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Geometric Langlands correspondence for SL(2), PGL(2) over the pair of pants

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Abstract

We establish the geometric Langlands correspondence for rank-one groups over the projective line with three points of tame ramification.

1. Introduction

1.1 Main result

Let $\mathbb{P}^1$ denote the complex projective line, and fix the three-element subset $S = \{0, 1, \infty\} \subset \mathbb{P}^1(\mathbb{C})$.

Let $\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)$ denote the moduli stack (over $\mathbb{C}$) of $G = \mathrm{PGL}(2)$-bundles on $\mathbb{P}^1$ with Borel reductions along S. In more classical language, it classifies rank-two vector bundles $\mathcal{E}$ with lines in the fibers $\ell_s \subset \mathcal{E}|_s$, $s \in S$, all up to tensoring with line bundles. It is locally of finite type with discretely many isomorphism classes of objects.

Let $Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S))$ be the $\mathbb{Q}$-linear dg category of constructible complexes of $\mathbb{Q}$-modules on $\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)$ that are extensions by zero off of finite type substacks.

Let $\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$ denote the moduli stack (over $\mathbb{Q}$) of $\mathrm{SL}(2)$-local systems on $\mathbb{P}^1 \backslash S$ equipped near S with a Borel reduction with unipotent monodromy. Thus a point of $\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$ consists of triples of pairs (A_s, ℓ_s), $s \in S$, consisting of a matrix $A_s \in \mathrm{SL}(2)$ and an eigenline $A_s(\ell_s) \subset \ell_s$ with trivial eigenvalue $A_s|_{\ell_s} = 1$, and the matrices satisfy the equation $A_0 A_1 A_\infty = 1$ inside of $\mathrm{SL}(2)$. It admits the presentation

$$\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S) \simeq (\widetilde{\mathcal{N}}^\vee)^{S, \prod=1}/\mathrm{SL}(2),$$

where $\widetilde{\mathcal{N}}^\vee \simeq T^*\mathbb{P}^1$ denotes the Springer resolution of the unipotent variety $\mathcal{N}^\vee$ of $G^\vee = \mathrm{SL}(2)$, and $(\widetilde{\mathcal{N}}^\vee)^{S, \prod=1}$ denotes the product of S copies of $\widetilde{\mathcal{N}}^\vee$ with the equation on the group elements $\prod = 1$ imposed inside of $\mathrm{SL}(2)$. Alternatively, it can be shown that $\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$ also admits a linear presentation

$$\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S) \simeq T^*((\mathbb{P}^1)^S/\mathrm{SL}(2)),$$

where the equation $\prod = 1$ is replaced by the zero-fiber of the moment map $\mu : T^*((\mathbb{P}^1)^S) \to \mathfrak{sl}(2)^*$.

Let $\mathrm{Coh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S))$ be the $\mathbb{Q}$-linear dg category of coherent complexes on $\mathrm{Loc}_{\mathrm{SL}(2)}$ $(\mathbb{P}^1, S)$.

One can similarly introduce the above objects with the roles of $\mathrm{PGL}(2)$ and $\mathrm{SL}(2)$ swapped. We will also need the slight variation where we write $\mathrm{Coh}^{\mathrm{SL}(2)-\mathrm{alt}}(\mathrm{Loc}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S))$ for the

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$\mathbb{Q}$-linear dg category of SL(2)-equivariant coherent complexes on $(\widetilde{\mathcal{N}}^\vee)^{S,\prod=1}$, where the equation $\prod = 1$ is imposed inside of PGL(2), and the center $\mu_2 \simeq Z(\mathrm{SL}(2)) \subset \mathrm{SL}(2)$ acts on coherent complexes by the alternating representation.

Our main theorem is the following geometric Langlands correspondence with tame ramification.

THEOREM 1.1.1. *There are equivalences of dg categories*

$$\mathrm{Coh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)) \xrightarrow{\sim} Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)), \tag{1.1}$$

$$\mathrm{Coh}^{\mathrm{SL}(2)-\mathrm{alt}}(\mathrm{Loc}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)) \xrightarrow{\sim} Sh_!(\mathrm{Bun}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)) \tag{1.2}$$

compatible with Hecke modifications and parabolic induction.

Remark 1.1.2. One can choose an equivalence

$$\mathrm{Coh}^{\mathrm{SL}(2)-\mathrm{alt}}(\mathrm{Loc}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)) \simeq \mathrm{Coh}(\mathrm{Loc}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)),$$

for example, by tensoring with a line bundle with an odd total twist, and thus reformulate the second assertion of the theorem in a more traditional form, but the formulation given in the theorem is more canonical and independent of choices.

Remark 1.1.3. It is also straightforward to use the theorem to deduce a similar result for GL(2).

Remark 1.1.4. One can view the theorem as an instance of the traditional de Rham geometric Langlands correspondence (see, for example, [BD]) or alternatively of the topological Betti geometric Langlands correspondence (see [BN18] for an outline of expectations).

On the automorphic side, the moduli $\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)$ has discretely many isomorphism classes of objects, hence all of their codirections are nilpotent. Thus, if we work specifically with $\mathbb{C}$-coefficients, via the Riemann–Hilbert correspondence, there is no difference in considering $\mathcal{D}$-modules or complexes of $\mathbb{C}$-modules (with nilpotent singular support).

Similarly, on the spectral side, the Betti moduli $\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$ is algebraically isomorphic to the analogous de Rham moduli $\mathrm{Conn}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$ of parabolic connections (see Corollary 4.2.4). Thus their coherent complexes coincide.

Remark 1.1.5. One can further impose nilpotent singular support (in the sense of [AG15]) on the coherent complexes on the spectral side. Under the equivalences of the theorem, this will correspond to requiring the stalks of the automorphic complexes to be torsion over the equivariant cohomology of automorphism groups. If one then passes to the ind-completions of these categories, what results are equivalences for all automorphic complexes without any constructibility or support restrictions

$$\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)) \xrightarrow{\sim} Sh(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)),$$

$$\mathrm{IndCoh}_{\mathcal{N}}^{\mathrm{SL}(2)-\mathrm{alt}}(\mathrm{Loc}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)) \xrightarrow{\sim} Sh(\mathrm{Bun}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)).$$

Remark 1.1.6. One can also pass on the automorphic side to monodromic complexes of any specified monodromy at the three ramification points. It is possible to find an equivalence with coherent complexes on the corresponding spectral stack of local systems with the same specified monodromy around the three ramification points. In the final section, we sketch the form this takes in the case of unipotent monodromy at all three ramification points. For monodromy with a more general semisimple part, the geometry only simplifies.

1.2 Sketch of proof

We highlight here some of the key structures in the proof of Theorem 1.1.1. The second equivalence (1.2) follows closely from the first (1.1) so we will focus on the first.

1.2.1 *Spectral action.* The category $Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S))$ is naturally acted upon by a large collection of commuting Hecke operators.

First, at each unramified point $x \in \mathbb{P}^1 \backslash S$, the symmetric monoidal Satake category $\mathrm{Sat}_{\mathrm{PGL}(2)} \simeq \mathrm{Rep}(\mathrm{SL}(2))$ of spherical perverse sheaves on the affine Grassmannian $\mathrm{Gr}_{\mathrm{PGL}(2)}$ acts via bundle modifications. It is a simple verification that the action is locally constant in $x \in \mathbb{P}^1 \backslash S$, and hence factors through the chiral homology

$$\int_{\mathbb{P}^1 \backslash S} \mathrm{Rep}(\mathrm{SL}(2)) \simeq \mathrm{Perf}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1 \backslash S)).$$

Second, at each point $s \in S$, the monoidal affine Hecke category of Iwahori-equivariant constructible complexes on the affine flag variety $\mathrm{Fl}_{\mathrm{PGL}(2)}$ acts via modifications of bundles with flags. In particular, its symmetric monoidal subcategory of Wakimoto operators acts, and hence via Bezrukavnikov's tame local Langlands correspondence [Bez16], the tensor category $\mathrm{Perf}(\widetilde{\mathcal{N}}^\vee/\mathrm{SL}(2))$ of equivariant perfect complexes on the Springer resolution acts at each point $s \in S$.

Thanks to the compatibility of Gaitsgory's central functor [Gai01], the above actions assemble into an action of the tensor category of perfect complexes on the spectral stack

$$\mathrm{Perf}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)) \otimes Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)) \longrightarrow Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)). \tag{1.3}$$

By continuity, this can be extended to an action of quasi-coherent complexes on all automorphic complexes and then further restricted to coherent complexes.

Remark 1.2.2. In the de Rham geometric Langlands program, the construction of such an action is a deep 'vanishing theorem' [Gai]. In the Betti geometric Langlands program, it is a geometric consequence of requiring automorphic complexes to have nilpotent singular support, see [NY16].

1.2.3 *Whittaker sheaf.* To construct the functor (1.1) from the action (1.3), we must choose an automorphic complex to act upon. It will be the object that the spectral structure sheaf $\mathcal{O} \in \mathrm{Perf}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S))$ maps to, and there is a well-known candidate given by the Whittaker sheaf $\mathrm{Wh}_S \in Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S))$.

In the situation at hand, the Whittaker sheaf takes the following simple form. Consider the open substack of $\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)$, where the underlying bundle is $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}$. Consider the further open substack, where the lines take the form

$$\{\ell_s, s \in S, \text{ generic}\} \xhookrightarrow{j} \{\ell_s \not\subset \mathcal{O}_{\mathbb{P}^1}(1), s \in S\} \xhookrightarrow{i} \mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S).$$

Here 'generic' in the first item means that, in addition to $\ell_s \not\subset \mathcal{O}_{\mathbb{P}^1}(1)$ for all $s \in S$, the three lines do not simultaneously lie in the image of any map $\mathcal{O}_{\mathbb{P}^1} \to \mathcal{E}$. Then the Whittaker sheaf is given by the simple topological construction

$$\mathrm{Wh}_S \simeq i_! j_* \mathbb{Q} \in Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)).$$

Remark 1.2.4. The most salient property of the Whittaker sheaf Wh_S, and indeed the only property we use, is that it corepresents the functor of vanishing cycles for a non-zero covector at the point given by the image of the natural induction map

$$\mathrm{Bun}_B^{-1}(\mathbb{P}^1) \xrightarrow{\sim} \{\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}, \ell_s \subset \mathcal{O}_{\mathbb{P}^1}, s \in S\} \subset \mathrm{Bun}_G(\mathbb{P}^1, S).$$

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1.2.5 *Compatibilities.* With the functor (1.1) in hand, to prove it is an equivalence, we first check that it behaves as expected with respect to certain distinguished objects.

First, we check that the functor (1.1) is compatible with induction from two points of tame ramification. (In fact, we check that it is equivariant for all affine Hecke symmetries at the ramification points.) Namely, for $s \in S$, we show that the functor (1.1) fits as the top arrow in the following commutative diagram.

$$\begin{tikzcd} \mathrm{Coh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)) \arrow[r] & Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)) \\ \mathrm{Coh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S\backslash s)) \arrow[u, "\eta_s^\ell"] \arrow[r, "\sim"] & Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S\backslash s)) \arrow[u, "\pi_s^*"'] \end{tikzcd} \tag{1.4}$$

Here the bottom arrow is the geometric Langlands correspondence for two points of tame ramification. A form of the Radon transform identifies it with Bezrukavnikov's tame local Langlands correspondence. The automorphic induction π_s^* is pullback along the natural $\mathbb{P}^1$-fibration

$$\pi_s : \mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S) \longrightarrow \mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S\backslash s),$$

where we forget the line at $s \in S$. The spectral induction η_s^* is the twisted integral transform

$$\eta_s^\ell(\mathcal{F}) = q_{s*}(p_s^*(\mathcal{F}) \otimes \mathcal{O}_{\mathbb{P}^1_s}(-1)[-1])$$

associated to the correspondence

$$\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S\backslash s) \xleftarrow{p_s} \mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S\backslash s) \times_{\{s\}/\mathrm{SL}(2)} \mathbb{P}^1_s/\mathrm{SL}(2) \xrightarrow{q_s} \mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S).$$

Alternatively, under the identification $\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S) \simeq T^*((\mathbb{P}^1)^S/\mathrm{SL}(2))$, the correspondence is simply the Lagrangian correspondence associated to the projection $(\mathbb{P}^1)^S \to (\mathbb{P}^1)^{S\backslash s}$.

Next, we check that the functor (1.1) is compatible with parabolic induction in the form of Eisenstein series. Namely, on the spectral side, consider the natural induction map

$$\mathrm{Loc}_{B^\vee}(\mathbb{P}^1, S) \xrightarrow{\sim} \widetilde{\mathcal{N}}^\vee_\Delta/\mathrm{SL}(2) \subset \mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$$

with the image the reduced total diagonal where all lines coincide. On the automorphic side, consider the natural induction map

$$q_{-1} : \mathrm{Bun}_B^{-1}(\mathbb{P}^1) \xrightarrow{\sim} \{\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}, \ell_s \subset \mathcal{O}_{\mathbb{P}^1}, s \in S\} \subset \mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S).$$

Then we show the functor (1.1) matches the objects

$$\mathcal{O}_{\widetilde{\mathcal{N}}^\vee_\Delta/\mathrm{SL}(2)} \longmapsto \mathrm{Eis}_{-1} := q_{-1!}\underline{\mathbb{Q}}_{\mathrm{Bun}_B^{-1}(\mathbb{P}^1)}[-1].$$

By applying Wakimoto operators on both sides, it follows that the functor (1.1) matches all Eisenstein objects $\mathcal{O}_{\widetilde{\mathcal{N}}^\vee_\Delta/\mathrm{SL}(2)}(n+1) \mapsto \mathrm{Eis}_n$, for all $n \in \mathbb{Z}$.

1.2.6 *New forms.* With the preceding compatibilities in hand, we are able to readily deduce that the functor (1.1) is an equivalence. The key idea is to focus on objects that are 'new forms' in that they do not come via induction from two points of tame ramification.

We introduce the full subcategories of 'old forms' as the images

$$C^{\mathrm{old}} = \langle \mathrm{Im}(\eta_s^\ell), s \in S \rangle \subset \mathrm{Coh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)),$$

$$Sh^{\mathrm{old}} = \langle \mathrm{Im}(\pi_s^*), s \in S \rangle \subset Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S))$$

and note that the compatibility (1.4) implies the functor (1.1) maps C^{old} essentially surjectively to Sh^{old}.

Thus to show that (1.1) is essentially surjective, it suffices to show it induces an essentially surjective functor on the quotient categories of new forms

$$C^{\mathrm{new}} = \mathrm{Coh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S))/C^{\mathrm{old}},$$

$$Sh^{\mathrm{new}} = Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S))/Sh^{\mathrm{old}}.$$

To achieve this, we first check that C^{new} and Sh^{new} are generated respectively by the Eisenstein objects $\mathcal{O}_{\widetilde{\mathcal{N}}^\vee/\mathrm{SL}(2)}(n+1)$ and Eis_n for $n \geqslant -1$. We do this by an explicit parameterization of objects on both sides.

Finally, to show the functor (1.1) is fully faithful, it suffices by evident Wakimoto symmetries and continuity to check it induces isomorphisms

$$\mathrm{Hom}^\bullet_{\mathrm{Coh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1,S))}(\mathcal{O}, \mathcal{O}_{\widetilde{\mathcal{N}}^\vee_\Delta/\mathrm{SL}(2)}(n+1)) \xrightarrow{\sim} \mathrm{Hom}^\bullet_{Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1,S))}(\mathrm{Wh}_S, \mathrm{Eis}_n), \quad n \geqslant -1.$$

For $n \geqslant 0$, we observe that both sides vanish, and for $n = -1$, both sides are scalars in degree 0 and the induced map is indeed an isomorphism.

1.3 Motivations

While the results of this paper can be viewed as an instance of the traditional de Rham geometric Langlands correspondence, our initial motivations grew out of our interest in the topological Betti geometric Langlands correspondence.

To recall the rough form of the Betti Geometric Langlands correspondence, let X be a smooth projective curve, and $S \subset X$ be a finite collection of points.

Let $\mathrm{Bun}_G(X, S)$ denote the moduli of G-bundles on X with a B-reduction along S.

Let $Sh_{\mathcal{N}}(\mathrm{Bun}_G(X, S))$ denote the dg category of complexes of sheaves with nilpotent singular support on $\mathrm{Bun}_G(\mathbb{P}^1, S)$.

Let $\mathrm{Loc}_{G^\vee}(X, S)$ denote the moduli of $G^\vee$-local systems on $X \backslash S$ equipped near S with a $B^\vee$-reduction with unipotent monodromy.

Let $\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Loc}_{G^\vee}(X, S))$ denote the dg category of ind-coherent sheaves with nilpotent singular support on $\mathrm{Loc}_{G^\vee}(X, S)$.

Conjecture 1.3.1 (Rough form of Betti geometric Langlands correspondence). There is an equivalence

$$\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Loc}_{G^\vee}(X, S)) \xrightarrow{\sim} Sh_{\mathcal{N}}(\mathrm{Bun}_G(X, S)) \tag{1.5}$$

compatible with Hecke modifications and parabolic induction.

Note that $\mathrm{Loc}_{G^\vee}(X,S)$ and hence the spectral side (1.5) depends only on the topological structure of the curve X and not its algebraic structure. Thus the automorphic side of (1.5) is also conjecturally a topological invariant, and hence the fiber at X of a locally constant family of categories over the moduli of curves. In particular, it makes sense to try to produce a 'Verlinde formula' calculating the automorphic side of (1.5) by degenerating to the boundary of the moduli of curves and replacing X with a nodal graph of genus zero curves. Such a gluing paradigm for the spectral side of (1.5) was established in [BN16].

Thus the Betti geometric Langlands correspondence admits the following two-step strategy.

(i) Produce a 'Verlinde formula' describing the automorphic category $Sh_{\mathcal{N}}(\mathrm{Bun}_G(X,S))$ in terms of the atomic building blocks, where $X=\mathbb{P}^1$, and S comprises 0, 1, 2, or 3 points.

(ii) Establish the Betti geometric Langlands correspondence for the atomic building blocks, where $X=\mathbb{P}^1$, and S comprises $0,1,2$, or 3 points.[1]

For $X=\mathbb{P}^1$, and S comprising 0, 1, and 2, the Betti geometric Langlands correspondence is equivalent via Radon transforms with the derived Satake correspondence and Bezrukavnikov's tame local Langlands correspondence. Thus the remaining challenge for step (ii) is to establish the Betti geometric Langlands correspondence for the 'pair of pants', where $X=\mathbb{P}^1$, and S comprises three points. This was our original motivation for pursuing the results of this paper.

Independently of the above considerations, the techniques of this paper also have immediate consequences for the geometric Langlands correspondence when $G=\mathrm{PGL}(2)$, $X=\mathbb{P}^1$, and S comprises four or more points. Note that $\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1,S)$ and $\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1,S)$ are of dimension $\#S-3$ and $2(\#S-3)$ respectively, and when $\#S\geqslant 4$, there are continuous moduli of objects within $\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1,S)$ and non-trivial global functions on $\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1,S)$. The techniques of this paper most directly apply to the expected correspondence between the full subcategory of $Sh_{\mathcal{N}}(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1,S))$ generated by complexes with unipotent monodromies, and the full subcategory of $\mathrm{Coh}_{\mathcal{N}}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1,S))$ of coherent complexes supported on local systems with global unipotent reductions. We hope to expand upon this in a subsequent paper.

1.4 Conventions

On the automorphic side, we will work with moduli stacks defined over the complex numbers $\mathbb{C}$ and sheaves of $\mathbb{Q}$-modules on them with respect to the classical topology.

Given a stack $\mathfrak{X}$ over $\mathbb{C}$, we write $Sh(\mathfrak{X})$, respectively $Sh_c(\mathfrak{X})$, for the $\mathbb{Q}$-linear dg category of complexes, respectively constructible complexes, of $\mathbb{Q}$-modules on $\mathfrak{X}$. When $\mathfrak{X}$ is locally of finite type, we write $Sh_!(\mathfrak{X})$ for the $\mathbb{Q}$-linear dg category of constructible complexes of $\mathbb{Q}$-modules on $\mathfrak{X}$ that are extensions by zero off of finite type substacks. Given an ind-stack $\mathfrak{X}$, we write $Sh_c(\mathfrak{X})$ for the $\mathbb{Q}$-linear dg category of constructible complexes of $\mathbb{Q}$-modules on $\mathfrak{X}$ that are extensions by zero off of substacks. (For dg categories of complexes of sheaves, in particular the extension of the standard six functor formalism, see [Sch18].)

On the spectral side, we will work with coherent sheaves over moduli stacks defined over $\mathbb{Q}$. All of our categories will be stable (= pretriangluated) $\mathbb{Q}$-linear dg categories, and all of our functors will be derived.

[1] There is also a non-orientable version for real reductive groups, which leads to the additional atomic building blocks, where '$X=\mathbb{RP}^2$', and S comprises 0 or 1 point.

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2. General constructions

In this section, we collect standard structures from the geometric Langlands program. Most of the materials in this section are known to experts.

2.1 Group theory

Let G be a reductive group, $B \subset G$ a Borel subgroup, $N \subset B$ its unipotent radical, and $T = B/N$ the universal Cartan. Let $\mathcal{B} \simeq G/B$ be the flag variety of G.

Let $(\Lambda_T, R_+^\vee, \Lambda_T^\vee, R_+)$ be the associated based root datum, where $\Lambda_T = \mathrm{Hom}(\mathbb{G}_m, T)$ is the coweight lattice, $R_+^\vee \subset \Lambda_T$ the positive coroots, $\Lambda_T^\vee = \mathrm{Hom}(T, \mathbb{G}_m)$ the weight lattice, and $R_+ \subset \Lambda_T^\vee$ the positive roots. Let W_f denote the Weyl group of G, and $W^{\mathrm{aff}} \simeq W_f \ltimes \Lambda_T$ its affine Weyl group. Let $\rho \in \Lambda_T^\vee$ (respectively $\rho^\vee \in \Lambda_T$) be half of the sum of elements in R_+ (respectively $R_+^\vee$).

Form the dual based root datum $(\Lambda_T^\vee, R_+, \Lambda_T, R_+^\vee)$, and construct the Langlands dual group $G^\vee$, with Borel subgroup $B^\vee \subset G^\vee$, unipotent radical $N^\vee \subset B^\vee$, and dual universal Cartan $T^\vee = B^\vee/N^\vee$. Let $\mathcal{B}^\vee \simeq G^\vee/B^\vee$ be the flag variety of $G^\vee$.

Let $\mathcal{N}^\vee$ be the nilpotent cone in the Lie algebra $\mathfrak{g}^\vee$. We identify $\mathcal{N}^\vee$ with the unipotent elements in $G^\vee$ via the exponential map.

Let $\mu : \widetilde{\mathcal{N}}^\vee \to \mathcal{N}^\vee$ be the Springer resolution. Recall that $\widetilde{\mathcal{N}}^\vee \subset G^\vee \times \mathcal{B}^\vee$ classifies pairs $(g, B_1^\vee)$ such that the class g lies in the unipotent radical of $B_1^\vee$. Note the isomorphism of adjoint quotients $N^\vee/B^\vee \simeq \widetilde{\mathcal{N}}^\vee/G^\vee$.

2.2 Hecke kernels

2.2.1 *Satake category.* Let $D = D_- = D_+ = \mathrm{Spec}\ k[[t]]$ be copies of the formal disk, $D^\times = \mathrm{Spec}\ k((t)) \subset D, D_-, D_+$ the punctured formal disk, and $\mathbb{D} = D_- \coprod_{D^\times} D_+$ the non-separated disk with two zeros $0_- \in D_-, 0_+ \in D_+$.

Let $\mathrm{Bun}_G(\mathbb{D})$ be the moduli of G-bundles on $\mathbb{D}$.

Introduce the Laurent series loop group $G((t)) = \mathrm{Maps}(D^\times, G)$, with its parahoric arc subgroup $G[[t]] = \mathrm{Maps}(D, G)$, and affine Grassmannian $\mathrm{Gr}_G = G((t))/G[[t]]$. The gluing presentation $\mathbb{D} = D_- \coprod_{D^\times} D_+$ induces a double-coset presentation

$$\mathrm{Bun}_G(\mathbb{D}) \simeq G[[t]]\backslash G((t))/G[[t]] \simeq G[[t]]\backslash \mathrm{Gr}_G.$$

Let $\mathcal{H}_G^{\mathrm{sph}} = Sh_c(\mathrm{Bun}_G(\mathbb{D}))$ be the dg spherical Hecke category of constructible complexes on $\mathrm{Bun}_G(\mathbb{D})$ with proper support, or equivalently $G[[t]]$-equivariant constructible complexes on Gr_G with proper support. Convolution and fusion equips $\mathcal{H}_G^{\mathrm{sph}}$ with an E_3-monoidal structure, which preserves the heart $\mathcal{H}_{G,\heartsuit}^{\mathrm{sph}} \subset \mathcal{H}_G^{\mathrm{sph}}$ with respect to the perverse t-structure. The E_3-monoidal structure on $\mathcal{H}_{G,\heartsuit}^{\mathrm{sph}}$ naturally lifts to a symmetric monoidal structure. Though we mention it for clarity, we will not need the E_3-monoidal structure on $\mathcal{H}_G^{\mathrm{sph}}$ but only the symmetric monoidal structure on $\mathcal{H}_{G,\heartsuit}^{\mathrm{sph}}$.

The geometric Satake correspondence [MV07, Gin95] provides a symmetric monoidal equivalence

$$\Phi^{\mathrm{sph}} : \mathrm{Rep}(G^\vee) \simeq \mathcal{H}_{G,\heartsuit,\rho^\vee}^{\mathrm{sph}}, \tag{2.1}$$

where $\mathcal{H}_{G,\heartsuit,\rho^\vee}^{\mathrm{sph}}$ denotes the same monoidal category $\mathcal{H}_{G,\heartsuit}^{\mathrm{sph}}$ but with its twisted commutativity constraint. There is also a derived geometric Satake correspondence but we will not need it.

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2.2.2 *Affine Hecke category.* Let $\mathrm{Bun}_G(\mathbb{D}, \{0_-, 0_+\})$ be the moduli of G-bundles on $\mathbb{D}$ with B-reductions at the points $0_-, 0_+ \in \mathbb{D}$. The natural projection $\mathrm{Bun}_G(\mathbb{D}, \{0_-, 0_+\}) \to \mathrm{Bun}_G(\mathbb{D})$ is a $\mathcal{B} \times \mathcal{B}$-fibration.

Let $I \subset G[[t]]$ be the Iwahori subgroup given by the inverse image of $B \subset G$ under the evaluation map at $0 \in D$, and $\mathrm{Fl}_G = G((t))/I$ the corresponding affine flag variety. The gluing presentation $\mathbb{D} = D_- \coprod_{D^\times} D_+$ induces a double-coset presentation

$$\mathrm{Bun}_G(\mathbb{D}, \{0_-, 0_+\}) \simeq I\backslash G((t))/I \simeq I\backslash \mathrm{Fl}_G.$$

Let $\mathcal{H}_G^{\mathrm{aff}} = Sh_c(\mathrm{Bun}_G(\mathbb{D}, \{0_-, 0_+\}))$ be the dg affine Hecke category of constructible complexes on $\mathrm{Bun}_G(\mathbb{D}, \{0_-, 0_+\})$ with proper support, or equivalently I-equivariant constructible complexes on Fl_G with proper support. Convolution equips $\mathcal{H}_G^{\mathrm{aff}}$ with a monoidal structure.

Recall we write $\mu : \widetilde{\mathcal{N}}^\vee \to \mathcal{N}^\vee$ for the Springer resolution, and identify $\mathcal{N}^\vee$ with the unipotent elements in $G^\vee$ via the exponential map. The Steinberg variety $St_{G^\vee}$ is the *derived scheme* given by the *derived* fiber product

$$St_{G^\vee} = \widetilde{\mathcal{N}}^\vee \times_{G^\vee} \widetilde{\mathcal{N}}^\vee.$$

Passing to adjoint quotients, we have

$$St_{G^\vee}/G^\vee = (\widetilde{\mathcal{N}}^\vee \times_{G^\vee} \widetilde{\mathcal{N}}^\vee)/G^\vee \simeq \widetilde{\mathcal{N}}^\vee/G^\vee \times_{G^\vee/G^\vee} \widetilde{\mathcal{N}}^\vee/G^\vee.$$

Let $\mathrm{Coh}^{G^\vee}(St_{G^\vee})$ be the dg derived category of coherent complexes on $St_{G^\vee}/G^\vee$, or equivalently $G^\vee$-equivariant coherent complexes on $St_{G^\vee}$. Convolution equips it with a monoidal structure.

