W$, the category $\mathcal{O}_0$ contains the *Verma module* M_w freely generated under $U(\mathfrak{n}^-)$ ($\mathfrak{n}^-$ is the span of negative root spaces) by a highest weight vector with weight $w \cdot 0 = w\rho - \rho$ (as usual, 2ρ is the sum of positive roots). Each M_w has a unique simple quotient L_w, and the collection $\{L_w\}_{w\in W}$ is the complete list of irreducible $U(\mathfrak{g})$-modules in $\mathcal{O}_0$.

A basic question is to compute how many times a simple module L_w appears in the composition series of M_y, for $w, y \in W$. Such knowledge can be used to compute the weight multiplicities of the simple modules, because the weight multiplicities of a Verma module is trivially expressible by the Kostant partition function. Kazhdan and Lusztig [31] proposed a conjectural formula for the multiplicity of L_w in M_y in terms of what are now called the *Kazhdan-Lusztig polynomials* $P_{y,w}(\mathbf{q})$. These polynomials are defined for any pair of elements in any Coxeter group, and Kazhdan and Lusztig give a recursive algorithm to compute them. They conjecture that the multiplicity $[M_y : L_w]$ of L_w in any composition series of M_y is equal to

$$[M_y : L_w] = P_{yw_0,ww_0}(1).$$

Moreover, they show that these polynomials $P_{y,w}(\mathbf{q})$ measure the singularities of Schubert varieties in the flag variety $\mathcal{B}$. We will discuss this aspect in more detail in §4.3.

The Kazhdan-Lusztig conjecture was proved independently by Beilinson–Bernstein [4] and by Brylinski–Kashiwara [14] through a series of profound reinterpretations:

$$\mathfrak{g}\text{-modules} \xrightarrow{\text{Localization}} D\text{-modules on } \mathcal{B} \xrightarrow{\text{Riemann-Hilbert}} \text{Perverse sheaves on } \mathcal{B} \tag{3.1}$$

The first step in the above diagram, the *Beilinson-Bernstein localization*, is the inverse of the following equivalence of categories. For any D-modules on the flag variety $\mathcal{B}$, taking global sections of the underlying quasi-coherent sheaf gives an equivalence of abelian categories

$$\Gamma : D\text{−mod}(\mathcal{B}) \xrightarrow{\sim} U(\mathfrak{g})_0\text{−mod}.$$

Here $U(\mathfrak{g})_0$ is the quotient of $U(\mathfrak{g})$ by the ideal generated by the kernel of the homomorphism $\chi_0 : Z(U(\mathfrak{g})) \to \mathbb{C}$ through which $Z(U(\mathfrak{g}))$ acts on the trivial $\mathfrak{g}$-module $\mathbb{C}$.

The second step in the (3.1) is the Riemann-Hilbert correspondence that sends a holonomic D-module on $\mathcal{B}$ to its solution complex, which is a perverse sheaf on the complex manifold $\mathcal{B}$. When starting with an object $M \in \mathcal{O}_0$, the composition of the two steps in (3.1) will produce a perverse sheaf on $\mathcal{B}$ that is constant along Schubert cells (B-orbits).

3.1.1. **Example.** In the case $\mathfrak{g} = \mathfrak{sl}_2$, under the functors in (3.1), the one-dimensional simple module $\mathbb{C}$ in the category $\mathcal{O}_0$ corresponds to the constant perverse sheaf $\mathbb{C}[1]$ on $\mathbb{P}^1$. The two Verma modules M_e (for the identity element $e \in W$, with highest weight 0) and M_s (with highest weight -2, which is also a simple module) correspond respectively to the perverse sheaves $j_!\mathbb{C}[1]$ and $i_*\mathbb{C}$, where $j : \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ and $i : \{\infty\} \hookrightarrow \mathbb{P}^1$ are open and closed embeddings.

The usefulness of the functors in (3.1) is not restricted to the study of category $\mathcal{O}$, but also to the category of $(\mathfrak{g}, K)$-modules, where $K \subset G$ is the fixed point of some involution θ. By Harish-Chandra's theory, $(\mathfrak{g}, K)$-modules are algebraic incarnations of infinite-dimensional representations of real forms of G corresponding to the Cartan involution θ. See [5] for more details.

The Kazhdan-Lusztig conjecture and its proof is a milestone in the development of geometric representation theory. Since then, D-modules and perverse sheaves became standard techniques in the subject.

3.2. **Geometric realization of discrete series representations.** Let G be a connected reductive group over the reals $\mathbb{R}$. Harish-Chandra [29] showed that a discrete series representation of $G(\mathbb{R})$ (direct summands of $L^2(G(\mathbb{R}))$) exists if and only if G has a compact maximal torus H. Let $\mathbb{X}^*(H) = \mathrm{Hom}_{\mathbb{R}}(H, \mathrm{U}(1)) = \mathrm{Hom}_{\mathbb{C}}(H_{\mathbb{C}}, \mathbb{G}_m)$ be the character lattice of H, and $W = W(G, H)$ be the Weyl group scheme $N_G(H)/H$ defined over $\mathbb{R}$. In this case, the isomorphism classes of discrete series representations π_λ of $G(\mathbb{R})$ are in bijection with $W(\mathbb{R})$-orbits on $\{\lambda \in \mathbb{X}^*(H) | \langle \lambda + \rho, \alpha^\vee \rangle \neq 0 \text{ for all coroots } \alpha^\vee \text{ of } (G_{\mathbb{C}}, H_{\mathbb{C}})\}$.

The following geometric realization of the discrete series representations π_λ was conjectured by Langlands [37] and proved by Schmid [53]. The homogeneous space $D = G(\mathbb{R})/H(\mathbb{R})$ can be identified with an open $G(\mathbb{R})$-orbit in the flag variety $\mathcal{B}$ of $G_{\mathbb{C}}$. Recall the line bundle $\mathcal{L}_\lambda$ on $\mathcal{B}$ from §2.3. When $\lambda + \rho$ is regular, π_λ appears as the unique non-vanishing L^2-cohomology group of the line bundle $\mathcal{L}_\lambda|_D$. The non-vanishing degree follows a similar pattern as in the Borel-Weil-Bott theorem.

3.2.1. **Example.** For $G = \mathrm{SL}_{2,\mathbb{R}}$, it is well-known that for each integer $n \geq 2$, $G(\mathbb{R})$ has a discrete series representation D_n^+ with weights $n, n+2, n+4, \cdots$ under its maximal compact $K = H \cong \mathrm{U}(1)$, and also a discrete series D_n^- with weights $-n, -(n+2), -(n+4), \cdots$. In Harish-Chandra's parametrization, D_n^+ corresponds to $\lambda = n-2 \in \mathbb{X}^*(H)$ and D_n^- corresponds to $\lambda = -n$. In the above realization, we identify the complex disc $D = G(\mathbb{R})/H(\mathbb{R})$ with the upper half of $\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$. Then D_n^- can be realized as square-integrable holomorphic sections of $\mathcal{O}(-n)|_D$, while D_n^+ is realized as the degree 1 L^2-cohomology of $\mathcal{O}(n-2)|_D$.

4. Finite groups of Lie type

4.1. **Deligne-Lusztig varieties.** Let G be a split reductive group over a finite field $\mathbb{F}_q$. In this section we are concerned with complex representations of the finite group $G(\mathbb{F}_q)$. For $G = \mathrm{GL}_n$, the irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$ were determined by Green [28], while the actual representations had to wait until the famous work of Deligne and Lusztig [15] to be constructed. The Deligne–Lusztig construction is via algebraic geometry, and it works for any reductive group G.

Recall that the abstract Weyl group $\mathbb{W}$ of G is defined to be the set of G-orbits on $\mathcal{B} \times \mathcal{B}$. For $w \in \mathbb{W}$, we denote the corresponding G-orbit by $O(w) \subset \mathcal{B} \times \mathcal{B}$. We say two Borel subgroups (B, B') are in relative position w if $(B, B') \in O(w)$. If T is a split maximal torus of G and B is a Borel subgroup containing T, then sending any $n \in N_G(T)$ to the relative position of (B, nBn^{-1}) defines a homomorphism $N_G(T) \to \mathbb{W}$ that factors through an isomorphism $W(G, T) \xrightarrow{\sim} \mathbb{W}$.

Let $F : \mathcal{B} \to \mathcal{B}$ be the Frobenius map that raises locally defined regular functions to the qth power. For $w \in \mathbb{W}$, let $X(w) \subset \mathcal{B}$ be the locally closed subscheme classifying those Borel subgroups B such that $(B, F(B))$ are in relative position w. The schemes $X(w)$ are smooth over $\mathbb{F}_q$, and they are called the *Deligne-Lusztig varieties*. The finite group $G(\mathbb{F}_q)$ acts on $X(w)$ by conjugating Borel subgroups. Therefore the étale cohomology groups $\mathrm{H}^*_c(X(w),\overline{\mathbb{Q}}_\ell)$ are $\overline{\mathbb{Q}}_\ell$-representations of $G(\mathbb{F}_q)$. Here, $\mathrm{H}^*_c(X(w),\overline{\mathbb{Q}}_\ell)$ denotes the compactly supported étale cohomoloy of the base change $X(w)_{\overline{\mathbb{F}}_q}$ with coefficients in $\overline{\mathbb{Q}}_\ell$; in particular, it also carries an action of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$.

