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Global Springer theory

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Abstract

We generalize Springer representations to the context of groups over a global function field. The global counterpart of the Grothendieck simultaneous resolution is the parabolic Hitchin fibration. We construct an action of the graded double affine Hecke algebra (DAHA) on the direct image complex of the parabolic Hitchin fibration. In particular, we get representations of the degenerate graded DAHA on the cohomology of parabolic Hitchin fibers, providing the first step towards a global Springer theory.

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1. Introduction

1.1. Springer theories: classical, local and global

In this subsection, we put the content of the paper in the appropriate historical context by briefly reviewing the classical and local versions of the Springer theories.

The classical Springer theory originated from Springer's study of Green functions for finite groups of Lie type (see [30]). Let G be a reductive group over an algebraically closed field k with Lie algebra $\mathfrak{g}$, and let $\mathcal{B}$ be the flag variety classifying Borel subgroups of G. Let $\tilde{\mathfrak{g}}$ be the scheme classifying pairs (γ, B) where $\gamma \in \mathfrak{g}$, $B \in \mathcal{B}$ such that $\gamma \in \operatorname{Lie} B$. Forgetting the choice of B we get the so-called *Grothendieck simultaneous resolution*:

$$\pi : \tilde{\mathfrak{g}} \to \mathfrak{g}. \tag{1.1}$$

For an element $\gamma \in \mathfrak{g}$, the fiber $\mathcal{B}_\gamma = \pi^{-1}(\gamma)$ is called the *Springer fiber* of γ. These are closed subschemes of the flag variety $\mathcal{B}$. If γ is regular semisimple, $\mathcal{B}_\gamma$ is simply a torsor under the Weyl group W; in general, $\mathcal{B}_\gamma$ has higher dimensions and singularities. Springer constructed representations of the Weyl group W of G on the top-dimensional cohomology of $\mathcal{B}_\gamma$.

Note that W does *not* act on the variety $\mathcal{B}_\gamma$, therefore the existence of the W-action on $\mathrm{H}^*(\mathcal{B}_\gamma)$ is quite mysterious. Later, a sheaf-theoretic construction of Springer representations was given by Lusztig [24]: he constructs an action of W on the shifted perverse sheaf $\mathbf{R}\pi_*\overline{\mathbb{Q}}_\ell$ on $\mathfrak{g}$, hence incorporating Springer representations into a family. This approach was further developed by Borho and MacPherson [5], and they showed that all irreducible representations of W arise as Springer representations. There are other constructions of Springer representations by Kazhdan and Lusztig [19] using the Coxeter presentation of W, and by Chriss and Ginzburg [9] using Steinberg correspondences. In [25], Lusztig extended the W action on the sheaf $\mathbf{R}\pi_*\overline{\mathbb{Q}}_\ell$ to an action of the graded affine Hecke algebra.

Since the classical Springer theory is related to the representation theory of $G(\mathbb{F}_q)$, we can think of it as a theory "over $\operatorname{Spec}\mathbb{F}_q$". It is then natural to ask whether there are corresponding theories over a local field such as $k((t))$, or over a global field such as the function field of an algebraic curve over a field k.

A local theory (for the local function field $F = k((t))$) already exists, by work of Lusztig [26]. In the local theory, the loop group $G((t))$ replaces the group G, the (extended) affine Weyl group replaces the finite Weyl group W, and *affine Springer fibers* (cf. [20]) replace Springer fibers. For an element $\gamma \in \mathfrak{g} \otimes k((t))$, the affine Springer fiber M_γ is the closed sub-ind-scheme of the *affine flag variety* Fl_G parametrizing Iwahori subgroups $\mathbf{I} \subset G((t))$ such that $\gamma \in \operatorname{Lie}\mathbf{I}$. Lusztig's construction works for individual affine Springer fibers. The work of Vasserot [33] and Varagnolo and Vasserot [32] extends this to an action of the double affine Hecke algebra (DAHA) on the homology of affine Springer fibers. However, there is not yet a sheaf-theoretic approach that organizes affine Springer fibers into geometrically manageable families. An essential difficulty is that the parameter space for affine Springer fibers has both infinite dimension and infinite codimension in $\mathfrak{g} \otimes k((t))$.

