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From conjugacy classes in the Weyl group to representations

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1. Definition of the map Ψ

1.1. Let G be a connected reductive algebraic group defined and split over the finite field $\mathbf{F}_p$ with p elements where p is a prime number which we assume to be good for G. Let W be the Weyl group of G (a Coxeter group with length function $w \mapsto |w|$), let $\underline{W}$ be the set of conjugacy classes of W and let $\mathcal{R}_W$ be the Grothendieck group of (finite dimensional) representations of W over $\bar{\mathbf{Q}}_l$; here l is a fixed prime number, $l \neq p$. Let $\mathcal{U}$ be the variety of unipotent elements of G. Let $\underline{\mathcal{U}}$ be the set of G-conjugacy classes of unipotent elements. Let $\mathcal{B}$ be the variety of Borel subgroups of G. We fix $B^+ \in \mathcal{B}$ such that B^+ is defined over $\mathbf{F}_p$.

In [**12**] a (surjective) map $\Phi : \underline{W} \to \underline{\mathcal{U}}$ was defined based on the study of intersections of unipotent classes in G with (B^+, B^+) double cosets in G.

Let $\mathbf{q}$ be an indeterminate. In this paper we define a map

$$\Psi : \underline{W} \to \mathbf{Z}[\mathbf{q}] \otimes \mathcal{R}_W$$

whose image is denoted by Z. We also define a (surjective) map $\Theta : Z \to \underline{\mathcal{U}}$ such that $\Phi(C) = \Theta(\Psi(C))$ for any $C \in \underline{W}$. Thus, Ψ is a refinement of Φ. The definition of Ψ again involves the study of intersections of unipotent classes in G with (B^+, B^+) double cosets in G. We also describe Ψ explicitly for G of low rank (see §3).

A map closely related to Ψ appears in the work of Minh-Tam Trinh [**14**].

1.2. For $(B, B') \in \mathcal{B} \times \mathcal{B}$ we denote by $pos(B, B') \in W$ the relative position of B, B' (see [**4**]). For $w \in W$ let $B^+wB^+ = \{g \in G; pos(B^+, gB^+g^{-1}) = w\}$. Let $F_1 : G \to G$ be the Frobenius map relative to the $\mathbf{F}_p$-structure. Let $q = p^e$ ($e \geq 1$) and let $F = F_1^e : G \to G$. This induces Frobenius maps $\mathcal{U} \to \mathcal{U}$, $\mathcal{B} \to \mathcal{B}$ denoted again by F. For $y \in W$, we set $X_y = \{B \in \mathcal{B}; pos(B, F(B)) = y\}$ (see [**4**]). If $i \in \mathbf{Z}$, the $\bar{\mathbf{Q}}_l$-cohomology space with compact support $H^i_c(X_y)$ has a natural action of the finite group G^F (see [**4**]); here $?^F$ denotes the fixed point set of $F : ? \to ?$. Let $R^1_y = \sum_i (-1)^i H^i_c(X_y)$, a virtual representation of G^F. For $g \in G^F$, $\mathrm{tr}(g, R^1_y)$ is an integer, see [**4**, §3.3]. For $w \in W, y \in W$ we set

$$a_{w,y} = \sum_{u \in \mathcal{U}^F \cap (B^+wB^+)} \mathrm{tr}(u, R^1_y) \in \mathbf{Z}. \tag{1.1}$$

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1.3. PROPOSITION. *Let $w \in W$. There is a unique element $\zeta_{w,q} \in \mathcal{R}_W$ such that for any $y \in W$ we have* $\operatorname{tr}(y, \zeta_{w,q}) = a_{w,y}$.

PROOF. For $u \in \mathcal{U}$ we set $\mathcal{B}_u = \{B \in \mathcal{B}; u \in B\}$. It is known (Springer) that W acts naturally on the l-adic cohomology space $H^i(\mathcal{B}_u)$. Let $\operatorname{Irr}(W)$ be a set of representatives for the isomorphism classes of irreducible representations of W over $\bar{\mathbf{Q}}_l$. We can identify $H^i(\mathcal{B}_u) = \oplus_{E \in \operatorname{Irr}(W)} E \otimes H^i(\mathcal{B}_u)_E$ where $H^i(\mathcal{B}_u)_E = \operatorname{Hom}_W(E, H^i(\mathcal{B}_u))$. From [**11**] (or [**7**] when $p \gg 0$), for $u \in \mathcal{U}^F$ and $y \in W$ we have

$$\operatorname{tr}(u, R_y^1) = \sum_i (-1)^i \operatorname{tr}(yF, H^i(\mathcal{B}_u)) = \sum_{E \in \operatorname{Irr}(W)} \operatorname{tr}(y, E) \operatorname{tr}(F, H^*(\mathcal{B}_u)_E). \tag{1.2}$$

Here $F : H^i(\mathcal{B}_u) \to H^i(\mathcal{B}_u)$ and $F : H^i(\mathcal{B}_u)_E \to H^i(\mathcal{B}_u)_E$ are induced by the restriction of $F : \mathcal{B} \to \mathcal{B}$ to $\mathcal{B}_u$ and $\operatorname{tr}(F, H^*(\mathcal{B}_u)_E)$ is defined to be

$$\sum_i (-1)^i \operatorname{tr}(F, H^i(\mathcal{B}_u)_E).$$

Thus we have

$$a_{w,y} = \sum_{u \in \mathcal{U}^F \cap (B^+ w B^+)} \sum_{E \in \operatorname{Irr}(W)} \operatorname{tr}(y, E) \operatorname{tr}(F, H^*(\mathcal{B}_u)_E).$$