4.1.1. **Example.** For $G=\mathrm{SL}_2$, $\mathbb{W}=\{1,s\}$. We have $X(1)=\mathbb{P}^1(\mathbb{F}_q)$ and $X(s)$ is the complement $\mathbb{P}^1-\mathbb{P}^1(\mathbb{F}_q)$. We see that $\dim \mathrm{H}^1_c(X(s),\overline{\mathbb{Q}}_\ell)=q$. The action of $\mathrm{SL}_2(\mathbb{F}_q)$ on $\mathrm{H}^1_c(X(s),\overline{\mathbb{Q}}_\ell)$ is the Steinberg representation.

More generally, there is a $G(\mathbb{F}_q)$-equivariant finite étale cover $\widetilde{X}(w)\to X(w)$ that is a torsor under $T_w(\mathbb{F}_q)$. Here T_w, although can be defined intrinsically, is isomorphic to a maximal torus of G of type w (up to $G(\mathbb{F}_q)$-conjugacy, maximal tori in G are classified by conjugacy classes in $\mathbb{W}$). The cohomology groups $\mathrm{H}^*_c(\widetilde{X}(w),\overline{\mathbb{Q}}_\ell)$ carry commuting actions of $G(\mathbb{F}_q)$ and $T_w(\mathbb{F}_q)$. For any character $\theta: T_w(\mathbb{F}_q)\to\overline{\mathbb{Q}}_\ell^\times$, one can take the subspace $\mathrm{H}^*_c(\widetilde{X}(w),\overline{\mathbb{Q}}_\ell)_\theta\subset \mathrm{H}^*_c(\widetilde{X}(w),\overline{\mathbb{Q}}_\ell)$ on which $T_w(\mathbb{F}_q)$ acts via θ. Let R^θ_w be the virtual $G(\mathbb{F}_q)$-module $\sum_i(-1)^i\mathrm{H}^i_c(\widetilde{X}(w),\overline{\mathbb{Q}}_\ell)_\theta$.

4.1.2. **Example.** For $G=\mathrm{SL}_2$ and $w=s$, the finite étale cover $\widetilde{X}(s)$ is the Drinfeld curve defined by the equation $xy^q-x^qy=1$ in $\mathbb{A}^2$. The map $\widetilde{X}(s)\to X(s)=\mathbb{P}^1-\mathbb{P}^1(\mathbb{F}_q)$ is given by $(x,y)\mapsto[x,y]$. The group $T_s(\mathbb{F}_q)$ is the kernel of the norm map $\mathbb{F}^\times_{q^2}\to\mathbb{F}^\times_q$, i.e., $T_s(\mathbb{F}_q)=\{t\in\mathbb{F}^\times_{q^2}|t^{q+1}=1\}$. The action of $t\in T_w(\mathbb{F}_q)\subset\mathbb{F}^\times_{q^2}$ on $\widetilde{X}(s)$ is by $(x,y)\cdot t=(xt,yt)$. For $\theta:T_w(\mathbb{F}_q)\to\overline{\mathbb{Q}}_\ell^\times$ such that $\theta^2\neq1$, $\mathrm{H}^1_c(\widetilde{X}(s))_\theta$ is an irreducible cuspidal representation of $\mathrm{SL}_2(\mathbb{F}_q)$.

Deligne and Lusztig show that every irreducible $\overline{\mathbb{Q}}_\ell$-representation of $G(\mathbb{F}_q)$ appears in R^θ_w for some $w\in\mathbb{W}$ and $\theta:T_w(\mathbb{F}_q)\to\overline{\mathbb{Q}}_\ell^\times$. The virtual representation R^θ_w only depends on the $\mathbb{W}$-conjugacy class of (w,θ), and they are otherwise orthogonal to each other. Moreover, each R^θ_w is close to being irreducible in the sense that its endomorphism ring has dimension independent of q. Building on [15], a complete classification of irreducible representations of $G(\mathbb{F}_q)$ was achieved by Lusztig in his book [41].

4.2. **Springer fibers and representations of Weyl groups.** Let k be an algebraically closed field and G be a reductive group over k. Springer [55] gave a geometric realization of irreducible representations of the (abstract) Weyl group $\mathbb{W}$ of G via the cohomology of *Springer fibers*. For $x\in\mathfrak{g}$, its Springer fiber is the closed subscheme $\mathcal{B}_x\subset\mathcal{B}$ consisting of Borel subgroups $B\subset G$ such that $x\in\mathrm{Lie}\,B$. Springer fibers are usually singular and reducible.

Let $\mathcal{N}$ be the nilpotent cone in $\mathfrak{g}$. Springer constructed an action of $\mathbb{W}$ on the cohomology $\mathrm{H}^*(\mathcal{B}_e,\overline{\mathbb{Q}}_\ell)$ of the Springer fiber $\mathcal{B}_e$ for $e\in\mathcal{N}$. This action does not come from an action of $\mathbb{W}$ on $\mathcal{B}_e$, but rather come from such an action on nearby Springer fibers. Indeed, there is a canonical $\mathbb{W}$-action on $\mathrm{H}^*(\mathcal{B}_x,\overline{\mathbb{Q}}_\ell)$ for all $x\in\mathfrak{g}$. Moreover, all irreducible representations ρ of $\mathbb{W}$ show up in the top cohomology of $\mathcal{B}_e$ for some $e\in\mathcal{N}$, and the nilpotent orbit of e is uniquely determined by ρ. This gives a map

$$\mathrm{Irr}(\mathbb{W})\to\{\text{nilpotent orbits of } G\}.$$

This map is surjective and can be upgraded to an injection from $\mathrm{Irr}(\mathbb{W})$ to the set of pairs $(\mathcal{O},\mathcal{L})$, where $\mathcal{O}$ is a nilpotent orbit of G, and $\mathcal{L}$ is an irreducible G-equivariant local system on $\mathcal{O}$.

4.2.1. **Example.** For the nilpotent element $e=0$, $\mathcal{B}_e=\mathcal{B}$, and the action of $\mathbb{W}$ on $\mathrm{H}^{top}(\mathcal{B})\cong\overline{\mathbb{Q}}_\ell$ is via the sign character of $\mathbb{W}$.

A quick way to construct the Springer actions on $\mathrm{H}^*(\mathcal{B}_x)$ for all $x\in\mathfrak{g}$ at once is to consider the direct image complex of the Grothendieck alteration $\pi:\widetilde{\mathfrak{g}}\to\mathfrak{g}$, where $\widetilde{\mathfrak{g}}=\{(x,B)\in\mathfrak{g}\times\mathcal{B}|x\in\mathrm{Lie}\,B\}$. Lusztig [40] showed that the complex $R\pi_*\overline{\mathbb{Q}}_\ell$ is (up to a shift) the intermediate extension of a local system $\mathcal{F}$ on the regular semisimple open subset $\mathfrak{g}^{\mathrm{rs}}$. The local system $\mathcal{F}=(R\pi_*\overline{\mathbb{Q}}_\ell)|_{\mathfrak{g}^{\mathrm{rs}}}$ carries an obvious $\mathbb{W}$-action because π becomes a $\mathbb{W}$-torsor when restricted over $\mathfrak{g}^{\mathrm{rs}}$. The functoriality of intermediate extension then gives a $\mathbb{W}$-action on $R\pi_*\overline{\mathbb{Q}}_\ell$ extending its action on $\mathcal{F}$. Taking stalks at $x\in\mathfrak{g}$ recovers Springer's $\mathbb{W}$-action on $\mathrm{H}^*(\mathcal{B}_e)$.

When $k = \overline{\mathbb{F}}_q$, the complex $R\pi_*\overline{\mathbb{Q}}_\ell$ and its direct summands are also closely related to characters of representations of $G(\mathbb{F}_q)$ via the Green functions. This was a key ingredient in Springer's original construction, and is a Lie algebra analog of the character sheaves which we will discuss in §4.4.

4.3. **Hecke category and geometrization.** The Hecke algebra H_q for $G(\mathbb{F}_q)$ is defined to be the space of $G(\mathbb{F}_q)$-invariant $\mathbb{C}$-valued functions on $\mathcal{B}(\mathbb{F}_q)\times\mathcal{B}(\mathbb{F}_q)$, with multiplication given by convolution. Choosing a Borel subgroup $B\subset G$ identifies H_q with functions on double cosets $B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/B(\mathbb{F}_q)$. There is a bijection between irreducible characters of $G(\mathbb{F}_q)$ that appear in the principal series representation $\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}\mathbb{C}$ and simple modules of H_q. As a vector space, H_q has a standard basis $\{T_w\}_{w\in\mathbb{W}}$ given by the indicator functions on the set of $\mathbb{F}_q$-points of each G-orbit $O(w)\subset\mathcal{B}\times\mathcal{B}$.

Let $S\subset\mathbb{W}$ be the set of simple reflections. The $\mathbb{C}$-algebra H_q is generated by T_s for $s\in S$ subject to two kinds of relations

- $(T_s+1)(T_s-q)=0$, for all $s\in S$.
- For any $s\neq t\in S$, denote by $m_{s,t}$ the order of st in $\mathbb{W}$. Then the braid relation $T_sT_t\cdots=T_tT_s\cdots$ holds, where both sides are alternating between T_s and T_t and both have $m_{s,t}$ terms.

In the above presentation, if we replace q with an indeterminate $\mathbf{q}$, and introduce a square root v of $\mathbf{q}$, we obtain the *generic Hecke algebra* $H_v(\mathbb{W})$ over $\mathbb{C}[v,v^{-1}]$. Specializing v to 1 we recover the group ring of $\mathbb{W}$; specializing v to $q^{1/2}$ recovers H_q.