The goal of this paper is to give a sheaf-theoretic construction of a *global* Springer theory. We will start with a complete smooth connected curve X over k and a reductive group G over k (in fact, our results easily extend to quasi-split group schemes over X). The global analog of the Grothendieck simultaneous resolution π is the *parabolic Hitchin fibration* (see Definition 2.1.1)

$$f^{\mathrm{par}} : \mathcal{M}^{\mathrm{par}} \to \mathcal{A}^{\mathrm{Hit}} \times X.$$

This is a modification of the usual Hitchin fibration [16, §4] by adding a Borel reduction at one point of X. The fibers of this map, called the *parabolic Hitchin fibers*, are the global analogs of the Springer fibers $\mathcal{B}_\gamma$. Over an open subset of $\mathcal{A}^{\mathrm{Hit}} \times X$, the parabolic Hitchin fibers are isomorphic to disjoint unions of abelian varieties. However, in general, the singularities of the parabolic Hitchin fibers are as complicated as affine Springer fibers (this can be made precise by the "product formula", see Section 2.4 and [29, §4.15]).

We will first construct an action of the extended affine Weyl group $\widetilde{W} = \mathbb{X}_*(T) \rtimes W$ on the *parabolic Hitchin complex* $\mathbf{R}f^{\text{par}}_*\overline{\mathbb{Q}}_\ell$. This is the global analog of Springer and Lusztig's W-action on $\mathbf{R}\pi_*\overline{\mathbb{Q}}_\ell$. In particular, taking stalkwise actions, we get $\widetilde{W}$-actions on the cohomology groups of parabolic Hitchin fibers, which we call *global Springer representations*. Note that in general, $\widetilde{W}$ does not act on the parabolic Hitchin fibers, the existence of this action is as mysterious as the classical Springer action.

Next, we will extend the $\widetilde{W}$-action into an action of the *graded double affine Hecke algebra* $\mathbb{H}$ on the parabolic Hitchin complex. This result is the global analog of Lusztig's main result in [25]. Here the extra ingredient is provided by certain natural line bundles on $\mathcal{M}^{\text{par}}$: their Chern classes act on $\mathbf{R}f^{\text{par}}_*\overline{\mathbb{Q}}_\ell$ by degree 2 endomorphisms. The Chern class action and the previously-defined $\widetilde{W}$-action satisfy certain commutation relations to make up an $\mathbb{H}$-action. Stalkwise we get $\mathbb{H}$-actions on the cohomology groups of parabolic Hitchin fibers, which gives a natural geometric construction of representations of the (degenerate) graded DAHA.

Global Springer theory carries a richer symmetry than classical or local Springer theory. There are at least three pieces of symmetry acting on the parabolic Hitchin fibration $f^{\text{par}} : \mathcal{M}^{\text{par}} \to \mathcal{A} \times X$: the first is the affine Weyl group action on $\mathbf{R}f^{\text{par}}_*\overline{\mathbb{Q}}_\ell$ to be constructed in this paper; the second is the *cup product* of Chern classes of certain line bundles on $\mathcal{M}^{\text{par}}$; the third is the action of a Picard stack $\mathcal{P}$ on $\mathcal{M}^{\text{par}}$ (see Section 2.3).

Global Springer theory, besides being an analog of the classical and local Springer theories, is also inspired by B.-C. Ngô's recent proof of the Fundamental Lemma ([29], see also [28]). In [29], he studies the relation between the *Hitchin fibration* $f^{\text{Hit}} : \mathcal{M}^{\text{Hit}} \to \mathcal{A}^{\text{Hit}}$, affine Springer fibers and orbital integrals. In particular, Ngô's product formula [29, §4.15] makes it clear that usual Hitchin fibers are the global analogs of affine Springer fibers in the affine Grassmannian. We push this analogy further to the case of parabolic Hitchin fibers and affine Springer fibers in the affine flag variety, which carry more symmetry than Ngô's situation.