For $E \in \operatorname{Irr}(W)$ we set

$$\zeta_{w,q;E} = \sum_{u \in \mathcal{U}^F \cap (B^+ w B^+)} \operatorname{tr}(F, H^*(\mathcal{B}_u)_E) \tag{1.3}$$

so that

$$a_{w,y} = \sum_{E \in \operatorname{Irr}(W)} \zeta_{w,q;E} \operatorname{tr}(y, E)$$

for any $y \in W$. Since $a_{w,y} \in \mathbf{Z}$ and the matrix $(\operatorname{tr}(y, E))$ (with y running through representatives for the conjugacy classes in W, and $E \in \operatorname{Irr}(W)$) has integer entries and nonzero determinant, we see that $\zeta_{w,q;E} \in \mathbf{Q}$ for any $E \in \operatorname{Irr}(W)$. On the other hand from the definition we see see that $\zeta_{w,q;E}$ is an algebraic integer. It follows that

$$\zeta_{w,q;E} \in \mathbf{Z}. \tag{1.4}$$

We set $\zeta_{w,q} = \sum_{E \in \operatorname{Irr}(W)} \zeta_{w,q;E} E \in \mathcal{R}_W$. This proves the existence statement in the proposition. The uniqueness is obvious. □

1.4. Let $\mathcal{F}$ be the vector space of functions $\mathcal{B}^F \to \bar{\mathbf{Q}}_l$. For $w \in W$ we define a linear map $T_w : \mathcal{F} \to \mathcal{F}$ by $T_w(f)(B) = \sum_{B' \in \mathcal{B}^F; pos(B,B')=w} f(B')$. Let $\mathcal{H}$ be the subspace of $End(\mathcal{F})$ with basis $\{T_w; w \in W\}$; this is a subalgebra of $End(\mathcal{F})$ (the Hecke algebra). Now G^F acts on $\mathcal{F}$ by $g : f \mapsto f'$ where $f'(B) = f(g^{-1} B g)$. This action commutes with the $\mathcal{H}$-action so that we can identify $\mathcal{F} = \oplus_{\mathcal{E} \in \operatorname{Irr}(\mathcal{H})} \mathcal{E} \otimes \mathcal{F}_{\mathcal{E}}$ where $\mathcal{F}_{\mathcal{E}} = \operatorname{Hom}_{\mathcal{H}}(\mathcal{E}, \mathcal{F})$ and $\operatorname{Irr}(\mathcal{H})$ is a set of representatives for the simple $\mathcal{H}$-modules; here $\mathcal{F}_{\mathcal{E}}$ is an irreducible G^F-module. For $u \in \mathcal{U}^F$ we have

$$\sharp(B \in \mathcal{B}^F; pos(B, uBu^{-1}) = w) = \operatorname{tr}(uT_w, \mathcal{F}) = \sum_{\mathcal{E} \in \operatorname{Irr}(\mathcal{H})} \operatorname{tr}(T_w, \mathcal{E}) \operatorname{tr}(u, \mathcal{F}_{\mathcal{E}}).$$

Hence for $y \in W$ we have

$$\sharp(\mathcal{B}^F)a_{w,y} = \sum_{u\in\mathcal{U}^F, B\in\mathcal{B}^F; pos(B,uBu^{-1})=w} \operatorname{tr}(u, R_y^1) = \sum_{\mathcal{E}\in\operatorname{Irr}(\mathcal{H})}\sum_{u\in\mathcal{U}^F} \operatorname{tr}(u,\mathcal{F}_{\mathcal{E}})\operatorname{tr}(u,R_y^1)\operatorname{tr}(T_w,\mathcal{E}). \tag{1.5}$$

From (1.5) and (1.2) for any $w \in W, E \in \operatorname{Irr}(W)$ we have:

$$\sharp(\mathcal{B}^F)\zeta_{w,q;E} = \sum_{u\in\mathcal{U}^F}\sum_{\mathcal{E}\in\operatorname{Irr}(\mathcal{H})} \operatorname{tr}(u,\mathcal{F}_{\mathcal{E}})\operatorname{tr}(F,H^*(\mathcal{B}_u)_E)\operatorname{tr}(T_w,\mathcal{E}). \tag{1.6}$$

1.5. Let $w \in W, E' \in \operatorname{Irr}(W)$. We set

$$c_{w,E',q} = \sharp(W)^{-1}\sum_{\mathcal{E}\in\operatorname{Irr}(\mathcal{H})}\sum_{z\in W}(\mathcal{F}_{\mathcal{E}} : R_z^1)_{G^F}\operatorname{tr}(z,E')\operatorname{tr}(T_w,\mathcal{E}) \in \mathbf{Q}. \tag{1.7}$$

Here $(:)_{G^F}$ is the standard inner product of virtual representations of G^F. We shall now regard q as variable. We show:

(1.8) *$c_{w,E',q}$ is the value at q of a polynomial $\mathbf{c}_{w,E'}(\mathbf{q})$ in $\mathbf{q}$ with rational coefficients independent of q.*

We can assume that G is adjoint simple. The only part of the right hand side of (1.7) which depends on q is $\operatorname{tr}(T_w,\mathcal{E})$. This is known to be the value at q of a polynomial in $\mathbf{q}$ with integral coefficients independent of q except when G is of type E_7, $\dim E' = 512$ or G is of type E_8, $\dim E' = 4096$. Assume now that G is of type E_7, $\dim E' = 512$. Let $\mathcal{E}', \mathcal{E}''$ be the two objects of $\operatorname{Irr}(\mathcal{E})$ which have dimension 512. We have $c_{w,E',q} = (1/2)(\operatorname{tr}(T_w,\mathcal{E}') + \operatorname{tr}(T_w,\mathcal{E}''))$ (see the proof of [**9**, §7.12]). Although the quantities $\operatorname{tr}(T_w,\mathcal{E}'), \operatorname{tr}(T_w,\mathcal{E}'')$ are separately not necessarily polynomials in q their sum is. This proves (1.8) in the case where G is of type $E_7, \dim E' = 512$. The proof in the case where G is of type $E_8, \dim E' = 4096$, is entirely similar.