Kazhdan and Lusztig [31] introduce a remarkable basis $\{C'_w\}_{w\in\mathbb{W}}$ of $H_v(\mathbb{W})$ indexed by $w\in\mathbb{W}$. The Kazhdan-Lusztig polynomials we mentioned in §3.1 are the entries of the change-of-basis matrix between the Kazhdan-Lusztig basis $\{C'_w\}_{w\in\mathbb{W}}$ and the standard basis $\{T_w\}_{w\in\mathbb{W}}$.

To explain the geometric meaning of the Kazhdan-Lusztig basis, we need to recall Grothendieck's sheaf-to-function correspondence. Consider a scheme X over $\mathbb{F}_q$, a constructible sheaf $\mathcal{F}$ (or more generally a constructible complex of sheaves) with $\overline{\mathbb{Q}}_\ell$-coefficients on X. We may define a $\overline{\mathbb{Q}}_\ell$-valued function $f_{\mathcal{F}}: X(\mathbb{F}_q)\to\overline{\mathbb{Q}}_\ell$ as follows. For any $x\in X(\mathbb{F}_q)$, the cohomology groups $\mathrm{H}^i(\mathcal{F}_x)$ of the stalk $\mathcal{F}_x$ carry actions of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, and in particular actions of the geometric Frobenius Fr_x (acting by $a\mapsto a^{1/q}$ for $a\in\overline{\mathbb{F}}_q$). We define $f_{\mathcal{F}}(x)$ to be the alternating trace

$$f_{\mathcal{F}}(x)=\sum_{i\in\mathbb{Z}}(-1)^i\mathrm{Tr}(\mathrm{Fr}_x,\mathrm{H}^i(\mathcal{F}_x)).$$

The Hecke category $\mathcal{H}=D_G(\mathcal{B}\times\mathcal{B})$ is defined to be the G-equivariant derived category of $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathcal{B}\times\mathcal{B}$. It is a monoidal category under convolution. Let $K_0(\mathcal{H})$ be the Grothendieck group of $\mathcal{H}$, which carries a multiplication coming from the monoidal structure of $\mathcal{H}$ making it into a ring. The sheaf-to-function correspondence gives a ring homomorphism (where we change the coefficient field of H_q from $\mathbb{C}$ to $\overline{\mathbb{Q}}_\ell$)

$$K_0(\mathcal{H})\to H_q. \tag{4.1}$$

Consider the intersection complex IC_w of the closure of $O(w)$ in $\mathcal{B}\times\mathcal{B}$. Kazhdan and Lusztig show that the image of IC_w under the map (4.1) is the specialization of $C'_w\in H_v(\mathbb{W})$ at $v=q^{1/2}$. In particular, the coefficients of the polynomials $P_{y,w}(\mathbf{q})$ are the dimensions of stalks of IC_w along a smaller orbit $O(y)$ in various degrees.

4.4. **Character sheaves.** Irreducible characters of $G(\mathbb{F}_q)$, as class functions on $G(\mathbb{F}_q)$, turn out to be of geometric origin: under the sheaf-to-function correspondence they are very close to being the functions associated to a collection of simple perverse sheaves on G. This remarkable collection of simple perverse sheaves are the *character sheaves* introduced by Lusztig [42].

One feature of character sheaves is that it can be defined on reductive groups over any field. Below we let k be any algebraically closed field and G be a reductive group over k.

Nowadays we have several equivalent definitions of character sheaves. For simplicity we restrict ourselves to *unipotent* character sheaves. First we recall Lusztig's original definition, which uses the diagram

$$\begin{array}{ccc} & G\times\mathcal{B} & \\ {}^{p}\swarrow & & \searrow^{q} \\ G & & \mathcal{B}\times\mathcal{B} \end{array} \tag{4.2}$$

Here p is the projection map and $q(g,B)=(B,gBg^{-1})$ for $g\in G$ and $B\in\mathcal{B}$. There is a G-action on each space in the above diagram by conjugation on all factors. Thus we have a functor

$$p_!q^*:\mathcal{H}=D_G(\mathcal{B}\times\mathcal{B})\to D_G(G) \tag{4.3}$$

the latter being the equivariant derived category of G under the conjugation action by itself. A unipotent character sheaf on G is by definition any simple perverse sheaf in $D_G(G)$ that appears as a summand of an object in the essential image of the functor $p_!q^*$. For example, starting with the constant sheaf C_Δ supported on the diagonal $\Delta(\mathcal{B})\subset\mathcal{B}\times\mathcal{B}$, $p_!q^*(C_\Delta)$ is the direct image complex of the group-theoretic Grothendieck alteration $\pi:\widetilde{G}\to G$. In this sense, the sheaf-theoretic construction of Springer representations is part of the theory of character sheaves. On the other hand, the more mysterious cuspidal character sheaves (those not appearing in the direct image of Grothendieck alteration or its parabolic analogs) also show up as a direct summand of $p_!q^*\mathcal{K}$ for some $\mathcal{K}\in\mathcal{H}$.

A second way of defining character sheaves is to turn Lusztig's deep classification result as a definition. For a parabolic subgroup $P\subset G$ with Levi quotient L, we have the parabolic induction functor

$$\mathrm{Ind}_P^G:D_L(L)\to D_G(G)$$

defined as the pull-push functor along the diagram

$$\begin{array}{ccc} & P/P & \\ \swarrow & & \searrow \\ G/G & & L/L \end{array}$$

where all quotients are by conjugation actions. Lusztig introduces the notion of cuspidal sheaves on a reductive group, which turns out to be clean extensions of local systems supported on a single conjugacy class. Then character sheaves on G can be defined as those simple perverse sheaves that appear as direct summands (up to shifts) in the parabolic induction of cuspidal sheaves of various Levi subquotients of G.

A third way to define (or to characterize) character sheaves is due to Mirković and Vilonen [47] via singular support. Let us assume $k=\mathbb{C}$. For a constructible sheaf $\mathcal{F}\in D_G(G)$ we may talk about its singular support, which is a conical Lagrangian subset $\mathbb{S}(\mathcal{F})\subset T^*G$. Identify T^*G with $G\times\mathfrak{g}^*$ using left invariant differentials, we say that $\mathcal{F}$ has *nilpotent singular support* if $\mathbb{S}(\mathcal{F})\subset G\times\mathcal{N}^*$, where $\mathcal{N}^*\subset\mathfrak{g}^*$ is the nilpotent cone in the dual of the Lie algebra. Mirkovic and Vilonen show that an $\mathrm{Ad}(G)$-equivariant simple perverse sheaf $\mathcal{F}$ on G is a character sheaf if and only if it has nilpotent singular support. In addition to its purely topological nature, this characterization motivates a similar definition in the context of automorphic sheaves (see §6.3) which plays a big role in the Betti geometric Langlands program and in connecting geometric Langlands to the arithmetic Langlands over function fields.

5. p-adic groups

5.1. **Representations of affine Hecke algebras.** Let F be a non-archimedean local field with residue field $\mathbb{F}_q$. It can be a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$. Let G be a reductive group over F which we again assume to be split. We are concerned with smooth representations of the locally compact totally disconnected group $G(F)$ on $\mathbb{C}$-vector spaces. Among the irreducible smooth representations of $G(F)$, the simplest ones are those with nonzero fixed vectors under an Iwahori subgroup (these are called *Iwahori-spherical* representations). An Iwahori subgroup $\mathbf{I}\subset G(F)$ is the preimage of a Borel subgroup $B(\mathbb{F}_q)$ under the reduction map $G(\mathcal{O}_F)\to G(\mathbb{F}_q)$. It can be viewed as the counterpart of Borel subgroups for

p-adic groups. Iwahori-spherical representations of $G(F)$ are in canonical bijection with simple modules of the *Iwahori-Hecke algebra* (or *affine Hecke algebra*) $H_{\mathrm{aff},q} = \mathrm{Fun}_c(\mathbf{I}\backslash G(F)/\mathbf{I}, \mathbb{C})$, the latter consisting of $\mathbb{C}$-valued functions on $G(F)$ that are left and right invariant under $\mathbf{I}$ and are supported on finitely many $\mathbf{I}$-double cosets.

For simplicity, assume G is almost simple and simply-connected. In this case, the set of $\mathbf{I}$-double cosets in $G(F)$ are parametrized by the affine Weyl group $\mathbb{W}_{\mathrm{aff}}$ of G. The group $\mathbb{W}_{\mathrm{aff}}$ is a Coxeter group with simple reflections $S_{\mathrm{aff}} = S \coprod \{s_0\}$, where S is the set of simple reflections in the abstract Weyl group $\mathbb{W}$ of G. Just as the Hecke algebra H_q, $H_{\mathrm{aff},q}$ admits a presentation with generators T_s for $s \in S_{\mathrm{aff}}$ and the same quadratic and braid relations as for H_q. Replacing q with an invertible indeterminate $\mathbf{q} = v^2$ we get the *generic affine Hecke algebra* $H_v(\mathbb{W}_{\mathrm{aff}})$ over $\mathbb{C}[v, v^{-1}]$.