Bezrukavnikov's theorem [Bez16, Theorem 1] provides a monoidal equivalence

$$\Phi^{\mathrm{aff}} : \mathrm{Coh}^{G^\vee}(St_{G^\vee}) \xrightarrow{\ \sim\ } \mathcal{H}_G^{\mathrm{aff}}. \tag{2.2}$$

Example 2.2.3 (Wakimoto sheaves, see [Bez16, § 3.3]). For $\lambda \in \Lambda_T = \{1\} \ltimes \Lambda_T \subset W_f \ltimes \Lambda_T = W^{\mathrm{aff}}$, we have the $G^\vee$-equivariant line bundle $\mathcal{O}_{\mathcal{B}^\vee}(\lambda)$ on the flag variety $\mathcal{B}^\vee = G^\vee/B^\vee$. It pulls back under the natural projection $\pi : \widetilde{\mathcal{N}}^\vee \to \mathcal{B}^\vee$ to a $G^\vee$-equivariant line bundle $\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda) = \pi^*\mathcal{O}_{\mathcal{B}^\vee}(\lambda)$.

Let $\Delta : \widetilde{\mathcal{N}}^\vee \to St_{G^\vee}$ be the diagonal map. Under the equivalence Φ^{aff}, the coherent sheaf $\Delta_*\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda)$ corresponds to the *Wakimoto sheaf* J_λ, which can be explicitly constructed as follows. Let $j_\lambda : \mathrm{Fl}_G^\lambda \hookrightarrow \mathrm{Fl}_G$ be the I-orbit indexed by λ regarded in W^{aff}. When λ is dominant, $J_\lambda \simeq j_{\lambda *}\mathbb{Q}[\langle 2\rho, \lambda\rangle]$; when λ is anti-dominant, $J_\lambda \simeq j_{\lambda !}\mathbb{Q}[\langle 2\rho, -\lambda\rangle] \simeq \mathbb{D}_{\mathrm{Fl}_G}\iota J_{-\lambda}$, where ι denotes the involution of $I\backslash\mathrm{Fl}_G$ induced by the inverse of G. In general, writing λ as $\lambda_1 - \lambda_2$, where λ_1 and λ_2 are both dominant, we have $J_\lambda \simeq J_{\lambda_1}J_{-\lambda_2}$ independently of the expression of λ as $\lambda_1 - \lambda_2$. One can check geometrically that the assignment $\lambda \mapsto J_\lambda$ gives a map of monoids $\Lambda_T \to \mathcal{H}_G^{\mathrm{aff}}$.

Example 2.2.4 ($G = \mathrm{SL}(2)$). The affine Weyl group W^{aff} can be identified with the infinite dihedral group acting on the real line $\mathbb{R}$ with fundamental domain $[0, 1]$. For $x \in \mathbb{Z}$, let $r_x \in W^{\mathrm{aff}}$ be the reflection with center x, then $W^{\mathrm{aff}} = \langle r_0, r_1\rangle$. Note that under the usual indexing scheme for affine Coxeter groups, r_0 (respectively r_1) corresponds to the simple reflection s_1 (respectively s_0) with respect to the simple root α_1 (respectively α_1) of the affine $\mathfrak{sl}(2)$.

Correspondingly there are two standard monoidal generators T_{0*}, T_{1*} for $\mathcal{H}^{\mathrm{aff}}$ given by the $*$-extensions of $\underline{\mathbb{Q}}_{\mathrm{Fl}^{r_0}}[1]$ and $\underline{\mathbb{Q}}_{\mathrm{Fl}^{r_1}}[1]$. Similarly define $T_{0!}$ and $T_{1!}$ using !-extensions instead of $*$-extensions. Then we have monoidal inverses $T_{0*}^{-1} \simeq T_{0!}$, $T_{1*}^{-1} \simeq T_{1!}$.

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For $k \in \mathbb{Z}$, the Wakimoto sheaf can be expressed as $J_{2k} \simeq (T_{0*}T_{1*})^k$, which corresponds under Φ^{aff} to the twist of the structure sheaf of the relative diagonal $\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(2k)$. This follows from the construction of the functor Φ^{aff}, see [Bez16, § 4.1].

The finite braid operator T_{0*} corresponds under Φ^{aff} to the *classical* (namely underived) structure sheaf $\mathcal{O}^{\mathrm{cl}}_{St_{G^\vee}}$. Its inverse $T_{0*}^{-1} \simeq T_{0!}$ corresponds to the twist $\mathcal{O}^{\mathrm{cl}}_{St_{G^\vee}}(-1,-1)$. We briefly indicate how to deduce $\Phi^{\mathrm{aff}}(T_{0*}) \cong \mathcal{O}^{\mathrm{cl}}_{St_{G^\vee}}$ from the results of [Bez16]. By [Bez16, Theorem 1], there is an equivalence

$$\Phi_{I^0I} : Sh_c(I^0\backslash \mathrm{Fl}_G) \xrightarrow{\sim} \mathrm{Coh}^{G^\vee}(St'),$$

where $I^0 = I \times_B N \subset I$, and $St' = \widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathcal{N}}^\vee$ ($\widetilde{\mathfrak{g}^\vee} \to \mathfrak{g}^\vee$ is the Grothendieck alteration). By [Bez16, Example 57], the object T_{0*}, viewed as an object in $Sh_c(I^0\backslash \mathrm{Fl}_G)$, corresponds to the structure sheaf of $\Gamma'_s = \Gamma_s \cap St'$ (where Γ_s is the closure of the graph of the non-trivial element $s \in W$ on the regular locus of $\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathfrak{g}^\vee}$, and Γ'_s is the scheme-theoretic intersection of Γ_s with St'). Under the natural embedding $i : St_{G^\vee} \hookrightarrow St'$, one checks that $i_*\mathcal{O}^{\mathrm{cl}}_{St_{G^\vee}} \cong \mathcal{O}_{\Gamma'_s}$. Since the forgetful functor $\mathcal{H}^{\mathrm{aff}}_G = Sh_c(I\backslash \mathrm{Fl}_G) \to Sh_c(I^0\backslash \mathrm{Fl}_G)$ corresponds to $i_* : \mathrm{Coh}^{G^\vee}(St_{G^\vee}) \to \mathrm{Coh}^{G^\vee}(St')$, we see that $i_*\Phi^{\mathrm{aff}}(T_{0*}) \cong i_*\mathcal{O}^{\mathrm{cl}}_{St_{G^\vee}}$. Using the right exactness of i_*, we conclude that $\Phi^{\mathrm{aff}}(T_{0*}) \cong \mathcal{O}^{\mathrm{cl}}_{St_{G^\vee}}$.

Below we will give more examples of how objects correspond to each other under the equivalence Φ^{aff}. The proof of the matchings follow from the two paragraphs above by easy calculations, which we omit here.

The affine braid operator T_{1*} corresponds under Φ^{aff} to the twisted classical structure sheaf $\mathcal{O}^{\mathrm{cl}}_{St_{G^\vee}}(-1,1)$. This follows from the fact that $J_2 = T_{0*}T_{1*}$ corresponds to $\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(2)$, and that T_{0*} corresponds to $\mathcal{O}^{\mathrm{cl}}_{St_{G^\vee}}$. Similarly $T_{1*}^{-1} \simeq T_{1!}$ corresponds to the twist $\mathcal{O}^{\mathrm{cl}}_{St_{G^\vee}}(-2,0)$. The conjugate $T_{0*}T_{1*}T_{0*}^{-1} \simeq J_2T_{1*}J_2^{-1}$ corresponds to the twist $\mathcal{O}^{\mathrm{cl}}_{St_{G^\vee}}(1,-1)$, and its inverse $T_{0*}T_{1!}T_{0*}^{-1} \simeq J_2T_{1!}J_2^{-1}$ to the twist $\mathcal{O}^{\mathrm{cl}}_{St_{G^\vee}}(0,-2)$.

Let Avg be the IC-sheaf of the closure of Fl^{r_0}. Then Avg corresponds to $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$ under Φ^{aff}. The natural distinguished triangles

$$\mathrm{Avg} \longrightarrow T_{0*} \longrightarrow \delta, \quad \delta \longrightarrow T_{0!} \longrightarrow \mathrm{Avg}$$

correspond to the natural distinguished triangles

$$\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1) \longrightarrow \mathcal{O}^{\mathrm{cl}}_{St_{G^\vee}} \longrightarrow \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}, \quad \mathcal{O}_{\widetilde{\mathcal{N}}^\vee} \longrightarrow \mathcal{O}^{\mathrm{cl}}_{St_{G^\vee}}(-1,-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1).$$

Example 2.2.5 ($G = \mathrm{PGL}(2)$). The morphism $\mathrm{SL}(2) \to \mathrm{PGL}(2)$ induces a canonical monoidal functor $\mathcal{H}^{\mathrm{aff}}_{\mathrm{SL}(2)} \to \mathcal{H}^{\mathrm{aff}}_{\mathrm{PGL}(2)}$. We use the same notation introduced in Example 2.2.4 for objects in $\mathcal{H}^{\mathrm{aff}}_{\mathrm{SL}(2)}$ to denote their images in $\mathcal{H}^{\mathrm{aff}}_{\mathrm{PGL}(2)}$. The description of $\Phi^{\mathrm{aff}}(\mathcal{F})$ for $\mathcal{F} \in \mathcal{H}^{\mathrm{aff}}_{\mathrm{SL}(2)}$ given in Example 2.2.4 is still valid in the case of $\mathrm{PGL}(2)$ for the same-named sheaf $\mathcal{F}$ but viewed as in $\mathcal{H}^{\mathrm{aff}}_{\mathrm{PGL}(2)}$ (note that the Steinberg variety is the same for $\mathrm{SL}(2)$ and $\mathrm{PGL}(2)$).

Now W^{aff} can be identified with the infinite dihedral group generated by r_0 and $r_{1/2}$ (reflection with center $1/2$). Correspondingly, $\mathcal{H}^{\mathrm{aff}}_G$ two standard monoidal generators T_{0*} and $T_{1/2}$, where $T_{1/2}$, the *Atkin–Lehner involution*, is the skyscraper sheaf of the point I-orbit $\mathrm{Fl}^{r_{1/2}}$, and $T_{1/2}^{-1} \simeq T_{1/2}$.

For $k \in \mathbb{Z}$, the Wakimoto sheaf can be expressed as $J_k \simeq (T_{0*}T_{1/2})^k$, which corresponds to the twist of the structure sheaf of the relative diagonal $\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(k)$. This again follows from the construction of Φ^{aff}, see [Bez16, § 4.1].

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The Atkin–Lehner involution $T_{1/2}$ corresponds under Φ^{aff} to the twisted classical structure sheaf $\mathcal{O}^{\mathrm{cl}}_{St_{G^\vee}}(-1,0)$. This follows from the fact that $J_1 = T_{0*}T_{1/2}$ corresponds to $\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(1)$ and that T_{0*} corresponds to $\mathcal{O}^{\mathrm{cl}}_{St_{G^\vee}}$. The conjugate $T_{0*}T_{1/2}T_{0*}^{-1} \simeq J_1 T_{1/2} J_1^{-1}$ corresponds to the twist $\mathcal{O}^{\mathrm{cl}}_{St_{G^\vee}}(0,-1)$.

2.2.6 *Compatibilty.* Gaitsgory's nearby cycles construction [Gai01] provides a central functor

$$Z : \mathcal{H}_G^{\mathrm{sph}} \longrightarrow \mathcal{H}_G^{\mathrm{aff}}.$$

Under the Satake equivalence (2.1) and Bezrukavnikov's equivalence Φ^{aff}, the central functor becomes the natural functor

$$\mathrm{Rep}(G^\vee) \longrightarrow \mathrm{Coh}^{G^\vee}(\widetilde{\mathcal{N}}^\vee) \xrightarrow{\Delta_*} \mathrm{Coh}^{G^\vee}(St_{G^\vee}),$$

where the first functor is the pullback along the projection $\widetilde{\mathcal{N}}^\vee/G^\vee \to \mathrm{pt}/G^\vee$. Its monodromy automorphism corresponds to the universal unipotent automorphism of the pullback.

2.2.7 *Finite and aspherical Hecke categories.* Let $\mathcal{H}_G^f = Sh_c(B\backslash G/B)$ be the finite Hecke category of B-equivariant constructible complexes on the flag variety $\mathcal{B} = G/B$, with monoidal structure defined by convolution. Pushforward along the closed embedding $\mathcal{B} = G/B \hookrightarrow G((t))/I = \mathrm{Fl}_G$ gives a fully fiathful monoidal functor $\mathcal{H}_G^f \to \mathcal{H}_G^{\mathrm{aff}}$.

Let $\Xi \in \mathrm{Perv}_N(\mathcal{B}) \subset Sh_c(G/B)$ be the tilting extension of the shifted constant sheaf $\underline{\mathbb{Q}}_{\mathcal{B}^{w_0}}[\dim \mathcal{B}]$ on the open N-orbit $\mathcal{B}^{w_0} \subset \mathcal{B}$. Equivalently, in the abelian category $\mathrm{Perv}_N(\mathcal{B})$, it is also the projective cover of the skyscraper sheaf on the closed N-orbit.

Consider the functor

$$\mathbb{V} = \mathrm{Hom}_{Sh_c(\mathcal{B})}(\Xi, q^*(-)) : \mathcal{H}_G^f \longrightarrow \mathrm{Vect},$$

where we first forget B-equivariance via the pullback $q^* : \mathcal{H}_G^f \to Sh_c(\mathcal{B})$ along $q : G/B \to B\backslash G/B$.

The functor $\mathbb{V}$ calculates the vanishing cycles at a generic covector at the closed N-orbit. It is the universal quotient of $\mathcal{H}_G^f$ with the kernel the full monoidal ideal $\langle \mathrm{IC}_w | w \neq 1 \in W_f\rangle$ generated by IC-sheaves of N-orbits $\mathcal{B}^w \subset \mathcal{B}$, for $w \neq 1 \in W_f$, that are not closed. It can be equipped with a monoidal structure (for the usual tensor product on Vect).

The aspherical affine Hecke category is defined to be the tensor product

$$\mathcal{H}_G^{\mathrm{asph}} := \mathcal{H}_G^{\mathrm{aff}} \otimes_{\mathcal{H}_G^f} \mathrm{Vect},$$

where the $\mathcal{H}_G^f$-module structure on Vect is given by $\mathbb{V}$. It has a natural $\mathcal{H}_G^{\mathrm{aff}}$-module structure via convolution on the left.

When the base field has positive characteristic, Bezrukavnikov [Bez16] realizes $\mathcal{H}_G^{\mathrm{asph}}$ as the dg category of Iwahori–Whittaker sheaves on the affine flag variety with the help of an Artin–Schreier sheaf. By [Bez16, Theorem 2], there is an equivalence of dg categories

$$\Phi^{\mathrm{asph}} : \mathrm{Perf}(\widetilde{\mathcal{N}}^\vee/G^\vee) = \mathrm{Coh}^{G^\vee}(\widetilde{\mathcal{N}}^\vee) \xrightarrow{\sim} \mathcal{H}_G^{\mathrm{asph}}. \tag{2.3}$$

Moreover, the $\mathcal{H}_G^{\mathrm{aff}}$-action on the right-hand side gets intertwined with the $\mathrm{Coh}^{G^\vee}(St_{G^\vee})$-action on the left-hand side by left convolution via the equivalence Φ^{aff}.

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The above equivalence also holds when the base field is $\mathbb{C}$. One way to see this is to work with D-modules (where the exponential D-module plays the role of an Artin–Schreier sheaf) to obtain an equivalence between the $\mathbb{C}$-linearizations of the two sides of (2.3), and then descend it to $\mathbb{Q}$. Another way is to use a $\mathbb{G}_m$-averaged version of an Artin–Schreier sheaf, as we do when introducing the Whittaker sheaf in § 2.5.2.

2.3 Hecke modifications

Let X be a connected smooth projective curve of genus g, and $S \subset X$ a finite subset.

Let $\mathrm{Bun}_G(X,S)$ be the moduli stack of G-bundles on X with B-reductions at S. This is an algebraic stack locally of finite type. Later we will focus on the case $G = \mathrm{PGL}(2)$ and $\mathrm{SL}(2)$. For more concrete modular interpretations of $\mathrm{Bun}_G(X,S)$ in these cases, see § 3.1.

Let $\mathit{Sh}(\mathrm{Bun}_G(X,S))$ be the dg derived category of all complexes on $\mathrm{Bun}_G(X,S)$. We will abuse terminology and use the term sheaves to refer to its objects.

Let $\mathit{Sh}_!(\mathrm{Bun}_G(X,S)) \subset \mathit{Sh}(\mathrm{Bun}_G(X,S))$ be the full dg subcategory of constructible complexes that are extensions by zero off of finite type substacks.

Introduce copies of the curve $X = X_- = X_+$, and for any $x \in X$, introduce the non-separated curve

$$\mathbb{X}_x = X_- \textstyle\coprod_{X\setminus\{x\}} X_+$$

with the two distinguished points $x_- \in X_-$, $x_+ \in X_+$, and the natural embeddings

$$\begin{tikzcd} & \mathbb{X}_x & \\ X_- \arrow[ur, "i_-"] & \mathbb{D}_x \arrow[u, "j"'] & X_+ \arrow[ul, "i_+"'] \end{tikzcd}$$

where $\mathbb{D}_x = D_{x_-} \coprod_{D_x^\times} D_{x_+}$ is the formal neighborhood of $\{x_-, x_+\} \subset X$. Note that for the choice of a local coordinate, we can identify $\mathbb{D}_x$ with the standard model $\mathbb{D}$.

2.3.1 *Spherical Hecke action.* For $x \in X \setminus S$, we may define the moduli stack $\mathrm{Bun}_G(\mathbb{X}_x, S)$ of G-bundles on $\mathbb{X}_x$ with B-reductions at S. We have the following diagram.

$$\begin{tikzcd} & \mathrm{Bun}_G(\mathbb{X}_x, S) \arrow[dl, "p_-"'] \arrow[d, "\kappa"] \arrow[dr, "p_+"] & \\ \mathrm{Bun}_G(X,S) & \mathrm{Bun}_G(\mathbb{D}_x) & \mathrm{Bun}_G(X,S) \end{tikzcd}$$

Passing to sheaves, and choosing a local coordinate to identify $\mathrm{Bun}_G(\mathbb{D}_x)$ with $G[[t]]\backslash G((t))/G[[t]]$, one obtains the spherical Hecke modifications

$$\mathrm{Hecke}_x^{\mathrm{sph}} : \mathcal{H}_G^{\mathrm{sph}} \otimes \mathit{Sh}(\mathrm{Bun}_G(X,S)) \longrightarrow \mathit{Sh}(\mathrm{Bun}_G(X,S)),$$

$$\mathrm{Hecke}_x^{\mathrm{sph}}(\mathcal{K}, \mathcal{F}) = (p_+)_!((p_-)^*\mathcal{F} \otimes \kappa^*(\mathcal{K})).$$

It evidently preserves the full dg subcategory $\mathit{Sh}_!(\mathrm{Bun}_G(X,S)) \subset \mathit{Sh}(\mathrm{Bun}_G(X,S))$.

Natural generalizations of the above constructions provide $\mathit{Sh}(\mathrm{Bun}_G(X,S))$ the requisite coherences of an $\mathcal{H}_G^{\mathrm{sph}}$-module.

Restricting to the heart of $\mathcal{H}_G^{\mathrm{sph}}$, one obtains a tensor action

$$\mathrm{Rep}(G^\vee) \otimes \mathit{Sh}(\mathrm{Bun}_G(X,S)) \longrightarrow \mathit{Sh}(\mathrm{Bun}_G(X,S)).$$

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Remark 2.3.2. It is straightforward to generalize the above from a point $x \in X \backslash S$ to a family of points parametrized by $Y \to X \backslash S$ to obtain a functor

$$\mathrm{Hecke}_Y^{\mathrm{sph}} : \mathcal{H}_G^{\mathrm{sph}} \otimes Sh(\mathrm{Bun}_G(X,S)) \longrightarrow Sh(\mathrm{Bun}_G(X,S) \times Y).$$

2.3.3 *Affine Hecke action.* For $s \in S$, let $S_\pm = S \coprod_{S\backslash\{s\}} S \subset \mathbb{X}_s$. We may similarly define the moduli stack $\mathrm{Bun}_G(\mathbb{X}_s, S_\pm)$ of G-bundles on $\mathbb{X}_s$ with B-reductions at $S_\pm$, and obtain a diagram

$$\begin{tikzcd} & \mathrm{Bun}_G(\mathbb{X}_s, S_\pm) \arrow[dl, "p_-"'] \arrow[d, "\kappa"] \arrow[dr, "p_+"] & \\ \mathrm{Bun}_G(X,S) & \mathrm{Bun}_G(\mathbb{D}_s, \{s_-, s_+\}) & \mathrm{Bun}_G(X,S) \end{tikzcd} \tag{2.4}$$

Passing to sheaves, and choosing a local coordinate to identify $\mathrm{Bun}_G(\mathbb{D}_s, \{s_-, s_+\})$ with $I\backslash G((t))/I$, one obtains the affine Hecke modifications

$$\mathrm{Hecke}_s^{\mathrm{aff}} : \mathcal{H}_G^{\mathrm{aff}} \otimes Sh(\mathrm{Bun}_G(X,S)) \longrightarrow Sh(\mathrm{Bun}_G(X,S)), \tag{2.5}$$

$$\mathrm{Hecke}_s^{\mathrm{aff}}(\mathcal{K}, \mathcal{F}) = (p_+)_!((p_-)^*\mathcal{F} \otimes \kappa^*(\mathcal{K})).$$

More often, we will use the binary notation $\star_s$ to denote the affine Hecke action

$$\mathcal{K} \star_s \mathcal{F} := \mathrm{Hecke}_s^{\mathrm{aff}}(\mathcal{K}, \mathcal{F}).$$

It evidently preserves the full dg subcategory $Sh_!(\mathrm{Bun}_G(X,S)) \subset Sh(\mathrm{Bun}_G(X,S))$.

Natural generalizations of the above constructions provide $Sh(\mathrm{Bun}_G(X,S))$ the requisite coherences of an $\mathcal{H}_G^{\mathrm{aff}}$-module structure. For different $s \in S$, the resulting $\mathcal{H}_G^{\mathrm{aff}}$-actions on $Sh(\mathrm{Bun}_G(X,S))$ commute with each other.

In particular, restricting the action of $\mathcal{H}_G^{\mathrm{aff}}$ to $\mathrm{Perf}(\widetilde{\mathcal{N}}^\vee/G^\vee)$ via the monoidal functor

$$\mathrm{Perf}(\widetilde{\mathcal{N}}^\vee/G^\vee) \xrightarrow{\Delta_*} \mathrm{Coh}^{G^\vee}(St_{G^\vee}) \xrightarrow{\Phi^{\mathrm{aff}}} \mathcal{H}_G^{\mathrm{aff}}, \tag{2.6}$$

where $\Delta : \widetilde{\mathcal{N}}^\vee/G^\vee \to St_{G^\vee}/G^\vee$ is the diagonal map, one obtains commuting tensor actions

$$\mathrm{Perf}(\widetilde{\mathcal{N}}^\vee/G^\vee)^{\otimes S} \otimes Sh(\mathrm{Bun}_G(X,S)) \longrightarrow Sh(\mathrm{Bun}_G(X,S)). \tag{2.7}$$

2.3.4 *Compatibility.* Let $s \in S$, and let $U_s \subset X\backslash(S\backslash\{s\})$ be a disk around s (in the classical topology). Let $U_s^\times = U_s\backslash\{s\}$ be the punctured disk. Recall the Hecke operators over $U_s^\times$ are defined as in Remark 2.3.2. By the construction of the central functor Z in [Gai01], there is a natural equivalence of bifunctors $\mathcal{H}_G^{\mathrm{sph}} \otimes Sh(\mathrm{Bun}_G(X,S)) \to Sh(\mathrm{Bun}_G(X,S))$

$$\mathrm{Hecke}_s^{\mathrm{aff}} \circ (Z \otimes \mathrm{id}_{Sh(\mathrm{Bun}_G(X,S))}) \simeq \Psi_s \circ \mathrm{Hecke}_{U_s^\times}^{\mathrm{sph}}, \tag{2.8}$$

where $\Psi_s : Sh(\mathrm{Bun}_G(X,S) \times U_s^\times) \to Sh(\mathrm{Bun}_G(X,S))$ denotes nearby cycles towards the s-fiber of $\mathrm{Bun}_G(X,S) \times U_s \to U_s$. Moreover, the monodromy of the central functor Z coincides with the monodromy of Ψ_s.

2.4 Eisenstein series

Consider the induction diagram

$$\mathrm{Bun}_T(X) \xleftarrow{p} \mathrm{Bun}_B(X) \xrightarrow{q} \mathrm{Bun}_G(X,S), \tag{2.9}$$

where p is the usual projection, and q assigns to a B-bundle the induced G-bundle with its given B-reduction remembered along S. Since $\mathrm{Bun}_T(X) \simeq \Lambda_T \otimes_{\mathbb{Z}} \mathrm{Pic}(X)$, for each $\lambda \in \Lambda_T$ we have a corresponding component $\mathrm{Bun}_T^\lambda(X)$ of $\mathrm{Bun}_T(X)$. Let $\mathrm{Bun}_B^\lambda(X)$ be the preimage of $\mathrm{Bun}_T^\lambda(X)$ under p. Restricting the diagram (2.9) to the λ-component we get

$$\mathrm{Bun}_T^\lambda(X) \xleftarrow{p_\lambda} \mathrm{Bun}_B^\lambda(X) \xrightarrow{q_\lambda} \mathrm{Bun}_G(X,S). \tag{2.10}$$

Example 2.4.1 ($G = \mathrm{PGL}(2)$). In this case, $T = \mathbb{G}_m$, with $\Lambda_T \simeq \mathbb{Z}$, and therefore $\mathrm{Bun}_T(X) \simeq \mathrm{Pic}(X)$. An object of $\mathrm{Bun}_B(X)$ is an inclusion $(\mathcal{L} \subset \mathcal{E})$ of a line bundle into a rank-two vector bundle on X up to simultaneous tensoring with a line bundle. Then p is given by $(\mathcal{L} \subset \mathcal{E}) \mapsto \mathcal{L}^{\otimes 2} \otimes (\det \mathcal{E})^{-1}$, and q is given by $(\mathcal{L} \subset \mathcal{E}) \mapsto (\mathcal{E}, \mathcal{L}|_S \subset \mathcal{E}|_S)$. An object $(\mathcal{L} \subset \mathcal{E}) \in \mathrm{Bun}_B(X)$ lies in the component $\mathrm{Bun}_B^n(X)$ if and only if $2\deg(\mathcal{L}) - \deg(\mathcal{E}) = n$.

Definition 2.4.2. For $\lambda \in \Lambda_T$, we define the (unipotent) *Eisenstein sheaf* to be

$$\mathrm{Eis}_\lambda = q_{\lambda !}\underline{\mathbb{Q}}_{\mathrm{Bun}_B^\lambda(X)}[\dim B \cdot (g-1) - \langle 2\rho, \lambda\rangle] \in Sh_!(\mathrm{Bun}_G(X,S)).$$

Note that the shift $\dim B \cdot (g-1) - \langle 2\rho, \lambda\rangle$ is the dimension of $\mathrm{Bun}_B^\lambda(X)$.