1.6. For $\mathcal{E} \in \operatorname{Irr}(\mathcal{H})$ there is a well defined class function $\xi_{\mathcal{E}}$ on G^F such that

$$\operatorname{tr}(g,\mathcal{F}_{\mathcal{E}}) = \sharp(W)^{-1}\sum_{z\in W}(\mathcal{F}_{\mathcal{E}} : R_z^1)_{G^F}\operatorname{tr}(g,R_z^1) + \xi_{\mathcal{E}}(g) \tag{1.9}$$

for all $g \in G^F$ and

(1.10) $\xi_{\mathcal{E}}$ is orthogonal to the character of any virtual representation R_T^θ of [**4**].

From (1.10) we deduce using results in [**10**] that for any $E \in \operatorname{Irr}(W)$ we have

$$\sum_{u\in\mathcal{U}^F}\xi_{\mathcal{E}}(u)\operatorname{tr}(F,H^*(\mathcal{B}_u)_E) = 0.$$

Using (1.9) and (1.6) we see that for any $w \in W, E \in \operatorname{Irr}(W)$ we have

$$\sharp(\mathcal{B}^F)\zeta_{w,q;E} = \sharp(W)^{-1}\sum_{u\in\mathcal{U}^F}\sum_{\mathcal{E}\in\operatorname{Irr}(\mathcal{H})}\sum_{z\in W}(\mathcal{F}_{\mathcal{E}} : R_z^1)_{G^F}\operatorname{tr}(u,R_z^1)\operatorname{tr}(F,H^*(\mathcal{B}_u)_E)\operatorname{tr}(T_w,\mathcal{E}).$$

Using (1.2) we deduce

$$\sharp(\mathcal{B}^F)\zeta_{w,q;E} = \sum_{E'\in \mathrm{Irr}(W)} c_{w,E',q} \sum_{u\in \mathcal{U}^F} \mathrm{tr}(F, H^*(\mathcal{B}_u)_{E'})\,\mathrm{tr}(F, H^*(\mathcal{B}_u)_E). \tag{1.11}$$

The right hand side of (1.11) is the value at $\mathbf{q} = q$ of a polynomial $\pi(\mathbf{q})$ (with the coefficients of π being rational numbers independent of q). This follows from (1.8) and the known properties of $\mathrm{tr}(F, H^i(\mathcal{B}_u)_E)$, see [**3**].

1.7. Let $w \in W, E \in \mathrm{Irr}(W)$. It is known that

(1.12) *$H^i(\mathcal{B}_u)_E$ can be interpreted as the stalk at $u \in \mathcal{U}$ of an intersection cohomology complex K_E on the closure of a unipotent class of G.*

(This was stated in [**8**, §3, Conj.2] and proved in [**1**].) We use this and the Grothendieck trace formula to rewrite the definition of $\zeta_{w,q;E}$ in the form

$$\zeta_{w,q;E} = \sum_i (-1)^i \,\mathrm{tr}(F, H^i(X, K_E|_X))$$

where $X = \mathcal{U} \cap (B^+wB^+)$ and the map $H^i(X, K_E|_X) \to H^i(X, K_E|_X)$ induced by $F : G \to G$ is denoted again by F. Replacing q by p^s with $s = 1, 2, \ldots$ we see that

$$\zeta_{w,p^s;E} = \sum_{\lambda \in V} n_\lambda \lambda^s$$

where V is a finite subset of $\bar{\mathbf{Q}}_l - \{0\}$ and n_λ are nonzero integers independent of s. It follows that

$$\left(\sum_{v\in W} p^{|v|s}\right)\zeta_{w,p^s;E} = \sum_{\lambda'\in V'} n'_{\lambda'}\lambda'^s$$

where V' is a finite subset of $\bar{\mathbf{Q}}_l - \{0\}$ and $n'_{\lambda'}$ are nonzero integers independent of s. By the results in §1.6 we have also

$$\left(\sum_{v\in W} p^{|v|s}\right)\zeta_{w,p^s;E} = \sum_{m\in[0,N]} n''_m p^{sm}$$

for some $N \geq 1$ where $n''_m \in \mathbf{Q}$ are independent of s. We deduce that

$$\sum_{\lambda'\in V'} n'_{\lambda'}\lambda'^s = \sum_{m\in[0,N]} n''_m p^{sm}$$

for $s = 1, 2, \ldots$. This forces V' to be a subset of $\{1, p, p^2, \ldots\}$. Thus we have the following result.

(1.13) *There exists a polynomial $\pi(\mathbf{q})$ in $\mathbf{q}$ with integer coefficients independent of s such that $(\sum_{v\in W} p^{|v|s})\zeta_{w,p^s;E} = \pi(p^s)$ for $s = 1, 2, \ldots$.*

1.8. For $C \in \underline{W}$ we denote by C_{min} the set of elements of minimal length in C. Let ρ be the reflection representation of W. For $w \in W$ let $n_w = \det(1 - w, \rho)$. We have $n_w \geq 0$. We say that w is elliptic if $n_w > 0$. Let $\underline{W}_{el}$ be the set of all $C \in \underline{W}$ such that for some/any $w \in C$, w is elliptic.

Let Z_G be the center of G. Let $\nu = \dim \mathcal{B}$. Let r be the rank of G/Z_G.

In this subsection we fix $C \in \underline{W}_{el}$ and $w \in C_{min}$. Let $n'_w \geq 1$ be the part prime to p of n_w. According to [**12**, §5.2], for any $g \in B^+wB^+$,

(1.14) *The group $\{b \in B^+; bgb^{-1} = g\}/Z_G$ is finite abelian of order dividing n'_w.*

(In the first line of [**12**, §5.1] one should add the sentence: *We fix $w \in C_{min}$.*) We show:

(1.15) *For any $E \in \mathrm{Irr}(W)$ we have $q^{-\nu}(q-1)^{-r}n'_w\zeta_{w,q;E} \in \mathbf{Z}$.*

We have $\zeta_{w,q;E} = \sum_S \sharp(S)\,\mathrm{tr}(F, H^*(\mathcal{B}_{u_S})_E)$ where S runs over the set of orbits of B^{+F} acting on $\mathcal{U}^F \cap B^+wB^+$ by conjugation and for each such S, u_S is an element of S. Here for any S in the sum, $\mathrm{tr}(F, H^*(\mathcal{B}_{u_S})_E)$ is an algebraic integer and, by (1.14), $\sharp(S) \in q^\nu(q-1)^r {n'_w}^{-1}\mathbf{Z}$. Hence $q^{-\nu}(q-1)^{-r}n'_w\zeta_{w,q;E}$ is an algebraic integer. By (1.4), this is also a rational number, hence an integer. This proves (1.15).