In order to classify simple modules of $H_{\mathrm{aff},q}$, Kazhdan and Lusztig [33] gave another description of $H_v(\mathbb{W}_{\mathrm{aff}})$ in terms of the Langlands dual group $G^\vee$. Let $\mathrm{St}_{G^\vee}$ be the Steinberg variety of $G^\vee$. Namely, it is the fiber square $\widetilde{\mathcal{N}}^\vee \times_{\mathfrak{g}^\vee} \widetilde{\mathcal{N}}^\vee$ ($\mathfrak{g}^\vee = \mathrm{Lie}\ G^\vee$, $\widetilde{\mathcal{N}}^\vee$ the Springer resolution of the nilpotent cone in $\mathfrak{g}^\vee$). Therefore $\mathrm{St}_{G^\vee}$ classifies triples (e, B, B') where e is a nilpotent element in $\mathfrak{g}^\vee$, B and B' are Borel subgroups of $G^\vee$ such that $e \in \mathfrak{n}_B \cap \mathfrak{n}_{B'}$. There is a canonical $G^\vee \times \mathbb{G}_m$ action on $\mathrm{St}_{G^\vee}$ where $g \in G^\vee$ acts by the adjoint action on $e \in \mathfrak{g}^\vee$ and by conjugation on B, B', and $\mathbb{G}_m$ acts by scaling e *by the square*. Let $K_0^{G^\vee \times \mathbb{G}_m}(\mathrm{St}_{G^\vee})_{\mathbb{C}}$ be the complexified Grothendieck group of the category of $G^\vee \times \mathbb{G}_m$-equivariant coherent sheaves on $\mathrm{St}_{G^\vee}$. It carries an algebra structure given by convolution because $\mathrm{St}_{G^\vee}$ is a fiber-square. In particular, $K_0^{G^\vee \times \mathbb{G}_m}(\mathrm{St}_{G^\vee})_{\mathbb{C}}$ is a $\mathbb{C}[v, v^{-1}] \cong K_0(\mathrm{Rep}(\mathbb{G}_m))_{\mathbb{C}}$-algebra. Kazhdan and Lusztig [33, Theorem 3.5] give an isomorphism of $\mathbb{C}[v, v^{-1}]$-algebras

$$H_v(\mathbb{W}_{\mathrm{aff}}) = K_0^{G^\vee \times \mathbb{G}_m}(\mathrm{St}_{G^\vee})_{\mathbb{C}}. \tag{5.1}$$

From this, they deduce that simple modules of $H_{\mathrm{aff},q}$ are parametrized by $G^\vee$-conjugacy classes of triples (s, e, ρ), where $s \in G^\vee$ is semisimple, $e \in \mathcal{N}^\vee$ such that $\mathrm{Ad}(s)e = qe$, and ρ is an irreducible representation of $A_{s,e} := \pi_0(C_{G^\vee}(s, e))$ such that ρ appears in the action of $A_{s,e}$ on $K_0(\mathcal{B}_e^s)$ (the s-fixed points on the Springer fiber $\mathcal{B}_e$). The simple module of $H_{\mathrm{aff},q}$ corresponding to (s, e, ρ) is the unique simple quotient of a certain specialization (using $(s, q) \in C_{G^\vee \times \mathbb{G}_m}(e)$) of $K_0^{C_{G^\vee \times \mathbb{G}_m}(e)}(\mathcal{B}_e)$, which carries a natural $H_{\mathrm{aff},q}$-action.

5.2. **Affine Hecke categories.** There is an affine version of the geometrization construction in §4.3. We work over a base field k, and G is a split reductive group k. Let Fl be the affine flag variety of G; it is the quotient of the loop group $G(\!(t)\!)$ (an ind-group over k) by an Iwahori subgroup $\mathbf{I}$ (now viewed as a pro-algebraic group over k) in the flat topology. Similar to the affine Grassmannian (see §2.4), Fl is an infinite union of projective schemes of increasing dimension. The affine Hecke category $\mathcal{H}_{\mathrm{aff}}$ is the derived category of $G(\!(t)\!)$-equivariant $\overline{\mathbb{Q}}_\ell$-sheaves, properly defined, on $\mathrm{Fl} \times \mathrm{Fl}$. As in the case of the finite Hecke category $\mathcal{H}$, it carries a monoidal structure given by convolution. It can also be identified with sheaves on the double stack of $\mathbf{I}$-double cosets $\mathbf{I}\backslash G(\!(t)\!)/\mathbf{I}$, or $\mathbf{I}$-equivariant sheaves on Fl.

When $k = \mathbb{F}_q$ is a finite field, the sheaf-to-function correspondence again gives an algebra homomorphism

$$K_0(\mathcal{H}_{\mathrm{aff}}) \to H_{\mathrm{aff},q}.$$

The intersection complexes of the closure of $\mathbf{I}$-orbits on Fl map to the (specialization to $v = q^{1/2}$ of) Kazhdan-Lusztig basis of $H_{\mathrm{aff},q}$ under the above map.

Now we switch to an algebraically closed field k. A theorem of Bezrukavnikov [11] categorifies the isomorphism (5.1). The left side of (5.1) is replaced by the monoidal category $\mathcal{H}_{\mathrm{aff}}$, whose Grothendieck group is identified with the specialization of $H_v(\mathbb{W}_{\mathrm{aff}})$ at $v = 1$; the right side of (5.1) by definition is the Grothendieck group of $\mathrm{Coh}^{G^\vee \times \mathbb{G}_m}(\mathrm{St}_{G^\vee})$, but we should drop $\mathbb{G}_m$-equivariance to match with the specialization $v = 1$ on the left side. Bezrukavnikov constructed a monoidal equivalence

$$\mathcal{H}_{\mathrm{aff}} \cong D^b\mathrm{Coh}^{G^\vee}(\mathrm{St}_{G^\vee})$$

where the geometric objects on the right side are now defined over $\overline{\mathbb{Q}}_\ell$.

The geometric Satake equivalence in §2.4 is a pillar of the geometric Langlands program which we will elaborate on in §6.3. The affine Hecke category is the next simplest category after the one studied in the geometric Satake equivalence, and it plays a basic role in the local geometric Langlands program.

5.3. **Affine Springer fibers.** Kazhdan and Lusztig [34] introduced an affine analog of Springer fibers. Let $F = k((t))$ be the Laurent series field over a field k, and $\gamma \in \mathfrak{g}(F)$ be regular semisimple. The affine Springer fiber Fl_γ of γ is the sub-ind-scheme of the affine flag variety Fl consisting of cosets $g\mathbf{I}$ such that $\mathrm{Ad}(g^{-1})\gamma \in \mathrm{Lie}\ \mathbf{I}$. They showed that Fl_γ is finite-dimensional (as an ind-scheme over k) but may have infinitely many irreducible components. Lusztig [43] constructed an action of the affine Weyl group on the homology of affine Springer fibers, giving an affine analog of Springer's action.

5.3.1. **Example.** Let $G = \mathrm{SL}_2$. Take $\gamma = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$, then Fl_γ is an infinite chain of projective lines, each intersecting two neighboring ones at a point.

One the other hand, for $\gamma = \begin{pmatrix} 0 & t \\ t^2 & 0 \end{pmatrix}$, Fl_γ is the union of two projective lines intersecting at one point.

In their original paper, Kazhdan and Lusztig noted a relationship between affine Springer fibers and certain characters of p-adic groups. Motivated by the Fundamental Lemma conjectured by Langlands and Shelstad in the endoscopy theory of automorphic forms, Goresky, Kottwitz and MacPherson [27] observed that the orbital integrals for Lie algebras over local function fields can be interpreted as point-counting on affine Springer fibers over finite fields (or rather their analogs inside the affine Grassmannian). Moreover, the phenomenon of *stability* for orbital integrals can be interpreted using the action of the centralizer of γ (in the loop group $G((t))$) on the cohomology of Fl_γ. Their work initiated the algebro-geometric approach to the Fundamental Lemma, which was completed by Ngô by combining it with a global perspective, see the discussions in §6.4. For an exposition of the connection between orbital integrals and affine Springer fibers, we refer to [62].

Recall from §5.1 that the K-groups of Springer fibers gives modules of the affine Hecke algebra, the K-groups and cohomology of affine Springer fibers also give modules of the *double affine Hecke algebra* (introduced by Cherednik) and its degenerations. For results in this direction, we refer to the work of Vasserot [58], Varagnolo–Vasserot [59] and Oblomkov–Y. [52].

Comparing to the discussions in §4, a complete theory of character sheaves for p-adic groups is still missing. See however the discussion in §7.2.

6. Automorphic representations

6.1. **Automorphic representations and the Langlands correspondence.** Let F be a global field, i.e., a number field or the function field of an algebraic curve over a finite field. Let $\mathbb{A}_F$ be the ring of adèles of F. Let G be a split reductive group over F, then $G(\mathbb{A}_F)$ is a locally compact topological group. An automorphic representation of $G(\mathbb{A}_F)$ is, roughly speaking, a representation π of $G(\mathbb{A}_F)$ that appears in the function space of the coset space $G(F)\backslash G(\mathbb{A}_F)$. Such a representation π is called cuspidal if it has zero image under the averaging map to the function space of $N_P(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$, where N_P is the unipotent radical of any proper parabolic subgroup $P \subsetneq G$.

Let Γ_F be the absolute Galois group of F. Let $G^\vee$ be the Chevalley group that is Langlands dual to G. Fix a prime ℓ not equal to the characteristic of F. The conjectural global Langlands correspondence, in rough terms, predicts a finite-to-one map

$$\{\text{Irr. auto. reps of } \mathrm{GL}_n(\mathbb{A}_F)\} \to \{\text{continuous homomorphisms } \rho_\ell : \Gamma_F \to G^\vee(\overline{\mathbb{Q}}_\ell)\}. \tag{6.1}$$

with several compatibilities, a crucial one being the equality of the L-functions defined from both sides.

The geometrization in the setting of automorphic representations has two goals

(1) Realizing automorphic representations in the cohomology of algebraic varieties or stacks.
(2) Finding Galois representations attached to automorphic representations, as predicted by the global Langlands correspondence, in the cohomology of algebraic varieties or stacks.