Example 2.4.3 ($X = \mathbb{P}^1, \lambda = 0$). In this case, using that $\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$, we see that $\mathrm{Bun}_B^0(\mathbb{P}^1) \simeq \mathrm{pt}/B$. The map $q_0 : \mathrm{Bun}_B^0(\mathbb{P}^1) \to \mathrm{Bun}_G(\mathbb{P}^1, S)$ is an isomorphism to its image, which is the point classifying the trivial G-bundle over $\mathbb{P}^1$ with the same B-reduction at all $s \in S$. The Eisenstein series sheaf Eis_0 is the constant sheaf $\mathbb{Q}[-\dim B]$ on this point extended by zero.

The next lemma shows that the Eisenstein sheaves are translated by Wakimoto sheaves.

Lemma 2.4.4. *For $\lambda, \mu \in \Lambda_T$, $s \in S$, there is a canonical isomorphism*

$$J_\mu \star_s \mathrm{Eis}_\lambda \simeq \mathrm{Eis}_{\mu+\lambda}.$$

Proof. We first treat the case when μ is anti-dominant. To make notation more convenient, let μ be dominant and consider the action of $J_{-\mu}$ on Eis_λ. By definition, $J_{-\mu}$ is the !-extension of the constant sheaf $\mathbb{Q}[\langle 2\rho, \mu\rangle]$ on $\mathrm{Fl}_G^{-\mu}$. Unravelling the definitions, in particular of the action (2.5), we may describe the Hecke operator $J_{-\mu}\star_s$ using the Hecke correspondence

$$\mathrm{Bun}_G(X,S) \xleftarrow{p_-} \Gamma_{-\mu} \xrightarrow{p_+} \mathrm{Bun}_G(X,S) \tag{2.11}$$

given by the subdiagram of the diagram (2.4) where $\Gamma_{-\mu} \subset \mathrm{Bun}_G(\mathbb{X}_s, S)$ classifies pairs of points in $\mathrm{Bun}_G(X,S)$ with relative position $-\mu$ at the point s. By definition, we have

$$J_{-\mu} \star_s \mathcal{F} = p_{+!}p_-{}^*\mathcal{F}[\langle 2\rho, \mu\rangle] \quad \text{for } \mathcal{F} \in Sh(\mathrm{Bun}_G(X,S)). \tag{2.12}$$

We first assume the following.

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CLAIM. *We have a commutative diagram*

$$\begin{tikzcd}
\mathrm{Bun}_B^{\lambda}(X) \arrow[d, "q_\lambda"] & {}^{\lambda}\Gamma'_{-\mu} \arrow[l, "\gamma_-"'] \arrow[r, "\gamma_+"] \arrow[d, "h"] & \mathrm{Bun}_B^{\lambda-\mu}(X) \arrow[d, "q_{\lambda-\mu}"] \\
\mathrm{Bun}_G(X,S) & \Gamma_{-\mu} \arrow[l, "p_-"'] \arrow[r, "p_+"] & \mathrm{Bun}_G(X,S)
\end{tikzcd}$$

with the left square Cartesian and γ_+ a homeomorphism.

From the claim and (2.12), we can conclude

$$\begin{aligned}
J_{-\mu} \star_s \mathrm{Eis}_\lambda &= p_{+!}p_-{}^* q_{\lambda,!}\mathbb{Q}[\dim B\cdot(g-1) - \langle 2\rho,\lambda\rangle][\langle 2\rho,\mu\rangle] \\
&\simeq p_{+!}h_!\mathbb{Q}[\dim B\cdot(g-1) - \langle 2\rho,\lambda-\mu\rangle] \\
&\simeq q_{\lambda-\mu!}\gamma_{+!}\mathbb{Q}[\dim B\cdot(g-1) - \langle 2\rho,\lambda-\mu\rangle] \\
&\simeq \mathrm{Eis}_{\lambda-\mu}\,.
\end{aligned} \tag{2.13}$$

This proves the lemma for μ anti-dominant.

Since $J_\mu\star_s$ is the inverse to $J_{-\mu}\star_s$, from (2.13) we obtain

$$J_\mu \star_s \mathrm{Eis}_{\lambda'} \simeq \mathrm{Eis}_{\mu+\lambda'} \quad \text{for } \mu \text{ dominant, } \lambda' \in \Lambda_T. \tag{2.14}$$

Finally, for general μ, write μ as $\mu_1 - \mu_2$, where μ_1, μ_2 are both dominant. Using (2.13) and (2.14), we conclude

$$J_\mu \star_s \mathrm{Eis}_\lambda \simeq J_{\mu_1} \star_s (J_{-\mu_2} \star_s \mathrm{Eis}_\lambda) \simeq J_{\mu_1} \star_s \mathrm{Eis}_{\lambda-\mu_2} \simeq \mathrm{Eis}_{\lambda-\mu_2+\mu_1} = \mathrm{Eis}_{\lambda+\mu}\,.$$

Now to prove the lemma, it remains to prove the claim. With the choice of $s \in S$, we claim there is a canonical morphism

$$b_\mu : \mathrm{Bun}_B^{\lambda-\mu}(X) \longrightarrow \mathrm{Bun}_B^{\lambda}(X). \tag{2.15}$$

Once this is in hand, a local calculation shows there is a homeomorphism

$$\gamma_+ : {}^{\lambda}\Gamma'_{-\mu} := \mathrm{Bun}_B^{\lambda}(X) \times_{\mathrm{Bun}_G(X,S)} \Gamma_{-\mu} \longrightarrow \mathrm{Bun}_B^{\lambda-\mu}(X)$$

respecting the maps to $\mathrm{Bun}_G(X,S)$.

Thus it remains to construct the map (2.15).

First, recall the following 'pushout' construction for filtered vector bundles. Suppose $\mathcal{E}$ is a vector bundle over X with a finite decreasing filtration $\{F^i\mathcal{E}\}_{i\in\Lambda}$ by subbundles indexed by i in some poset Λ. Let $i \mapsto \mathcal{L}_i$ be a functor $\Lambda \to \mathrm{Pic}(X)^{\hookrightarrow}$, where $\mathrm{Pic}(X)^{\hookrightarrow}$ is the category of line bundles on X with injective sheaf maps as morphisms. Then there is a canonical vector bundle $\mathcal{E}'$ equipped with a decreasing filtration $\{F^i\mathcal{E}'\}_{i\in\Lambda}$ such that

$$\mathrm{Gr}^i_F\mathcal{E}' \simeq \mathrm{Gr}^i_F\mathcal{E} \otimes \mathcal{L}_i \quad \text{for all } i \in \Lambda.$$

The construction is by induction on the number of steps in the filtration, and we omit the details.

Next, the fiber of the natural projection $\mathrm{Bun}_B(X) \to \mathrm{Bun}_T(X)$ above a point $\mathcal{L} \in \mathrm{Bun}_T(X)$ classifies the following data.

– A tensor functor $\mathcal{E} : \mathrm{Rep}(G) \to \mathrm{Vect}(X)$ (the tensor category of vector bundles on X) denoted by $V \mapsto \mathcal{E}_V$.
– For $V \in \mathrm{Rep}(G)$, a decreasing filtration $\{F^\beta \mathcal{E}_V\}_{\beta \in \Lambda_T^\vee}$ indexed by the poset $\Lambda_T^\vee$ (where $\beta \leqslant \beta' \in \Lambda_T^\vee$ iff $\beta' - \beta$ is a $\mathbb{Z}_{\geqslant 0}$-combination of simple roots), along with isomorphisms $\mathrm{Gr}_F^\beta \mathcal{E}_V \simeq \mathcal{L}_\beta^{\oplus \dim V(\beta)}$ (where $V(\beta)$ denotes the β-weight space of V, and $\mathcal{L}_\beta \in \mathrm{Pic}(X)$ the induction of $\mathcal{L} \in \mathrm{Bun}_T(X)$ along $\beta : \Lambda_T \to \mathbb{Z}$).
– Moreover, the filtrations $\{F^\beta \mathcal{E}_V\}_{\beta \in \Lambda_T^\vee}$ and the tensor structure of $V \mapsto \mathcal{E}_V$ are compatible in the following sense: if $V, V' \in \mathrm{Rep}(G)$, then under the isomorphism $\mathcal{E}_{V \otimes V'} \simeq \mathcal{E}_V \otimes \mathcal{E}_{V'}$, we have $F^{\beta''} \mathcal{E}_{V \otimes V'} = \sum_{\beta + \beta' \geqslant \beta''} F^\beta \mathcal{E}_V \otimes F^{\beta'} \mathcal{E}_{V'}$.

Now we are ready to define the map (2.15). Starting with a point $(\mathcal{E}_V; F^\beta \mathcal{E}_V)_{V \in \mathrm{Rep}(G)}$ of $\mathrm{Bun}_B^{\lambda-\mu}(X)$. Let $\mathcal{E}'_V$ be the pushout of $\mathcal{E}_V$ with respect to the line bundles $\beta \mapsto \mathcal{O}_X(\langle \beta, \mu \rangle \cdot s)$. Since μ is dominant, for $\beta \leqslant \beta' \in \Lambda_T^\vee$, we have $\langle \beta, \mu \rangle \leqslant \langle \beta', \mu \rangle$ hence a natural inclusion $\mathcal{O}_X(\langle \beta, \mu \rangle \cdot s) \hookrightarrow \mathcal{O}_X(\langle \beta', \mu \rangle \cdot s)$, therefore the pushout is defined. The data $(\mathcal{E}'_V; F^\beta \mathcal{E}'_V)_{V \in \mathrm{Rep}(G)}$ then defines a point in $\mathrm{Bun}_B^\lambda(X)$.

This completes the proof of the claim and thus that of the lemma. □

Example 2.4.5 ($G = \mathrm{PGL}(2)$). We explain the stacks that appear in the proof above in the case $G = \mathrm{PGL}(2)$. Let $\mu = n \geqslant 0$. The Hecke correspondence Γ_{-n} in the proof above can be described as follows. Let $\widetilde{\Gamma}_{-n}$ be the moduli stack of $(\mathcal{E}_{-1} \hookrightarrow \mathcal{E}_0 \hookrightarrow \cdots \hookrightarrow \mathcal{E}_n; \{\ell_{s'}\}_{s' \in S \setminus \{s\}})$, where each $\mathcal{E}_i$ is a rank-two vector bundle on X, each arrow $\mathcal{E}_i \hookrightarrow \mathcal{E}_{i+1}$ is an upper modification of degree 1 at s, such that $\mathcal{E}_{i-1}(s) \neq \mathcal{E}_{i+1}$ for $i = 0, 1, \ldots, n-1$; finally, for $s' \neq s$, $\ell_{s'}$ is a line of the fiber of $\mathcal{E}_0$ at s'. Then we define $\Gamma_{-n} = \widetilde{\Gamma}_{-n}/\mathrm{Pic}(X)$, where $\mathrm{Pic}(X)$ acts by simultaneous tensoring on $\mathcal{E}_i$. The map p_- sends $(\mathcal{E}_{-1} \hookrightarrow \mathcal{E}_0 \hookrightarrow \cdots \hookrightarrow \mathcal{E}_n; \{\ell_{s'}\}_{s' \in S \setminus \{s\}})$ to $(\mathcal{E}_0; \{\ell_{s'}\}_{s' \in S})$, where ℓ_s is the image of $\mathcal{E}_{-1}$ in the fiber of $\mathcal{E}_0$ at s. The map p_+ sends $(\mathcal{E}_{-1} \hookrightarrow \mathcal{E}_0 \hookrightarrow \cdots \hookrightarrow \mathcal{E}_n; \{\ell_{s'}\}_{s' \in S \setminus \{s\}})$ to $(\mathcal{E}_n, \{\ell'_{s'}\}_{s' \in S})$, where ℓ'_s is the image of $\mathcal{E}_{n-1}$ in the fiber of $\mathcal{E}_n$ at s, $\ell'_{s'}$ for $s' \neq s$ is induced from $\ell_{s'}$ after identifying $\mathcal{E}_0|_{X \setminus \{s\}}$ and $\mathcal{E}_n|_{X \setminus \{s\}}$.

Let $\lambda = m \in \mathbb{Z}$. The stack ${}^m\Gamma'_{-n}$ defined in the proof above has the following moduli interpretation. It classifies $(\mathcal{L} \subset \mathcal{E}_0 \hookrightarrow \cdots \hookrightarrow \mathcal{E}_n)$, where the chain $\mathcal{E}_0 \hookrightarrow \cdots \mathcal{E}_n$ is as before, $\mathcal{L}$ is a line subbundle of $\mathcal{E}_0$, which is also saturated in $\mathcal{E}_1$ (the last condition is equivalent to $\mathcal{E}_{-1}(s) \neq \mathcal{E}_1$, if we define $\mathcal{E}_{-1}$ to be the lower modification of $\mathcal{E}_0$ at s determined by the line $\mathcal{L}_s$ of the fiber of $\mathcal{E}_0$ at s). It is easy to see inductively that $\mathcal{L}$ is saturated in $\mathcal{E}_2, \ldots, \mathcal{E}_n$. Therefore, $(\mathcal{L} \subset \mathcal{E}_n)$ defines a point in $\mathrm{Bun}_B^{m-n}(X)$. This gives the map $\gamma_+ : {}^m\Gamma'_{-n} \to \mathrm{Bun}_B^{m-n}(X)$, which is an isomorphism: the pair $\mathcal{L} \subset \mathcal{E}_n$ determines the chain $\mathcal{E}_0 \hookrightarrow \cdots \hookrightarrow \mathcal{E}_n$ because $\mathcal{E}_{i-1}$ can be inductively identified with the pullback of $(\mathcal{E}_i/\mathcal{L})(-s)$ under the quotient $\mathcal{E}_i \twoheadrightarrow \mathcal{E}_i/\mathcal{L}$.

2.5 Whittaker sheaf

In this subsection, we assume in addition that $\rho^\vee \in \Lambda_T$, for example G is adjoint.

2.5.1 *Twisted N-bundles.* Consider the distinguished T-bundle

$$\omega(S) := \rho^\vee \otimes \omega_X(S) \in \mathrm{Bun}_T(X) \simeq \Lambda_T \otimes_{\mathbb{Z}} \mathrm{Pic}(X).$$

Define the moduli

$$\mathrm{Bun}_N^{\omega(S)}(X, S) = \mathrm{Bun}_N^{\omega(S)}(X) \times_{\mathrm{Bun}_G(X)} \mathrm{Bun}_G(X, S)$$

classifying triples $(\mathcal{E}_B, \tau, \{\mathcal{F}_s\}_{s \in S})$, where $\mathcal{E}_B$ is a B-torsor, $\tau : \mathcal{E}_B/N \to \omega(S)$ is an isomorphism of T-torsors, and $\mathcal{F}_s$ is a B-reduction of the fiber $\mathcal{E}_G|_s$ of the G-bundle induced by $\mathcal{E}_B$. In other words, the choice of $\mathcal{F}_s$ is equivalent to the choice of a point of the twisted flag variety $\mathcal{B}_{\mathcal{E}_B|_s} = \mathcal{E}_B|_s \overset{B}{\times} \mathcal{B}$.

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Observe that there is an open substack

$$\mathrm{Bun}_N^{\omega(S),\circ}(X,S) \subset \mathrm{Bun}_N^{\omega(S)}(X,S),$$

where the B-reductions $\mathcal{E}_B|_s$ and $\mathcal{F}_s$ of the fiber $\mathcal{E}_G|_s$ are transverse, for each $s \in S$. If we let $\mathcal{B}^\circ \subset \mathcal{B}$ be the open B-orbit, then the choice of $\mathcal{F}_s$ is now equivalent to the choice of a point of the twisted open cell

$$\mathcal{B}^\circ_{\mathcal{E}_B|_x} \subset \mathcal{B}_{\mathcal{E}_B|_s}.$$

Note since $B\backslash\mathcal{B}^\circ \to T\backslash pt$ is an equivalence, the choice of such $\mathcal{F}_s$ is in turn equivalent to a splitting of $\mathcal{E}_B|_s \to \omega(S)|_s$.

Thus the abelianization map $N \to N/[N,N] \simeq \prod_{i=1}^r \mathbb{G}_a$, where r is the rank, induces a map

$$\mathrm{Bun}_N^{\omega(S),\circ}(X,S) \longrightarrow \prod_{i=1}^r \mathrm{Bun}_{\mathbb{G}_a,S}^{\omega_X(S)}(X), \tag{2.16}$$

where $\mathrm{Bun}_{\mathbb{G}_a,S}^{\omega_X(S)}(X)$ classifies extensions $\omega_X(S) \to \mathcal{E} \to \mathcal{O}_X$ with a splitting at each $s \in S$.

Pushout of extensions along the inclusion $\omega_X \to \omega_X(S)$ provides a canonical equivalence

$$\mathrm{Bun}_{\mathbb{G}_a}^{\omega_X}(X) \xrightarrow{\sim} \mathrm{Bun}_{\mathbb{G}_a,S}^{\omega_X(S)}(X), \quad (\omega_X \to \mathcal{E} \to \mathcal{O}_X) \longmapsto (\omega_X(S) \to \mathcal{E}' \to \mathcal{O}_X) \tag{2.17}$$

since the inclusion $\mathcal{E}_x \to \mathcal{E}'_x$ factors through $\mathcal{E}_x \to \mathcal{O}_{X,x}$, and hence its image gives a splitting of $\mathcal{E}'_x \to \mathcal{O}_{X,x}$.

Composing (2.16) with the inverse of (2.17) and taking the sum of the canonical evaluations

$$\mathrm{Bun}_{\mathbb{G}_a}^{\omega_X}(X) \simeq H^1(X,\omega_X) \simeq \mathbb{G}_a$$

we obtain the total evaluation

$$\mathrm{ev} : \mathrm{Bun}_N^{\omega(S),\circ}(X,S) \longrightarrow \mathbb{G}_a.$$

Note that the total evaluation is $\mathbb{G}_m$-equivariant for the action on $\mathrm{Bun}_N^{\omega(S),\circ}(X,S)$ induced via $\rho^\vee : \mathbb{G}_m \to T$ from the adjoint T-action and the usual rotation action on $\mathbb{G}_a$. Therefore, it descends to a map

$$\overline{\mathrm{ev}} : \mathrm{Bun}_N^{\omega(S),\circ}(X)/\mathbb{G}_m \longrightarrow \mathbb{G}_a/\mathbb{G}_m.$$

We also have the natural induction map

$$p : \mathrm{Bun}_N^{\omega(S),\circ}(X,S) \longrightarrow \mathrm{Bun}_G(X,S),$$

which descends to a map

$$\overline{p} : \mathrm{Bun}_N^{\omega(S),\circ}(X,S)/\mathbb{G}_m \longrightarrow \mathrm{Bun}_G(X,S),$$

where again the $\mathbb{G}_m$-action on $\mathrm{Bun}_N^{\omega(S),\circ}(X,S)$ is induced via $\rho^\vee : \mathbb{G}_m \to T$ from the adjoint T-action.

2.5.2 $\mathbb{G}_m$*-averaged Artin–Schreier sheaf.* Let us write $j : pt = \mathbb{G}_m/\mathbb{G}_m \to \mathbb{G}_a/\mathbb{G}_m$ for the open inclusion. Let

$$\Psi := j_*\underline{\mathbb{Q}}_{\mathrm{pt}}[-1] \in D_{\mathbb{G}_m}(\mathbb{G}_a).$$

This sheaf should be thought of as a $\mathbb{G}_m$-equivariant version of an Artin–Schreier sheaf over $\mathbb{G}_a$ if we worked over a base field of finite characteristic, or a $\mathbb{G}_m$-equivariant version of the exponential D-module over $\mathbb{G}_a$ if we worked in the D-module setting.

Definition 2.5.3. The Whittaker sheaf is the object

$$\mathrm{Wh}_S = \overline{p}_!\overline{\mathrm{ev}}^*\Psi[-d_S] \in Sh_!(\mathrm{Bun}_G(X,S)),$$

where

$$d_S = \dim B \cdot (g-1) + \langle 2\rho, \rho^\vee\rangle(2g-2+\#S)$$

is the dimension of $\mathrm{Bun}_B^{-\omega(S)}(X)$.

Example 2.5.4 ($X = \mathbb{P}^1, S = \{0,\infty\}$). In this case, we have $\omega(S) \simeq \mathcal{O}_{\mathbb{P}^1}$, and hence the Whittaker sheaf is supported on the open locus, where the underlying G-bundle is semistable or equivalently trivializable

$$\mathrm{Bun}_G^{\mathrm{triv}}(\mathbb{P}^1,\{0,\infty\}) \simeq G\backslash(\mathcal{B}\times\mathcal{B}).$$

On the other hand, let $\mathcal{B}^\circ$ be the open N-orbit in $\mathcal{B}$, then we have

$$\mathrm{Bun}_N^{\omega(S),\circ}(\mathbb{P}^1,\{0,\infty\}) \simeq N\backslash(\mathcal{B}^\circ\times\mathcal{B}^\circ).$$

If we choose a point $B^- \in \mathcal{B}^\circ$ represented by a Borel opposite to B, then we have $G\backslash(\mathcal{B}\times\mathcal{B}) \simeq B^-\backslash\mathcal{B}$ by fixing the first coordinate to be B^-; similarly, we have $N\backslash(\mathcal{B}^\circ\times\mathcal{B}^\circ) \simeq \mathcal{B}^\circ$ by fixing the first coordinate to be B^-. Under the above isomorphisms, the map $p : \mathrm{Bun}_N^{\omega(S),\circ}(\mathbb{P}^1,\{0,\infty\}) \to \mathrm{Bun}_G(\mathbb{P}^1,\{0,\infty\})$ is the evident composition

$$\mathcal{B}^\circ \xhookrightarrow{i} \mathcal{B} \xrightarrow{q} B^-\backslash\mathcal{B}.$$

Let $\Xi \in \mathrm{Perv}_N(\mathcal{B})$ be the tilting extension to $\mathcal{B}$ of the constant perverse sheaf $\underline{\mathbb{Q}}_{\mathcal{B}^\circ}[\dim\mathcal{B}^\circ]$. We claim that

$$\mathrm{Wh}_{\{0,\infty\}} \simeq u_!q_!\Xi[\dim B]. \tag{2.18}$$

Here $u : B^-\backslash\mathcal{B} \simeq \mathrm{Bun}_G^{\mathrm{triv}}(\mathbb{P}^1,\{0,\infty\}) \hookrightarrow \mathrm{Bun}_G(\mathbb{P}^1,\{0,\infty\})$ is the open inclusion. To see this, we only need to note that both sides of (2.18), up to appropriate shifts, corepresent the functor of vanishing cycles at a generic covector at the image of $\mathrm{Bun}_B^0(\mathbb{P}^1) \to \mathrm{Bun}_G(\mathbb{P}^1,\{0,\infty\})$.

The Whittaker sheaf Wh_S enjoys an asphericity property, as we spell out now. For $s \in S$ and a parabolic subgroup $P \subset G$, we may define a moduli stack $\mathrm{Bun}_G(X,S)_{s,P}$, where the level structure at s is changed to a P-reduction. We have a proper smooth projection

$$\pi_{s,P} : \mathrm{Bun}_G(X,S) \to \mathrm{Bun}_G(X,S)_{s,P},$$

which induces adjoint functors

$$\begin{tikzcd} Sh_!(\mathrm{Bun}_G(X,S)) \arrow[r, "\pi_{s,P,*}=\pi_{s,P!}" description] & Sh_!(\mathrm{Bun}_G(X,S)_{s,P}). \arrow[l, bend right, "\pi_{s,P}^*"'] \arrow[l, bend left, "\pi_{s,P}^!"] \end{tikzcd}$$

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Lemma 2.5.5. *Let $s \in S$ and $P \subset G$ be a parabolic subgroup, which is not a Borel. Then*

$$\pi_{s,P!}\mathrm{Wh}_S \simeq 0.$$

Proof. Let P_i be the standard parabolic whose Levi only has simple root α_i. Then each P, which is not a Borel, contains some P_i, and $\pi_{s,P}$ factors as

$$\mathrm{Bun}_G(X,S) \xrightarrow{\pi_{s,P_i}} \mathrm{Bun}_G(X,S)_{s,P_i} \to \mathrm{Bun}_G(X,S)_{s,P}.$$

Therefore, it suffices to show that $\pi_{s,P_i!}\mathrm{Wh}_S \simeq 0$, for each P_i.

We denote $\mathrm{Bun}_G(X,S)_{s,P_i}$ simply by $\mathrm{Bun}_G(X,S)_{s,i}$, and denote π_{s,P_i} similarly by $\pi_{s,i}$, which is a $\mathbb{P}^1$-fibration.

Let us extend the maps in the definition of Wh_S to a commutative (but not Cartesian) diagram

$$\begin{tikzcd}
\mathrm{pt} = \mathbb{G}_m/\mathbb{G}_m \arrow[r, hook, "j"] & \mathbb{G}_a/\mathbb{G}_m & \mathrm{Bun}_N^{\omega(S),\circ}(X,S)/\mathbb{G}_m \arrow[l, "\overline{\mathrm{ev}}"'] \arrow[r, "\overline{p}"] \arrow[d, "\pi'_{s,i}"] & \mathrm{Bun}_G(X,S) \arrow[d, "\pi_{s,i}"] \\
& & \mathrm{Bun}_N^{\omega(S),\circ}(X,S)_{s,i}/\mathbb{G}_m \arrow[r, "\overline{p}_{s,i}"] & \mathrm{Bun}_G(X,S)_{s,i}
\end{tikzcd}$$

where we denote by $\mathrm{Bun}_N^{\omega(S),\circ}(X,S)_{s,i}/\mathbb{G}_m$ the moduli, where we replace the B-reduction at s with a P_i-reduction in general position with the given N-structure, and $\pi'_{s,i}$ is the natural $\mathbb{A}^1$-fibration, where we forget the B-reduction at s to a P_i-reduction.

Now returning to the definition of Wh_S, we have

$$\pi_{s,i!}\mathrm{Wh}_S = \pi_{s,i!}\overline{p}_!\overline{\mathrm{ev}}^* j_*\underline{\mathbb{Q}}_{\mathrm{pt}}[-1-d_S] \simeq \overline{p}_{s,i!}\pi'_{s,i!}\overline{\mathrm{ev}}^* j_*\underline{\mathbb{Q}}_{\mathrm{pt}}[-1-d_S]$$

and so it suffices to show

$$\pi'_{s,i!}\overline{\mathrm{ev}}^* j_*\underline{\mathbb{Q}}_{\mathrm{pt}} \simeq 0.$$

Fix a point $\xi : \mathrm{pt} \to \mathrm{Bun}_N^{\omega(S),\circ}(X,S)_{s,i}$, and consider the following base-changed Cartesian diagram.

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r, hook, "j"] & \mathbb{G}_a & \mathrm{Bun}_N^{\omega(S),\circ}(X,S) \arrow[l, "\mathrm{ev}"'] \arrow[d, "\pi'_{s,i}"] & \mathbb{A}^1 \arrow[l, "\widetilde{\xi}"'] \arrow[d] \\
& & \mathrm{Bun}_N^{\omega(S),\circ}(X,S)_{s,i} & \mathrm{pt} \arrow[l, "\xi"']
\end{tikzcd}$$

Then it suffices to show

$$\xi^*\pi'_{s,i!}\mathrm{ev}^* j_*\underline{\mathbb{Q}}_{\mathbb{G}_m} \simeq 0.$$

Finally, observe that $\mathrm{ev} \circ \widetilde{\xi} : \mathbb{A}^1 \to \mathbb{G}_a$ is an isomorphism of schemes, and so

$$\widetilde{\xi}^*\mathrm{ev}^* j_*\underline{\mathbb{Q}}_{\mathbb{G}_m} \simeq j'_*\underline{\mathbb{Q}}_U,$$

where $j' : U = (\mathrm{ev} \circ \widetilde{\xi})^{-1}(\mathbb{G}_m) \hookrightarrow \mathbb{A}^1$ is the complement of one point in $\mathbb{A}^1$. Thus we have the required vanishing

$$\xi^*\pi'_{s,i!}\mathrm{ev}^* j_*\underline{\mathbb{Q}}_{\mathbb{G}_m} \simeq \pi'_{s,i!}\widetilde{\xi}^*\mathrm{ev}^* j_*\underline{\mathbb{Q}}_{\mathbb{G}_m} \simeq \mathrm{H}_c^*(\mathbb{A}^1, j'_*\underline{\mathbb{Q}}_U) \simeq 0. \qquad \square$$

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COROLLARY 2.5.6. *Let $s \in S$.*

(i) *For any $w \neq 1 \in W_f$, we have $\mathrm{IC}_w \star_s \mathrm{Wh}_S \simeq 0$.*

(ii) *The action of $\mathcal{H}_G^f \subset \mathcal{H}_G^{\mathrm{aff}}$ on $Sh(\mathrm{Bun}_G(X,S))$ by Hecke modification at s factors through the monoidal functor $\mathbb{V} : \mathcal{H}_G^f \to \mathrm{Vect}$ in that for any $\mathcal{K} \in \mathcal{H}_G^f$, there is a canonical isomorphism*

$$\mathcal{K} \star_s \mathrm{Wh}_S \simeq \mathbb{V}(\mathcal{K}) \otimes \mathrm{Wh}_S$$

compatible with the monoidal structures in the obvious sense.

