18.706 HOMEWORK 10

DUE NOV. 18, 2020

Theorems

In this part you will see theorems that we studied in the class. Please supply a detailed proof of each.

Theorem 1. Let \( R \) be a central simple algebra of finite dimension over a field \( k \). Let \( S \) be a simple \( k \)-algebra and \( \varphi_1, \varphi_2 : S \to R \) be two \( k \)-linear ring homomorphisms. Then there exists an invertible element \( u \in R \) such that \( \varphi_2(s) = u\varphi_1(s)u^{-1} \) for all \( s \in S \).

Exercises

Problem 1. Let \( k \) be a field with \( \text{char}(k) \neq 2 \). For \( a, b \in k^\times \) let \( \left( \frac{a,b}{k} \right) \) denote the cyclic algebra of dimension 4 over \( k \) given by

\[
\left( \frac{a,b}{k} \right) = k\langle x, y \rangle/(x^2 - a, y^2 - b, xy + yx).
\]

1. Show that every 4-dimensional central simple algebra over \( k \) is isomorphic to \( \left( \frac{a,b}{k} \right) \) for some \( a, b \in k^\times \).
2. Show that \( \left( \frac{a,b}{k} \right) \cong M_2(k) \) if and only if \( u^2 - bv^2 = a \) has a solution \((u,v) \in k^2 \). In particular, show that \( \left( \frac{a,1-a}{k} \right) \) is isomorphic to \( M_2(k) \).
3. Let \( k(\sqrt{c}) \) be a quadratic extension of \( k \). When does \( k(\sqrt{c}) \) embed into \( \left( \frac{a,b}{k} \right) \)?
4. Show that \( \left( \frac{a,b}{k} \right) \otimes_k \left( \frac{a,c}{k} \right) \cong \left( \frac{a,bc}{k} \right) \otimes M_2(k) \). Hence in the Brauer group \( Br(k) \), the sum of the classes of \( \left( \frac{a,b}{k} \right) \) and \( \left( \frac{a,c}{k} \right) \) is the class of \( \left( \frac{a,bc}{k} \right) \).

Hint: construct a module of \( \left( \frac{a,b}{k} \right) \otimes_k \left( \frac{a,c}{k} \right) \) that is 8-dimensional over \( k \), and identify its endomorphism algebra with \( \left( \frac{a,bc}{k} \right) \).