For number fields, the geometric objects involved are the *Shimura varieties*; for function fields, they are the moduli stacks of *Drinfeld Shtukas*. Below we will concentrate on the function field case.

6.2. **Moduli of Drinfeld Shtukas.** Now let $F = \mathbb{F}_q(X)$ be the function field of a smooth, projective and geometrically connected curve X over $\mathbb{F}_q$. In the function field case, we replace the Galois group Γ_F by the Weil group W_F. A continuous homomorphisms $\rho : W_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, up to conjugacy, is the same datum as a rank n Weil local system on some non-empty Zariski open subset U of X (i.e., a local system on $U_{\overline{\mathbb{F}}_q}$ together with an equivariant structure under the pullback by the Frobenius map). For $G = \mathrm{GL}_n$, L.Lafforgue [35] proved a strong version of the Langlands conjecture by establishing a bijection

$$\{\text{Irr. cuspidal auto. reps of } \mathrm{GL}_n(\mathbb{A}_F)\} \leftrightarrow \{\text{Irr. rank } n \text{ Weil local systems on an open of } X\}. \tag{6.2}$$

This generalizes the pioneering work of Drinfeld [18] for GL_2.

For a general reductive group G over a function field F, continuous homomorphisms $\rho : W_F \to G^\vee(\overline{\mathbb{Q}}_\ell)$ can again be interpreted as Weil $G^\vee$-local systems on some open subset of X. V.Lafforgue [36] constructed a semisimplified version of the map (6.1):

$$\{\text{Irr. cuspidal auto. reps of } G(\mathbb{A}_F)\} \to \{\text{Semisimple Weil } G^\vee\text{-local systems on an open of } X\}.$$

The geometric object that plays a fundamental role in the above developments is the notion of *Shtukas* introduced by Drinfeld [17]. For any scheme S over $\mathbb{F}_q$, let $\mathrm{Fr}_S : S \to S$ be the Frobenius morphism over $\mathbb{F}_q$ that raises local functions to the qth power. We recall Drinfeld's original version of rank n Shtukas. An S-family of rank n Shtukas of Drinfeld type is the datum of

$$(x_1, x_2, \mathcal{F}_0 \xrightarrow{\alpha} \mathcal{F}_1 \xleftarrow{\beta} \mathcal{F}_2 \overset{\iota}{\cong} (\mathrm{id} \times \mathrm{Fr}_S)^* \mathcal{F}_0). \tag{6.3}$$

Here

- $x_1, x_2 : S \to X$ are maps called *legs*;
- $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$ are rank n vector bundles over $X \times S$ (fiber product over $\mathbb{F}_q$);
- The maps α and β are called modifications. Here, α is an injective map of coherent sheaves with cokernel scheme-theoretically supported on the graph $\Gamma(x_1) \subset X \times S$ of x_1, and is locally free of rank one there. The same properties hold for β, whose cokernel is supported on the graph of x_2.
- $\iota : \mathcal{F}_2 \xrightarrow{\sim} (\mathrm{id} \times \mathrm{Fr}_S)^* \mathcal{F}_0$ is an isomorphism of vector bundles over $X \times S$.

Shtukas generalize the earlier notion of *elliptic modules* introduced by Drinfeld [16], which are analogs of elliptic curves. Shtukas can be viewed as global counterparts of Deligne-Lusztig varieties. Shtukas can also be thought of as a version of motives not defined in characteristic p but whose coefficient ring is of characteristic p. For more discussions on this analogy, we refer to [64].

There is an algebraic stack $\mathrm{Sht}_n^{\mathrm{DrL}}$ (the superscript stands for Drinfeld-Lafforgue) whose S-points is the groupoid of S-families of rank n Shtukas of Drinfeld type defined above. The stack $\mathrm{Sht}_n^{\mathrm{DrL}}$ is a Deligne-Mumford stack locally of finite type over $\mathbb{F}_q$. The map $\pi : \mathrm{Sht}_n^{\mathrm{DrL}} \to X^2$ recording the two legs x_1, x_2 is smooth of relative dimension $2(n-1)$.

For a reductive group G over $\mathbb{F}_q$, more general versions of Shtukas and their moduli stacks are defined and studied by Varshavsky [57]: vector bundles are replaced by (principal) G-bundles; there can be an arbitrary finite number of legs (say r), and the modification at each leg (analogs of α and β in (6.3)) can be arbitrary, parametrized by a sequence of dominant coweights $\lambda = (\lambda_1, \cdots, \lambda_r)$ of G. We denote by Sht_G^λ the moduli stack of G-Shtukas with r legs and modification type λ. We have a leg map $\pi : \mathrm{Sht}_G^\lambda \to X^r$. When $r = 1$, we should think of π as the analogue of the structure map $\mathrm{Sh}_G \to \mathrm{Spec}\, \mathcal{O}_E$ for a Shimura variety Sh_G defined over a number ring $\mathcal{O}_E$. Having Shtukas with more than one leg is a crucial extra structure in the function field case, which allows V.Lafforgue to construct Galois representations attached to automorphic representations.

6.3. **The geometric Langlands correspondence.** Automorphic forms for groups over the function field of a curve X admits a geometrization in the spirit of sheaf-to-function correspondence (see §4.4). In this geometrization automorphic forms are upgraded to *automorphic sheaves*, and it leads to a categorical upgrading of the map (6.1).

The starting point is an interpretation of the coset space $G(F)\backslash G(\mathbb{A}_F)$ in terms of G-bundles on X by Weil. Let $|X|$ be the set of closed points of X, and for each $x \in |X|$ let $\mathcal{O}_x$ be the completed local ring of

X at x, with fraction field F_x and residue field k_x. Let $K = \prod_{x \in |X|} G(\mathcal{O}_x)$. Then the groupoid of double cosets $G(F)\backslash G(\mathbb{A}_F)/K$ is canonically equivalent to the groupoid of principal G-bundles on the curve X.

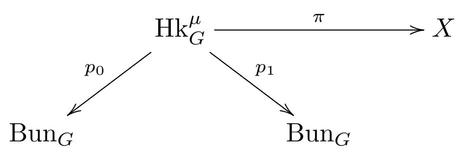
The space of *unramified automorphic functions* are by definition $\mathbb{C}$-valued function space

$$\mathcal{A}_K := \mathrm{Fun}(G(F)\backslash G(\mathbb{A}_F)/K).$$

Hence $\mathcal{A}_K$ can be interpreted as functions on the isomorphism classes of G-bundles on X. On the space $\mathcal{A}_K$ there is an action of the global Hecke algebra $\mathcal{H}_{G,K} = \mathrm{Fun}_c(K\backslash G(\mathbb{A}_F)/K)$. The study of unramified automorphic representations is equivalent to the study of $\mathcal{A}_K$ as a module of $\mathcal{H}_{G,K}$. In particular, since $\mathcal{H}_{G,K}$ is commutative, we are interested in diagonalizing its action on $\mathcal{A}_K$ and describing its spectrum as a subset among ring homomorphisms $\mathcal{H}_{G,K} \to \mathbb{C}$.

Let Bun_G be the moduli stack of G-bundles on X; this is an algebraic stack locally of finite type and smooth of dimension $(g-1)\dim G$ over $\mathbb{F}_q$. Since $\mathcal{A}_K$ can be interpreted as the space of functions on $\mathrm{Bun}_G(\mathbb{F}_q)$, it is natural to geometrize $\mathcal{A}_K$ into the category $D(\mathrm{Bun}_G)$ of $\overline{\mathbb{Q}}_\ell$-sheaves on the stack Bun_G. The action of $\mathcal{H}_{G,K}$ on $\mathcal{A}_K$ can be geometrized into certain pull-push functors on sheaves along the Hecke correspondences over Bun_G, called *Hecke functors*. Moreover, eigenfunctions under $\mathcal{H}_{G,K}$ should geometrize to *eigensheaves* under the Hecke functors. We give an example below for $G = \mathrm{GL}_2$.

6.3.1. **Example.** When $G = \mathrm{GL}_2$, the simplest kind of Hecke correspondence, which is analogous to the family of operators $\{T_p\}$ on modular forms (for varying primes p), is given by the Hecke stack



Here Hk_G^μ is the moduli stack whose S-points are tuples $(x, \mathcal{F}_0 \xrightarrow{\alpha} \mathcal{F}_1)$ where $x : S \to X$, $\mathcal{F}_0$ and $\mathcal{F}_1$ are rank two vector bundles over $X \times S$, and α is an injective map of coherent sheaves on $X \times S$ whose cokernel is scheme-theoretically supported on the graph of x and is locally free of rank one there. The map p_i records $\mathcal{F}_i$ for $i = 0, 1$, and π records the point x. An eigensheaf in this case is an object $E \in D(\mathrm{Bun}_G)$ such that $(\pi \times p_1)_! p_0^* E \cong L \boxtimes E \in D(X \times \mathrm{Bun}_G)$ for some $L \in D(X)$. If E is cuspidal, it turns out that L is necessarily a rank two local system on X, which is the local system attached to the unramified automorphic representation with K-fixed vector given by the Frobenius trace function f_E of E, under the global Langlands correspondence (6.2).

Drinfeld [19] used the above geometrization to prove the unramified Langlands correspondence for GL_2 over function fields. He constructed a cuspidal eigensheaf E starting from an irreducible rank two local system L on X. This approach was generalized by Laumon [38] to GL_n, and cultimated in the work of Frenkel–Gaitsgory–Vilonen [21] who completed the construction of cuspidal eigensheaves on $\mathrm{Bun}_{\mathrm{GL}_n}$ for any n.