Proof. (i) Since any $w \neq 1$ can be written as a product of simple reflections σ_i, it suffices to show $\mathrm{IC}_{\sigma_i} \star_s \mathrm{Wh}_S \simeq 0$, for the simple reflections $\sigma_i \in W_f$. Let P_i be the standard parabolic of G whose Levi has only simple root α_i. Then

$$\mathrm{IC}_{\sigma_i} \star_s \mathrm{Wh}_S \simeq \pi^*_{s,P_i}\pi_{s,P_i,!}\mathrm{Wh}_S[1],$$

which vanishes by Lemma 2.5.5. Therefore, (i) is proved.

Since $\mathbb{V} : \mathcal{H}_G^f \to \mathrm{Vect}$ is monoidal and the universal quotient functor with the kernel the monoidal ideal $\langle \mathrm{IC}_w | w \neq 1 \in W_f \rangle$, (ii) follows from (i). □

2.5.7 *Wakimoto action on Whittaker sheaf.* For $s \in S$, we have an action of $\mathrm{Perf}(\widetilde{\mathcal{N}}^\vee/G^\vee)$ on $Sh_!(\mathrm{Bun}_G(X,S))$ as the restriction of the affine Hecke action at s, see (2.6) and (2.7). By acting on Wh_S, we obtain a functor

$$\alpha_s : \mathrm{Perf}(\widetilde{\mathcal{N}}^\vee/G^\vee) \longrightarrow Sh_!(\mathrm{Bun}_G(X,S)) \tag{2.19}$$

such that line bundles go to translations of Wh_S by Wakimoto operators

$$\alpha_s(\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(\lambda)) = J_\lambda \star_s \mathrm{Wh}_S, \quad \lambda \in \Lambda_T.$$

PROPOSITION 2.5.8. *The functor α_s intertwines the action of $\mathrm{Coh}^{G^\vee}(St_{G^\vee})$ on the left side and the $\star_s$-action of $\mathcal{H}_G^{\mathrm{aff}}$ on the right side under the monoidal equivalence Φ^{aff}.*

Proof. By Corollary 2.5.6, the $\star_s$-action of $\mathcal{H}_G^{\mathrm{aff}}$ on the object Wh_S factors through the aspherical quotient $\mathcal{H}_G^{\mathrm{asph}}$, or in other words, we have a functor

$$\alpha'_s : \mathcal{H}_G^{\mathrm{asph}} \longrightarrow Sh_!(\mathrm{Bun}_G(X,S))$$

and a canonical equivalence $\mathcal{K} \star_s \mathrm{Wh}_S \simeq \alpha'_s(\overline{\mathcal{K}})$, where $\mathcal{K} \in \mathcal{H}_G^{\mathrm{aff}}$, and $\overline{\mathcal{K}} \in \mathcal{H}_G^{\mathrm{asph}}$ is its image. By construction, the functor α'_s is a $\mathcal{H}_G^{\mathrm{aff}}$-module map.

Now we claim that α_s and α'_s are the same functors under the equivalence Φ^{asph}. By the construction in [Bez16], Φ^{asph} is the composition $\mathrm{Perf}(\widetilde{\mathcal{N}}^\vee/G^\vee) \to \mathcal{H}_G^{\mathrm{aff}} \to \mathcal{H}_G^{\mathrm{asph}}$ given by $\mathcal{K} \mapsto \overline{\Delta_*\mathcal{K}}$. Thus we have canonical equivalences

$$\alpha_s(\mathcal{K}) \simeq (\Delta_*\mathcal{K}) \star_s \mathrm{Wh}_S \simeq \alpha'_s(\overline{\Delta_*\mathcal{K}}), \quad \mathcal{K} \in \mathrm{Perf}(\widetilde{\mathcal{N}}^\vee/G^\vee).$$

Finally, since Φ^{asph} intertwines the $\mathcal{H}_G^{\mathrm{aff}}$-action and the $\mathrm{Coh}^{G^\vee}(St_{G^\vee})$-action via the monoidal equivalence Φ^{aff}, and α'_s is a $\mathcal{H}_G^{\mathrm{aff}}$-module map, the lemma follows. □

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2.6 Two point ramification

In this section, we specialize to the case $X = \mathbb{P}^1$ and $S = \{0, \infty\}$. We elaborate on the principle that '$Sh_!(\mathrm{Bun}_G(\mathbb{P}^1, \{0,\infty\}))$ is the same as $\mathcal{H}_G^{\mathrm{aff}}$'.

We have the two commuting actions $\star_0, \star_\infty$ of $\mathcal{H}_G^{\mathrm{aff}}$ on $Sh_!(\mathrm{Bun}_G(\mathbb{P}^1, \{0,\infty\}))$ by Hecke modifications at respectively 0, ∞. We have the Eisenstein series sheaf Eis_0 described in Example 2.4.3. Acting by $\mathcal{H}_G^{\mathrm{aff}}$ on Eis_0 at 0, we obtain a functor

$$\Phi'_{0,\infty} : \mathcal{H}_G^{\mathrm{aff}} \longrightarrow Sh_!(\mathrm{Bun}_G(\mathbb{P}^1, \{0,\infty\}), \quad \Phi'_{0,\infty}(\mathcal{K}) = \mathcal{K} \star_0 \mathrm{Eis}_0 .$$

LEMMA 2.6.1. $\Phi'_{0,\infty}$ *is an equivalence.*

Proof. Let us relate $\Phi'_{0,\infty}$ to the Radon transform.

Let $j : \mathrm{pt}/T \hookrightarrow \mathrm{Bun}_G(\mathbb{P}^1, \{0,\infty\})$ be the open substack, where the underlying bundle is trivial and the two Borel reductions at $0, \infty$ are transverse. Acting by $\mathcal{H}_G^{\mathrm{aff}}$ on $j_!\mathbb{Q}[-\dim T]$ at 0 we recover the Radon transform

$$R : \mathcal{H}_G^{\mathrm{aff}} \longrightarrow Sh_!(\mathrm{Bun}_G(\mathbb{P}^1, \{0,\infty\}), \quad R(\mathcal{K}) = \mathcal{K} \star_0 j_!\mathbb{Q}[-\dim T].$$

It is well-known that R is an equivalence (see [Yun09, Corollary 4.1.5 and § 5.2] for example). Let $T_{w_0*} \in \mathcal{H}_G^f$ denote the perverse sheaf, which is the $*$-extension of the shifted constant sheaf from the open B-orbit in $\mathcal{B}$. Then $T_{w_0*} \star_0 j_!\mathbb{Q}[-\dim T] \simeq \mathrm{Eis}_0$. Therefore,

$$\Phi'_{0,\infty}(\mathcal{K}) = \mathcal{K} \star_0 \mathrm{Eis}_0 \simeq (\mathcal{K} \star T_{w_0*}) \star_0 j_!\mathbb{Q}[-\dim T] = R(\mathcal{K} \star T_{w_0*}).$$

In other words, $\Phi'_{0,\infty}$ is the composition of first convolution on $\mathcal{H}^{\mathrm{aff}}$ on the right by T_{w_0*} (which is an equivalence with inverse given by convolution on the right by $T_{w_0!}$), and then the Radon transform R (which is again also an equivalence). This shows that $\Phi'_{0,\infty}$ is an equivalence. □

Let $\mathrm{Loc}_{G^\vee}(\mathbb{P}^1, \{0,\infty\})$ denote the (derived) moduli stack (over $\mathbb{Q}$) of $G^\vee$-local systems on $\mathbb{P}^1 \backslash \{0,\infty\}$ equipped near $\{0,\infty\}$ with a Borel reduction with unipotent monodromy. Then $\mathrm{Loc}_{G^\vee}(\mathbb{P}^1, \{0,\infty\})$ admits the presentation as the substack of $(\widetilde{\mathcal{N}}^\vee \times \widetilde{\mathcal{N}}^\vee)/G^\vee$ given by imposing on pairs $(\widetilde{A}_0, \widetilde{A}_\infty) \in \widetilde{\mathcal{N}}^\vee \times \widetilde{\mathcal{N}}^\vee$ the equation $A_0 A_\infty = 1$ on the underlying group elements inside of $G^\vee$. Therefore, we have an isomorphism

$$\iota : St_{G^\vee}/G^\vee = (\widetilde{\mathcal{N}}^\vee \times_{G^\vee} \widetilde{\mathcal{N}}^\vee)/G^\vee \xrightarrow{\sim} \mathrm{Loc}_{G^\vee}(\mathbb{P}^1, \{0,\infty\}), \quad \iota(\widetilde{A}_0, \widetilde{A}_\infty) = (\widetilde{A}_0, \widetilde{A}_\infty^{-1}),$$

where $\widetilde{A}_\infty^{-1}$ means we invert the group element A_∞ while keeping the Borel containing it unchanged.

Now introduce the equivalence given by the composition of equivalences

$$\Phi_{0,\infty} : \mathrm{Coh}(\mathrm{Loc}_{G^\vee}(\mathbb{P}^1, \{0,\infty\})) \xrightarrow{\iota^*} \mathrm{Coh}^{G^\vee}(St_{G^\vee}) \xrightarrow{\Phi^{\mathrm{aff}}} \mathcal{H}_G^{\mathrm{aff}}$$

$$\xrightarrow{\Phi'_{0,\infty}} Sh_!(\mathrm{Bun}_G(\mathbb{P}^1, \{0,\infty\}).$$

By construction, $\Phi_{0,\infty}$ intertwines the $\mathrm{Coh}^{G^\vee}(St_{G^\vee})$-action on $\mathrm{Coh}(\mathrm{Loc}_{G^\vee}(\mathbb{P}^1, \{0,\infty\}))$ by convolution at 0 and the $\mathcal{H}_G^{\mathrm{aff}}$-action on $Sh_!(\mathrm{Bun}_G(\mathbb{P}^1, \{0,\infty\}))$ by the Hecke modifications $\star_0$, under the monoidal equivalence Φ^{aff}. One can also show that $\Phi_{0,\infty}$ similarly intertwines the $\mathrm{Coh}^{G^\vee}(St_{G^\vee})$-action on $\mathrm{Coh}(\mathrm{Loc}_{G^\vee}(\mathbb{P}^1, \{0,\infty\}))$ by convolution at ∞ and the $\mathcal{H}_G^{\mathrm{aff}}$-action on $Sh_!(\mathrm{Bun}_G(\mathbb{P}^1, \{0,\infty\}))$ by the Hecke modifications $\star_\infty$. We will not use this statement in the rest of the paper, only the following compatibilities.

Lemma 2.6.2.

(i) *Let* $\Delta^- : \widetilde{\mathcal{N}}^\vee/G^\vee \to \mathrm{Loc}_{G^\vee}(\mathbb{P}^1, \{0,\infty\})$ *be the anti-diagonal* $\Delta^-(\widetilde{A}_0) = (\widetilde{A}_0, \widetilde{A}_0^{-1})$*. Then we have*

$$\Phi_{0,\infty}(\Delta^-_* \mathcal{O}_{\widetilde{\mathcal{N}}}) \simeq \mathrm{Eis}_0 .$$

(ii) *For* $\mathcal{O}_{\mathrm{Loc}}$ *the derived structure sheaf of* $\mathrm{Loc}_{G^\vee}(\mathbb{P}^1, \{0,\infty\})$*, we have*

$$\Phi_{0,\infty}(\mathcal{O}_{\mathrm{Loc}}) \simeq \mathrm{Wh}_{0,\infty}.$$

Proof. (i) Under the equivalence Φ^{aff}, the monoidal unit $\delta \in \mathcal{H}_G^{\mathrm{aff}}$, given by the constant sheaf on the closed I-orbit in $\mathrm{Fl}_G = G((t))/I$, corresponds to $\Delta_* \mathcal{O}_{\widetilde{\mathcal{N}}^\vee} \in \mathrm{Coh}^{G^\vee}(St_{G^\vee})$ (see § 2.2.3). By construction, we also have $\iota^*(\Delta_* \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}) \simeq \Delta^-_* \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}$, and $\Phi'_{0,\infty}(\delta) = \delta \star_0 \mathrm{Eis}_0 \simeq \mathrm{Eis}_0$, therefore $\Phi_{0,\infty}(\Delta^-_* \mathcal{O}_{\widetilde{\mathcal{N}}}) \simeq \mathrm{Eis}_0$.

(ii) First, we claim that under the equivalence Φ^{aff}, the derived structure sheaf $\mathcal{O}_{St_{G^\vee}} \in \mathrm{Coh}^{G^\vee}(St_{G^\vee})$ corresponds to $q_! \Xi[2\dim B] \in \mathcal{H}_G^f \subset \mathcal{H}_G^{\mathrm{aff}}$ (see Example 2.5.4 for notation). To see this, we use the equivalence established in [Bez16, Theorem 1]

$$\Phi_{I^0 I} : Sh_c(I^0 \backslash \mathrm{Fl}_G) \xrightarrow{\sim} \mathrm{Coh}^{G^\vee}(St'),$$

where $I^0 = I \times_B N \subset I$, and $St' = \widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathcal{N}}^\vee$ ($\widetilde{\mathfrak{g}^\vee} \to \mathfrak{g}^\vee$ is the Grothendieck alteration). By [Bez16, Example 57], $\Phi_{I^0 I}(\Xi) = \mathcal{O}_{St'}$. On the other hand, the equivalences $\Phi_{I^0 I}$ and Φ^{aff} are compatible: the forgetful functor $\mathrm{Forg} : \mathcal{H}_G^{\mathrm{aff}} \to Sh_c(I^0 \backslash \mathrm{Fl}_G)$ corresponds to pushforward along $i : St_{G^\vee} \hookrightarrow St'$. Therefore, i^* corresponds to the left adjoint of Forg, and this is given by $q_![2\dim B]$ when restricted to $Sh_c(N \backslash \mathcal{B}) \subset Sh_c(I^0 \backslash \mathrm{Fl}_G)$. Hence $\mathcal{O}_{St_{G^\vee}} \simeq i^* \mathcal{O}_{St'}$ corresponds to $q_! \Xi[2\dim B]$ under the equivalence Φ^{aff}.

Therefore, we have

$$\Phi_{0,\infty}(\mathcal{O}_{\mathrm{Loc}}) = \Phi'_{0,\infty}(\Phi^{\mathrm{aff}}(\mathcal{O}_{St_{G^\vee}})) \simeq \Phi'_{0,\infty}(q_! \Xi[2\dim B]) = q_! \Xi[2\dim B] \star_0 \mathrm{Eis}_0.$$

Finally, if we view $q_! \Xi[2\dim B] \star_0 \mathrm{Eis}_0$ as an object of $Sh_c(B \backslash G/B) \xrightarrow{u_!} Sh_!(\mathrm{Bun}_G(\mathbb{P}^1, \{0,\infty\}))$, it is equivalent to $q_! \Xi[2\dim B] \star \delta[-\dim B] \simeq q_! \Xi[\dim B]$. Thus $\Phi_{0,\infty}(\mathcal{O}_{\mathrm{Loc}}) \simeq u_! q_! \Xi[\dim B]$, and in turn $u_! q_! \Xi[\dim B] \simeq \mathrm{Wh}_{0,\infty}$ as seen in (2.18). □

3. Automorphic side: $\mathbb{P}^1$, three ramification points, $G = \mathrm{PGL}(2), \mathrm{SL}(2)$

Let $\mathbb{P}^1 = \mathrm{Proj}\mathbb{C}[x,y]$ be the projective line with homogeneous coordinates $[x,y]$ and coordinate $t = y/x$.

Fix the three points $S = \{0,1,\infty\} \subset \mathbb{P}^1$, where the coordinate t takes the respective value.

3.1 Moduli of bundles

Let $\mathrm{Pic}(\mathbb{P}^1) \simeq \mathrm{Bun}_{\mathrm{GL}(1)}(\mathbb{P}^1)$ denote the Picard stack of line bundles on $\mathbb{P}^1$, and $\mathrm{Vect}_2(\mathbb{P}^1) \simeq \mathrm{Bun}_{\mathrm{GL}(2)}(\mathbb{P}^1)$ the moduli of rank-two vector bundles on $\mathbb{P}^1$.

3.1.1 $G = \mathrm{PGL}(2)$. By the exact sequence $1 \to \mathrm{GL}(1) \to \mathrm{GL}(2) \to \mathrm{PGL}(2) \to 1$ and the vanishing of the Brauer group of a curve over $\mathbb{C}$, we have an isomorphism

$$\mathrm{Vect}_2(\mathbb{P}^1)/\mathrm{Pic}(\mathbb{P}^1) \xrightarrow{\sim} \mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1).$$

Thus we can represent PGL(2)-bundles by rank-two vector bundles up to tensoring with a line bundle. There is a disjoint union decomposition

$$\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1) = \mathrm{Bun}^{\overline{0}}_{\mathrm{PGL}(2)}(\mathbb{P}^1) \coprod \mathrm{Bun}^{\overline{1}}_{\mathrm{PGL}(2)}(\mathbb{P}^1)$$

given by the parity of the degree of a rank-two vector bundle.

The stack $\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)$ is the moduli of PGL(2)-bundles on $\mathbb{P}^1$ with B-reductions at the points of $S = \{0, 1, \infty\}$. We can represent objects of $\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)$ by $(\mathcal{E}, \{\ell_s\}_{s\in S})$, where $\mathcal{E}$ is a rank-two vector bundles on $\mathbb{P}^1$ up to tensoring with a line bundle, and ℓ_s is a line in the fiber $\mathcal{E}_s$ for each $s \in S$.

Let us list the isomorphism classes of objects of $\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)$. For each isomorphism class of $\mathcal{E} \in \mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1)$, we describe the poset of points in $\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)$ over it, where an arrow $x \to y$ means y lies in the closure of x.

(i) $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$, $\mathrm{Aut}(\mathcal{E}) \simeq \mathrm{PGL}(2)$, with the poset of configurations of lines

$$\begin{tikzcd}
 & c_0(\emptyset) \arrow[dl] \arrow[d] \arrow[dr] & & \mathrm{Aut} \simeq \{1\} \\
c_0(0,1) \arrow[dr] & c_0(0,\infty) \arrow[d] & c_0(1,\infty) \arrow[dl] & \mathrm{Aut} \simeq T \simeq \mathbb{G}_m \\
 & c_0(S) & & \mathrm{Aut} \simeq B \simeq \mathbb{G}_m \ltimes \mathbb{G}_a
\end{tikzcd}$$

where $c_0(R)$ denotes the locus where two lines ℓ_s and $\ell_{s'}$ at distinct points $s, s' \in S$ are equal if and only if $s, s' \in R$.

(ii) $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}$, $\mathrm{Aut}(\mathcal{E}) \simeq \mathbb{G}_m \ltimes \mathbb{G}_a^2$, with the poset of configurations of lines

$$\begin{tikzcd}
 & & c_1(*) \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] & & \mathrm{Aut} \simeq \{1\} \\
c_1(\emptyset) \arrow[dr] \arrow[drr] \arrow[drrr] & c_1(0) \arrow[d] \arrow[dr] & c_1(1) \arrow[dl] \arrow[dr] & c_1(\infty) \arrow[dl] \arrow[d] & \mathrm{Aut} \simeq T \simeq \mathbb{G}_m \\
 & c_1(0,1) \arrow[dr] & c_1(0,\infty) \arrow[d] & c_1(1,\infty) \arrow[dl] & \mathrm{Aut} \simeq B \simeq \mathbb{G}_m \ltimes \mathbb{G}_a \\
 & & c_1(S) & & \mathrm{Aut} \simeq \mathbb{G}_m \ltimes \mathbb{G}_a^2
\end{tikzcd}$$

where $c_1(R)$ denotes where the lines ℓ_r lie in the summand $\mathcal{O}_{\mathbb{P}^1}(1)$, for $r \in R \subset S$, and in the summand $\mathcal{O}_{\mathbb{P}^1}$, for $r \notin R \subset S$. The generic configuration $c_1(*)$ denotes where none of the lines ℓ_s lie in $\mathcal{O}_{\mathbb{P}^1}(1)$, for $s \in S$, and also they do not all lie in the image of any map $\mathcal{O}_{\mathbb{P}^1} \to \mathcal{E}$ (as in the configuration $c_1(\emptyset)$).

(iii) $k \geqslant 2$, $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}$, and we have an exact sequence

$$1 \longrightarrow \mathbb{G}_a^{k-2} \longrightarrow \mathrm{Aut}(\mathcal{E}) \xrightarrow{\mathrm{ev}_S} \mathbb{G}_m \ltimes (\mathbb{G}_a)^S \longrightarrow 1.$$

The poset of configurations of lines is the product

$$\textstyle\prod_{s\in S}(\{\ell_s \subset \mathcal{O}_{\mathbb{P}^1}\} \longrightarrow \{\ell_s \subset \mathcal{O}_{\mathbb{P}^1}(k)\})$$

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with automorphisms

$$1 \longrightarrow \mathbb{G}_a^{k-2} \longrightarrow \mathrm{Aut} \xrightarrow{\mathrm{ev}_S} \mathbb{G}_m \ltimes (\mathbb{G}_a)^R \longrightarrow 1, \quad R = \{s \in S \mid \ell_s \subset \mathcal{O}_{\mathbb{P}^1}(k)\}.$$

Let us denote by $c_k(R)$ where the lines ℓ_r lie in the summand $\mathcal{O}_{\mathbb{P}^1}(k)$, for $r \in R \subset S$, and in the summand $\mathcal{O}_{\mathbb{P}^1}$, for $r \notin R \subset S$.

3.1.2 $G = \mathrm{SL}(2)$. Note that $1 \to \mathrm{SL}(2) \to \mathrm{GL}(2) \to \mathrm{GL}(1) \to 1$ allows us to represent SL(2)-bundles by rank-two vector bundles with trivialized determinant.

Let $\mathrm{Bun}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$ denote the moduli of SL(2)-bundles on $\mathbb{P}^1$ with B-reductions at the points of $S = \{0, 1, \infty\}$. We can represent objects of $\mathrm{Bun}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$ by $(\mathcal{E}, \tau, \{\ell_s\}_{s\in S})$ where $\mathcal{E}$ is a rank-two vector bundle on $\mathbb{P}^1$, $\tau : \mathcal{O}_{\mathbb{P}^1} \xrightarrow{\sim} \det(\mathcal{E})$, and ℓ_s is a line in the fiber $\mathcal{E}_s$ for $s \in S$.

Let us list the isomorphism classes of objects of $\mathrm{Bun}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$ according to the isomorphism type of the underlying rank-two bundles.

(i) $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ $\mathrm{Aut}(\mathcal{E}) \simeq \mathrm{SL}(2)$, with the poset of configurations of lines

$$\begin{tikzcd}
& c_0(\emptyset) \arrow[dl] \arrow[d] \arrow[dr] & & \mathrm{Aut} \simeq Z(\mathrm{SL}(2)) \simeq \mu_2 \\
c_0(0,1) \arrow[dr] & c_0(0,\infty) \arrow[d] & c_0(1,\infty) \arrow[dl] & \mathrm{Aut} \simeq T \simeq \mathbb{G}_m \\
& c_0(S) & & \mathrm{Aut} \simeq B \simeq \mathbb{G}_m \ltimes \mathbb{G}_a
\end{tikzcd}$$

where $c_0(R)$ denotes where the lines ℓ_r coincide, for $r \in R \subset S$.

(ii) $k \geqslant 1$, $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(-k)$, and we have an exact sequence

$$1 \longrightarrow \mathbb{G}_a^{2k-2} \longrightarrow \mathrm{Aut}(\mathcal{E}) \xrightarrow{\mathrm{ev}_S} \mathbb{G}_m \ltimes (\mathbb{G}_a)^S \longrightarrow 1.$$

The poset of of configurations of lines is product

$$\textstyle\prod_{s\in S}(\{\ell_s \subset \mathcal{O}_{\mathbb{P}^1}(-k)\} \longrightarrow \{\ell_s \subset \mathcal{O}_{\mathbb{P}^1}(k)\})$$

with automorphisms

$$1 \longrightarrow \mathbb{G}_a^{2k-2} \longrightarrow \mathrm{Aut} \xrightarrow{\mathrm{ev}_S} \mathbb{G}_m \ltimes (\mathbb{G}_a)^R \longrightarrow 1, \quad R = \{s \in S \mid \ell_s \subset \mathcal{O}_{\mathbb{P}^1}(k)\}.$$

Let us denote by $c_{2k}(R)$ where the lines ℓ_r lie in the summand $\mathcal{O}_{\mathbb{P}^1}(k)$, for $r \in R \subset S$, and in the summand $\mathcal{O}_{\mathbb{P}^1}(-k)$, for $r \notin R \subset S$.

3.2 Coarse symmetries

3.2.1 *Atkin–Lehner modifications for* $G = \mathrm{PGL}(2)$. Atkin–Lehner modifications provide involutions exchanging the two connected components of $\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)$. For $r \in S$, define the involution

$$AL_r : \mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S) \longrightarrow \mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S), \quad AL_r(\mathcal{E}, \{\ell_s\}_{s\in S}) = (\mathcal{E}', \{\ell'_s\}_{s\in S}),$$

where $\mathcal{E}' \subset \mathcal{E}$ is the lower modification at $r \in \mathbb{P}^1$ so that $\ell_r \subset \mathcal{E}_r$ factors through $\mathcal{E}'_r \subset \mathcal{E}_r$, the resulting map $\mathcal{E} \to \mathcal{E}_r$ induces an isomorphism

$$\mathcal{E}/\mathcal{E}' \xrightarrow{\sim} \mathcal{E}_r/\ell_r$$

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and $\ell'_r \subset \mathcal{E}'_r$ is the image of the map $\mathcal{E}(-r)_r \to \mathcal{E}'_r$, and the other lines are unchanged $\ell'_s = \ell_s \subset \mathcal{E}'_s = \mathcal{E}_s$, for $s \neq r \in S$. Note the involution AL_r exchanges the open points

$$c_0(\emptyset) \longleftrightarrow c_1(*).$$

The Atkin–Lehner modifications generate a group $(\mathbb{Z}/2\mathbb{Z})^S$ of order 8. For $R \subset S$ of even size, the Atkin–Lehner modifications $AL_R = \prod_{r\in R} AL_r$ preserve the two connected components, and generate a subgroup $(\mathbb{Z}/2\mathbb{Z})^{S,ev}$ of order 4.

3.2.2 *Central automorphisms for $G = \mathrm{SL}(2)$.* The inclusion $\mu_2 \simeq Z(\mathrm{SL}(2)) \subset \mathrm{SL}(2)$ of the center induces an automorphisms of the identity functor of $\mathrm{Bun}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$.

3.3 Constructible sheaves

3.3.1 $G = \mathrm{PGL}(2)$. Recall the points of $\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)$ are discretely parameterized and their automorphism groups are connected. We have the corresponding generating set of objects of $Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S))$ given by the respective extensions by zero of constant sheaves (in the following j denotes the inclusion of $c_k(R)$ into $\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)$):

$$\begin{gathered}\mathcal{F}_0(R) = j_!\underline{\mathbb{Q}}_{c_0(R)}, \quad R = \emptyset, \{0,1\}, \{0,\infty\}, \{1,\infty\}, S;\\ \mathcal{F}_1(R) = j_!\underline{\mathbb{Q}}_{c_1(R)}, \quad R \subset S \text{ or } R = *;\\ \mathcal{F}_k(R) = j_!\underline{\mathbb{Q}}_{c_k(R)}, \quad R \subset S, k \geqslant 2.\end{gathered}$$

Another generating set for $Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S))$ is defined as follows. For $k \geqslant 0$, let $i_k : B_k \hookrightarrow \mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)$ be the locally closed substack where the underlying bundle is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}$. Let $j_k^R : c_k(R) \hookrightarrow B_k$ be the inclusion map. We define

$$\mathrm{IC}_k(R) := i_{k,!} j^R_{k,!*} \underline{\mathbb{Q}}_{c_k(R)}[-\dim \mathrm{Aut}(c_k(R))] \tag{3.1}$$

to be the IC-sheaf of the closure of $c_k(R)$ in B_k, extended by zero off of B_k.