1.9. Proposition. *Let $C \in \underline{W}_{el}$ and $w \in C_{min}$. Let $E \in \mathrm{Irr}(W)$. Then there exists a polynomial $\pi_1(\mathbf{q}) \in \mathbf{Z}[\mathbf{q}]$ such that $\zeta_{w,p^s;E}/(p^{s\nu}(p^s-1)^r) = \pi_1(p^s)$ for $s = 1$, $2, \ldots$. In particular, $\zeta_{w,q;E}/(q^\nu(q-1)^r) \in \mathbf{Z}$.*

Proof. Let $\pi_0(\mathbf{q}) = (\sum_{v\in W}\mathbf{q}^{|v|_s})\mathbf{q}^\nu(\mathbf{q}-1)^r$. This is a monic polynomial in $\mathbf{q}$ of degree $2\nu + r$ with integer coefficients. Let $\pi(\mathbf{q})$ be as in (1.13). We have $\pi(\mathbf{q}) = \pi_1(\mathbf{q})\pi_0(\mathbf{q}) + \pi_2(\mathbf{q})$ where $\pi_1(\mathbf{q}), \pi_2(\mathbf{q})$ are polynomials with integer coefficients and $\pi_2(\mathbf{q})$ is either 0 or has degree $< 2\nu + r$. By (1.13) and (1.15) for $s = 1, 2, \ldots$ we have $\pi(q^s)n'_w/\pi_0(q^s) \in \mathbf{Z}$ hence $\pi_1(q^s)n'_w + \pi_2(q^s)n'_w/\pi_0(q^s) \in \mathbf{Z}$ so that $\pi_2(q^s)n'_w/\pi_0(q^s) \in \mathbf{Z}$. If $\pi_2(\mathbf{q}) \neq 0$ this is impossible for large s since $\deg\pi_2 < \deg\pi_0$. We see that $\pi(\mathbf{q}) = \pi_1(\mathbf{q})\pi_0(\mathbf{q})$. Setting $\mathbf{q} = q$ we see that $\zeta_{w,q;E}/(q^\nu(q-1)^r) = \pi_1(q) \in \mathbf{Z}$. The proposition is proved. $\square$

1.10. Let P be a parabolic subgroup of G containing B^+. Let L be an F-stable Levi subgroup of P and let U be the unipotent radical of P. Then $B_L^+ := B^+ \cap L$ is an F-stable Borel subgroup of L and we have $B^+ = B_L^+U$. Let $\mathcal{U}_L$ be the set of unipotent elements of L. Let W_L be the Weyl group of L. We can view W_L as a subgroup of W and as an indexing set for the (B_L^+, B_L^+) double cosets of L, so that for $w \in W_L$ the double coset $B_L^+wB_L^+$ satisfies $(B_L^+wB_L^+)U = B^+wB^+$. For $y' \in W_L$ we define $X_{y',L}$ in the same way as X_y, but replacing G, y by L, y'; then $R^1_{y',L} = \sum_i(-1)^iH^i_c(X_{y',L})$ is naturally a (virtual) L^F-module.

Let σ be an irreducible L^F-module. We can view σ as a P^F-module on which U^F acts trivially. Let $y \in W$. The following identity is a reformulation of a special case of a result in [**5**]:

$$(\mathrm{ind}_{P^F}^{G^F}(\sigma) : R^1_y)_{G^F} = \sharp(W_L)^{-1}\sum_{z\in W;zyz^{-1}\in W_L}(\sigma : R^1_{zyz^{-1},L})_{L^F}. \tag{1.16}$$

Here $(:)_{L^F}$ denote the standard inner product of virtual representations of L^F. The left hand side of (1.16) is equal to $(\sigma : \sum_i(-1)^iH^i_c(X_y)^{U^F})_{L^F}$ where $H^i_c(X_y)^{U^F}$ is the space of U^F-invariants on $H^i_c(X_y)$ viewed as an L^F-module. It follows that

$$\sharp(W_L)\sum_i(-1)^iH^i_c(X_y)^{U^F} = \sum_{z\in W;zyz^{-1}\in W_L}R^1_{zyz^{-1},L}$$

as virtual L^F-modules.

For $w \in W_L, y' \in W_L$ we define $a_{w,y';L}$ as in (1.1), in terms of L instead of G that is,

$$a_{w,y';L} = \sum_{u \in \mathcal{U}_L^F \cap (B_L^+ w B_L^+)} \mathrm{tr}(u, R^1_{y',L}) \in \mathbf{Z}.$$

Now let $w \in W_L, y \in W$. We have

$$\begin{aligned}
a_{w,y} &= \sum_{u \in \mathcal{U}^F \cap (B_L^+ w B_L^+) U} \mathrm{tr}(u, R^1_y) \\
&= \sum_{(u',u'') \in \mathcal{U}_L^F \times U^F; u' \in B_L^+ w B_L^+} \mathrm{tr}(u'u'', R^1_y) \\
&= \sharp(U^F) \sum_{u' \in \mathcal{U}_L^F (B_L^+ w B_L^+)} \sum_i (-1)^i \mathrm{tr}(u', H^i_c(X_y)^{U^F}) \\
&= \sharp(W_L)^{-1} \sharp(U^F) \sum_{u' \in \mathcal{U}_L^F (B_L^+ w B_L^+)} \sum_{z \in W; zyz^{-1} \in W_L} \mathrm{tr}(u', R^1_{zyz^{-1},L}) \\
&= \sharp(W_L)^{-1} \sharp(U^F) \sum_{z \in W; zyz^{-1} \in W_L} a_{w,zyz^{-1};L}.
\end{aligned} \tag{1.17}$$