1.2. Main results

Let X be a projective smooth connected curve over an algebraically closed field k. Let G be a reductive group over k. Fix a Borel subgroup B of G. Fix a divisor D on X with $\deg(D) \geqslant 2g_X$ (g_X is the genus of X). The *parabolic Hitchin moduli stack* $\mathcal{M}^{\text{par}} = \mathcal{M}^{\text{par}}_{G,X,D}$ classifies quadruples $(x, \mathcal{E}, \varphi, \mathcal{E}^B_x)$, where x is a point on X, $\mathcal{E}$ is a G-torsor over X, φ is a global section of the vector bundle $\text{Ad}(\mathcal{E}) \otimes \mathcal{O}_X(D)$ on X, and $\mathcal{E}^B_x$ is a B-reduction of the restriction of $\mathcal{E}$ at x compatible with φ. For a concrete description in the case of $G = \text{GL}(n)$, see Example 2.2.5.

Let T be the quotient torus of B (the universal Cartan) and $\mathfrak{t}$ be its Lie algebra. Let $\mathfrak{c} = \mathfrak{t} /\!/ W = \mathfrak{g} /\!/ G = \operatorname{Spec} k[f_1, \dots, f_n]$ be the GIT adjoint quotient, with fundamental invariants $f_1, \dots, f_n$ of degree $d_1, \dots, d_n$. Let $\mathcal{A}^{\text{Hit}} = \mathcal{A}^{\text{Hit}}_{G,X,D}$ be the *Hitchin base*:

$$\mathcal{A}^{\text{Hit}} = \bigoplus_{i=1}^{n} \mathrm{H}^0\big(X, \mathcal{O}_X(d_i D)\big).$$

The morphism

$$f^{\text{par}} : \mathcal{M}^{\text{par}} \to \mathcal{A}^{\text{Hit}} \times X,$$

$$\left(x,\mathcal{E},\varphi,\mathcal{E}^B_x\right)\mapsto\left(f_1(\varphi),\dots,f_n(\varphi),x\right)$$

is called the *parabolic Hitchin fibration*. The direct image complex $\mathbf{R}f^{\mathrm{par}}_*\overline{\mathbb{Q}}_\ell$ of the constant sheaf $\overline{\mathbb{Q}}_\ell$ under f^{par} is called the *parabolic Hitchin complex*.

In the case of $\mathrm{GL}(n)$, the fibers of f^{par} can be described in terms of the compactified Picard stack of the spectral curves, see Example 2.2.5. In general, the fibers of f^{par} (called the *parabolic Hitchin fibers*) are mixtures of abelian varieties and affine Springer fibers (see the product formula Proposition 2.4.1).

Let $\mathcal{A}=\mathcal{A}^{\mathrm{ani}}\subset\mathcal{A}^{\mathrm{Hit}}$ be the anisotropic locus (see [29, §6.1] and Definition 2.3.6 below). In Section 2.5, we show that $\mathcal{M}^{\mathrm{par}}|_{\mathcal{A}}$ is a smooth Deligne–Mumford stack and f^{par} is a proper morphism over $\mathcal{A}\times X$. Our first main result will justify the phrase "global Springer theory" in the title.

Theorem A. *(See Theorem 3.3.3 and 5.1.2 for two constructions.) There is a natural $\widetilde{W}$-action on the complex $\mathbf{R}f^{\mathrm{par}}_*\overline{\mathbb{Q}}_\ell\in D^b(\mathcal{A}\times X)$. Here $\widetilde{W}=\mathbb{X}_*(T)\rtimes W$ is the extended affine Weyl group of G.*

In particular, taking the stalk of $\mathbf{R}f^{\mathrm{par}}_*\overline{\mathbb{Q}}_\ell$ at each geometric point $(a,x)\in\mathcal{A}\times X$, we get an action of $\widetilde{W}$ on $\mathrm{H}^*(\mathcal{M}^{\mathrm{par}}_{a,x})$. This action is the global analog of Springer representations.