The decomposition into connected components

$$\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S) = \mathrm{Bun}^{\overline{0}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S) \coprod \mathrm{Bun}^{\overline{1}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)$$

provides a direct sum decomposition

$$Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)) \simeq Sh_!(\mathrm{Bun}^{\overline{0}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)) \oplus Sh_!(\mathrm{Bun}^{\overline{1}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)).$$

The above basis of objects $\mathcal{F}_k(R)$ belongs to $Sh_!(\mathrm{Bun}^{\overline{k}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S))$, where $\overline{k} = k \mod 2$. For $r \in S$, note the Atkin–Lehner involution AL_r exchanges the basis elements

$$\mathcal{F}_0(\emptyset) \longleftrightarrow \mathcal{F}_1(*).$$

3.3.2 *Whittaker sheaf for $G = \mathrm{PGL}(2)$.* Let us record the form of the Whittaker sheaf. Consider the open substacks of the odd component

$$c_1(*) \xhookrightarrow{j} c_1(*) \cup c_1(\emptyset) \xhookrightarrow{i} \mathrm{Bun}^{\overline{1}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)$$

classifying respectively bundles $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}$ with generic lines $\ell_0, \ell_1, \ell_\infty$, and more generally, lines $\ell_0, \ell_1, \ell_\infty$ with none contained within $\mathcal{O}_{\mathbb{P}^1}(1)$. Then the Whittaker sheaf is given by

$$\mathrm{Wh}_S = i_! j_* \underline{\mathbb{Q}}_{c_1(*)} \in Sh_!(\mathrm{Bun}^{\overline{1}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)).$$

Note the twist in the definition of Wh_S disappears because $d_S = -1$ in this situation.

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3.3.3 $G = \mathrm{SL}(2)$. Recall the points of $\mathrm{Bun}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$ are discretely parameterized and their automorphism groups are connected except for the configuration $c_0(\emptyset)$ with $\mathrm{Aut} \simeq Z(\mathrm{SL}(2)) \simeq \mu_2$. Let $\mathbb{Q}^{\mathrm{alt}}_{c_0(\emptyset)}$ denote the rank-one local system on $c_0(\emptyset)$, where the automorphism group μ_2 acts by the sign character. We have the corresponding basis of objects of $Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S))$ given by the respective extensions by zero of constant sheaves and one additional sheaf $\mathbb{Q}^{\mathrm{alt}}_{c_0(\emptyset)}$:

$$\mathcal{F}_0(R) = j_! \underline{\mathbb{Q}}_{c_0(R),} \quad R = \emptyset, \{0,1\}, \{0,\infty\}, \{1,\infty\}, S;$$
$$\mathcal{F}_0(\emptyset)^{\mathrm{alt}} = j_! \mathbb{Q}^{\mathrm{alt}}_{c_0(\emptyset)};$$
$$\mathcal{F}_{2k}(R) = j_! \underline{\mathbb{Q}}_{c_{2k}(R),} \quad R \subset S, k \geqslant 1.$$

The canonical automorphisms of the identity functor of $\mathrm{Bun}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$ given by $\mu_2 \simeq Z(\mathrm{SL}(2))$ provides a direct sum decomposition

$$Sh_!(\mathrm{Bun}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)) \simeq Sh_!^{\mathrm{triv}}(\mathrm{Bun}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)) \oplus Sh_!^{\mathrm{alt}}(\mathrm{Bun}_{\mathrm{SL}(2)}(\mathbb{P}^1, S))$$

determined by whether the induced action of $\mu_2 \simeq Z(\mathrm{SL}(2))$ on sheaves is trivial or alternating.

The second summand admits an equivalence

$$Sh_!^{\mathrm{alt}}(\mathrm{Bun}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)) \simeq \mathrm{Vect},$$

since all of its objects are finite complexes built out of shifts of $\mathcal{F}_0(\emptyset)^{\mathrm{alt}}$ whose automorphisms are scalars.

3.3.4 *Relation between $G = \mathrm{PGL}(2)$ and $G = \mathrm{SL}(2)$.* The natural map $\mathrm{SL}(2) \to \mathrm{PGL}(2)$ induces a map

$$p : \mathrm{Bun}_{\mathrm{SL}(2)}(\mathbb{P}^1, S) \longrightarrow \mathrm{Bun}^{\bar{0}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S) \subset \mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S),$$

which sends $c_{2k}(R) \in \mathrm{Bun}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$ to the same-named point $c_{2k}(R)$ in $\mathrm{Bun}^{\bar{0}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)$, for any $k \geqslant 0$ and $R \subset S$.

Pullback provides an equivalence

$$p^* : Sh_!(\mathrm{Bun}^{\bar{0}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)) \xrightarrow{\sim} Sh_!^{\mathrm{triv}}(\mathrm{Bun}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)) \tag{3.2}$$

that acts on the above basis by

$$p^*(\mathcal{F}_0(R)) \simeq \mathcal{F}_0(R), \quad R = \emptyset, \{0,1\}, \{0,\infty\}, \{1,\infty\}, S;$$
$$p^*(\mathcal{F}_{2k}(R)) \simeq \mathcal{F}_{2k}(R), \quad R \subset S, k \geqslant 1.$$

Thus using the prior decompositions and Atkin–Lehner involutions, we see that to understand any of the above categories, it suffices to understand, for example, $Sh_!(\mathrm{Bun}^{\bar{1}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S))$. We prefer the odd component of $\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)$ since it supports the Whittaker sheaf.

4. Spectral side: $\mathbb{P}^1$, three ramification points, $G^\vee = \mathrm{SL}(2), \mathrm{PGL}(2)$

Continue with $\mathbb{P}^1 = \mathrm{Proj}(k[x, y])$ the projective line with homogeneous coordinates $[x, y]$ and coordinate $t = y/x$, and the three points $S = \{0, 1, \infty\} \subset \mathbb{P}^1$, where the coordinate t takes the respective value.

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4.1 Moduli of local systems

4.1.1 *General definition.* We start with a reductive group $G^\vee$ over $\mathbb{Q}$. Let $\mathrm{Loc}_{G^\vee}(\mathbb{P}^1, S)$ be the moduli of (Betti) $G^\vee$-local systems on $\mathbb{P}^1 \backslash S$ with $B^\vee$-reductions near S with trivial induced $T^\vee$-monodromy. By choosing a point u_0 in $\mathbb{P}^1 \backslash S$ and a based loop γ_s around $s \in S$ for each $s \in S$ such that $\gamma_0\gamma_1\gamma_\infty = 1$ in $\pi_1(\mathbb{P}^1 \backslash S, u_0)$, we obtain the presentation

$$\mathrm{Loc}_{G^\vee}(\mathbb{P}^1, S) \simeq (\widetilde{\mathcal{N}}^\vee)^{S,\prod=1}/G^\vee.$$

Here, $(\widetilde{\mathcal{N}}^\vee)^{S,\prod=1}$ is the derived fiber of 1 of the map

$$(\widetilde{\mathcal{N}}^\vee)^S \xrightarrow{\mu^S} (\mathcal{N}^\vee)^S \xrightarrow{\mathrm{mult}} G^\vee \tag{4.1}$$

and the map 'mult' takes (A_0, A_1, A_∞) to $A_0A_1A_\infty$.

4.1.2 $G^\vee = \mathrm{SL}(2)$. In this case, $(\widetilde{\mathcal{N}}^\vee)^{S,\prod=1}$ is the derived subscheme of $(\widetilde{\mathcal{N}}^\vee)^S$ classifying triples of pairs $(A_s, \ell_s)_{s\in S}$ consisting of a matrix $A_s \in \mathrm{SL}(2)$ and an eigenline $A_s(\ell_s) \subset \ell_s$ with trivial eigenvalue $A_s|_{\ell_s} = 1$, and the matrices satisfy the equation $A_0A_1A_\infty = 1$ inside of $\mathrm{SL}(2)$.

To write explicit local equations for $(\widetilde{\mathcal{N}}^\vee)^{S,\prod=1}$, we may apply the SL(2)-symmetry to assume without loss of generality that $\ell_\infty = [1:0]$, $\ell_0 = [1:x]$ and $\ell_1 = [1:y]$. Then the three matrices take the form

$$A_0 = \begin{pmatrix} 1-ax & a \\ -ax^2 & 1+ax \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1-by & b \\ -by^2 & 1+by \end{pmatrix}$$

$$A_\infty^{-1} = A_0A_1 = \begin{pmatrix} 1-ax-by+aby(x-y) & a+b-ab(x-y) \\ -ax^2-by^2+abxy(x-y) & 1+ax+by-abx(x-y) \end{pmatrix}$$

such that A_∞ is of the form $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$. Since $\det(A_\infty) = 1$, we need only impose the equations

$$1-ax-by+aby(x-y) = 1, \quad -ax^2-by^2+abxy(x-y) = 0.$$

These in turn are equivalent to the equations

$$ax+by = 0, \quad ax^2+by^2 = 0.$$

We conclude that $(\widetilde{\mathcal{N}}^\vee)^{S,\prod=1}$ is a lci classical scheme (i.e., not derived) with five irreducible components.

(i) $A_0 = A_1 = A_\infty = 1$. This component is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Local equation:

$$a = b = 0.$$

(ii) $A_0 = 1$ (hence $A_1 = A_\infty^{-1}$) and $\ell_1 = \ell_\infty$. This component is isomorphic to $\mathbb{P}^1 \times \widetilde{\mathcal{N}}^\vee$. Local equation:

$$a = 0, \quad y = 0.$$

(iii) $A_1 = 1$ (hence $A_0 = A_\infty^{-1}$) and $\ell_0 = \ell_\infty$. This component is isomorphic to $\mathbb{P}^1 \times \widetilde{\mathcal{N}}^\vee$. Local equation:

$$b = 0, \quad x = 0.$$

(iv) $A_\infty = 1$ (hence $A_0 = A_1^{-1}$) and $\ell_0 = \ell_1$. This component is isomorphic to $\mathbb{P}^1 \times \widetilde{\mathcal{N}}^\vee$. Local equation:

$$a+b = 0, \quad x = y.$$

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(v) A_0, A_1, A_∞ all lie in a single Borel. Note this does not mean that $\ell_0, \ell_1, \ell_\infty$ are the same; in fact, this component is non-reduced since A_0, A_1, A_∞ fix $\ell_0, \ell_1, \ell_\infty$ respectively. Local equation:

$$x^2 = 0, \quad y^2 = 0, \quad xy = 0, \quad ax + by = 0.$$

Note for a, b not both zero, there is a unique infinitesimal direction for (x, y). The reduced structure of this component is isomorphic to the total space of $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$ over $\mathbb{P}^1$, and we denote it as $\widetilde{\mathcal{N}}^\vee_\Delta$.

If we view $(\widetilde{\mathcal{N}}^\vee)^S$ as the cotangent bundle of $(\mathbb{P}^1)^S$, the five components listed above, after passing to reduced structures, are exactly the conormal bundles of various partial diagonals in $(\mathbb{P}^1)^S$. For this reason, we introduce the following notation. For a subset $R \subset S$ with $\#R \neq 1$, we denote by Δ_R the partial diagonal of $(\mathbb{P}^1)^S$ where the R-components are equal. For example, $\Delta_\emptyset = (\mathbb{P}^1)^S$. Let $\Lambda_R \subset T^*(\mathbb{P}^1)^S \simeq (\widetilde{\mathcal{N}}^\vee)^S$ be the conormal bundle of Δ_R. Then the reduced structure of the five components of $((\widetilde{\mathcal{N}}^\vee)^S)^{\prod=1}$ are, in the order listed above, $\Lambda_\emptyset, \Lambda_{1,\infty}, \Lambda_{0,\infty}, \Lambda_{0,1}$ and $\Lambda_S = \widetilde{\mathcal{N}}^\vee_\Delta$.

4.1.3 $G^\vee = \mathrm{PGL}(2)$. The stack $\mathrm{Loc}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)$ has two connected components. In fact, for $G^\vee = \mathrm{PGL}(2)$, the Springer resolution $\widetilde{\mathcal{N}}^\vee$ and the unipotent variety $\mathcal{N}^\vee$ are the same as those of $\mathrm{SL}(2)$. Therefore, the map 'mult' in (4.1) factorizes as

$$(\mathcal{N}^\vee)^S \xrightarrow{\widetilde{\mathrm{mult}}} \mathrm{SL}(2) \longrightarrow \mathrm{PGL}(2).$$

Hence, according to whether the product of three elements in $\mathcal{N}^\vee$ is 1 or -1 in $\mathrm{SL}(2)$, we have a decomposition of $\mathrm{Loc}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)$

$$\mathrm{Loc}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S) = \mathrm{Loc}^{\bar{0}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S) \coprod \mathrm{Loc}^{\bar{1}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S), \tag{4.2}$$

where

$$\mathrm{Loc}^{\bar{0}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S) = (\widetilde{\mathcal{N}}^\vee)^{S, \widetilde{\prod}=1}/\mathrm{PGL}(2),$$

$$\mathrm{Loc}^{\bar{1}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S) = (\widetilde{\mathcal{N}}^\vee)^{S, \widetilde{\prod}=-1}/\mathrm{PGL}(2).$$

The natural map $\mathrm{SL}(2) \to \mathrm{PGL}(2)$ induces a map

$$p : \mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S) \longrightarrow \mathrm{Loc}^{\bar{0}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S) \subset \mathrm{Loc}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S), \tag{4.3}$$

which in turn induces an equivalence

$$\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S) \xrightarrow{\sim} \mathrm{Loc}^{\bar{0}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S) \times_{\mathrm{pt}/\mathrm{PGL}(2)} (\mathrm{pt}/\mathrm{SL}(2)).$$

The odd component of $\mathrm{Loc}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)$ actually reduces to a single point.

Lemma 4.1.4. *The derived scheme $(\widetilde{\mathcal{N}}^\vee)^{S, \widetilde{\prod}=-1}$ is a trivial torsor for* $\mathrm{PGL}(2)$*. In particular,*

$$\mathrm{Loc}^{\bar{1}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S) \cong \mathrm{Spec}\ \mathbb{Q}.$$

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Proof. Let $(A_s, \ell_s)_{s\in S}$ be a point of $(\widetilde{\mathcal{N}}^\vee)^{S,\widetilde{\prod}=-1}$. We view A_s as unipotent elements in SL(2), then $A_0A_1A_\infty = -1 \in \mathrm{SL}(2)$. It is easy to see that none of A_s can be 1, hence each line ℓ_s is determined by A_s. It is also easy to see that no two lines are equal, hence using the PGL(2)-action we may arrange $\ell_0 = [1:0], \ell_1 = [0:1]$ and using the remaining $T^\vee$-conjugacy we may arrange uniquely

$$A_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}.$$

Then we have

$$A_\infty = -A_1^{-1}A_0^{-1} = \begin{pmatrix} -1 & 1 \\ c & -1-c \end{pmatrix},$$

which is unipotent if and only if $c=-4$. This shows that $(\widetilde{\mathcal{N}}^\vee)^{S,\widetilde{\prod}=-1}$ is a torsor for PGL(2) with a rational point. ∎

Remark 4.1.5. The unique point in $\mathrm{Loc}^{\overline{1}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)$ corresponds to a rank-two local system on $\mathbb{P}^1\backslash S$ with non-trivial unipotent monodromy at 0 and 1, and monodromy with a single Jordan block of eigenvalue -1 at ∞. This local system arises from the universal Tate module of the Legendre family of elliptic curves over $\mathbb{P}^1\backslash\{0,1,\infty\}$ given by $y^2 = x(x-1)(x-t), t \in \mathbb{P}^1\backslash\{0,1,\infty\}$.

4.2 Comparison with linear and de Rham moduli

In this subsection $G^\vee = \mathrm{SL}(2)$. We will show that $\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$ is isomorphic to its linearized version and its de Rham version, which traditionally appears in the formulation of the geometric Langlands correspondence.

4.2.1 *Linearized version.* Let $\sum_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$ denote the linearized version of $\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$ defined by the presentation

$$\textstyle\sum_{\mathrm{SL}(2)}(\mathbb{P}^1, S) := T^*((\mathbb{P}^1)^S/\mathrm{SL}(2)) = (\widetilde{\mathcal{N}}^\vee)^{S,\sum=0}/\mathrm{SL}(2),$$

where we regard $\widetilde{\mathcal{N}}^\vee$ as the Springer resolution of the nilpotent cone in $\mathfrak{g}^\vee = \mathfrak{sl}(2)$, and impose that the sum of the Lie algebra elements be zero. Thus a point of $\sum_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$ is a triple of pairs $(B_s, \ell_s)_{s\in S}$ consisting of a matrix $B_s \in \mathfrak{sl}(2)$ and an eigenline $B_s(\ell_s) \subset \ell_s$ with trivial eigenvalue $B_s|_{\ell_s} = 0$, and the matrices satisfy the equation $B_0 + B_1 + B_\infty = 0$ inside of $\mathfrak{sl}(2)$.

The local equations for $(\widetilde{\mathcal{N}}^\vee)^{S,\sum=0}$ are exactly the same as those derived above for $(\widetilde{\mathcal{N}}^\vee)^{S,\prod=1}$ except now B_0, B_1, B_∞ are nilpotent rather than unipotent matrices

$$B_0 = \begin{pmatrix} -ax & a \\ -ax^2 & ax \end{pmatrix}, \quad B_1 = \begin{pmatrix} -by & b \\ -by^2 & by \end{pmatrix}$$
$$-B_\infty = B_0 + B_1 = \begin{pmatrix} -ax-by & a+b \\ -ax^2-by^2 & +ax+by \end{pmatrix}$$

with the requirement that B_∞ is of the form $\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)$ imposing the equations

$$ax + by = 0, \quad ax^2 + by^2 = 0.$$

Thus we can construct an SL(2)-equivariant isomorphism

$$(\widetilde{\mathcal{N}}^\vee)^{S,\prod=1} \xrightarrow{\sim} (\widetilde{\mathcal{N}}^\vee)^{S,\sum=0}$$

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by the assignment

$$(A_0, A_1, A_\infty, \ell_0, \ell_1, \ell_\infty) \longmapsto (A_0 - 1, A_1 - 1, 2 - A_0 - A_1, \ell_0, \ell_1, \ell_\infty).$$

Note that $A_\infty - 1 \neq 2 - A_0 - A_1$ as they differ in local coordinates by

$$\begin{pmatrix} 0 & 0 \\ ab(x-y) & 0 \end{pmatrix}$$

though nevertheless $(2 - A_0 - A_1)\ell_\infty = 0$.

We could just as well choose either of the alternative isomorphisms given by the assignments

$$(A_0, A_1, A_\infty, \ell_0, \ell_1, \ell_\infty) \longmapsto (2 - A_1 - A_\infty, A_1 - 1, A_\infty - 1, \ell_0, \ell_1, \ell_\infty),$$

$$(A_0, A_1, A_\infty, \ell_0, \ell_1, \ell_\infty) \longmapsto (A_0 - 1, 2 - A_0 - A_\infty, A_\infty - 1, \ell_0, \ell_1, \ell_\infty).$$

They give different isomorphisms reflecting the fact that $(\widetilde{\mathcal{N}}^\vee)^{S,\Pi=1}$ has automorphisms that infinitesimally move points in its non-reduced component.

4.2.2 *de Rham moduli.* Let $\mathrm{Conn}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$ denote the de Rham version of $\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$ classifying data $(\mathcal{E}, \tau, \{\ell_s\}_{s\in S}, \nabla)$, where $\mathcal{E}$ is a rank-two vector bundle on $\mathbb{P}^1$ equipped with a line $\ell_s \subset \mathcal{E}_s$ at each $s \in S$, and a meromorphic connection

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega_{\mathbb{P}^1}(S)$$

with regular singularity at each $s \in S$, whose residue $\mathrm{Res}_s\nabla$ is trivial when restricted to ℓ_s, and

$$\tau : \mathcal{O}_{\mathbb{P}^1} \xrightarrow{\sim} \det(\mathcal{E})$$

is a ∇-flat trivialization of the determinant.

LEMMA 4.2.3. *There is canonical isomorphism from the de Rham moduli to linearized moduli*

$$\mathrm{Conn}_{\mathrm{SL}(2)}(\mathbb{P}^1, S) \xrightarrow{\sim} \textstyle\sum_{\mathrm{SL}(2)}(\mathbb{P}^1, S).$$

Proof. First, for any $((\mathcal{E}, \tau, \{\ell_s\}_{s\in S}, \nabla) \in \mathrm{Conn}_{\mathrm{SL}(2)}(\mathbb{P}^1, S))$, we have $\mathcal{E} \simeq \mathcal{O}^2_{\mathbb{P}^1}$. Otherwise, there is an embedding $\mathcal{O}_{\mathbb{P}^1}(n) \hookrightarrow \mathcal{E}$ with quotient $\mathcal{O}_{\mathbb{P}^1}(-n)$, for some $n > 0$. The composition

$$\mathcal{O}_{\mathbb{P}^1}(n) \hookrightarrow \mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes \Omega_{\mathbb{P}^1}(S) \longrightarrow \mathcal{E}/\mathcal{O}_{\mathbb{P}^1}(n) \otimes \Omega_{\mathbb{P}^1}(S) \simeq \mathcal{O}_{\mathbb{P}^1}(1-n)$$

is $\mathcal{O}_{\mathbb{P}^1}$-linear, hence must be zero since $n > 1 - n$. Thus ∇ restricts to a connection on $\mathcal{O}_{\mathbb{P}^1}(n)$ without poles (because the residues of ∇ are nilpotent), which is impossible since $n \neq 0$.

Next, fix an isomorphism $(\mathcal{E}, \tau) \simeq (\mathcal{O}^2_{\mathbb{P}^1}, \tau_0)$ with the trivial bundle (such choices form an $\mathrm{SL}(2)$-torsor). The trivial bundle carries the de Rham connection d, and any $((\mathcal{E}, \tau, \{\ell_s\}_{s\in S}, \nabla) \in \mathrm{Conn}_{\mathrm{SL}(2)}(\mathbb{P}^1, S))$ is equivalent to one of the form $(\mathcal{O}^2_{\mathbb{P}^1}, \{\ell_s\}_{s\in S}, \tau_0, \nabla = d + \varphi) \in \mathrm{Conn}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$, where $\varphi : \mathcal{O}^2_{\mathbb{P}^1} \to \mathcal{O}^2_{\mathbb{P}^1} \otimes \Omega_{\mathbb{P}^1}(S)$ is a traceless $\mathcal{O}$-linear map whose restriction to ℓ_s, for each $s \in S$, is trivial.

Now, define the sought-after isomorphism by the $\mathrm{SL}(2)$-equivariant assignment

$$(\mathcal{O}^2_{\mathbb{P}^1}, \{\ell_s\}_{s\in S}, \tau_0, \nabla = d + \varphi) \longrightarrow (\mathrm{Res}_0\varphi, \ell_0, \mathrm{Res}_1\varphi, \ell_1, \mathrm{Res}_\infty\varphi, \ell_\infty) \in (\widetilde{\mathcal{N}}^\vee)^S$$

whose image lies in $(\widetilde{\mathcal{N}}^\vee)^{S,\Sigma=0}$ thanks to the residue theorem for curves

$$\mathrm{Res}_0\varphi + \mathrm{Res}_1\varphi + \mathrm{Res}_\infty\varphi = 0. \qquad \square$$

COROLLARY 4.2.4. *The Betti moduli $\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$, its linearized version $\sum_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$, and the de Rham moduli $\mathrm{Conn}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$ are all isomorphic as stacks over the classifying stack of* $\mathrm{SL}(2)$.

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4.3 Coherent sheaves

Given a stack Z, recall we write $\mathrm{Coh}(Z)$ to denote the dg derived category of coherent complexes on Z. We abuse terminology and use the term coherent sheaves to refer to its objects.

4.3.1 *Affine Hecke action.* Let $G^\vee$ be any reductive group over $\mathbb{Q}$. Fix $s \in S$, then the monoidal category $\mathrm{Coh}^{G^\vee}(St_{G^\vee})$ acts on $\mathrm{Coh}(\mathrm{Loc}_{G^\vee}(\mathbb{P}^1, S))$ as follows. Recall the curve $\mathbb{X}_s = \mathbb{P}^1_- \coprod_{\mathbb{P}^1\setminus\{s\}} \mathbb{P}^1_+$ in § 2.3 with the point s doubled. The moduli stack $\mathrm{Loc}_{G^\vee}(\mathbb{X}_x, S_\pm)$ can be similarly defined as $\mathrm{Loc}_{G^\vee}(\mathbb{P}^1, S)$, with $B^\vee$-reductions at both s_- and s_+. The Steinberg stack $St_{G^\vee}/G^\vee$ can be identified with the moduli stack $\mathrm{Loc}_{G^\vee}(\mathbb{D}_s, \{s_-, s_+\})$ of $G^\vee$-local systems on the doubled disk $\mathbb{D}_s$ with unipotent monodromy and $B^\vee$-reductions at s_- and s_+. We have the following diagram.

$$\begin{tikzcd}[column sep=small]
 & \mathrm{Loc}_{G^\vee}(\mathbb{X}_s, S_\pm) \arrow[dl, "p_-"'] \arrow[d, "\kappa"] \arrow[dr, "p_+"] & \\
\mathrm{Loc}_{G^\vee}(\mathbb{P}^1, S) & \mathrm{Loc}_G(\mathbb{D}_s, \{s_-, s_+\}) = St_{G^\vee}/G^\vee & \mathrm{Loc}_{G^\vee}(\mathbb{P}^1, S)
\end{tikzcd}$$

Passing to quasi-coherent sheaves, one obtains the affine Hecke action

$$\star_s : \mathrm{Coh}^{G^\vee}(St_{G^\vee}) \otimes \mathrm{QCoh}(\mathrm{Loc}_{G^\vee}(\mathbb{P}^1, S)) \longrightarrow \mathrm{QCoh}(\mathrm{Loc}_{G^\vee}(\mathbb{P}^1, S))$$

$$\mathcal{K} \star_s \mathcal{F} = (p_+)_!((p_-)^*\mathcal{F} \otimes \kappa^*(\mathcal{K}))\ ,$$

which preserves the subcategory $\mathrm{Coh}(\mathrm{Loc}_{G^\vee}(\mathbb{P}^1, S))$ because p_+ is proper.

Natural generalizations of the above constructions provide $\mathrm{Coh}(\mathrm{Loc}_{G^\vee}(\mathbb{P}^1, S))$ the requisite coherences of a $\mathrm{Coh}^{G^\vee}(St_{G^\vee})$-module structure.

4.3.2 $G^\vee = \mathrm{SL}(2)$. The center $Z(\mathrm{SL}(2)) \simeq \mu_2$ acts trivially on $(\widetilde{\mathcal{N}}^\vee)^{S,\prod=1}$, therefore it acts on the underlying coherent sheaf of each object in $\mathrm{Coh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S))$. This provides a direct sum decomposition

$$\mathrm{Coh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)) \simeq \mathrm{Coh}^{\mathrm{triv}}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)) \oplus \mathrm{Coh}^{\mathrm{alt}}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S))$$

determined by whether the action of $\mu_2 \simeq Z(\mathrm{SL}(2))$ is trivial or by the alternating representation.

For any $s \in S$, the corresponding Atkin–Lehner involution $\mathcal{O}^{\mathrm{cl}}_{St_{G^\vee}}(-1, 0) \in \mathrm{Coh}^{G^\vee}(St_{G^\vee})$ exchanges the two summands.