Let $\mathcal{R}_{W_L}$ be the Grothendieck group of W_L-modules over $\bar{\mathbf{Q}}_l$. We define $\zeta_{w,L,q} \in \mathcal{R}_{W_L}$ as in Proposition 1.3 in terms of L instead of G. Using (1.17) we have

$$\mathrm{tr}(y, \zeta_{w,q}) = \sharp(W_L)^{-1} \sharp(U^F) \sum_{z \in W; zyz^{-1} \in W_L} \mathrm{tr}(zyz^{-1}, \zeta_{w,L,q}),$$

that is,

$$\zeta_{w,q} = \sharp(U^F)\, \mathrm{ind}_{W_L}^W(\zeta_{w,L,q}). \tag{1.18}$$

1.11. Let $C \in \underline{W}$. For $w \in C_{min}, w' \in C_{min}$ we show:

$$\zeta_{w,q} = \zeta_{w',q}. \tag{1.19}$$

It is enough to show that for any $y \in W$ we have $a_{w,y} = a_{w',y}$. We write (1.5) for w, y and for w', y. We see that it is enough to show that $\mathrm{tr}(T_w, \mathcal{E}) = \mathrm{tr}(T_{w'}, \mathcal{E})$ for any $\mathcal{E} \in \mathrm{Irr}(\mathcal{H})$; this follows from results in [**6**].

1.12. Let $C \in \underline{W}$. Let $m(C)$ be the multiplicity of the eigenvalue 1 of w in ρ (see §1.8) for some/any $w \in C$. We define an element $\underline{\zeta}_{C,q} \in \mathbf{Q} \otimes \mathcal{R}_W$ by

$$\underline{\zeta}_{C,q} = \zeta_{w,q}/(q^\nu (q-1)^{r-m(C)})$$

where $w \in C_{min}$. This is independent of the choice of w by 1.11. We show that

$$\underline{\zeta}_{C,q} \in \mathcal{R}_W. \tag{1.20}$$

If $m(C) = 0$ (so that $C \in \underline{W}_{el}$) this follows from Proposition 1.9. Assume now that $m(C) > 0$. In this case C_{min} contains an element w which is contained and elliptic in W_L where P, U, L, W_L are as in §1.10 and $P \neq G$. By Proposition 1.9 for L instead of G we have $\zeta_{w,L,q}/(q^{\nu-\omega}(q-1)^{r-m(C)}) \in \mathcal{R}_{W_L}$ where $\sharp(U) = q^\omega$. It follows that $\mathrm{ind}_{W_L}^W(\zeta_{w,L,q}/(q^{\nu-\omega}(q-1)^{r-m(C)})) \in \mathcal{R}_W$. Using (1.18) we deduce $\zeta_{w,q}/(q^\omega q^{\nu-\omega}(q-1)^{r-m(C)}) \in \mathcal{R}_W$. This proves (1.20).

1.13. Let $C \in \underline{W}$. From Proposition 1.9 and the definitions we see that there is a well defined element $\Psi(C) \in \mathbf{Z}[\mathbf{q}] \otimes \mathcal{R}_W$ such that for any $s = 1, 2, \dots,$ the specialization $\Psi(C)|_{\mathbf{q}=p^s} \in \mathcal{R}_W$ is equal to $\underline{\zeta}_{C,p^s}$ as in §1.12. We can write $\Psi(C) = \sum_{i \ge 0} \mathbf{q}^i \Psi_i(C)$ where $\Psi_i(C) \in \mathcal{R}_W$ are zero for $i \gg 0$. Thus we have a map $\Psi : \underline{W} \to \mathbf{Z}[\mathbf{q}] \otimes \mathcal{R}_W$ and maps $\Psi_i : \underline{W} \to \mathcal{R}_W$.

2. Properties of the map Ψ

2.1. Let $w \in W, E \in \mathrm{Irr}(W)$. We show:

(2.1) *$|\mathcal{B}^F|\zeta_{w,E,q}$ is a polynomial in q of degree $\le |w| + 2\nu$;*

(2.2) *$|\mathcal{B}^F|\zeta_{w,1,q}$ is a monic polynomial in q of degree $|w| + 2\nu$.*

From (1.8) and the definitions, for any $E' \in \mathrm{Irr}(W)$,

(2.3) *$c_{w,E',q}$ is a polynomial in q of degree $\le |w|$.*

Moreover, we have

$$c_{w,1,q} = \sharp(W)^{-1} \sum_{\mathcal{E} \in \mathrm{Irr}(\mathcal{H})} \sum_{z \in W} (\mathcal{F}_{\mathcal{E}} : R_z^1)_{G^F} \operatorname{tr}(T_w, \mathcal{E})$$
$$= \sum_{\mathcal{E} \in \mathrm{Irr}(\mathcal{H})} (\mathcal{F}_{\mathcal{E}} : 1)_{G^F} \operatorname{tr}(T_w, \mathcal{E}) = \operatorname{tr}(T_1, \mathcal{E}_1) = q^{|w|} \tag{2.4}$$

where $\mathcal{E}_1 \in \mathrm{Irr}(\mathcal{H})$ is such that T_y acts on it as $q^{|y|}$ for any $y \in W$.