In the next main result, we extend the $\widetilde{W}$-action on $\mathbf{R}f^{\mathrm{par}}_*\overline{\mathbb{Q}}_\ell$ to an action of a bigger algebra. Assume G is almost simple. We can write $\widetilde{W}=W_{\mathrm{aff}}\rtimes\Omega$, where W_{aff} is an affine Coxeter group with simple reflections $\{s_0,s_1,\dots,s_n\}$, and the finite group Ω is the stabilizer of the fundamental alcove. Let $\widetilde{T}=\mathbb{G}^{\mathrm{cen}}_m\times T\times\mathbb{G}^{\mathrm{rot}}_m$ be the Cartan torus of the affine Kac–Moody group associated to G (see Section 6.2), where $\mathbb{G}^{\mathrm{cen}}_m$ is the one-dimensional central torus and $\mathbb{G}^{\mathrm{rot}}_m$ is the one-dimensional "loop rotation" torus. Let $\delta\in\mathbb{X}^*(\mathbb{G}^{\mathrm{rot}}_m)$ and $\Lambda_0\in\mathbb{X}^*(\mathbb{G}^{\mathrm{cen}}_m)_{\mathbb{Q}}$ be the generators (here we are using Kac's notation for affine Kac–Moody groups, see [17, 6.5]).

We recall the graded *double affine Hecke algebra* $\mathbb{H}$ (graded DAHA) defined by Cherednik [8, §2.12.3]. As a vector space, $\mathbb{H}$ is the tensor product of the group ring $\overline{\mathbb{Q}}_\ell[\widetilde{W}]$ with the polynomial algebra $\mathrm{Sym}_{\overline{\mathbb{Q}}_\ell}(\mathbb{X}^*(\widetilde{T})_{\overline{\mathbb{Q}}_\ell})\otimes\overline{\mathbb{Q}}_\ell[u]$. The graded algebra structure of $\mathbb{H}$ is determined by

- $\overline{\mathbb{Q}}_\ell[\widetilde{W}]$ is a subalgebra of $\mathbb{H}$ in degree 0;
- $\mathrm{Sym}_{\overline{\mathbb{Q}}_\ell}(\mathbb{X}^*(\widetilde{T})_{\overline{\mathbb{Q}}_\ell})$ is a subalgebra of $\mathbb{H}$ with $\xi\in\mathbb{X}^*(\widetilde{T})$ in degree 2;
- u has degree 2, and is central in $\mathbb{H}$;
- For each affine simple reflection s_i and $\xi\in\mathbb{X}^*(\widetilde{T})$,

$$s_i\xi-{}^{s_i}\xi s_i=\langle\xi,\alpha_i^\vee\rangle u.\tag{1.2}$$

  Here $\alpha_i^\vee\in\mathbb{X}_*(\widetilde{T})$ is the affine coroot corresponding to s_i;
- For any $\omega\in\Omega$ and $\xi\in\mathbb{X}^*(\widetilde{T})$,

$$\omega\xi={}^{\omega}\xi\omega.$$

Theorem B. *(See Theorem 6.1.6.) There is a graded algebra homomorphism*

$$\mathbb{H}\to\bigoplus_{i\in\mathbb{Z}}\mathrm{End}^{2i}_{\mathcal{A}\times X}\left(\mathbf{R}f^{\mathrm{par}}_*\overline{\mathbb{Q}}_\ell\right)(i)$$

extending the $\widetilde{W}$-action in Theorem A. *Here* $\mathrm{End}^{2i}_{\mathcal{A}\times X}(\mathbf{R}f^{\mathrm{par}}_{*}\overline{\mathbb{Q}}_\ell)(i)$ *means* $\mathrm{Hom}(\mathbf{R}f^{\mathrm{par}}_{*}\overline{\mathbb{Q}}_\ell, \mathbf{R}f^{\mathrm{par}}_{*}\overline{\mathbb{Q}}_\ell[2i](i))$ *in the derived category* $D^b(\mathcal{A}\times X)$; (i) *means Tate twist.*

This result is a global analog of Lusztig's main result in [25]. In particular, for any point $(a,x)\in\mathcal{A}\times X$, we get an action of the degenerate DAHA $\mathbb{H}/(u,\delta)$ on the cohomology $\mathrm{H}^*(\mathcal{M}^{\mathrm{par}}_{a,x})$. This gives geometric realizations of representations of the degenerate DAHA.