4.3.3 $G^\vee = \mathrm{PGL}(2)$. The decomposition into connected components (4.2) provides a direct sum decomposition

$$\mathrm{Coh}(\mathrm{Loc}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)) \simeq \mathrm{Coh}(\mathrm{Loc}^{\overline{0}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)) \oplus \mathrm{Coh}(\mathrm{Loc}^{\overline{1}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)).$$

By Lemma 4.1.4, the second summand admits an equivalence

$$\mathrm{Coh}(\mathrm{Loc}^{\overline{1}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)) \simeq \mathrm{Vect}.$$

4.3.4 *Relation between* $G^\vee = \mathrm{SL}(2)$ *and* $G^\vee = \mathrm{PGL}(2)$. Pullback along the map p in (4.3) provides an equivalence

$$p^* : \mathrm{Coh}(\mathrm{Loc}^{\overline{0}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)) \xrightarrow{\ \sim\ } \mathrm{Coh}^{\mathrm{triv}}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S))$$

as $\mathrm{Perf}(\mathrm{Loc}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S))$-module categories.

Thus using the prior decompositions and Atkin–Lehner involutions, we see that to understand any of the above categories, it suffices to understand for example $\mathrm{Coh}^{\mathrm{triv}}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S))$.

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5. Langlands duality

In this section we give the proof of our main theorem. For most of this section we focus on $G = \mathrm{PGL}(2)$ and $G^\vee = \mathrm{SL}(2)$. We will establish results in this case first, and then use them to deduce the case of $G = \mathrm{SL}(2)$ and $G^\vee = \mathrm{PGL}(2)$.

5.1 Dictionary: matching objects

Before proceeding to the construction and proof of the equivalence, let us record here various distinguished objects that will be matched by it.

Let $U^{\overline{0}}, U^{\overline{1}} \subset \mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)$ denote the open substacks classifying parabolic bundles with respectively underlying bundle $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$, $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}$. In what follows, all sheaves will be extensions by zero off of $U^{\overline{0}}$ or $U^{\overline{1}}$. In particular, we recall the objects $\mathrm{IC}_0(R)$ introduced in (3.1), which is the IC-sheaf of the closure of $c_0(R)$ in $U^{\overline{0}}$ then extended by zero.

5.1.1 $U^{\overline{1}}$. Within $U^{\overline{1}}$, consider the open substacks

$$c_1(*) \overset{j}{\hookrightarrow} c_1(*) \cup c_1(\emptyset) \overset{i}{\hookrightarrow} U^{\overline{1}}$$

classifying respectively bundles $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}$ with generic lines $\ell_0, \ell_1, \ell_\infty$, and more generally, lines $\ell_0, \ell_1, \ell_\infty$ with none contained within $\mathcal{O}_{\mathbb{P}^1}(1)$.

We have the following distinguished objects:

$$\mathrm{Wh}_S = i_! j_* \underline{\mathbb{Q}}_{c_1(*)} \longleftrightarrow \mathcal{O}_{\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)} \quad \text{(by construction)},$$

$$\mathrm{Eis}_{-1} = i_! \underline{\mathbb{Q}}_{c_1(\emptyset)}[-1] \longleftrightarrow \mathcal{O}_{\widetilde{\mathcal{N}}^\vee_\Delta} \quad \text{(by Proposition 5.4.3)},$$

$$\mathrm{Eis}_1 = \underline{\mathbb{Q}}_{c_1(S)}[-3] \longleftrightarrow \mathcal{O}_{\widetilde{\mathcal{N}}^\vee_\Delta}(2) \quad \text{(by Proposition 5.4.3)}.$$

5.1.2 $U^{\overline{0}}$. On $U^{\overline{0}}$, we have the following distinguished objects:

$$\mathrm{IC}_0(\emptyset) \longleftrightarrow \mathcal{O}_{(\mathbb{P}^1)^3}(-1,-1,-1) \quad \text{(by Proposition 5.3.3)},$$

$$\mathrm{IC}_0(0,1) \longleftrightarrow \mathcal{O}_{\Lambda_{0,1}}(0,0,-1) \quad \text{(by Proposition 5.3.3)}, \tag{5.1}$$

$$\mathrm{IC}_0(0,\infty) \longleftrightarrow \mathcal{O}_{\Lambda_{0,\infty}}(0,-1,0) \quad \text{(by Proposition 5.3.3)},$$

$$\mathrm{IC}_0(1,\infty) \longleftrightarrow \mathcal{O}_{\Lambda_{1,\infty}}(-1,0,0) \quad \text{(by Proposition 5.3.3)}, \tag{5.2}$$

$$\mathrm{Eis}_0 = \underline{\mathbb{Q}}_{c_0(S)}[-2] \longleftrightarrow \mathcal{O}_{\widetilde{\mathcal{N}}^\vee_\Delta}(1) \quad \text{(by Proposition 5.4.3)}.$$

We will also use the object $J_1 \star_1 \mathrm{Wh}_S$. Consider the open substacks

$$c_0(\emptyset) \overset{j}{\hookrightarrow} c_0(\emptyset) \cup c_0(0,\infty) \overset{i}{\hookrightarrow} U^{\overline{0}}$$

classifying bundles $\mathcal{E} \simeq \mathcal{O}^2_{\mathbb{P}^1}$ with respectively generic lines $\ell_0, \ell_1, \ell_\infty$, and more generally, lines $\ell_0, \ell_1, \ell_\infty$ with the only possible coincidence $\ell_0 = \ell_\infty$. Then we have

$$J_1 \star_1 \mathrm{Wh}_S \simeq i_* j_! \underline{\mathbb{Q}}_{c_0(\emptyset)} \longleftrightarrow \mathcal{O}_{\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)}(0,1,0) \quad \text{(by calculating } J_1 \star_1\text{)}.$$

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5.2 Construction of functor

In order to construct the functor in (1.1), we first construct an action of the monoidal category $\mathrm{Perf}(\mathrm{Loc}_{G^\vee}(\mathbb{P}^1, S))$ (under $\otimes$) on the automorphic category $Sh(\mathrm{Bun}_G(X, S))$.

Note the natural inclusion and projection

$$\mathrm{Loc}_{G^\vee}(\mathbb{P}^1, S) \xrightarrow{\sim} (\widetilde{\mathcal{N}}^\vee)^{S,\Pi=1}/G^\vee \longrightarrow (\widetilde{\mathcal{N}}^\vee)^S/G^\vee \longrightarrow (\widetilde{\mathcal{N}}^\vee/G^\vee)^S.$$

Passing to perfect complexes, we obtain a composite pullback functor

$$\mathrm{Perf}(\widetilde{\mathcal{N}}^\vee/G^\vee)^{\otimes S} \xrightarrow{\sim} \mathrm{Perf}((\widetilde{\mathcal{N}}^\vee/G^\vee)^S) \longrightarrow \mathrm{Perf}((\widetilde{\mathcal{N}}^\vee)^S/G^\vee) \longrightarrow \mathrm{Perf}(\mathrm{Loc}_{G^\vee}(\mathbb{P}^1, S)). \tag{5.3}$$

Recall from (2.7) that we have an action of $\mathrm{Perf}(\widetilde{\mathcal{N}}^\vee/G^\vee)^{\otimes S}$ on $Sh(\mathrm{Bun}_G(\mathbb{P}^1, S))$ coming from Wakimoto operators at each $s \in S$. This action preserves the subcategory $Sh_!(\mathrm{Bun}_G(\mathbb{P}^1, S))$.

Theorem 5.2.1. *For $G = \mathrm{PGL}(2)$ or $\mathrm{SL}(2)$, the $\mathrm{Perf}(\widetilde{\mathcal{N}}^\vee/G^\vee)^{\otimes S}$-action on $Sh(\mathrm{Bun}_G(\mathbb{P}^1, S))$ in (2.7) factors through the functor in (5.3)*

$$\mathrm{Perf}(\widetilde{\mathcal{N}}^\vee/G^\vee)^{\otimes S} \longrightarrow \mathrm{Perf}(\mathrm{Loc}_{G^\vee}(\mathbb{P}^1, S)).$$

Thus there is an action of $\mathrm{Perf}(\mathrm{Loc}_{G^\vee}(\mathbb{P}^1, S))$ on $Sh(\mathrm{Bun}_G(\mathbb{P}^1, S))$ preserving the subcategory $Sh_!(\mathrm{Bun}_G(\mathbb{P}^1, S))$. Moreover, for any $x \in \mathbb{P}^1\backslash S$, the $\mathrm{Rep}(G^\vee)$-action on $Sh(\mathrm{Bun}_G(\mathbb{P}^1, S))$ given via the evaluation map

$$\mathrm{ev}_x^* : \mathrm{Rep}(G^\vee) \to \mathrm{Perf}(\mathrm{Loc}_{G^\vee}(\mathbb{P}^1, S)) \tag{5.4}$$

coincides with the spherical Hecke action at x via the geometric Satake equivalence (2.1).

Proof. In [NY16, Theorem 6.3.9], we prove a more general theorem, where G is any reductive group and $\mathbb{P}^1\backslash S$ can be replaced by any punctured curve $X\backslash S$. The result there says that there is an action of $\mathrm{Perf}(\mathrm{Loc}_{G^\vee}(\mathbb{P}^1, S))$ on $Sh_{\mathcal{N}_G(X,S)}(\mathrm{Bun}_G(X, S))$, the full subcategory of objects whose singular support is contained in the global nilpotent cone $\mathcal{N}_G(X, S) \subset T^*\mathrm{Bun}_G(X, S)$. In the case G is semisimple of rank one, $X = \mathbb{P}^1$ and $\#S = 3$, the Hitchin base for $T^*\mathrm{Bun}_G(X, S)$ reduces to a point, therefore the nilpotent singular support condition is vacuous, i.e., $Sh_{\mathcal{N}_G(X,S)}(\mathrm{Bun}_G(X, S))$ is equal to $Sh(\mathrm{Bun}_G(X, S))$. The theorem then follows from [NY16, Theorem 6.3.9]. □

Remark 5.2.2. A key ingredient in the proof of [NY16, Theorem 6.3.9] is the local constancy of the spherical Hecke action, namely [NY16, Theorem 1.2.1]. This is the only place where the nilpotent singular support condition is used. In the situation of Theorem 5.2.1, one can give a more elementary proof, which we sketch below.

Let $G = \mathrm{PGL}(2)$ or $\mathrm{SL}(2)$. First, let us be precise about the meaning of local constancy of the spherical Hecke action. Write $U = \mathbb{P}^1\backslash S$. Consider the family version of Hecke modifications $\mathrm{Hecke}_U^{\mathrm{sph}}$ introduced in Remark 2.3.2. For any $\mathcal{F} \in Sh_!(\mathrm{Bun}_G(\mathbb{P}^1, S))$ and $\mathcal{K} \in \mathcal{H}_G^{\mathrm{sph}}$, the complex $\mathrm{Hecke}_U^{\mathrm{sph}}(\mathcal{K}, \mathcal{F}) \in Sh_!(\mathrm{Bun}_G(\mathbb{P}^1, S) \times U)$ is called *locally constant in the U-direction* if its singular support $SS(\mathrm{Hecke}_U^{\mathrm{sph}}(\mathcal{K}, \mathcal{F}))$, a conical Lagrangian in $T^*\mathrm{Bun}_G(\mathbb{P}^1, S) \times T^*U$, is contained in $T^*\mathrm{Bun}_G(\mathbb{P}^1, S) \times U$ (the second factor is the zero section of the cotangent bundle T^*U).

Claim. *For any $\mathcal{F} \in Sh_!(\mathrm{Bun}_G(\mathbb{P}^1, S))$ and any $\mathcal{K} \in \mathcal{H}_G^{\mathrm{sph}}$, the complex $\mathrm{Hecke}_U^{\mathrm{sph}}(\mathcal{K}, \mathcal{F})$ on $\mathrm{Bun}_G(\mathbb{P}^1, S) \times U$ is locally constant in the U-direction in the above sense.*

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Below we only sketch the proof of this claim in the case $G = \mathrm{PGL}(2)$ and $\mathcal{K}$ corresponds to the standard representation of $G^\vee = \mathrm{SL}(2)$ (i.e., $\mathrm{Hecke}^{\mathrm{sph}}_x(\mathcal{K}, -)$ corresponds to the lower modification at x).

For $x \in U$, the Hecke correspondence of a lower modification at x is given by the diagram

$$\mathrm{Bun}_G(\mathbb{P}^1, S) \xleftarrow{p_{x-}} \mathrm{Bun}_G(\mathbb{P}^1, S \cup \{x\}) \xrightarrow{p_{x+}} \mathrm{Bun}_G(\mathbb{P}^1, S). \tag{5.5}$$

Here p_{x-} sends a point $(\mathcal{E}, \ell_0, \ell_1, \ell_\infty, \ell_x)$ to $(\mathcal{E}, \ell_0, \ell_1, \ell_\infty)$, and p_{x+} sends it to $(\mathcal{E}', \ell'_0, \ell'_1, \ell'_\infty)$, where $\mathcal{E}'$ fits into a short exact sequence $\mathcal{E}' \to \mathcal{E} \to \ell_x$, and $\ell'_s = \ell_s$ for $s \in S$ after identifying $\mathcal{E}'_s$ with $\mathcal{E}_s$.

Since $\mathrm{Bun}_G(\mathbb{P}^1, S)$ is stratified by points, we only need to show that for any object $\mathcal{F} \in Sh_!(\mathrm{Bun}_G(\mathbb{P}^1, S))$, the stalk of $p_{x+!}p^*_{x-}\mathcal{F}$ at any point $b' \in \mathrm{Bun}_G(\mathbb{P}^1, S)$ is locally constant as x varies in U. It suffices to check this for $\mathcal{F} = \mathcal{F}_k(R)$, one of the basis objects. Let $b = c_k(R)$ and let $H_x(b, b') = (p_{x-}, p_{x+})^{-1}(b, b') \subset \mathrm{Bun}_G(\mathbb{P}^1, S \cup \{x\})$. As x varies, the $H_x(b, b')$ form a family $h_{b,b'} : H(b, b') \to U$. Since the stalk of $p_{x+!}p^*_{x-}\mathcal{F}_k(R)$ at b' is simply $\mathrm{H}^*_c(H_x(b, b'), \mathbb{Q})$, it suffices to show that h is a fibration.

Therefore, we fix $b, b' \in \mathrm{Bun}_G(\mathbb{P}^1, S)$, viewed as classifying spaces of their respective automorphism groups. Consider the two projections restricted from (5.5):

$$b \xleftarrow{h_{x-}} H_x(b, b') \xrightarrow{h_{x+}} b'.$$

The fibers of h_{x-} and h_{x+} are subsets of $\mathbb{P}^1$, hence $\dim \mathrm{Aut}(b)$ and $\dim \mathrm{Aut}(b')$ differ at most by 1. We have the following two cases.

(i) If $\dim \mathrm{Aut}(b)$ and $\dim \mathrm{Aut}(b')$ differ by 1, one of the arrows h_{x-} or h_{x+} has to be an isomorphism. Therefore, in this case, $H(b, b') \simeq b \times U$ or $H(b, b') \simeq b' \times U$, hence $h_{b,b'}$ is a trivial fibration.

(ii) If $\dim \mathrm{Aut}(b) = \dim \mathrm{Aut}(b')$. Inspecting the list of points in $\mathrm{Bun}_G(\mathbb{P}^1, S)$ given in § 3.1.1, we see this happens only for the following pairs (b, b').

- $(b, b') = (c_0(S - \{s\}), c_1(s))$ or $(c_1(s), c_0(S - \{s\}))$ for $s \in S$. In this case both h_{x-} and h_{x+} are isomorphisms, therefore $h_{b,b'}$ is a trivial fibration.
- $(b, b') = (c_1(\emptyset), c_2(\emptyset))$ or $(c_2(\emptyset), c_1(\emptyset))$. In this case both h_{x-} and h_{x+} are isomorphisms, therefore $h_{b,b'}$ is a trivial fibration.
- $(b, b') = (c_0(\emptyset), c_1(*))$ or $(c_1(*), c_0(\emptyset))$. In this case, $\mathrm{Aut}(b) = \mathrm{Aut}(b') = 1$. If $b = (\mathcal{E} = \mathcal{O}^2_{\mathbb{P}^1}, \ell_0, \ell_1, \ell_\infty)$, then

$$H(b, b') = \{(x, \ell_x) \in U \times \mathbb{P}^1 \mid \text{ there is no map } \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{E} \text{ containing } \ell_0, \ell_1, \ell_\infty \text{ and } \ell_x\}.$$

One can check that $H(b, b') \subset U \times \mathbb{P}^1$ is the complement of the graph of an open embedding $U \hookrightarrow \mathbb{P}^1$. Therefore, $h_{b,b'}$ is an $\mathbb{A}^1$-fibration.

This proves the claim in the special case of lower modification.

5.2.3 *The functor* Φ. Let $G = \mathrm{PGL}(2)$ and $G^\vee = \mathrm{SL}(2)$. By making $\mathrm{Perf}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S))$ act on the Whittaker sheaf $\mathrm{Wh}_S \in Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S))$, we obtain a functor

$$\Phi_{\mathrm{Perf}} : \mathrm{Perf}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)) \longrightarrow Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)).$$

Since $Sh(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S))$ is cocomplete, we can take the continuous extension of Φ_{Perf} to get a functor

$$\Phi : \mathrm{QCoh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)) \longrightarrow Sh(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)).$$

Consider as well the restriction

$$\Phi_{\mathrm{Coh}} = \Phi|_{\mathrm{Coh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1,S))}.$$

We will defer the proof of the following until Proposition 5.6.1 below but mention it here for clarity.

Proposition 5.2.4. *The functor Φ_{Coh} lands in the full dg subcategory $Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S))$.*

5.3 Compatibilities with changing level structure

In this section, we will see why it is important that we act upon the Whittaker sheaf to construct the functor Φ and its elaborations.

5.3.1 *Changing level structure on the automorphic side.* On the automorphic side, for $s \in S$, consider the natural $\mathbb{P}^1$-fibration

$$\pi_s : \mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S) \longrightarrow \mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S \backslash \{s\}),$$

where we forget the flag at $s \in S$. It provides an adjoint triple

$$\begin{tikzcd} Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)) \arrow[r, "\pi_{s*}=\pi_{s!}" description] & Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S\backslash\{s\})). \arrow[l, bend right, "\pi_s^*"'] \arrow[l, bend left, "\pi_s^!"] \end{tikzcd}$$

5.3.2 *Changing level structure on the spectral side.* We seek the corresponding adjoint triple on the spectral side.

First, introduce the intermediate stack

$$\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S\backslash\{s\}, \{s\}) = \mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S\backslash\{s\}) \times_{\{s\}/\mathrm{SL}(2)} \mathbb{P}^1/\mathrm{SL}(2)$$

classifying an SL(2)-local system on $\mathbb{P}^1\backslash(S\backslash\{s\})$ with $B^\vee$-reductions near $S\backslash\{s\}$ with trivial induced $T^\vee$-monodromy, and an additional $B^\vee$-reduction at $s \in \mathbb{P}^1$.

Next, consider the natural correspondence

$$\begin{tikzcd} & \mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S\backslash\{s\}, \{s\}) \arrow[dl, "q_s"'] \arrow[d, "\kappa_s"] \arrow[dr, "p_s"] & \\ \mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S) & \mathbb{P}^1/\mathrm{SL}(2) & \mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S\backslash\{s\}) \end{tikzcd}$$

where p_s is the evident $\mathbb{P}^1$-fibration forgetting the flag at $s \in \mathbb{P}^1$, κ_s forgets all of the data except the flag at $s \in \mathbb{P}^1$, and q_s is the evident inclusion fitting into the Cartesian square.

$$\begin{tikzcd} \mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S) \arrow[d] & \mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S\backslash\{s\}, \{s\}) \arrow[l, "q_s"'] \arrow[d] \\ (T^*\mathbb{P}^1)/\mathrm{SL}(2) & \mathbb{P}^1/\mathrm{SL}(2) \arrow[l] \end{tikzcd}$$

Note, in particular, that the pullback q_s^* preserves coherent complexes since up to base change it is given by tensoring with the perfect complex $\mathcal{O}_{\mathbb{P}^1} = \mathrm{Cone}(\mathcal{O}_{T^*\mathbb{P}^1}(2) \to \mathcal{O}_{T^*\mathbb{P}^1})$.

Passing to coherent complexes, define the adjoint triple

$$\begin{tikzcd} \mathrm{Coh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)) \arrow[r, "\eta_s" description] & \mathrm{Coh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S\backslash\{s\})) \arrow[l, bend right, "\eta_s^\ell"'] \arrow[l, bend left, "\eta_s^r"] \end{tikzcd}$$

$$\eta_s(\mathcal{F}) = p_{s*}(q_s^*\mathcal{F} \otimes \kappa_s^*\mathcal{O}_{\mathbb{P}^1}(-1))$$

$$\eta_s^\ell = q_{s*}(p_s^*\mathcal{F} \otimes \kappa_s^*\mathcal{O}_{\mathbb{P}^1}(-1)[-1]), \quad \eta_s^r = q_{s*}(p_s^*\mathcal{F} \otimes \kappa_s^*\mathcal{O}_{\mathbb{P}^1}(-1)[1]).$$

Finally, recall from § 2.6 that there is an equivalence (denoted by $\Phi_{0,\infty}$ there)[2]

$$\Phi_{S\backslash\{s\}} : \mathrm{Coh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S\backslash\{s\})) \xrightarrow{\sim} \mathit{Sh}_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S\backslash\{s\})).$$

PROPOSITION 5.3.3. *For each pair of vertical arrows (η_s^ℓ, π_s^*), $(\eta_s, \pi_{s!})$, and $(\eta_s^r, \pi_s^!)$, the following diagram commutes by a canonical isomorphism.*

$$\begin{tikzcd} \mathrm{Coh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)) \arrow[r, "\Phi"] \arrow[d, shift left, "\eta_s"] & \mathit{Sh}_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)) \arrow[d, shift left, "\pi_{s!}"] \\ \mathrm{Coh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S\backslash\{s\})) \arrow[r, "\Phi_{S\backslash\{s\}}"] \arrow[u, shift left=3, "\eta_s^\ell"] \arrow[u, shift right=3, "\eta_s^r"'] & \mathit{Sh}_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S\backslash\{s\})) \arrow[u, shift left=3, "\pi_s^*"] \arrow[u, shift right=3, "\pi_s^!"'] \end{tikzcd}$$

Proof. (1) We first prove the commutativity for the pair (η_s^ℓ, π_s^*).

By construction, both compositions

$$\Phi \circ \eta_s^\ell, \pi_s^* \circ \Phi_{S\backslash\{s\}} : \mathrm{Coh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S\backslash\{s\})) \longrightarrow \mathit{Sh}_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S))$$

are naturally equivariant for the tensor action of $\mathrm{Perf}(\widetilde{\mathcal{N}}^\vee/G^\vee)$ at the points $S\backslash\{s\}$. Therefore. it suffices to give a natural isomorphism when evaluated on the structure sheaf

$$\pi_s^*(\Phi_{S\backslash\{s\}}(\mathcal{O}_{\mathrm{Loc}})) \xrightarrow{\sim} \Phi(\eta_s^\ell(\mathcal{O}_{\mathrm{Loc}^s})), \tag{5.6}$$

where we use the short-handed notation $\mathcal{O}_{\mathrm{Loc}}$ (respectively $\mathcal{O}_{\mathrm{Loc}^s}$) to denote the structure sheaf of $\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$ (respectively $\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S\backslash\{s\})$).[3]

On the one hand, note the isomorphism

$$\eta_s^\ell(\mathcal{O}_{\mathrm{Loc}^s}) \simeq \mathrm{Cone}(\mathcal{O}_{\mathrm{Loc}}(1,0,0) \longrightarrow \mathcal{O}_{\mathrm{Loc}}(-1,0,0))[-1],$$

where we order the twists with s as the first component. We can recast this as an isomorphism

$$\eta_s^\ell(\mathcal{O}_{\mathrm{Loc}^s}) \simeq \mathrm{Cone}(\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(1) \star_s \mathcal{O}_{\mathrm{Loc}} \longrightarrow \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(-1) \star_s \mathcal{O}_{\mathrm{Loc}})[-1].$$

[2] Note that the definition of $\Phi_{0,\infty}$ in § 2.6 is asymmetric with respect to 0 and ∞: its definition uses the Hecke action at 0. Therefore, in defining $\Phi_{S\backslash\{s\}}$, we need to make a choice of one of the two points in $S\backslash\{s\}$. The results involving $\Phi_{S\backslash\{s\}}$ will be valid for any such choice. In fact, one can show that for different choices of points in $S\backslash\{s\}$, the resulting functors are canonically isomorphic to each other, but we do not need this statement in the following.

[3] There is a general argument that works for any G, but here we give a more down-to-earth argument for $G = \mathrm{PGL}(2)$.

Moreover, the morphism of the cone is induced by the natural morphism of Wakimoto kernels

$$\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(1) \longrightarrow \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(-1).$$

Thus we have an isomorphism

$$\Phi(\eta_s^\ell(\mathcal{O}_{\mathrm{Loc}^s})) \simeq \mathrm{Cone}((J_1 \longrightarrow J_{-1}) \star_s \mathrm{Wh}_S)[-1].$$

Expanding in terms of the standard basis, we have the reformulation

$$\Phi(\eta_s^\ell(\mathcal{O}_{\mathrm{Loc}^s})) \simeq \mathrm{Cone}((T_{0*}T_{1/2} \longrightarrow T_{1/2}T_{0!}) \star_s \mathrm{Wh}_S)[-1].$$

Thanks to the distinguished triangles in $\mathcal{H}^{\mathrm{aff}}_{\mathrm{PGL}(2)}$ given by

$$\delta \longrightarrow T_{0!} \longrightarrow \mathrm{Avg}, \quad \mathrm{Avg} \longrightarrow T_{0*} \longrightarrow \delta$$

and the fact that $\mathrm{Avg} \star_s \mathrm{Wh}_S = 0$ as seen in Corollary 2.5.6, we have the further reformulation

$$\Phi(\eta_s^\ell(\mathcal{O}_{\mathrm{Loc}^s})) \simeq \mathrm{Avg} \star_s T_{1/2} \star_s \mathrm{Wh}_S = \pi_s^* \pi_{s*}(T_{1/2} \star_s \mathrm{Wh}_S)[1].$$

Finally, we have an isomorphism

$$T_{1/2} \star_s \mathrm{Wh}_S \simeq i_! j_* \underline{\mathbb{Q}}_{c_0(\emptyset)}$$

in terms of the open substacks

$$c_0(\emptyset) \overset{j}{\hookrightarrow} c_0(\emptyset) \cup c_0(0,\infty) \overset{i}{\hookrightarrow} \mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)$$

classifying trivial bundles $\mathcal{O}^2_{\mathbb{P}^1}$ with distinct lines $\ell_0, \ell_1, \ell_\infty$, and with ℓ_1 alone distinct. From this, we observe an isomorphism

$$\pi_{s*}(T_{1/2} \star_s \mathrm{Wh}_S)[1] \simeq \mathrm{Wh}_{S \backslash \{s\}}$$

from which (5.6) follows.

(2) The proof for the pair $(\eta_s^r, \pi_s^!)$ is completely the same as that for (η_s^ℓ, π_s^*).

(3) By adjunction and the known canonical isomorphism $\pi_s^* \circ \Phi_{S\backslash\{s\}} \simeq \Phi \circ \eta_s^\ell$, we get a natural transformation $\Phi_{S\backslash\{s\}} \Rightarrow \pi_{s*} \circ \Phi \circ \eta_s^\ell$. Precomposing with η_s, we get a natural transformation

$$\theta : \Phi_{S\backslash\{s\}} \circ \eta \Rightarrow \pi_{s*} \circ \Phi \circ \eta_s^\ell \circ \eta \Rightarrow \pi_{s*} \circ \Phi.$$

We will show that θ is an equivalence. Note it suffices to show that $\pi_s^*\theta$ is an equivalence because π_s^* is conservative.