For any $\gamma \in \underline{\mathcal{U}}$ and any $E' \in \mathrm{Irr}(W)$ we consider the sum

$$M_{E,E',\gamma} = \sum_{u \in \gamma^F} \operatorname{tr}(F, H^*(\mathcal{B}_u)_{E'}) \operatorname{tr}(F, H^*(\mathcal{B}_u)_E).$$

It is known that

(2.5) *This is a polynomial in q of degree $\le 2 \dim \mathcal{B}_u + \dim \gamma = 2\nu$ (where $u \in \gamma$) with the inequality being strict unless E, E' appear in the top cohomology of $\mathcal{B}_u, u \in \gamma$.*

We have

$$\sharp(\mathcal{B}^F)\zeta_{w,q;E} = \sum_{E' \in \mathrm{Irr}(W)} c_{w,E',q} \sum_{\gamma \in \underline{\mathcal{U}}} M_{E,E',\gamma}.$$

(See (1.11).) Using this together with (2.5) and (2.3) we see that (2.1) holds. Now assume that $E = 1$. If γ is not the regular unipotent class then E does not appear in the top cohomology of $\mathcal{B}_u, u \in \gamma$ hence by (2.5), $M_{E,E',\gamma}$ is a polynomial in q of degree $< 2\nu$. If γ is the regular unipotent class and $E' \ne 1$ then $M_{E,E',\gamma} = 0$. If γ is the regular unipotent class and $E' = 1$ then $M_{E,E',\gamma} = |\gamma^F|$ is a monic polynomial in q of degree 2ν. Combining this with (2.4) we see that (2.2) holds.

Now let $C \in \underline{W}$ and let $w \in C_{min}$. From (2.1) and (2.2) we deduce

(2.6) *If $i > |w| - (r - m(C))$ then $\Psi_i(C) = 0$.*

(2.7) *If $i = |w| - (r - m(C))$ then $\Psi_i(C) \ne 0$; more precisely, the multiplicity of 1 in $\Psi_i(C)$ is 1.*

Note that $|w| - (r - m(C))$ is even.

We make the following conjecture.

2.2. CONJECTURE. *If $i, i' \in \mathbf{N}$ satisfy $i + i' = |w| - (r - m(C))$, then $\Psi_i(C) = \Psi_{i'}(C)$. In particular the multiplicity of* 1 *in $\Psi_0(C)$ is* 1.

This is supported by the examples in §3.

2.3. For $\gamma \in \underline{\mathcal{U}}$ we set $d(\gamma) = \dim \mathcal{B}_u$ where $u \in \gamma$. According to Springer, for any $E \in \mathrm{Irr}(W)$ there is a unique $\gamma \in \underline{\mathcal{U}}$ such that $H^{2d(\gamma)}(\mathcal{B}_u)_E \neq 0$ for some/any $u \in \gamma$; we set $\Xi(E) = \gamma$, $d'(E) = d(\gamma)$. The map $\Xi : \mathrm{Irr}(W) \to \underline{\mathcal{U}}$ is surjective. We shall need the following property of Ξ.

(2.8) *Let $\gamma \in \underline{\mathcal{U}}$, $u \in \gamma$ and let $E \in \mathrm{Irr}(W)$ be such that $H^i(\mathcal{B}_u)_E \neq 0$ for some i. Then $d'(E) = d(\Xi(E)) \leq d(\gamma)$. If in addition $i < 2d(\gamma)$ then $d'(E) < d(\gamma)$.*

This follows from (1.12).

2.4. Let $\Phi : \underline{W} \to \underline{\mathcal{U}}$ be the (surjective) map defined in [**12**]. By definition, if $C \in \underline{W}$ and $w \in C_{min}$, then $\Phi(C) \cap (B^+wB^+) \neq \emptyset$; moreover, if $\gamma' \in \underline{\mathcal{U}}$ satisfies $\gamma' \cap (B^+wB^+) \neq \emptyset$ then $\Phi(C)$ is contained in the closure of γ' in $\mathcal{U}$.

Let $C \in \underline{W}$ and let $\gamma = \Phi(C)$. Let $w \in C_{min}$. We show:

(2.9) *If $E \in \mathrm{Irr}(W)$ appears in $\underline{\zeta}_{C,q}$ then $d'(E) \leq d(\gamma)$.*

From (1.3) we see that $H^i(\mathcal{B}_u)_E \neq 0$ for some $u \in \mathcal{U} \cap (B^+wB^+)$. Now (2.9) follows from (2.8).

We show:

(2.10) *If $E \in \mathrm{Irr}(W)$ appears in $\underline{\zeta}_{C,q}$ and $\Xi(E) \neq \gamma$ then $d'(E) < d(\gamma)$.*

From (1.3) we see that $H^i(\mathcal{B}_u)_E \neq 0$ for some $\gamma' \in \underline{\mathcal{U}}$ and some $u \in \gamma' \cap (B^+wB^+)$. If $i < 2d(\gamma')$ then by (2.8) we have $d'(E) < d(\gamma')$. Using (2.9) we deduce $d(\gamma) < d(\gamma')$; but by the definition of Φ, γ is contained in the closure of γ' so that $d(\gamma) \geq d(\gamma')$, a contradiction. If $i = 2d(\gamma')$ then $\Xi(E) = \gamma'$ hence $d'(E) = d(\gamma')$. Since γ is contained in the closure of γ' and $\gamma' \neq \gamma$ we have $d(\gamma') < d(\gamma)$ hence $d'(E) < d(\gamma)$. This proves (2.10).

(2.11) *If $E_0 \in \mathrm{Irr}(W)$ is the part of $H^{2d(\gamma)}(\mathcal{B}_{u_0})$ which is fixed by the action of the centralizer of u_0 ($u_0 \in \gamma$) then E_0 appears in $\underline{\zeta}_{C,q}$.*

We must show that $\sum_{u \in \mathcal{U}^F \cap (B^+wB^+)} \mathrm{tr}(F, H^*(\mathcal{B}_u)_{E_0}) \neq 0$. Assume that

$$\sum_{u \in \mathcal{U}^F \cap (B^+wB^+)} \mathrm{tr}(F, H^*(\mathcal{B}_u)_{E_0}) = 0. \tag{2.12}$$

If $\gamma' \in \underline{\mathcal{U}}$ is such that γ is contained in the closure of γ' and $\gamma \neq \gamma'$ then $H^i(\mathcal{B}_u)_{E_0} = 0$ for all i hence from (2.12) we deduce $\sum_{u \in \gamma^F \cap (B^+wB^+)} \mathrm{tr}(F, H^*(\mathcal{B}_u)_{E_0}) = 0$. It follows that $\sharp(u \in \gamma^F \cap (B^+wB^+))q^{d(\gamma)} = 0$. This contradicts $\gamma^F \cap (B^+wB^+) \neq \emptyset$. This proves (2.11).