Our main theorems generalize to *parahoric* Hitchin fibrations $f_{\mathbf{P}}:\mathcal{M}_{\mathbf{P}}\to\mathcal{A}\times X$ (here $\mathbf{P}\subset G(k((t)))$ is a standard parahoric subgroup; for details see Section 4.3). We spell this out in the case $\mathcal{M}_{\mathbf{P}}=\mathcal{M}^{\mathrm{Hit}}\times X$ (i.e., when $\mathbf{P}=G(k[\![t]\!])$).

Theorem B′ *(Special case of Theorem 6.6.1). Let* $\mathbb{H}_{\mathrm{sph}}=\mathbf{1}_W\mathbb{H}\mathbf{1}_W$ *be the "spherical graded DAHA", where* $\mathbf{1}_W\in\overline{\mathbb{Q}}_\ell[\widetilde{W}]$ *is the characteristic function of the finite Weyl group* W. *Then there is a graded algebra homomorphism*:

$$\mathbb{H}_{\mathrm{sph}}\to\bigoplus_{i\in\mathbb{Z}}\mathrm{End}^{2i}_{\mathcal{A}\times X}\big(\mathbf{R}f^{\mathrm{Hit}}_{*}\overline{\mathbb{Q}}_\ell\boxtimes\overline{\mathbb{Q}}_{\ell,X}\big)(i).$$

In particular, $\overline{\mathbb{Q}}_\ell[\mathbb{X}_*(T)]^W$ acts on the complex $\mathbf{R}f^{\mathrm{Hit}}_{*}\overline{\mathbb{Q}}_\ell\boxtimes\overline{\mathbb{Q}}_{\ell,X}$. This gives an algebro-geometric construction of the so-called *'t Hooft operators* considered by Kapustin–Witten in their gauge-theoretic approach to the geometric Langlands program (see [18]).

Finally, we give a nontrivial example of global Springer representations in Section 7. In this example, we take $G=\mathrm{SL}(2)$, $X=\mathbb{P}^1$. We consider a "subregular" parabolic Hitchin fiber $\mathcal{M}^{\mathrm{par}}_{a,x}$ which is the union of two $\mathbb{P}^1$'s intersecting at two points. Using the basis of $\mathrm{H}^2(\mathcal{M}^{\mathrm{par}}_{a,x})$ dual to the cycle classes of the two $\mathbb{P}^1$'s, the action of $\widetilde{W}$ on $\mathrm{H}^2(\mathcal{M}^{\mathrm{par}}_{a,x})$ takes the form:

$$s_1=\begin{pmatrix}1&2\\0&-1\end{pmatrix};\qquad s_0=\begin{pmatrix}-1&0\\2&1\end{pmatrix};\qquad \alpha^\vee=\begin{pmatrix}-1&-2\\2&3\end{pmatrix}.$$

Here s_0,s_1 are the simple reflections in $\widetilde{W}$, and $\alpha^\vee=s_0s_1$ the generator of the lattice $\mathbb{X}_*(T)$. In particular, we see that the action of $\alpha^\vee$ is *unipotent* (but not identity). This is a new feature of the global Springer representations compared to the classical one. The unipotent part of the $\widetilde{W}$-action can be partially understood by using the parabolic Hitchin fibration for the Langlands dual group (see [37]).

1.3. Methods of construction

The parabolic Hitchin complex $\mathbf{R}f^{\mathrm{par}}_{*}\overline{\mathbb{Q}}_\ell$ does not lie in a single perverse degree as the Springer sheaf $\mathbf{R}\pi_*\overline{\mathbb{Q}}_\ell$ does, hence the middle extension approach of Lusztig in [24] does not apply in this situation. We give two constructions for the $\widetilde{W}$-action on $\mathbf{R}f^{\mathrm{par}}_{*}\overline{\mathbb{Q}}_\ell$.