On the other hand, for general reductive groups G over $\mathbb{C}$, Beilinson and Drinfeld [8] formulated an unramified categorical geometric Langlands conjecture. The automorphic side is the derived category of D-modules on Bun_G, and the spectral (Galois) side is the the derived category of quasi-coherent sheaves on the moduli stack $\mathrm{Loc}_{G^\vee}$ of (de Rham) $G^\vee$-local systems on X. They developed a global analog of the Beilinson-Bernstein localization [8] and used it to construct eigensheaves from $G^\vee$-opers.

In [1], Arinkin and Gaitsgory gave a precise formulation of the unramified categorical geometric Langlands conjecture by introducing the notion of singular support for coherent sheaves. The conjecture predicts an equivalence of dg categories over $\mathbb{C}$

$$D\mathrm{-mod}(\mathrm{Bun}_G) \cong \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{G^\vee}) \tag{6.4}$$

satisfying, among other things, compatibilities with Hecke actions on both sides. In a recent breakthrough [23], Gaitsgory, Raskin and their collaborators prove the equivalence (6.4), completing the vast project that spanned forty years.

We continue to assume G and X are defined over $\mathbb{C}$. In [10], Ben-Zvi and Nadler proposed another version of the categorical geometric Langlands conjecture. On the spectral (Galois) side the moduli stack of de Rham G-local systems on X is replaced by its Betti version, namely the moduli stack of homomorphisms $\pi_1(X) \to G^\vee$. On the automorphic side they consider sheaves on Bun_G in the topological sense, but a crucial new ingredient is that they impose the condition that sheaves should have singular support in the *global nilpotent cone*. This is a global analog of the nilpotent singular support property of character sheaves. This singular support condition allows Nadler–Y. [50] to construct the spectral action (the action of $\mathrm{QCoh}(\mathrm{Loc}_{G^\vee})$ on the automorphic side, as predicted by the categorical conjecture).

The singular support condition in the Betti version inspired analogous constructions in other versions of the geometric Langlands correspondence, and eventually led to a satisfying bridge from the geometric to the classical Langlands correspondence for function fields by the work of Arinkin–Gaitsgory–Kazhdan–Raskin–Rozenblyum–Varshavsky [2], which we will say more about in §7.1.

Going beyond the unramified context, one should replace Bun_G with the moduli of G-bundles on X with level structures at a finite number of points S, and replace $\mathrm{Loc}_{G^\vee}$ by an appropriate moduli stack of local systems on the punctured curve $X - S$. Not much is known for general ramifications, especially when the level structures are deeper than the Iwahori level. However, there are ramified cases where the automorphic side is simple to describe, the so-called *rigid* cases, and their consequences on the Galois side turn out to be quite useful in solving arithmetic questions such as the inverse Galois problem. For works in this direction, see [30, 60, 61] and the survey article [63].

6.4. **Trace formula and Hitchin moduli space.** For applications to automorphic forms it is useful to have a formula for the trace of Hecke operators on the space of automorphic forms in terms of group-theoretic data. Such a formula is known as the trace formula. A comprehensive introduction to the trace formula can be found in [3].

6.4.1. **Example.** We start with a toy model of the trace formula. Let Γ be a finite group acting on a set S. We would like to count the number of Γ-orbits in S. By Burnside's lemma, we have

$$|S/\Gamma| = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}|S^\gamma|. \tag{6.5}$$

We can formulate this counting problem intrinsically in terms of the groupoid $\mathfrak{X} = S/\Gamma$. For a finite groupoid $\mathfrak{X}$ we let $|\mathfrak{X}|$ be its cardinality weighted by the reciprocal of the automorphisms

$$|\mathfrak{X}| = \sum_{x\in \mathrm{Ob}(\mathfrak{X})/\cong}\frac{1}{|\mathrm{Aut}(x)|}.$$

Let $I(\mathfrak{X})$ be the inertia groupoid of $\mathfrak{X}$: its objects are pairs (x,α) where $x \in \mathrm{Ob}(\mathfrak{X})$ and α is an automorphism of x. Morphisms between (x,α) and (y,β) in $I(\mathfrak{X})$ are isomorphisms $f : x \xrightarrow{\sim} y$ in $\mathfrak{X}$ such that $f\alpha = \beta f$. Then Burnside's lemma can be generalized to give

$$|\mathrm{Ob}(\mathfrak{X})/_\cong| = |I(\mathfrak{X})|. \tag{6.6}$$

When $\mathfrak{X} = S/\Gamma$, we may further rewrite (6.5) as

$$|S/\Gamma| = \sum_{\gamma\in\Gamma/\Gamma}\frac{|S^\gamma|}{|C_\Gamma(\gamma)|}. \tag{6.7}$$

where the sum runs over a set of representatives of conjugacy classes in Γ. The index set of the summation is now not intrinsic to the groupoid $\mathfrak{X}$, but depends on the presentation $\mathfrak{X} = S/\Gamma$.

Now we apply the same principle to count the number of isomorphism classes of G-bundles on the curve X over $\mathbb{F}_q$ (which will be infinite, but let us ignore this issue for now). Note that this is not computing $|\mathrm{Bun}_G(\mathbb{F}_q)|$, which is a weighted counting of G-bundles. Rather we should be counting $|I(\mathrm{Bun}_G)(\mathbb{F}_q)|$ by (6.6). Now the inertia stack $I(\mathrm{Bun}_G)$ classifies $(\mathcal{E},\alpha)$ where $\mathcal{E}$ is a G-bundle on X and α is an automorphism of $\mathcal{E}$. By Weil's interpretation, the dimension of the space of unramified automorphic functions $\mathcal{A}_K$ (see §6.3) is the number of isomorphism classes of G-bundles, which we have seen to be the same as $|I(\mathrm{Bun}_G)(\mathbb{F}_q)|$. The latter can be computed using the Grothendieck-Lefschetz trace formula, which expresses $|I(\mathrm{Bun}_G)(\mathbb{F}_q)|$ as the alternating trace of Frobneius on the cohomology of $I(\mathrm{Bun}_G)$.

Taking one step further, as we did in (6.7) for the toy model, we may break up the counting in terms of "conjugacy classes" of the automorphism α. For example, if $G = \mathrm{GL}_n$, $\alpha : \mathcal{E} \xrightarrow{\sim} \mathcal{E}$ is an automorphism of a vector bundle $\mathcal{E}$, its characteristic polynomial is a well-defined degree n monic polynomial in $\mathbb{F}_q[t]$. In general, the notion of characteristic polynomial is replaced by the invariant theory quotient $G /\!\!/ G$. This gives a map

$$\chi : I(\mathrm{Bun}_G) \to G /\!\!/ G. \tag{6.8}$$

Then $|I(\mathrm{Bun}_G)(\mathbb{F}_q)|$ is decomposed into the contribution of each fiber of χ. For example, when $G = \mathrm{GL}_n$ and $a(t) \in \mathbb{F}_q(t)$ is a degree n polynomial with n distinct roots in $\mathbb{F}_q$, then $\chi^{-1}(a)$ can be identified with the moduli stack of n-tuples of line bundles on X, because any rank n vector bundle $\mathcal{E}$ with an automorphism α whose characteristic polynomial is $a(t)$ is canonically a direct sum of eigen-line bundles under α. To summarize, we get a meta-trace formula for the identity operator on $\mathcal{A}_K$, in which both sides are infinite

$$\dim \mathcal{A}_K \text{ “ = ” } \sum_{a \in (G/\!\!/G)(\mathbb{F}_q)} |\chi^{-1}(a)|.$$

The actual (stable) trace formula concerns the trace of Hecke operators on $\mathcal{A}_K$ on the left side, and the right side will be a sum of *global (stable) orbital integrals* indexed by F-points of $G /\!\!/ G$. The quantities $|\chi^{-1}(a)|$ are examples of global (stable) orbital integrals. Making both sides convergent is a complicated question and constitutes the technical heart of Arthur's work on trace formula, see [3].

In Langlands's theory of endoscopy, a certain conjectural identity of orbital integrals called the Fundamental Lemma plays a pivotal role. The Fundamental Lemma is proved by B.C.Ngô [51] after the work of many mathematicians. The key new insight of Ngô is a geometric interpretation of the (Lie algebra analog of) global orbital integrals as point-counting on the fibers of the Hitchin map. Indeed, the Hitchin moduli stack can be thought of as a Lie algebra analog of $I(\mathrm{Bun}_G)$, and the Hitchin map an analog of the map χ in (6.8). Ngô's proof of the fundamental lemma was achieved through a deep study of the cohomology of fibers of the Hitchin map.

6.5. **Local Langlands correspondence and Fargues-Scholze geometrization.** Now let us go back to the setting of §5 and let F be a non-archimedean local field: a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p(\!(t)\!)$. Let W_F be the Weil group of F. The local Langlands conjecture predicts a parametrization of all irreducible smooth representations of a p-adic group $G(F)$ on $\overline{\mathbb{Q}}_\ell$-vector spaces in terms the Langlands dual group $G^\vee$ over $\overline{\mathbb{Q}}_\ell$. In particular, it predicts the existence of a map

$$\{\text{Irr. smooth reps of } G(F)\} \to \{\text{Continuous homomorphisms } \rho : W_F \to G^\vee(\overline{\mathbb{Q}}_\ell)\}$$

with strong compatibilities with many representation-theoretic constructions and invariants, such as parabolic induction and characters.