2.5. Let $Z = \Psi(\underline{W}) \subset \mathbf{Z}[\mathbf{q}] \otimes \mathcal{R}_W$. For $\xi \in Z$ there is a unique $\gamma \in \underline{\mathcal{U}}$ such that the following holds:

If $E \in \mathrm{Irr}(W)$ appears in ξ then $d'(E) \leq d(\gamma)$. If $E \in \mathrm{Irr}(W)$ appears in ξ and $\Xi(E) \neq \gamma$ then $d'(E) < d(\gamma)$. If $E_0 \in \mathrm{Irr}(W)$ is the part of $H^{2d(\gamma)}(\mathcal{B}_{u_0})$ which is fixed by the action of the centralizer of u_0 ($u_0 \in \gamma$), then E_0 appears in ξ.

The existence of γ follows from §2.4; the uniqueness is obvious. Thus $E \mapsto \gamma$ is a well defined map $\Theta : Z \to \underline{\mathcal{U}}$. From §2.4 we have that $\Phi(C) = \Theta(\Psi(C))$ for any $C \in \underline{W}$. Since Φ is surjective, we see that Θ is surjective.

2.6. Let $G_{ad} = G/Z_G$. From (1.2) for any $u \in \mathcal{U}^F$ and $E \in \mathrm{Irr}(W)$ we have

$$\mathrm{tr}(F, H^*(\mathcal{B}_u)_E) = \sharp(W)^{-1} \sum_{y \in W} \mathrm{tr}(y, E)\,\mathrm{tr}(u, R_y^1). \tag{2.13}$$

For y, y' in W we have

$$\sum_{u \in \mathcal{U}^F} \mathrm{tr}(u, R_y^1)\,\mathrm{tr}(u, R_{y'}^1) = \sharp(z \in W; zyz^{-1} = y')\det(q - y, \rho)^{-1}\sharp(G_{ad}^F),$$

see [**4**, Theorem 6.9]. Using this and (2.13) we obtain

$$\begin{aligned} &\sum_{u \in \mathcal{U}^F} \mathrm{tr}(F, H^*(\mathcal{B}_u)_{E'})\,\mathrm{tr}(F, H^*(\mathcal{B}_u)_E) \\ &= \sharp(W)^{-2} \sum_{y \in W, z \in W} \mathrm{tr}(y, E')\,\mathrm{tr}(zyz^{-1}, E)\det(q - y, \rho)^{-1}\sharp(G_{ad}^F) \\ &= \sharp(W)^{-1} \sum_{y \in W} \mathrm{tr}(y, E \otimes E')\det(q - y, \rho)^{-1}\sharp(G_{ad}^F). \end{aligned} \tag{2.14}$$

Using this and (1.11) we see that if $C \in \underline{W}$, $w \in C_{min}$ then the coefficient of $E \in \mathrm{Irr}(W)$ in $\underline{\zeta}_{C,q}$ is

$$\sum_{\mathcal{E} \in \mathrm{Irr}(\mathcal{H}), E' \in \mathrm{Irr}(W)} A_{C,\mathcal{E}} A'_{\mathcal{E},E'} A''_{E',E}, \tag{2.15}$$

where

$$\begin{aligned} A_{C,\mathcal{E}} &= (q-1)^{m(C)}\,\mathrm{tr}(T_w, \mathcal{E}), \\ A'_{\mathcal{E},E'} &= \sharp(W)^{-1} \sum_{z \in W} (\mathcal{F}_{\mathcal{E}} : R_z^1)_{G^F}\,\mathrm{tr}(z, E'), \\ A''_{E',E} &= \sharp(W)^{-1} \sum_{y \in W} \mathrm{tr}(y, E \otimes E')\det(q - y, \rho)^{-1}. \end{aligned}$$

Thus this coefficient is an entry of a product of three square matrices of size $\sharp(\underline{W}) = \sharp(\mathrm{Irr}(W))$. The matrices $(A_{C,\mathcal{E}}), (A''_{E',E})$ are known to be invertible when regarded as matrices with entries in $\mathbf{Q}$. We see that

(2.16) *$\{\underline{\zeta}_{C,q}; C \in \underline{W}\}$ is a basis of $\mathbf{Q} \otimes \mathcal{R}_W$ if and only if the matrix $(A'_{\mathcal{E},E'})$ (with entries independent of q) is invertible.*

The entries of the last matrix are some of the entries of the nonabelian Fourier transform [**9**] for the various two-sided cells of W. This matrix is not necessarily invertible. For example if W is of type B_n or C_n this matrix is invertible if $n \le 11$ but is not invertible if $n = 12$. Also if W is of type E_6 this matrix is not invertible.

2.7. Proposition. *For any $y \in W$ we have*

$$a_{w,y} = \sum_{\mathcal{E} \in \mathrm{Irr}(W)} (-1)^{|y|}(\mathcal{F}_{\mathcal{E}} : \dim(R_y^1)R_y^1)\,\mathrm{tr}(T_w, \mathcal{E})q^{\nu}\sharp(\mathcal{B}^F)^{-1}.$$

PROOF. Combining (1.7),(1.11) and (2.14), we obtain the identity

$$\sharp(\mathcal{B}^F)a_{w,y} = \sharp(W)^{-2} \sum_{E,E',\mathcal{E}\in\mathrm{Irr}(W)} \sum_{z,y'\in W}$$
$$(\mathcal{F}_{\mathcal{E}} : R_z^1)\,\mathrm{tr}(z,E')\,\mathrm{tr}(T_w,\mathcal{E})\,\mathrm{tr}(y',E)\,\mathrm{tr}(y',E')\det(q-y',\rho)^{-1}\sharp(G_{ad}^F)\,\mathrm{tr}(y,E),$$

where $(\mathcal{F}_{\mathcal{E}} :?)$ is the multiplicity of $\mathcal{F}_{\mathcal{E}}$ in the virtual representation ? of G^F, ρ is the reflection representation of W and G_{ad} is the adjoint group of G.