Observe that

$$\eta_s^\ell \eta(\mathcal{F}) = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1,-1)[-1] \star_s \mathcal{F} \quad \text{for } \mathcal{F} \in \mathrm{Coh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)).$$

By the affine Hecke equivariance of Φ (see Proposition 2.5.8), and the fact that $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$ corresponds to Avg under the equivalence Φ^{aff}, we have

$$\Phi \circ \eta_s^\ell \circ \eta = \Phi \circ (\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)[-1]\star_s) \simeq \mathrm{Avg}[-1] \star_s \Phi.$$

On the other hand, the above equivalence of functors is the composition

$$\Phi \circ \eta_s^\ell \circ \eta \xrightarrow{\quad\sim\quad} \pi_s^* \circ \Phi_{S\backslash\{s\}} \circ \eta \xrightarrow{\quad\pi_s^*\theta\quad} \pi_s^* \circ \pi_{s*} \circ \Phi = \mathrm{Avg}[-1] \star_s \Phi\ ,$$

where the first equivalence is the composition of the identity of η and the equivalence established above in (1). Therefore, $\pi_s^*\theta$ is an isomorphism. This completes the proof. □

This proposition allows us to calculate the image of $\mathrm{IC}_0(\emptyset), \mathrm{IC}_0(1,\infty), \mathrm{IC}_0(0,\infty)$, and $\mathrm{IC}_0(0,1)$ under Φ, as listed in § 5.1.2. For example, $\mathrm{IC}_0(0,\infty) \cong \pi_1^* \mathrm{Eis}_{0,\{0,\infty\}}$ (the Eisenstein series Eis_0 for $\mathbb{P}^1\backslash\{0,\infty\}$).

5.4 Compatibility with Eisenstein series

In the case $G = \mathrm{PGL}(2)$, we have $\Lambda_T = \mathbb{Z}$. For $n \in \mathbb{Z}$, recall the subdiagram

$$\mathrm{Bun}_T^n(\mathbb{P}^1) \xleftarrow{p_n} \mathrm{Bun}_B^n(\mathbb{P}^1) \xrightarrow{q_n} \mathrm{Bun}_{\mathrm{PGL}(2)}^{\bar{n}}(\mathbb{P}^1, S),$$

where we fix $n = 2\deg(\mathcal{L}) - \deg(\mathcal{E})$ (and $\bar{n} = n \mod 2$).

Recall the Eisenstein series sheaf

$$\mathrm{Eis}_n = q_{n!}\underline{\mathbb{Q}}_{\mathrm{Bun}_B^n(\mathbb{P}^1)}[-n-2].$$

To describe it, recall we write $j : c_n(S) \to \mathrm{Bun}_G^{\bar{n}}(\mathbb{P}^1, S)$ for the point, where $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}$ with lines $\ell_0, \ell_1, \ell_\infty \subset \mathcal{O}_{\mathbb{P}^1}(n)$, and write $\mathcal{F}_n(S) = j_! \underline{\mathbb{Q}}_{c_n(S)} \in Sh_!(\mathrm{Bun}_G^{\bar{n}}(\mathbb{P}^1, S))$ for the extension by zero of the constant sheaf. Recall also the special point $j : c_1(\emptyset) \to \mathrm{Bun}_G^{\bar{1}}(\mathbb{P}^1, S)$ where $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}$ with collinear lines $\ell_0, \ell_1, \ell_\infty \subset \mathcal{O}_{\mathbb{P}^1}$, and $\mathcal{F}_1(\emptyset) = j_! \underline{\mathbb{Q}}_{c_1(\emptyset)} \in Sh_!(\mathrm{Bun}_G^{\bar{1}}(\mathbb{P}^1, S))$ is the extension by zero of the constant sheaf.

LEMMA 5.4.1.

(i) *When $n \geqslant 0$, we have an isomorphism*

$$p_n : \mathrm{Bun}_B^n(\mathbb{P}^1) \xrightarrow{\sim} c_n(S) \subset \mathrm{Bun}_{\mathrm{PGL}(2)}^{\bar{n}}(\mathbb{P}^1, S)$$

and hence an isomorphism

$$\mathrm{Eis}_n \simeq \mathcal{F}_n(S)[-n-2].$$

(ii) *When $n = -1$, we have an isomorphism*

$$p_1 : \mathrm{Bun}_B^{-1}(\mathbb{P}^1) \xrightarrow{\sim} c_1(\emptyset) \subset \mathrm{Bun}_{\mathrm{PGL}(2)}^{\bar{1}}(\mathbb{P}^1, S)$$

and hence an isomorphism

$$\mathrm{Eis}_{-1} \simeq \mathcal{F}_1(\emptyset)[-1].$$

Proof. For $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}$ with $n \geqslant -1$, and lines $\ell_0, \ell_1, \ell_\infty \subset \mathcal{O}_{\mathbb{P}^1}(n)$, there exists a unique inclusion $\mathcal{O}_{\mathbb{P}^1}(n) \subset \mathcal{E}$ such that $\mathcal{O}_{\mathbb{P}^1}(n)|_S$ coincides with the given lines. (In fact, for $n \geqslant 1$, there exists a unique inclusion independently of the lines.) $\square$

5.4.2 *Spectral Eisenstein series.* We seek the objects on the spectral side corresponding to Eisenstein sheaves.

Consider the substack $\mathrm{Loc}_{B^\vee}(\mathbb{P}^1, S) \subset \mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$ classifying $B^\vee$-local systems on $\mathbb{P}^1 \backslash S$ with trivial induced $T^\vee$-monodromy near S (which in this case implies trivial induced $T^\vee$-monodromy globally). It admits the presentation as a quotient

$$\mathrm{Loc}_{B^\vee}(\mathbb{P}^1, S) \simeq \widetilde{\mathcal{N}}_\Delta^\vee / \mathrm{SL}(2)$$

of the reduced subscheme $\widetilde{\mathcal{N}}_\Delta^\vee \subset (\widetilde{\mathcal{N}}^\vee)^{S, \prod = 1}$ of the irreducible component Λ_S from the list of § 4.1.2. In particular, we have the natural $\mathrm{SL}(2)$-equivariant projection $\pi : \widetilde{\mathcal{N}}_\Delta^\vee / \mathrm{SL}(2) \to \mathbb{P}^1 / \mathrm{SL}(2)$.

For $n \in \mathbb{Z}$, define the spectral Eisenstein series coherent sheaf to be

$$\mathcal{O}_\Delta(n) = \mathcal{O}_{\widetilde{\mathcal{N}}_\Delta^\vee / \mathrm{SL}(2)}(n) \simeq \pi^* \mathcal{O}_{\mathbb{P}^1 / \mathrm{SL}(2)}(n).$$

Proposition 5.4.3. *For $n \in \mathbb{Z}$, we have an isomorphism*

$$\Phi(\mathcal{O}_\Delta(n+1)) \simeq \mathrm{Eis}_n\,.$$

Proof. We denote $(\widetilde{\mathcal{N}}^\vee)^{S,\Pi=1}$ simply by Λ during the proof. Also, we will denote objects in $\mathrm{Coh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)) \simeq \mathrm{Coh}^{\mathrm{SL}(2)}(\Lambda)$ by their pullbacks to Λ.

By the construction of Φ, we have $\Phi(\mathcal{O}_\Delta(n+1)) = \Phi(\mathcal{O}_\Lambda(n,0,0) \otimes_{\mathcal{O}_\Lambda} \mathcal{O}_\Delta(1)) \simeq J_n \star_0 \Phi(\mathcal{O}_\Delta(1))$; on the other hand, by Lemma 2.4.4, $J_n \star_0 \mathrm{Eis}_0 \simeq \mathrm{Eis}_n$. Therefore, it suffices to show that

$$\Phi(\mathcal{O}_\Delta(1)) \simeq \mathrm{Eis}_0\,.$$

One direct strategy would be to write $\mathcal{O}_\Delta(1)$ as a complex of vector bundles, then apply Φ to the complex, and show the resulting complex is isomorphic to Eis_0. Unfortunately, since $\mathcal{O}_\Delta(1)$ is coherent but not perfect, this would involve infinite complexes. To avoid this complication, we will instead bootstrap off of Proposition 5.3.3 and express $\mathcal{O}_\Delta(1)$ in terms of the structure sheaf $\mathcal{O}_\Lambda$ and objects coming from two points of ramification.

First, by construction we have

$$\Phi(\mathcal{O}_\Lambda(0,1,0)) \simeq \Phi(J_1 \star_1 \mathcal{O}_\Lambda) \simeq J_1 \star_1 \Phi(\mathcal{O}_\Lambda) \simeq J_1 \star_1 \mathrm{Wh}_S.$$

Let us describe this sheaf explicitly. Consider the open substacks

$$c_0(\emptyset) \xhookrightarrow{j} c_0(\emptyset) \cup c_0(0,\infty) \xhookrightarrow{i} U^{\overline{0}} \xhookrightarrow{u} \mathrm{Bun}^{\overline{0}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)$$

classifying bundles $\mathcal{E} \simeq \mathcal{O}^2_{\mathbb{P}^1}$ with respectively distinct lines $\ell_0, \ell_1, \ell_\infty$, more generally, lines $\ell_0, \ell_1, \ell_\infty$ with the only possible coincidence $\ell_0 = \ell_\infty$, and finally most generally, any configuration of lines $\ell_0, \ell_1, \ell_\infty$. Then a simple calculation, for example via the identity $J_1 = T_{0*}T_{1/2}$, shows that

$$J_1 \star_1 \mathrm{Wh}_S \simeq u_! i_* j_! \underline{\mathbb{Q}}_{c_0(\emptyset)}.$$

From here on, we will only consider the open substack $U^{\overline{0}} \subset \mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)$, and all sheaves will be understood to be extensions by zero off of $U^{\overline{0}}$.

Let Y be the preimge of the partial diagonals $\Delta_{0,1} \cup \Delta_{1,\infty} \subset (\mathbb{P}^1)^S$ in $(\widetilde{\mathcal{N}}^\vee)^{S,\Pi=1}$. Under the local coordinates introduced in § 4.1.2, Y is given locally by the equation $xy = 0$. Therefore, $Y = \Lambda_{0,1} \cup \Lambda_{1,\infty} \cup \widetilde{\Lambda}_S$, where $\widetilde{\Lambda}_S$ denotes the non-reduced component (5) in § 4.1.2 whose reduced structure is $\Lambda_S \simeq \widetilde{\mathcal{N}}^\vee_\Delta$. Since $\Delta_{0,1} \cup \Delta_{1,\infty}$ have ideal sheaves $\mathcal{O}_{(\mathbb{P}^1)^S}(-1,-1,0) \otimes \mathcal{O}_{(\mathbb{P}^1)^S}(0,-1,-1) = \mathcal{O}_{(\mathbb{P}^1)^S}(-1,-2,-1)$ within $(\mathbb{P}^1)^S$, the ideal sheaf $\mathcal{I}_Y$ is a quotient of $\mathcal{O}_\Lambda(-1,-2,-1)$. Using local coordinates, we see that the ideal sheaf of Y in Λ is generated by one equation $(a+b)$, which defines the components $\Lambda_\emptyset$ and $\Lambda_{0,\infty}$. This gives in the heart of $\mathrm{Coh}^{\mathrm{SL}(2)}(\Lambda)$ a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\Lambda_\emptyset \cup \Lambda_{0,\infty}}(-1,-2,-1) \longrightarrow \mathcal{O}_\Lambda \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

and its twist

$$0 \longrightarrow \mathcal{O}_{\Lambda_\emptyset \cup \Lambda_{0,\infty}}(-1,-1,-1) \longrightarrow \mathcal{O}_\Lambda(0,1,0) \longrightarrow \mathcal{O}_Y(0,1,0) \longrightarrow 0. \tag{5.7}$$

By a similar process, using a Koszul-like resolution of $\mathcal{O}_\Lambda$ as a quotient of $\mathcal{O}_Y$ (locally defined by the equations $x = 0, y = 0$), we get a filtration of $\mathcal{O}_Y(0,1,0)$ by SL(2)-equivariant coherent subsheaves with associated-graded (from sub to quotient)

$$\mathcal{O}_\Delta(1), \quad \mathcal{O}_{\Lambda_{0,1}}(0,0,-1) \oplus \mathcal{O}_{\Lambda_{1,\infty}}(-1,0,0), \quad \mathcal{O}_\Delta(1).$$

In particular, $\mathcal{O}_Y(0,1,0)$ carries an endomorphism $\epsilon : \mathcal{O}_Y(0,1,0) \twoheadrightarrow \mathcal{O}_\Delta(1) \hookrightarrow \mathcal{O}_Y(0,1,0)$ such that $\epsilon^2 = 0$.

Next we consider the automorphic side. Consider the respective open and closed substacks

$$a : A = \{\ell_0 \neq \ell_\infty\} \hookrightarrow U, \quad b : B = \{\ell_0 = \ell_1\} \cup \{\ell_1 = \ell_\infty\} \hookrightarrow U.$$

We have a short exact sequence of perverse sheaves

$$0 \longrightarrow a_! \underline{\mathbb{Q}}_A \longrightarrow i_* j_! \underline{\mathbb{Q}}_{c_0(\emptyset)} \longrightarrow b_! T_B \longrightarrow 0, \tag{5.8}$$

where T_B is a perverse sheaf on B. It is easy to see that $b_! T_B$ has a filtration (as a perverse sheaf) with associated-graded (from sub to quotient)

$$\mathrm{Eis}_0, \quad \mathrm{IC}_0(0,1) \oplus \mathrm{IC}_0(1,\infty), \quad \mathrm{Eis}_0 .$$

In particular, $b_! T_B$ carries an endomorphism $\epsilon' : b_! T_B \twoheadrightarrow \mathrm{Eis}_0 \hookrightarrow b_! T_B$ such that $\epsilon'^2 = 0$.

Recall there is an isomorphism

$$\Phi(\mathcal{O}_\Lambda(0,1,0)) \simeq i_* j_! \underline{\mathbb{Q}}_{c_0(\emptyset)}.$$

Claim. *There is an isomorphism*

$$\Phi(\mathcal{O}_{\Lambda_\emptyset \cup \Lambda_{0,\infty}}(-1,-1,-1)) \simeq a_! \underline{\mathbb{Q}}_A.$$

Proof. In the case of $\mathbb{P}^1$ with two punctures 0 and ∞, we may identify $\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, \{0,\infty\})$ with the adjoint quotient $St_{\mathrm{SL}(2)}/\mathrm{SL}(2)$ of the derived Steinberg variety. In the following we write $\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, \{0,\infty\})$ simply as $\mathrm{Loc}(0,\infty)$. Recall from Example 2.2.5, Φ^{aff} sends the twisted classical structure sheaf $\mathcal{O}^{\mathrm{cl}}_{St}(-1,-1)$ to $T_{0!}$. Therefore, by the definition of $\Phi_{0,\infty}$ we have

$$\begin{aligned}\Phi_{0,\infty}(\mathcal{O}^{\mathrm{cl}}_{\mathrm{Loc}(0,\infty)}(-1,-1)) &= \Phi^{\mathrm{aff}}(\mathcal{O}^{\mathrm{cl}}_{St}(-1,-1)) \star_0 \mathrm{Eis}_{0,\{0,\infty\}} \\ &\simeq T_{0!} \star_0 \mathrm{Eis}_{0,\{0,\infty\}} \\ &\simeq j_{0,\infty !} \underline{\mathbb{Q}}_{U_{0,\infty}}[-1].\end{aligned}$$

Here $j_{0,\infty} : U_{0,\infty} \simeq \mathrm{pt}/T \hookrightarrow \mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, \{0,\infty\})$ is the open point $\mathcal{E} = \mathcal{O}^2_{\mathbb{P}^1}$ with two distinct lines ℓ_0, ℓ_∞.

Since $\eta^\ell_1(\mathcal{O}^{\mathrm{cl}}_{\mathrm{Loc}(0,\infty)}(-1,-1)) \simeq \mathcal{O}_{\Lambda_\emptyset \cup \Lambda_{0,\infty}}(-1,-1,-1)[-1]$, by Proposition 5.3.3, we have that

$$\begin{aligned}\Phi(\mathcal{O}_{\Lambda_\emptyset \cup \Lambda_{0,\infty}}(-1,-1,-1)) &\simeq \Phi(\eta^\ell_1(\mathcal{O}^{\mathrm{cl}}_{\mathrm{Loc}(0,\infty)}(-1,-1)))[1] \\ &\simeq \pi_1^* \Phi_{0,\infty}(\mathcal{O}^{\mathrm{cl}}_{\mathrm{Loc}(0,\infty)}(-1,-1))[1] \\ &\simeq \pi_1^* j_{0,\infty !} \underline{\mathbb{Q}}_{U_{0,\infty}} \\ &\simeq a_! \underline{\mathbb{Q}}_A.\end{aligned}$$

□

Taking direct sums, we obtain an isomorphism

$$\Phi(\mathcal{O}_{\Lambda_\emptyset \cup \Lambda_{0,\infty}}(-1,-1,-1) \oplus \mathcal{O}_\Lambda(0,1,0)) \simeq a_! \underline{\mathbb{Q}}_A \oplus i_* j_! \underline{\mathbb{Q}}_{c_0(\emptyset)}.$$

Claim. *The functor Φ induces a quasi-isomorphism*

$$\mathrm{End}(\mathcal{O}_{\Lambda_\emptyset \cup \Lambda_{0,\infty}}(-1,-1,-1) \oplus \mathcal{O}_\Lambda(0,1,0)) \xrightarrow{\sim} \mathrm{End}(a_! \underline{\mathbb{Q}}_A \oplus i_* j_! \underline{\mathbb{Q}}_{c_0(\emptyset)}).$$

Proof. First, one can calculate

$$\mathrm{End}(\mathcal{O}_\Lambda(0,1,0)) \simeq \mathbb{Q}, \quad \mathrm{End}(i_* j_! \underline{\mathbb{Q}}_{c_0(\emptyset)}) \simeq \mathbb{Q}.$$

Since both are generated by the identity morphism, Φ must induce a quasi-isomorphism on them.

Next, we have seen in the previous claim that

$$\mathcal{O}_{\Lambda_\emptyset \cup \Lambda_{0,\infty}}(-1,-1,-1) \simeq \eta_1^\ell(\mathcal{O}^{\mathrm{cl}}_{\mathrm{Loc}(0,\infty)}(-1,-1))[1],$$

$$a_! \underline{\mathbb{Q}}_A \simeq \pi_1^* j_{0,\infty !} \underline{\mathbb{Q}}_{U_{0,\infty}}.$$

Thus for any object $\mathcal{M}$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{O}_{\Lambda_\emptyset \cup \Lambda_{0,\infty}}(-1,-1,-1), \mathcal{M}) \arrow[r, "\Phi"] \arrow[d, "\sim"'] & \mathrm{Hom}(a_! \underline{\mathbb{Q}}_A, \Phi(\mathcal{M})) \arrow[d, "\sim"'] \\
\mathrm{Hom}(\mathcal{O}^{\mathrm{cl}}_{\mathrm{Loc}(0,\infty)}(-1,-1)[1], \eta_1 \mathcal{M}) \arrow[r, "\Phi_{0,\infty}"] & \mathrm{Hom}(j_{0,\infty !} \underline{\mathbb{Q}}_{U_{0,\infty}}, \pi_{1*}\Phi(\mathcal{M}))
\end{tikzcd}$$

where the vertical equivalences are by adjunction. Since the bottom arrow is an equivalence, the top arrow must be as well. In particular, we can apply this for $\mathcal{M} \simeq \mathcal{O}_{\Lambda_\emptyset \cup \Lambda_{0,\infty}}(-1,-1,-1) \oplus \mathcal{O}_\Lambda(0,1,0)$.

Finally, a similar argument using the respective right adjoints $\eta_1^r, \pi^!$, shows for any object $\mathcal{M}$, that Φ induces an equivalence

$$\mathrm{Hom}(\mathcal{M}, \mathcal{O}_{\Lambda_\emptyset \cup \Lambda_{0,\infty}}(-1,-1,-1)) \xrightarrow{\sim} \mathrm{Hom}(\Phi(\mathcal{M}), a_! \underline{\mathbb{Q}}_A).$$

Again, we can apply this for $\mathcal{M} \simeq \mathcal{O}_{\Lambda_\emptyset \cup \Lambda_{0,\infty}}(-1,-1,-1) \oplus \mathcal{O}_\Lambda(0,1,0)$.

This concludes the proof of the claim. □

Claim. *The functor Φ applied to the sequence (5.7) gives the sequence (5.8). In particular, we have an isomorphism*

$$\Phi(\mathcal{O}_Y(0,1,0)) \simeq b_! T_B. \tag{5.9}$$

Moreover, the functor Φ takes the endomorphism ϵ of $\mathcal{O}_Y(0,1,0)$ to a non-zero multiple of the endomorphism ϵ' of $b_! T_B$.

Proof. We have seen that

$$\Phi(\mathcal{O}_{\Lambda_\emptyset \cup \Lambda_{0,\infty}}(-1,-1,-1)) \simeq a_! \underline{\mathbb{Q}}_A, \quad \Phi(\mathcal{O}_\Lambda(0,1,0)) \simeq i_* j_! \underline{\mathbb{Q}}_{c_0(\emptyset)}.$$

One can calculate

$$\mathrm{Hom}(\mathcal{O}_{\Lambda_\emptyset \cup \Lambda_{0,\infty}}(-1,-1,-1), \mathcal{O}_\Lambda(0,1,0)) \simeq \mathbb{Q} \oplus \mathbb{Q}[-1],$$

$$\mathrm{Hom}(a_! \underline{\mathbb{Q}}_A, i_* j_! \underline{\mathbb{Q}}_{c_0(\emptyset)}) \simeq \mathbb{Q} \oplus \mathbb{Q}[-1].$$

Note that each is one-dimensional in degree 0.

By the previous claim, the first morphism of (5.7) is taken to a non-zero morphism. Since this morphism and the first morphism of (5.8) are non-zero elements of a one-dimensional vector

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space, each is a non-zero scale of the other. This implies Φ takes the sequence (5.7) to the sequence (5.8), and in particular, passing to cones gives the isomorphism (5.9).

Furthermore, the previous claim also implies the functor Φ induces a quasi-isomorphism on endomorphisms of the cones

$$\mathrm{End}(\mathcal{O}_Y(0,1,0)) \xrightarrow{\sim} \mathrm{End}(b_! T_B). \tag{5.10}$$

One can calculate the degree 0 endomorphisms on both sides of (5.10) to see each is isomorphic to the dual numbers with respective generators ϵ and ϵ'. Thanks to the quasi-isomorphism (5.10), this implies Φ takes ϵ to a non-zero multiple of ϵ'.

This completes the proof of the claim. □

To complete the proof of the proposition, introduce the quotient categories

$$\overline{C} = \mathrm{QCoh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S))/\langle \mathcal{O}_{\Lambda_{0,1}}(0,0,-1), \mathcal{O}_{\Lambda_{1,\infty}}(-1,0,0)\rangle,$$

$$\overline{Sh} = Sh(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S))/\langle \mathrm{IC}_0(0,1) \oplus \mathrm{IC}_0(1,\infty)\rangle.$$

By (5.1) and (5.2), Φ induces a continuous functor

$$\overline{\Phi} : \overline{C} \longrightarrow \overline{Sh}.$$

Let $\mathcal{K}$ be the image of $\mathcal{O}_Y(0,1,0)$ in $\overline{C}_1$; let $\mathcal{T}$ be the image of $b_! T_B$ in $\overline{Sh}$. By (5.9), we have

$$\overline{\Phi}(\mathcal{K}) \simeq \mathcal{T}.$$

Inside of $\overline{C}$, the image of $\mathcal{O}_\Delta(1)$ is represented by the infinite complex (the last non-zero entry is in degree 0)

$$\cdots \to \mathcal{K} \xrightarrow{\bar{\epsilon}} \mathcal{K} \xrightarrow{\bar{\epsilon}} \mathcal{K} \xrightarrow{0} 0 \to \cdots, \tag{5.11}$$

where $\bar{\epsilon}$ is the endomorphism of $\mathcal{K}$ induced by the endomorphism ϵ of $\mathcal{O}_Y(0,1,0)$.

Inside $\overline{Sh}$, the image of Eis_0 is represented by the infinite complex of perverse sheaves (the last non-zero entry is in degree 0)

$$\cdots \to \mathcal{T} \xrightarrow{\bar{\epsilon}'} \mathcal{T} \xrightarrow{\bar{\epsilon}'} \mathcal{T} \xrightarrow{0} 0 \to \cdots, \tag{5.12}$$

where $\bar{\epsilon}'$ is the endomorphism of $\mathcal{T}$ induced by the endomorphism ϵ' of $b_! T_B$.

By the previous claim, the continuous functor $\overline{\Phi}$ sends (5.11) to (5.12). Therefore, the image of $\Phi(\mathcal{O}_\Delta(1))$ in $\overline{Sh}$ is the same as the image of Eis_0. In particular,

$$\Phi(\mathcal{O}_\Delta(1)) \subset \langle \mathrm{IC}_0(0,1), \mathrm{IC}_0(1,\infty), \mathrm{Eis}_0\rangle. \tag{5.13}$$

The same argument can be applied when the point $1 \in S$ is replaced by 0 or ∞, and we get

$$\Phi(\mathcal{O}_\Delta(1)) \subset \langle \mathrm{IC}_0(0,1), \mathrm{IC}_0(0,\infty), \mathrm{Eis}_0\rangle, \tag{5.14}$$
$$\Phi(\mathcal{O}_\Delta(1)) \subset \langle \mathrm{IC}_0(0,\infty), \mathrm{IC}_0(1,\infty), \mathrm{Eis}_0\rangle. \tag{5.15}$$

Since the intersection of the categories on the right sides of (5.13)–(5.15) consists of sheaves supported at the point $c_0(S)$, we conclude that $\Phi(\mathcal{O}_\Delta(1))$ is supported at $c_0(S)$.

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Finally, using the compatibility of Φ with changing levels, we can calculate the push forward of $\Phi(\mathcal{O}_\Delta(1))$ under $\pi_1 : Sh(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)) \to Sh(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, \{0, \infty\}))$. By Proposition 5.3.3 we have

$$\pi_{1*}\Phi(\mathcal{O}_\Delta(1)) \simeq \Phi(\eta_1(\mathcal{O}_\Delta(1))). \tag{5.16}$$

Since we will be changing the level structure, we use $\widetilde{\mathcal{N}}^\vee_{\Delta,\{0,\infty\}}$ and $\mathcal{O}_{\Delta,\{0,\infty\}}$ to denote the analogues of $\widetilde{\mathcal{N}}^\vee_\Delta$ and $\mathcal{O}_\Delta$ when S is replaced by $\{0, \infty\}$. We have the following commutative diagram where the left parallelogram is derived Cartesian.

$$\begin{tikzcd}
 & \widetilde{\mathcal{N}}^\vee_{\Delta,\{0,\infty\}}/\mathrm{SL}(2) \arrow[r, "\theta'"] \arrow[dl, "q_1'"'] \arrow[drrr, "p_1'"] & \mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, \{0,\infty\}, \{1\}) \arrow[dl, "q_1"'] \arrow[drr, "p_1"] & & \\
\widetilde{\mathcal{N}}^\vee_\Delta/\mathrm{SL}(2) \arrow[r, "\theta"] & \mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S) & & & \mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, \{0,\infty\})
\end{tikzcd}$$

Then we have

$$\eta_1(\mathcal{O}_\Delta(1)) \simeq p_{1*}(q_1^*\theta_*\mathcal{O}_\Delta(1) \otimes \mathcal{O}_\Lambda(0, -1, 0)) \simeq p_{1*}(\theta'_* q_1'^* \mathcal{O}_\Delta) \simeq \mathcal{O}_{\Delta,\{0,\infty\}}. \tag{5.17}$$

By Lemma 2.6.2, $\Phi_{0,\infty}$ sends $\mathcal{O}_{\Delta,\{0,\infty\}}$ (which is the same as $\Delta^-_*\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}$ in the notation of Lemma 2.6.2) to the Eisenstein sheaf $\mathrm{Eis}_{0,\{0,\infty\}}$. Combining (5.16) and (5.17), we have

$$\pi_{1*}\Phi(\mathcal{O}_\Delta(1)) \simeq \mathrm{Eis}_{0,\{0,\infty\}}\,.$$

Since $\Phi(\mathcal{O}_\Delta(1))$ is supported on $c_0(S)$, which is mapped isomorphically onto its image under π_1, we conclude that $\Phi(\mathcal{O}_\Delta(1)) \simeq \mathrm{Eis}_0$. This completes the proof of the proposition. □

5.5 Newforms

On the automorphic side, for $s \in S$, define $Sh_s \subset Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S))$ to be the full subcategory generated by the image of π_s^*.