The first construction (Theorem 3.3.3) uses Hecke correspondences, which resembles the Steinberg correspondences in classical Springer theory. For each $\tilde{w}\in\widetilde{W}$, there is a self-correspondence $\mathcal{H}_{\tilde{w}}$ of $\mathcal{M}^{\mathrm{par}}$ over $\mathcal{A}\times X$ which is generically a graph. The fundamental class $[\mathcal{H}_{\tilde{w}}]$ of $\mathcal{H}_{\tilde{w}}$ gives an endomorphism $[\mathcal{H}_{\tilde{w}}]_\#$ of $\mathbf{R}f^{\mathrm{par}}_{*}\overline{\mathbb{Q}}_\ell$ in the derived category $D^b(\mathcal{A}\times X)$. Here $[\mathcal{H}_{\tilde{w}}]_\#$ is the sheaf-theoretic version of "convolution with a kernel function", see Appendix A.1. The assignment $\tilde{w}\mapsto[\mathcal{H}_{\tilde{w}}]_\#$ gives an action of $\widetilde{W}$ on $\mathbf{R}f^{\mathrm{par}}_{*}\overline{\mathbb{Q}}_\ell$.

A key geometric fact that makes this construction work is that the Hecke correspondence is "graph-like" (see Appendix A.6). This property in turn relies on the codimension estimate of certain strata in $\mathcal{A} \times X$ (see Proposition 2.6.3), proved by Ngô [29, §5.7] on the base of the work of Goresky, Kottwitz and MacPherson [14]. For technical reason (see Remark 2.6.4), we will restrict $\mathbf{R}f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell$ to an open subset $(\mathcal{A} \times X)' \subset \mathcal{A} \times X$. If $\mathrm{char}(k) = 0$, this restriction is not necessary.

The second construction (Construction 5.1.1) uses the Coxeter presentation of $\widetilde{W}$, which resembles the construction of Lusztig in [26]. We construct the action of each affine simple reflection, and show that the braid relations hold. For this we need to introduce Hitchin moduli stacks with *parahoric* structures, depending on a parahoric subgroup $\mathbf{P} \subset G(k((t)))$. We show in Proposition 5.2.1 that the two constructions give the same $\widetilde{W}$-action on $\mathbf{R}f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell$.

The construction of the DAHA action on $\mathbf{R}f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell$ makes use of certain line bundles on $\mathcal{M}^{\mathrm{par}}$. Let $\mathrm{Bun}_G^{\mathrm{par}}$ be the moduli stack of G-torsors on X with a Borel reduction at some (varying) point. It carries a tautological T-torsor. In particular, for any $\xi \in \mathbb{X}^*(T)$, we have a line bundle $\mathcal{L}(\xi)$ on $\mathrm{Bun}_G^{\mathrm{par}}$, which we also view as a line bundle on $\mathcal{M}^{\mathrm{par}}$ via the forgetful map $\mathcal{M}^{\mathrm{par}} \to \mathrm{Bun}_G^{\mathrm{par}}$. We also have the *determinant line bundle* on $\mathcal{M}^{\mathrm{par}}$, which is (up to a power) the pull-back of the canonical bundle ω_{Bun} of Bun_G.

Recall the degree 2 generators of $\mathbb{H}$ are $u, \xi \in \mathbb{X}^*(T)$, δ and Λ_0. In Theorem B, these elements act as cup products with the Chern classes:

$$\bigcup c_1\big(\mathcal{O}_X(D)\big), \bigcup c_1\big(\mathcal{L}(\xi)\big), \bigcup c_1(\omega_X), \bigcup \frac{1}{h^\vee} c_1(\omega_{\mathrm{Bun}}) : \mathbf{R}f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell \to \mathbf{R}f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell[2](1)$$

where D is the divisor on X that we used to define $\mathcal{M}^{\mathrm{par}}$, and $h^\vee$ is the dual Coxeter number of G.

To prove that the actions of these Chern classes and the $\widetilde{W}$-action satisfy the relations in the graded DAHA, we again use parahoric versions of Hitchin stacks to reduce to an SL(2) calculation.