We now replace $\sum_{E'\in\mathrm{Irr}(W)}\mathrm{tr}(z,E')\,\mathrm{tr}(y',E')$ by $\sharp(e\in W; eze^{-1}=y')$ and we obtain

$$\sharp(\mathcal{B}^F)a_{w,y} = \sharp(W)^{-1} \sum_{E,\mathcal{E}\in\mathrm{Irr}(W)} \sum_{z\in W}$$
$$(\mathcal{F}_{\mathcal{E}} : R_z^1)\,\mathrm{tr}(T_w,\mathcal{E})\,\mathrm{tr}(z,E)\det(q-z,\rho)^{-1}\sharp(G_{ad}^F)\,\mathrm{tr}(y,E).$$

We now replace $\sum_{E\in\mathrm{Irr}(W)}\mathrm{tr}(z,E)\,\mathrm{tr}(y,E)$ by $\sharp(e'\in W; e'ze'^{-1}=y)$ and we obtain

$$a_{w,y} = \sum_{\mathcal{E}\in\mathrm{Irr}(W)} (\mathcal{F}_{\mathcal{E}} : R_y^1)\,\mathrm{tr}(T_w,\mathcal{E})\det(q-y,\rho)^{-1}\sharp(G_{ad}^F)\sharp(\mathcal{B}^F)^{-1}.$$

We now replace $\det(q-y,\rho)^{-1}\sharp(G_{ad}^F)$ by $\dim(R_y^1)q^\nu(-1)^{|y|}$ and we obtain the desired identity. □

2.8. In this subsection we diverge from the setup of §1.1; we assume instead that W is a finite Coxeter group. Then all ingredients of (2.13) make sense. The (constant) matrix $(A'_{\mathcal{E},E'})$ makes sense as a matrix whose entries involve the appropriate generalization of the nonabelian Fourier transform. Hence the map Ψ can be defined in this generality (although the ring of coefficients $\mathbf{Z}$ may have to be increased).

3. Examples

3.1. We return to the setup in §1.1. We shall denote by sgn the sign representation of W, by 1 the unit representation of W. Let ρ be as in §1.8. In this section we shall sometime write w instead of C when w is an element of $C\in\underline{W}$.

Assume first that W is of type A_1. The elements of $\underline{W}$ are represented by $1,s$ where s is the simple reflection. The objects of $\mathrm{Irr}(W)$ are $1,\rho=\mathrm{sgn}$. We have

$\Psi(1)=1+\mathrm{sgn}$,
$\Psi(s)=1$.

3.2. We now assume that W is of type A_2. The elements of $\underline{W}$ are represented by $1,s_1,c$ where s_1,s_2 are the simple reflections and $c=s_1s_2$. The objects of $\mathrm{Irr}(W)$ are $1,\rho,\mathrm{sgn}$. We have

$\Psi(1)=1+2\rho+\mathrm{sgn}$,
$\Psi(s_1)=1+\rho$,
$\Psi(c)=1$.

3.3. We now assume that W is of type B_2. The elements of $\underline{W}$ are represented by $1,s_1,s_2,c,c^2$ where s_1,s_2 are the simple reflections and $c=s_1s_2$. The objects of $\mathrm{Irr}(W)$ are $1,\rho,\epsilon',\epsilon'',\mathrm{sgn}$ where ϵ',ϵ'' are the one dimensional representations other than $1,\mathrm{sgn}$. The following result was obtained by making use of [**13**]. We can arrange notation so that:

$\Psi(1) = 1 + 2\rho + \epsilon' + \epsilon'' + \mathrm{sgn}$,
$\Psi(s_1) = 1 + \rho + \epsilon'$,
$\Psi(s_2) = 1 + \rho + \epsilon''$,
$\Psi(c) = 1$,
$\Psi(c^2) = \mathbf{q}^2 1 + \mathbf{q}\rho + 1$.

3.4. We now assume that W is of type G_2. The elements of $\underline{W}$ are represented by $1, s_1, s_2, c, c^2, c^3$ where s_1, s_2 are the simple reflections and $c = s_1 s_2$. The objects of $\mathrm{Irr}(W)$ are $1, \rho, \epsilon', \epsilon'', \rho' = \rho \otimes \epsilon' = \rho \otimes \epsilon''$, sgn where ϵ', e'' are the one dimensional representations other than 1, sgn. The following result was obtained by making use of [**2**]. We can arrange notation so that:

$\Psi(1) = 1 + 2\rho + 2\rho' + \epsilon' + \epsilon'' + \mathrm{sgn}$,
$\Psi(s_1) = 1 + \rho + \rho' + \epsilon'$,
$\Psi(s_2) = 1 + \rho + \rho' + \epsilon''$,
$\Psi(c) = 1$,
$\Psi(c^2) = \mathbf{q}^2 1 + \mathbf{q}\rho + 1$,
$\Psi(c^3) = \mathbf{q}^4 1 + \mathbf{q}^3\rho + \mathbf{q}^2(\rho' + 1) + \mathbf{q}\rho + 1$.

3.5. In the examples above, $\Psi_i(C)$ is always an actual representation of W. This is not so in higher rank. We have calculated $\Psi(C)$ for G of type F_4 on a computer using the formulas in §2.6, by the same method as in [**12**, §1.2]; we thank Gongqin Li for programming this in GAP. If C consists of the longest element in W of type F_4 and E is a one-dimensional representation of W other than 1, sgn then the coefficient of E in $\Phi(C)$ is $-2\mathbf{q}^{10}$; thus, $\Phi_{10}(C)$ is not an actual representation of W. A similar thing happens in type B_3 and C_3.

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