A slightly weaker version of the such a map has been constructed by Genestier–V.Lafforgue [24] when F has equal characteristic, and by Fargues–Scholze [20] for general F. These constructions attach a *semisimple* homomorphism $\rho : W_F \to G^\vee(\overline{\mathbb{Q}}_\ell)$ to an irreducible smooth representation of $G(F)$, which should be the semisimplification of the conjectured Langlands parameter. The construction of Genestier–V.Lafforgue uses a local version of the moduli stack of Shtukas. The construction of Fargues–Scholze builds on the amazing insight that ideas from the geometric Langlands correspondence can be borrowed to work for the *Fargues–Fontaine curve*, an analytic curve that governs p-adic Hodge structures. Along the way they also construct moduli stack of Shtukas for mixed characteristic local fields F, generalizing Rapoport–Zink spaces (local version of Shimura varieties). Both works rely on the idea of *excursion operators* discovered in V.Lafforgue's work [36] on the global Langlands parameters for function fields.

7. The trace principle

It has been a guiding principle in representation theory that representations of a group (or an algebra) are determined by its trace up to semisimplification (of course when traces make sense, so finiteness conditions are needed). In recent years, a vast generalization of the above principle, partly motivated by topological quantum field theory, has emerged in geometric representation theory, which can be called the *categorical trace principle*. The general principle is that taking trace allows one to go one step down the categorical level: starting with a vector space with an endomorphism, taking trace gives a number; starting with a category with an endofunctor, taking trace gives a vector space; starting with a 2-category with an endo-2-functor, taking trace gives a usual category, and so one. For an introduction to the trace formalism, we refer to [22, §3].

The categorical trace principle has become a powerful tool in the subject. It puts many known constructions in the same framework, and suggests deep new constructions.

7.1. **Automorphic forms as trace.** In the setup of the unramified categorical geometric Langlands correspondence of §6.3, let us assume G and X are defined over $\mathbb{F}_q$. On the automorphic side we consider the category $\mathrm{Shv}_{\mathcal{N}}(\mathrm{Bun}_G)$ of $\overline{\mathbb{Q}}_\ell$-sheaves on the $\overline{\mathbb{F}}_q$-stack $\mathrm{Bun}_G(X)_{\overline{\mathbb{F}}_q}$, with singular support contained in the global nilpotent cone. In [2], the authors prove a surprisingly clean formula relating the categorical and classical Langlands correspondence for function fields. Namely, the Frobenius morphism $\mathrm{Fr} : \mathrm{Bun}_G(X) \to \mathrm{Bun}_G(X)$ induces an endo-functor Fr_* of $\mathrm{Shv}_{\mathcal{N}}(\mathrm{Bun}_G)$. The authors prove that the trace $\mathrm{Tr}(\mathrm{Fr}_*, \mathrm{Shv}_{\mathcal{N}}(\mathrm{Bun}_G))$ is defined (which amounts to the dualizability of $\mathrm{Shv}_{\mathcal{N}}(\mathrm{Bun}_G)$) as a complex of $\overline{\mathbb{Q}}_\ell$-vector spaces, and as such there is a quasi-isomorphism

$$\mathrm{Tr}(\mathrm{Fr}_*, \mathrm{Shv}_{\mathcal{N}}(\mathrm{Bun}_G) \cong \mathrm{Fun}_c(\mathrm{Bun}_G(\mathbb{F}_q)),$$

the right side being compactly supported $\overline{\mathbb{Q}}_\ell$-valued functions on the set of isomorphism classes of G-bundles on X. In view of Weil's interpretation, the right side is precisely compactly supported functions in $\mathcal{A}_K = \mathrm{Fun}(G(F)\backslash G(\mathbb{A}_F)/K)$, where $K = \prod_{x\in|X|} G(\mathcal{O}_x)$. This is a significant enrichment of the sheaf-to-function correspondence. If one has the categorical geometric Langlands correspondence available over $\mathbb{F}_q$, it would gives a precise description of the Hecke module $\mathcal{A}_K$ in terms of the Langlands dual group $G^\vee$.

7.2. **Character sheaves as a categorical trace.** It was observed in several sheaf-theoretic contexts, see [9, 12, 44], that the category of character sheaves for a reductive group G is equivalent to the categorical trace (or Hochschild homology) of the Hecke category $\mathcal{H}$ introduced in §4.3. More precisely, the image of the functor (4.3), after idempotent completion, is the derived category of unipotent character sheaves on G. The functor (4.3) realizes the category of unipotent character sheaves as the categorical trace of the Hecke category $\mathcal{H}$. If one replaces $\mathcal{H}$ by its monodromic versions, namely replacing the left and right B-equivariance by equivariance with respect to various rank one local systems pulled back from the Cartan T, the same procedure realizes the entire category of character sheaves on G as the categorical trace of monodromic Hecke categories.

Passing from a reductive group G to its loop group LG, one can reverse the logic and *define* the notion of *affine character sheaves* (or rather the depth zero part) as the categorical trace of the affine Hecke category of G and its monodromic versions. An in-depth study of affine character sheaves in the spirit of Lusztig's work [42] on character sheaves is still lacking.

7.3. **Representations as a twisted categorical trace.** Representations of groups themselves can be realized as the trace of a monoidal category. In [45], Lusztig made the following key observation: for a split reductive group G over $\mathbb{F}_q$, the category of unipotent representations of $G(\mathbb{F}_q)$ can be realized as the Frobenius twisted categorical trace of the Hecke category $\mathcal{H}$ of G. Indeed, in the diagram (4.2), instead of considering G-action on G by conjugation, we let G act on G by Frobenius-twisted conjugation: $h\cdot g = \mathrm{Fr}(h)gh^{-1}$, where $\mathrm{Fr} : G \to G$ is the Frobenius morphism over $\mathbb{F}_q$. To make the map $q : G\times\mathcal{B} \to \mathcal{B}\times\mathcal{B}$ equivariant under G, the G-action on the first factor $\mathcal{B}$ should also be pre-composed by the Frobenius on G, while the action on second factor remain unchanged. Then $p_!q^*$ gives a functor

$$\mathcal{H} \to D\left(\frac{G}{\mathrm{Ad}_{\mathrm{Fr}}(G)}\right) \tag{7.1}$$

where $\mathrm{Ad}_{\mathrm{Fr}}(G)$ denotes the Frobenius-twisted conjugation of G on itself. By Lang's theorem, $\frac{G}{\mathrm{Ad}_{\mathrm{Fr}}(G)}$ is isomorphic to $\mathrm{pt}/G(\mathbb{F}_q)$ as a stack, therefore the target of (7.1) is equivalent to the derived category of representations of $G(\mathbb{F}_q)$. In the current setup, the image of (7.1) is the derived category of unipotent representations of $G(\mathbb{F}_q)$. The Deligne-Lusztig virtual representation R^1_w in §4.1 is the alternating sum of the cohomologies of $p_!q^*\Delta_w$, where $\Delta_w \in \mathcal{H}$ is the constant sheaf on the G-orbit $O(w) \subset \mathcal{B} \times \mathcal{B}$ extended by zero. The functor (7.1) realizes the derived category of unipotent representations of $G(\mathbb{F}_q)$ as the categorical trace of Frobenius on the Hecke category $\mathcal{H}$. To realize all representations $G(\mathbb{F}_q)$ as categorical traces, one replaces $\mathcal{H}$ with its monodromic version, see [46].

In a recent paper [66], X. Zhu gives a vast generalization of the above picture to p-adic groups. Roughly speaking, the depth zero representations of a p-adic group is realized as the Frobenius-twisted categorical trace of the affine Hecke category (and its monodromic versions) of G. From this, he deduces a *categorical local Langlands correspondence* for depth zero representations.

Acknowledgment The author thanks Nanhua Xi for the invitation to write a survey article for Scientia Sinica Mathematica. He also thanks George Lusztig and the referees for their careful reading and many helpful suggestions.