1.4. Organization of the paper

In Section 1.6, we fix notations. In Section 2, we introduce the parabolic Hitchin fibration. The key geometric facts are the smoothness of $\mathcal{M}^{\mathrm{par}}$ in Section 2.5 and the codimension estimate in Section 2.6. This section relies heavily on Ngô's works [29,28].

From Section 3 on, we work over the anisotropic locus $\mathcal{A}$. In Section 3, we give the first construction of the $\widetilde{W}$-action using Hecke correspondences. The technical part is Section 3.2, in which we prove that the Hecke correspondence is a union of graphs over the regular semisimple locus (in fact over a larger locus).

In Section 4, we define parahoric versions of Hitchin stacks. For this we need to define the notion of bundles with parahoric level structures at a varying point in Section 4.2; otherwise many results are parallel to Section 2.

In Section 5, we give the second construction of the $\widetilde{W}$-action using the Coxeter presentation. In Section 5.2, we prove that the two constructions give the same action.

In Section 6, we state and prove Theorem B. The proof will occupy Sections 6.2–6.5. In Section 6.6, we generalize our main results to parahoric Hitchin stacks.

In Sections 2–6, whenever a key geometric object is introduced, examples in the case $G = \mathrm{GL}(n)$ will be given using the language of compactified Picard stacks.

In Section 7, we calculate an example for $G = \mathrm{SL}(2)$.

In Appendix A, we review the general formalism of cohomological correspondences, with emphasis on *graph-like* correspondences in Appendix A.6.

1.5. Further remarks and applications

This paper is a completely re-organized version of the most part of the preprints “Towards a global Springer theory I, II” [34,35] and the example section of the preprint “Towards a global Springer theory III” [36]. While the three preprints together form the author’s PhD thesis, this paper is self-contained.

Some problems in local Springer theory can be solved by using global Springer theory. For example, in the work in progress of Bezrukavnikov and Varshavsky on stable distributions on p-adic groups, the main result relies on a compatibility statement for local Springer actions which used to be open. In [38], we will prove this compatibility using global Springer theory.

Notions such as endoscopy and Langlands duality from the modern theory of automorphic forms will naturally show up in global Springer theory. Just as the usual Hitchin fibration is closely related to endoscopy (see [28]), an analogous endoscopic decomposition result holds for the $\widetilde{W}$-action on $\mathbf{R}f^{\mathrm{par}}_*\overline{\mathbb{Q}}_\ell$, see [36]. As an application, we will obtain a proof of Kottwitz’s conjecture on the endoscopic transfer for Green functions, generalizing the Fundamental Lemma of Langlands–Shelstad.

Also, inspired by the conjectural mirror symmetry between Hitchin fibrations for Langlands dual groups, it would be interesting to compare global Springer actions on parabolic Hitchin complexes for Langlands dual groups. This is the content of [37].

As mentioned earlier, the work [33] and [32] systematically studies representations of DAHA via the homology of affine Springer fibers. In joint work in progress with A. Oblomkov, we use the construction of the current paper to study finer structures (such as “perverse filtrations”) on representations of the rational DAHA.

1.6. Notations and conventions

1.6.1. Notations concerning geometry

Throughout this paper, we work over a fixed algebraically closed field k. We fix a prime ℓ different from $\mathrm{char}(k)$.

For a Deligne–Mumford stack $\mathfrak{X}$ over k (see [23]), let $D^b(\mathfrak{X})$ denote the derived category of constructible $\overline{\mathbb{Q}}_\ell$-complexes on $\mathfrak{X}$. Let $\mathcal{F}\mapsto\mathcal{F}(1)$ be the Tate twist in $D^b(\mathfrak{X})$.

For a morphism $f:\mathfrak{X}\to\mathfrak{Y}$, we have derived functors $\mathbf{R}f^*$, $\mathbf{R}f_*$, $\mathbf{R}f_!$, $\mathbf{R}f^!$ between $D^b(\mathfrak{X})$ and $D^b(\mathfrak{Y})$. In the main body of the paper, we will simply write them as f^*, f_*, $f_!$, $f^!$; **all such functors are understood to be derived**.