Define the dg category of newforms to be the dg quotient

$$Sh^{\mathrm{new}} = Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S))/\langle Sh_0, Sh_1, Sh_\infty\rangle,$$

where we kill all 'old forms' coming from fewer points of ramification.

Similarly, on the spectral side, define $C_s \subset \mathrm{Coh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S))$ to be the full subcategory generated by the image of η_s^ℓ.

Define the dg quotient category

$$C^{\mathrm{new}} = \mathrm{Coh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S))/\langle C_0, C_1, C_\infty\rangle.$$

Lemma 5.5.1.

(i) *Sh^{new} is generated by Eis_n, for $n \geqslant -1$.*
(ii) *C^{new} is generated by $\mathcal{O}_\Delta(n)$, for $n \geqslant 0$.*

Proof. (i) Set $Sh^{\mathrm{old}} = \langle Sh_0, Sh_1, Sh_\infty\rangle$. We only need to exhibit a set of generators for the category $Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S))$ whose members are either in Sh^{old}, or a (shifted) Eisenstein sheaf. Such a set of generators is given by the following.

– For $n \geqslant 2$, $\mathrm{IC}_n(R) \in Sh^{\mathrm{old}}$ when $R \neq S$ (see (3.1)); by Lemma 5.4.1, we also have $\mathcal{F}_n(S) \simeq \mathrm{Eis}_n[-n-2]$.

– For $n=1$, $\mathrm{IC}_1(R)\in Sh^{\mathrm{old}}$ when $R\neq\emptyset$ or S; by Lemma 5.4.1, we also have $\mathcal{F}_1(\emptyset)\simeq\mathrm{Eis}_{-1}[-1]$ and $\mathcal{F}_1(S)\simeq\mathrm{Eis}_1[-3]$.
– For $n=0$, $\mathrm{IC}_0(R)\in Sh^{\mathrm{old}}$ when $R\neq S$; by Lemma 5.4.1, we also have $\mathcal{F}_0(S)\simeq\mathrm{Eis}_0[-2]$.

(ii) Let $C^{\mathrm{old}}=\langle C_0,C_1,C_\infty\rangle$ and let $C'=\langle C^{\mathrm{old}},\mathcal{O}_\Delta(n);n\geqslant 0\rangle$. Our goal is to show that $C'=\mathrm{Coh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1,S))$.

We will use the following well-known fact. Let Y be a stack of finite type, $i:Z\hookrightarrow Y$ a closed substack and $j:U=Y-Z\hookrightarrow Y$ the open complement of Z. Then j^* induces an equivalence $\mathrm{Coh}(Y)/\mathrm{Coh}_Z(Y)\simeq\mathrm{Coh}(U)$, where $\mathrm{Coh}_Z(Y)$ is the dg subcategory of $\mathrm{Coh}(Y)$ generated by the image of $i_*:\mathrm{Coh}(Z)\to\mathrm{Coh}(Y)$.

Using the above fact and induction, one can show the following statement, which we label (†):

Suppose a stack Y of finite type is stratified into a union of finitely many strata $Y_\alpha\subset Y$, for α in some index set A. Suppose for each $\alpha\in A$, we have a collection of objects $\mathcal{F}_\alpha^{(i)}\in\mathrm{Coh}(\overline{Y}_\alpha)$, for i in some index set I_α, such that $\{\mathcal{F}_\alpha^{(i)}|_{Y_\alpha};i\in I_\alpha\}$ generate $\mathrm{Coh}(Y_\alpha)$. Then the collection $\{\mathcal{F}_\alpha;\alpha\in A,i\in I_\alpha\}$ generate $\mathrm{Coh}(Y)$. (†)

We will also use the following additional simple observation we label (‡):

Let Y be an affine scheme with an action of an affine group H. Then $\mathrm{Coh}^H(Y)$ is generated by objects of the form $V\otimes\mathcal{O}_Y$, where V runs over all finite-dimensional irreducible representations of H, and the H-equivariant structure on $V\otimes\mathcal{O}_Y$ is given by the diagonal action of H. (‡)

In § 4.1.2 we listed the irreducible components of $(\widetilde{\mathcal{N}}^\vee)^{S,\prod=1}$, and denoted their reduced structure by Λ_R, for subsets $R\subset S$ such that $\#R\neq 1$. We know that Λ_R is the conormal bundle to the partial diagonal $\Delta_R\subset(\mathbb{P}^1)^S$.

Let us now stratify $(\widetilde{\mathcal{N}}^\vee)^{S,\prod=1}$ by taking the intersections of the components Λ_R. By the above statements (†) and (‡), it is enough to exhibit a collection of objects in C' on the closure of each stratum whose restrictions to that stratum generate the $G^\vee$-equivariant derived category of coherent sheaves on that stratum.

The three-dimensional strata are the opens $\Lambda_R^\circ=\Lambda_R\backslash\bigcup_{R'\neq R}\Lambda_{R'}$, for $R\subset S$, $\#R\neq 1$. Let us describe the quotients $\Lambda_R^\circ/G^\vee$, along with a set of objects of C' whose restrictions generate $\mathrm{Coh}^{G^\vee}(\Lambda_R^\circ)$.

(i) $\Lambda_\emptyset^\circ/G^\vee\simeq\mathrm{pt}/\mu_2$, where μ_2 is the center of $G^\vee$. By (‡), $\mathrm{Coh}^{G^\vee}(\Lambda_R^\circ)$ is generated by two elements $\mathcal{O}_{\Lambda_R^\circ}$ and $\mathrm{sgn}\otimes\mathcal{O}_{\Lambda_R^\circ}$, where sgn denotes the sign representation of μ_2. Therefore, for $R=\emptyset$, the restrictions of $\mathcal{O}_{\Lambda_\emptyset}(-1,-1,0),\mathcal{O}_{\Lambda_\emptyset}(-1,0,0)\in C_0\subset C^{\mathrm{old}}\subset C'$ to $\Lambda_\emptyset^\circ$ generate $\mathrm{Coh}^{G^\vee}(\Lambda_\emptyset^\circ)$.

(ii) When $\#R=2$, $\Lambda_R^\circ/G^\vee\simeq(N^\vee\backslash\{1\})/T^\vee\simeq\mathrm{pt}/\mu_2$. Note $\Lambda_R\simeq\mathbb{P}^1\times\widetilde{\mathcal{N}}^\vee$. By the same argument as in the previous case, the restrictions of $\mathcal{O}_{\mathbb{P}^1}(-1)\boxtimes\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(1),\mathcal{O}_{\mathbb{P}^1}(-1)\boxtimes\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}\in C^{\mathrm{old}}\subset C'$ to Λ_R° generate $\mathrm{Coh}^{G^\vee}(\Lambda_R^\circ)$.

(iii) $\Lambda_S^\circ/G^\vee\simeq Y'/B^\vee$, where $Y'=(N^\vee\backslash\{1\})^{S,\prod=1}$, and the action of $B^\vee$ factors through $T^\vee$. Therefore, $\Lambda_S^\circ/G^\vee\simeq(\mathbb{A}^1\backslash\{0,1\})\times(\mathrm{pt}/(\mathbb{G}_a\times\mu_2))$. Again by (‡), $\mathrm{Coh}^{G^\vee}(\Lambda_S^\circ)$ is generated by two elements $\mathcal{O}_{\Lambda_S^\circ}$ and $\mathrm{sgn}\otimes\mathcal{O}_{\Lambda_S^\circ}$. Therefore, the restrictions of $\mathcal{O}_\Delta$ and $\mathcal{O}_\Delta(1)\in C'$ to Λ_S° generate $\mathrm{Coh}^{G^\vee}(\Lambda_S^\circ)$.

The one-dimensional stratum is the intersection of all Λ_R given by the diagonal $\Delta_S\subset(\mathbb{P}^1)^S$. We will return to it momentarily.

For $\#R = 2$, we have $\Delta_R = \Lambda_\emptyset \cap \Lambda_R$, and set $\Delta_R^\circ = \Delta_R \backslash \Delta_S$; we also set $\Theta_R = \Lambda_S \cap \Lambda_R \simeq \widetilde{\mathcal{N}}^\vee$, and $\Theta_R^\circ = \Theta_R \backslash \Delta_S$. Then the two-dimensional strata are $\Delta_R^\circ, \Theta_R^\circ$, for $\#R = 2$. Let us describe their quotients by $G^\vee$, along with a set of objects of C' whose restrictions generate equivariant coherent sheaves.

(i) $\Delta_R^\circ/G^\vee \simeq \mathrm{pt}/T^\vee$. Write Δ_R as $\mathbb{P}^1 \times \mathbb{P}^1$. By (‡), $\mathrm{Coh}^{G^\vee}(\Delta_R^\circ)$ is generated by the restrictions of $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1, n)$, for all $n \in \mathbb{Z}$, which all lie in $C^{\mathrm{old}} \subset C'$.

(ii) $\Theta_R^\circ/G^\vee \simeq \mathrm{pt}/(\mathbb{G}_a \times \mu_2)$. Note the canonical projection $\Theta_R \simeq \widetilde{\mathcal{N}}^\vee \to \mathbb{P}^1$, providing the line bundles $\mathcal{O}_{\Theta_R}(n)$, for $n \in \mathbb{Z}$. By (‡), $\mathrm{Coh}^{G^\vee}(\Theta_R^\circ)$ is generated by the restrictions of $\mathcal{O}_{\Theta_R}$ and $\mathcal{O}_{\Theta_R}(1)$. Note that $\Theta_R \subset \Lambda_S = \widetilde{\mathcal{N}}_\Delta$ is a $G^\vee$-invariant line sub-bundle in the two-dimensional vector bundle $\widetilde{\mathcal{N}}_\Delta \simeq \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 2}$ over $\mathbb{P}^1$. Therefore, we have an exact sequence of $G^\vee$-equivariant coherent sheaves $0 \to \mathcal{O}_\Delta(2) \to \mathcal{O}_\Delta \to \mathcal{O}_{\Theta_R} \to 0$. This shows that $\mathcal{O}_{\Theta_R} \in C'$. Similarly, $0 \to \mathcal{O}_\Delta(n+2) \to \mathcal{O}_\Delta(n) \to \mathcal{O}_{\Theta_R}(n) \to 0$ implies $\mathcal{O}_{\Theta_R}(n) \in C'$, for any $n \geqslant 0$.

Finally let us show that $\mathcal{O}_{\Delta_S}(n) \in C'$, for all $n \in \mathbb{Z}$. This will complete the proof by providing a generating set for $\mathrm{Coh}^{G^\vee}(\Delta_S)$, where recall $\Delta_S \subset (\mathbb{P}^1)^S$ is the closed one-dimensional stratum. Since Δ_S is the zero section of Θ_R, for any $\#R = 2$, we have a $G^\vee$-equivariant exact sequence $0 \to \mathcal{O}_{\Theta_R}(n+2) \to \mathcal{O}_{\Theta_R}(n) \to \mathcal{O}_{\Delta_S}(n) \to 0$. Since we have already shown that $\mathcal{O}_{\Theta_R}(n) \in C'$, for all $n \geqslant 0$, we also have $\mathcal{O}_{\Delta_S}(n) \in C'$, for all $n \geqslant 0$. Now pick any $\#R = 2$ and write $\Delta_R = \mathbb{P}^1 \times \mathbb{P}^1$ and regard Δ_S as the diagonal. Consider the $G^\vee$-equivariant exact sequence

$$0 \longrightarrow \mathcal{O}_{\Delta_R}(-1, -n-2) \longrightarrow \mathcal{O}_{\Delta_R}(n, -1) \longrightarrow \mathcal{E}_n \longrightarrow 0 \tag{5.18}$$

obtained by restricting $\mathcal{O}_{\Delta_R}(n, -1)$ to the nth infinitesimal neighborhood of the diagonal Δ_S. Then $\mathcal{E}_n$ (which is topologically supported on Δ_S) is a successive extension of $\mathcal{O}_{\Delta_S}(n-1)$, $\mathcal{O}_{\Delta_S}(n-3), \ldots, \mathcal{O}_{\Delta_S}(-n-1)$ (each time the twisting decreases by 2). We have $\mathcal{E}_n \in C^{\mathrm{old}}$, for any $n \geqslant 0$, by (5.18) because the first two terms are in C^{old}. We have already shown that $\mathcal{O}_{\Delta_S}(n) \in C'$, for any $n \geqslant 0$. Using that $\mathcal{E}_n \in C'$, for any $n \geqslant 0$, we conclude that $\mathcal{O}_{\Delta_S}(n) \in C'$, for all $n < 0$. This completes the proof. □

5.6 Equivalence

Proposition 5.6.1. *Proposition 5.2.4 holds:* Φ_{Coh} *lands in* $Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S))$.

Moreover, Φ_{Coh} *is essentially surjective onto* $Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S))$.

Proof. We continue with the notation introduced in the previous section.

By Proposition 5.3.3, we have $\Phi|_{C_s} : C_s \to Sh_s$, for $s \in S$. Moreover, it is essentially surjective since $\Phi_{S\backslash\{s\}}$ is essentially surjective.

Therefore Φ induces a functor

$$\Phi^{\mathrm{new}} : C^{\mathrm{new}} \longrightarrow Sh(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S))/\langle Sh_0, Sh_1, Sh_\infty\rangle.$$

It suffices to show that Φ^{new} has image exactly equal to Sh^{new}. By Proposition 5.4.3, we have

$$\Phi^{\mathrm{new}}(\mathcal{O}_\Delta(n+1)) \simeq \mathrm{Eis}_n, \quad n \geqslant -1.$$

Thus by Lemma 5.5.1(ii), the image of Φ^{new} lies in Sh^{new}, and by Lemma 5.5.1(i), it is essentially surjective onto Sh^{new}. □

Now we are ready to prove our main theorem for $G = \mathrm{PGL}(2)$.

Theorem 5.6.2. *The functor Φ_{Coh} provides an equivalence*

$$\mathrm{Coh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)) \xrightarrow{\sim} Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S))$$

compatible with the affine Hecke actions at $s \in S$.

It restricts to equivalences

$$\mathrm{Coh}^{\mathrm{triv}}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)) \xrightarrow{\sim} Sh_!(\mathrm{Bun}^{\overline{1}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)),$$
$$\mathrm{Coh}^{\mathrm{alt}}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)) \xrightarrow{\sim} Sh_!(\mathrm{Bun}^{\overline{0}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)). \tag{5.19}$$

Proof. Compatibility of Φ_{Coh} with the affine Hecke actions follow from Proposition 2.5.8.

Thanks to Proposition 5.6.1, it remains to show the following: for $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S))$, the natural homomorphism

$$\mathrm{Hom}_{\mathrm{Coh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1,S))}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}_{Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1,S))}(\Phi\mathcal{F}, \Phi\mathcal{G}) \tag{5.20}$$

is a quasi-isomorphism.

We will make a series of reductions. We use the abbreviation $Sh_! = Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S))$ and $\mathrm{Loc} = \mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$.

First, by continuity, we may assume that $\mathcal{F} = \mathcal{O}_{\mathrm{Loc}}(a, b, c) \otimes V$, i.e., the tensor of a line bundle and an $\mathrm{SL}(2)$-representation. Then the left-hand side of (5.20) takes the form

$$\mathrm{Hom}_{\mathrm{Coh}(\mathrm{Loc})}(\mathcal{F}, \mathcal{G}) \simeq \Gamma(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S), \mathcal{O}_{\mathrm{Loc}}(-a, -b, -c) \otimes V^\vee \otimes \mathcal{G}).$$

Second, by construction, we have

$$\begin{aligned}\mathrm{Hom}_{Sh_!}(\Phi\mathcal{F}, \Phi\mathcal{G}) &\simeq \mathrm{Hom}_{Sh_!}(\mathrm{Hecke}^{\mathrm{sph}}_{u_0}(V, J_a \star_0 J_b \star_1 J_c \star_\infty \mathrm{Wh}_S), \Phi\mathcal{G})\\ &\simeq \mathrm{Hom}_{Sh_!}(\mathrm{Wh}_S, \mathrm{Hecke}^{\mathrm{sph}}_{u_0}(V^\vee, J_{-a} \star_0 J_{-b} \star_1 J_{-c} \star_\infty \Phi\mathcal{G}))\\ &\simeq \mathrm{Hom}_{Sh_!}(\mathrm{Wh}_S, \Phi(\mathcal{O}_{\mathrm{Loc}}(-a, -b, -c) \otimes V^\vee \otimes \mathcal{G})),\end{aligned}$$

where $u_0 \in \mathbb{P}^1 \backslash S$ is a base point.

Thus we may reduce to the case $\mathcal{F} = \mathcal{O}_{\mathrm{Loc}}$, and would like to show that the natural map

$$\Gamma(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S), \mathcal{G}) \longrightarrow \mathrm{Hom}_{Sh_!}(\mathrm{Wh}_S, \Phi\mathcal{G})$$

is a quasi-isomorphism.

Now the global sections' functor $\Gamma(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S), -)$ factors through C^{new} since objects in C_s, for $s \in S$, have a factor $\mathcal{O}_{\mathbb{P}^1}(-1)$ whose global sections must vanish.

Similarly, since $\pi_{s!}\mathrm{Wh}_S = 0$ by Lemma 2.5.5, the functor $\mathrm{Hom}_{Sh_!}(\mathrm{Wh}_S, -)$ factors through Sh^{new}. Furthermore, by Proposition 5.3.3, we have $\Phi(C_s) \subset Sh_s$, for $s \in S$. Thus the functor $\mathrm{Hom}_{Sh_!}(\mathrm{Wh}_S, \Phi(-))$ factors through C^{new}.

Hence by Lemma 5.5.1(2), it suffices to assume $\mathcal{G} = \mathcal{O}_\Delta(n)$, for $n \geqslant 0$.

For $n = 0$, both sides are canonically quasi-isomorphic to $\mathbb{Q}$ and we claim the morphism is a quasi-isomorphism. Equivalently, applying Wakimoto symmetry, we claim the induced morphism

$$\mathrm{Hom}_{\mathrm{Coh}(\mathrm{Loc})}(\mathcal{O}(0, 1, 0), \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(1)) \longrightarrow \mathrm{Hom}_{Sh_!}(J_1 \star_1 \mathrm{Wh}_S, \mathrm{Eis}_0) \tag{5.21}$$

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is a quasi-isomorphism. Returning to the proof and notation of Proposition 5.4.3, observe the left-hand side of (5.21) is generated by the composition

$$\mathcal{O}(0,1,0) \xrightarrow{\sigma} \mathcal{O}_Y(0,1,0) \xrightarrow{\widetilde{\epsilon}} \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(1)$$

such that the endomorphism $\epsilon : \mathcal{O}_Y(0,1,0) \to \mathcal{O}_Y(0,1,0)$ therein is the composition of the surjection $\widetilde{\epsilon}$ and the inclusion $\mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(1) \hookrightarrow \mathcal{O}_Y(0,1,0)$. Similarly, the right-hand side of (5.21) is generated by the composition

$$J_1 \star_1 \mathrm{Wh}_S \xrightarrow{\sigma'} b_! T_B \xrightarrow{\widetilde{\epsilon}'} \mathcal{O}_{\widetilde{\mathcal{N}}^\vee}(1)$$

such that the endomorphism $\epsilon' : b_! T_B \to b_! T_B$ is the composition of the surjection $\widetilde{\epsilon}'$ and the inclusion $\mathrm{Eis}_0 \hookrightarrow b_! T_B$. Moreover, in the two claims in the proof of Proposition 5.4.3, we saw that $\Phi(\sigma) = \sigma'$, and $\Phi(\epsilon)$ is a non-zero multiple of ϵ'. Thus $\Phi(\widetilde{\epsilon})$ is a non-zero multiple of $\widetilde{\epsilon}'$, since both lie in one-dimensional spaces, and we have confirmed (5.21) is a quasi-isomorphism.

For $n > 0$, both sides of (5.21) vanish. On the one hand, $\Gamma(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S), \mathcal{O}_\Delta(n))$ is a direct sum of the $\mathrm{SL}(2)$-invariants in $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n+2i))$, for $i \geqslant 0$, and hence vanishes for $n > 0$. On the other hand, the support of $\Phi(\mathcal{O}_\Delta(n)) = \mathrm{Eis}_{n-1}$ is disjoint from the support of Wh_S, and hence they are orthogonal. □

By invoking the identifications and symmetries for the automorphic and spectral categories recorded in §§ 3.3.4 and 4.3.4, we can conclude from the theorem an additional equivalence.

Corollary 5.6.3. *There is an equivalence*

$$\mathrm{Coh}^{\mathrm{SL}(2)-\mathrm{alt}}(\mathrm{Loc}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)) \xrightarrow{\sim} Sh_!(\mathrm{Bun}_{\mathrm{SL}(2)}(\mathbb{P}^1, S))$$

compatible with affine Hecke actions at $s \in S$. Here we write $\mathrm{Coh}^{\mathrm{SL}(2)-\mathrm{alt}}(\mathrm{Loc}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S))$ for the dg category of $\mathrm{SL}(2)$-equivariant coherent complexes on $(\widetilde{\mathcal{N}}^\vee)^{S, \prod=1}$, where the equation $\prod = 1$ is imposed inside of $\mathrm{PGL}(2)$, and such that the center $\mu_2 \simeq Z(\mathrm{SL}(2)) \subset \mathrm{SL}(2)$ acts by the alternating representation on coherent complexes.

It restricts to equivalences

$$\mathrm{Coh}^{\mathrm{SL}(2)-\mathrm{alt}}(\mathrm{Loc}^{\bar{0}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)) \xrightarrow{\sim} Sh_!^{\mathrm{triv}}(\mathrm{Bun}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)), \tag{5.22}$$

$$\mathrm{Coh}^{\mathrm{SL}(2)-\mathrm{alt}}(\mathrm{Loc}^{\bar{1}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)) \xrightarrow{\sim} Sh_!^{\mathrm{alt}}(\mathrm{Bun}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)). \tag{5.23}$$

Proof. The equivalence (5.22) follows by combining (5.19) and (3.2), and the fact that

$$\mathrm{Coh}^{\mathrm{SL}(2)-\mathrm{alt}}(\mathrm{Loc}^{\bar{0}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)) = \mathrm{Coh}^{\mathrm{alt}}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)).$$

By §§ 3.3.3 and 4.3.3 that both sides of (5.23) are equivalent to Vect; the equivalence (5.23) then follows immediately. □

Remark 5.6.4. The sheaf $\mathcal{F}_0(\emptyset)^{\mathrm{alt}}$ on $\mathrm{Bun}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$ is a cuspidal Hecke eigensheaf with eigenvalue given by the unique 'odd' PGL(2)-local system on $\mathbb{P}^1 \backslash S$ given in Lemma 4.1.4. See Remark 4.1.5 for a description of this local system.

Remark 5.6.5. Though we will not discuss the details here, the above equivalences further restrict to equivalences from those coherent sheaves with nilpotent singular support to those constructible sheaves that are point-wise compact

$$\mathrm{Coh}_{\mathcal{N}}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)) \xrightarrow{\sim} Sh_{\dagger}(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)),$$
$$\mathrm{Coh}_{\mathcal{N}}^{\mathrm{SL}(2)-\mathrm{alt}}(\mathrm{Loc}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)) \xrightarrow{\sim} Sh_{\dagger}(\mathrm{Bun}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)).$$

5.7 Unipotently monodromic version

We record here the monodromic form of the prior equivalence. Its construction and proof are similar.

Let $\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, \widetilde{S})$ denote the moduli of PGL(2)-bundles on $\mathbb{P}^1$ with N-reductions at the points of $S = \{0, 1, \infty\}$. Note the natural map $\pi : \mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, \widetilde{S}) \to \mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)$ is a $T^S = T^3$-torsor.

Let $Sh_!^{\mathrm{mon}}(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, \widetilde{S}))$ denote the full dg subcategory of $Sh_!(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, \widetilde{S}))$ generated by pullbacks along π.

Let $\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, \widetilde{S})$ denote the Betti moduli of SL(2)-local systems on $\mathbb{P}^1 \backslash S$ with $B^\vee$-reductions near S with arbitrary induced $T^\vee$-monodromy. Thus it admits a presentation

$$\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, \widetilde{S}) \simeq (\widetilde{\mathrm{SL}(2)})^{S, \prod = 1}/\mathrm{SL}(2),$$

where $\widetilde{\mathrm{SL}(2)}$ is the Grothendieck alteration of SL(2), and the equation on the product of the group elements $\prod = 1$ is imposed inside of SL(2).

Let $\mathrm{Coh}_{\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, \widetilde{S}))$ be the full subcategory of $\mathrm{Coh}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, \widetilde{S}))$ consisting of coherent complexes set-theoretically supported on the substack $\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$.

Theorem 5.7.1. *There is an equivalence*

$$\widetilde{\Phi}_{\mathrm{Coh}} : \mathrm{Coh}_{\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)}(\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, \widetilde{S})) \xrightarrow{\sim} Sh_!^{\mathrm{mon}}(\mathrm{Bun}_{\mathrm{PGL}(2)}(\mathbb{P}^1, \widetilde{S}))$$

compatible with Hecke modifications.

The proof is similar to the equivariant version with the following changes.

The monodromic version of the Whittaker sheaf $\widehat{\mathrm{Wh}}_S$ corresponds to the structure sheaf of the completion of $\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, \widetilde{S})$ along $\mathrm{Loc}_{\mathrm{SL}(2)}(\mathbb{P}^1, S)$. It can be constructed as follows. Consider the diagram of Cartesian squares of open substacks

$$\begin{tikzcd}
\widetilde{c_1(*)} \arrow[r, hook, "\widetilde{j}"] \arrow[d] & \widetilde{c_1(*)} \cup \widetilde{c_1(\emptyset)} \arrow[r, hook, "\widetilde{i}"] \arrow[d, "\pi"'] & \mathrm{Bun}^{\bar{1}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, \widetilde{S}) \arrow[d] \\
c_1(*) \arrow[r, hook, "j"] & c_1(*) \cup c_1(\emptyset) \arrow[r, hook, "i"] & \mathrm{Bun}^{\bar{1}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, S)
\end{tikzcd}$$

where the vertical maps are T^S-torsors. In particular, since $c_1(*)$ is simply a point, $\widetilde{c_1(*)}$ is itself a T^S-torsor. Then the free-monodromic Whittaker sheaf is given by

$$\widehat{\mathrm{Wh}}_S = \widetilde{i}_! \widetilde{j}_* \mathcal{L}_{c_1(*)}[2\#S \cdot \dim T] = \widetilde{i}_! \widetilde{j}_* \mathcal{L}_{c_1(*)}[6] \in Sh_!^{\mathrm{mon}}(\mathrm{Bun}^{\bar{1}}_{\mathrm{PGL}(2)}(\mathbb{P}^1, \widetilde{S})),$$

where $\mathcal{L}_{c_1(*)}$ denotes the free-monodromic unipotent local system on $\widetilde{c_1(*)}$: its monodromy representation is the completion of the regular representation of $\pi_1(\widetilde{c_1(*)}) \cong \pi_1(T^S)$ at the augmentation ideal. By construction we have

$$\pi_! \widehat{\mathrm{Wh}}_S \simeq \mathrm{Wh}_S.$$

The functor $\widetilde{\Phi}_{\mathrm{Coh}}$ is constructed by acting on the monodromic Whittaker sheaf. Its essential surjectivity follows from that of the equivariant case, and its fully faithfulness comes down to the calculation

$$\mathrm{Hom}(\widehat{\mathrm{Wh}}_S, \pi^! \operatorname{Eis}_{-1}) \simeq \mathrm{Hom}(\pi_! \widehat{\mathrm{Wh}}_S, \operatorname{Eis}_{-1}) \simeq \mathrm{Hom}(\mathrm{Wh}_S, \operatorname{Eis}_{-1}) \simeq \mathbb{Q}.$$

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