References

[1] Arinkin, D.; Gaitsgory, D. Singular support of coherent sheaves and the geometric Langlands conjecture. Selecta Math. (N.S.) 21 (2015), no. 1, 1–199.
[2] Arinkin, D.; Gaitsgory, D.; Kazhdan, D.; Raskin, S.; Rozenblyum, N.; Varshavsky,Y. Automorphic functions as the trace of Frobenius, arXiv:2102.07906.
[3] Arthur, J. An introduction to the trace formula. In *Harmonic analysis, the trace formula, and Shimura varieties*, 1–263. Clay Math. Proc., 4. American Mathematical Society, Providence, RI, 2005.
[4] Beilinson, A.; Bernstein, J. Localisation de $\mathfrak{g}$-modules. C. R. Acad. Sci. Paris Sér. I Math. 292 (1981), no. 1, 15–18.
[5] Beilinson, A.; Bernstein, J. A proof of Jantzen conjectures. I. M. Gelfand Seminar, 1–50. Adv. Soviet Math., 16, Part 1. American Mathematical Society, Providence, RI, 1993.
[6] Beilinson, A.; Bernstein, J.; Deligne, P.; Gabber, O. Faisceaux pervers. Astérisque 2018, no. 100, vi+180 pp.
[7] Bernstein, I.N.; Gelfand, I.M.; Gelfand, S. I. A certain category of $\mathfrak{g}$-modules. Funkcional. Anal. i Prilozen. 10 (1976), no. 2, 1–8.
[8] Beilinson, A.; Drinfeld, V. Quantization of Hitchin's integrable system and Hecke eigensheaves. Available at `https://math.uchicago.edu/ drinfeld/langlands/QuantizationHitchin.pdf`
[9] Ben-Zvi, D.; Nadler, D. The Character Theory of a Complex Group. arXiv:0904.1247.
[10] Ben-Zvi, D.; Nadler, D. Betti geometric Langlands. Proc. Sympos. Pure Math., 97.2 American Mathematical Society, Providence, RI, 2018, 3–41.
[11] Bezrukavnikov, R. On two geometric realizations of an affine Hecke algebra. Publ. Math. Inst. Hautes Études Sci. 123 (2016), 1–67.
[12] Bezrukavnikov, R.; Finkelberg, M.; Ostrik, V. Character D-modules via Drinfeld center of Harish-Chandra bimodules. Invent. Math. 188 (2012), no. 3, 589–620.
[13] Bott, R. Homogeneous vector bundles. Ann. of Math. (2) 66 (1957), 203–248.
[14] Brylinski, J.-L.; Kashiwara, M. Kazhdan-Lusztig conjecture and holonomic systems. Invent. Math. 64 (1981), no. 3, 387–410.
[15] Deligne, P.; Lusztig, G. Representations of reductive groups over finite fields. Ann. of Math. (2) 103 (1976), no. 1, 103–161.
[16] Drinfeld, V.G. Elliptic modules. Mat. Sb. (N.S.) 94(136) (1974), 594–627, 656.
[17] Drinfeld, V.G. Moduli varieties of F-sheaves. Funktsional. Anal. i Prilozhen. 21 (1987), no. 2, 23–41.
[18] Drinfeld, V. G. Proof of the Petersson conjecture for GL(2) over a global field of characteristic p. Funktsional. Anal. i Prilozhen. 22 (1988), no. 1, 34–54, 96; translation in Funct. Anal. Appl. 22 (1988), no. 1, 28–43.
[19] Drinfeld, V.G. Two-dimensional ℓ-adic representations of the fundamental group of a curve over a finite field and automorphic forms on GL(2). Amer. J. Math. 105 (1983), no. 1, 85–114.
[20] Fargues, L.; Scholze, S. Geometrization of the local Langlands correspondence. arXiv:2102.13459.
[21] Frenkel, E.; Gaitsgory, D.; Vilonen, K. On the geometric Langlands conjecture. J. Amer. Math. Soc. 15 (2002), no. 2, 367–417.
[22] Gaitsgory, D; Kazhdan, D.; Rozenblyum, N.; Varshavsky, Y. A toy model for the Drinfeld-Lafforgue shtuka construction. Indag. Math. (N.S.) 33 (2022), no. 1, 39–189.
[23] Gaitsgory, et al. Proof of the geometric Langlands conjecture. Available `https://people.mpim-bonn.mpg.de/gaitsgde/GLC/`.
[24] Genestier, A.; Lafforgue, V. Chtoucas restreints pour les groupes réductifs et paramétrisation de Langlands locale. arXiv:1709.00978.
[25] Ginzburg, V. Lagrangian construction of the enveloping algebra $U(\mathfrak{sl}_n)$. C. R. Acad. Sci. Paris Sér I Math. 312 (1991), 907–912.
[26] Ginzburg, V. Perverse sheaves on a Loop group and Langlands' duality. arXiv:alg-geom/9511007.
[27] Goresky, M.; Kottwitz, R.; Macpherson, R. Homology of affine Springer fibers in the unramified case. Duke Math. J. 121 (2004), no. 3, 509–561.
[28] Green, J. A. The characters of the finite general linear groups. Trans. Amer. Math. Soc. 80 (1955), 402–447.

[29] Harish-Chandra, Discrete series: I, Acta Math. 113 (1965), 241–318; II, Acta Math. 116 (1966), 1–111.
[30] Heinloth, J.; Ngô, B.C.; Yun, Z. Kloosterman sheaves for reductive groups. Ann. of Math. (2) 177 (2013), no. 1, 241–310.
[31] Kazhdan, D.; Lusztig, G. Representations of Coxeter groups and Hecke algebras Invent. Math. 53 (1979), no. 2, 165–184.
[32] Kazhdan, D.; Lusztig, G. Schubert varieties and Poincaré duality. Proc. Sympos. Pure Math., XXXVI American Mathematical Society, Providence, RI, 1980, pp. 185–203.
[33] Kazhdan, D.; Lusztig, G. Proof of the Deligne-Langlands conjecture for Hecke algebras. Invent. Math. 87 (1987), no. 1, 153–215.
[34] Kazhdan, D.; Lusztig, G. Fixed point varieties on affine flag manifolds. Israel J. Math. 62 (1988), no. 2, 129–168.
[35] Lafforgue, L. Chtoucas de Drinfeld et correspondance de Langlands. Invent. Math. 147 (2002), no. 1, 1–241.
[36] Lafforgue, V. Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale Shtukas for the reductive groups and global Langlands parametrization. J. Amer. Math. Soc. 31 (2018), no. 3, 719–891.
[37] R.Langlands, Dimension of spaces of automorphic forms, Proc. Symposia in Pure Math. IX (1966), 253–257.
[38] Laumon, G. Correspondance de Langlands géométrique pour les corps de fonctions. Duke Math. J. 54 (1987), no. 2, 309–359.
[39] Lusztig, G. Singularities, character formulas, and a q-analog of weight multiplicities. Astérisque, 101-102. Société Mathématique de France, Paris, 1983, 208–229.
[40] Lusztig, G. Green polynomials and singularities of unipotent classes Adv. in Math. 42 (1981), no. 2, 169–178.
[41] Lusztig, G. Characters of reductive groups over a finite field. Ann. of Math. Stud., 107. Princeton University Press, Princeton, NJ, 1984, xxi+384 pp.
[42] Lusztig, G. Character sheaves, I. Adv. in Math. 56 (1985), no. 3, 193–237. II. Adv. in Math. 57 (1985), no. 3, 226–265. III. Adv. in Math. 57 (1985), no. 3, 266–315. IV. Adv. in Math. 59 (1986), no. 1, 1–63. V. Adv. in Math. 61 (1986), no. 2, 103–155.
[43] Lusztig, G. Affine Weyl groups and conjugacy classes in Weyl groups. Transform. Groups 1 (1996), no. 1–2, 83–97.
[44] Lusztig, G. Truncated convolution of character sheaves. Bull. Inst. Math. Acad. Sin. (N.S.) 10 (2015), no. 1, 1–72.
[45] Lusztig, G. Unipotent representations as a categorical centre. Represent. Theory 19 (2015), 211–235.
[46] Lusztig, G. Non-unipotent representations and categorical centres Bull. Inst. Math. Acad. Sin. (N.S.) 12 (2017), no. 3, 205–296.
[47] Mirković, I.; Vilonen, K. Characteristic varieties of character sheaves. Invent. Math. 93 (1988), no. 2, 405–418.
[48] Mirković, I.; Vilonen, K. Geometric Langlands duality and representations of algebraic groups over commutative rings. Ann. of Math. (2) 166 (2007), no. 1, 95–143.
[49] Nakajima, H. Quiver varieties and Kac-Moody algebras. Duke Math. J. 91 (1998), no. 3, 515–560.
[50] Nadler, D.; Yun, Z. Spectral action in Betti geometric Langlands. Israel J. Math. 232 (2019), no. 1, 299–349.
[51] Ngo, B.C. Le lemme fondamental pour les algèbres de Lie. Publ. Math. Inst. Hautes Études Sci. No. 111 (2010), 1–169.
[52] Oblomkov, A.; Yun, Z. Geometric representations of graded and rational Cherednik algebras. Adv. Math. 292 (2016), 601–706.
[53] Schmid, W. L^2-cohomology and the discrete series Ann. of Math. (2) 103 (1976), no. 2, 375–394.
[54] Serre, J.P. Représentations liéaires et espaces homogènes kählériens des groupes de Lie compacts (d'après Armand Borel et André Weil). Séminaire Bourbaki 1954, Exp. No. 100. Société Mathématique de France, Paris, 1995, 447–454.
[55] Springer, T. Trigonometric sums, Green functions of finite groups and representations of Weyl groups. Invent. Math. 36 (1976), 173–207.
[56] Springer, T. Linear algebraic groups. (Second edition) Progr. Math., 9. Birkhäuser Boston, Inc., Boston, MA, 1998. xiv+334 pp.
[57] Varshavsky, Y. Moduli spaces of principal F-bundles. Selecta Math. (N.S.) 10 (2004), no. 1, 131–166.
[58] Vasserot, E. Induced and simple modules of double affine Hecke algebras. Duke Math. J. 126 (2005), no. 2, 251–323.
[59] Varagnolo, M.; Vasserot, E. Finite-dimensional representations of DAHA and affine Springer fibers: the spherical case. Duke Math. J. 147 (2009), no. 3, 439–540.
[60] Yun, Z. Motives with exceptional Galois groups and the inverse Galois problem. Invent. Math. 196 (2014), no. 2, 267–337.
[61] Yun, Z. Rigidity in automorphic representations and local systems. In *Current Development in Mathematics 2013*, International Press, Somerville, MA, 2014, 73–168.
[62] Yun, Z. Lectures on Springer theories and orbital integrals. IAS/Park City Math. Ser., 24. American Mathematical Society, Providence, RI, 2017, 155–215.
[63] Yun, Z. Rigidity method for automorphic forms over function fields. Math. Surveys Monogr., 279. American Mathematical Society, Providence, RI, 2024, 147–187.
[64] Yun, Z. Introduction to Shtukas and their moduli. arXiv:2411.10248.
[65] Zhu, X. An introduction to affine Grassmannians and the geometric Satake equivalence. IAS/Park City Math. Ser., 24. American Mathematical Society, Providence, RI, 2017, 59–154.
[66] Zhu, X. Tame categorical local Langlands correspondence. arXiv:2504.07482.

Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Ave, Cambridge, MA 02139

Email address: zyun@mit.edu