We use $\mathbb{D}_{\mathfrak{X}/\mathfrak{Y}}$ or $\mathbb{D}_f$ to denote the relative dualizing complex $f^!\overline{\mathbb{Q}}_\ell$. When $\mathfrak{Y}=\operatorname{Spec}k$, we simply write $\mathbb{D}_{\mathfrak{X}}$ for the dualizing complex of $\mathfrak{X}$. We recall that the Borel–Moore homology is defined as

$$\mathrm{H}^{\mathrm{BM}}_i(\mathfrak{X}):=\mathrm{H}^{-i}(\mathfrak{X},\mathbb{D}_{\mathfrak{X}}).$$

All torsors are right torsors unless otherwise stated.

Suppose $\mathfrak{X}$ (resp. $\mathfrak{Y}$) is a stack with right (resp. left) action of a group scheme A over k, then we write the contracted product

$$\mathfrak{X}\overset{A}{\times}\mathfrak{Y}:=\big[(\mathfrak{X}\times\mathfrak{Y})/A\big]$$

for the stack quotient of $\mathfrak{X} \times \mathfrak{Y}$ by the anti-diagonal right A-action: $a \in A$ acts on $\mathfrak{X} \times \mathfrak{Y}$ by $(x, y) \mapsto (xa, a^{-1}y)$.

1.6.2. The curve X

For the most part of this paper (except Appendix A), we fix X to be a smooth connected projective curve over k of genus g_X.

For any k-algebra R, let $X_R = \operatorname{Spec} R \times_{\operatorname{Spec} k} X$. If $x \in X(R)$ is an R-point of X, let $\Gamma(x) \subset X_R$ be the graph of $x : \operatorname{Spec} R \to X$. We let

$$\mathfrak{D}_x = \operatorname{Spec} \widehat{\mathcal{O}}_x; \qquad \mathfrak{D}_x^\times = \operatorname{Spec} \widehat{\mathcal{O}}_x^{\text{punc}} = \mathfrak{D}_x - \Gamma(x)$$

be the formal disc and the punctured formal disc of X_R along $\Gamma(x)$.

For a divisor D on X and a quasi-coherent sheaf $\mathcal{F}$ on X, we write $\mathcal{F}(D) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$.

For a line bundle $\mathcal{L}$ (or a divisor D) over X, we let $\rho_{\mathcal{L}}$ (or ρ_D) denote the complement of the zero section in the total space of the line bundle $\mathcal{L}$ (or $\mathcal{O}_X(D)$). This is naturally a $\mathbb{G}_m$-torsor.

Let D be a divisor on X and let $\mathfrak{Y}$ be a stack over X with a $\mathbb{G}_m$-action such that the structure morphism $\mathfrak{Y} \to X$ is $\mathbb{G}_m$-invariant. Then we define

$$\mathfrak{Y}_D := \rho_D \overset{\mathbb{G}_m}{\times}_X \mathfrak{Y} = \big[(\rho_D \times_X \mathfrak{Y})/\mathbb{G}_m\big].$$

If $\mathfrak{Y}$ is a stack over k, with no specified morphism to X from the context, we also write $\mathfrak{Y}_D$ for $\rho_D \overset{\mathbb{G}_m}{\times} \mathfrak{Y} = [(\rho_D \times \mathfrak{Y})/\mathbb{G}_m]$.

1.6.3. The group G

We fix G to be a connected reductive group over k of semisimple rank n, and we fix a Borel subgroup B of G with universal quotient torus T. Let $\mathbb{X}_*(T)$ and $\mathbb{X}^*(T)$ be the cocharacter and character groups of T.

Let W be the canonical Weyl group given by (G, B). This is a Coxeter group with simple reflections $\Sigma = \{s_1, \ldots, s_n\}$. Let $\Phi^+ \subset \Phi \subset \mathbb{X}^*(T)$ and $\Phi^{\vee,+} \subset \Phi^\vee \subset \mathbb{X}_*(T)$ be the based root and coroot systems. Throughout the paper, we assume that char(k) is either 0 or is great than $2h$, where h is the maximal Coxeter number of all simple factors of G (see [29, §1.1]).

Let $\mathbb{Z}\Phi^\vee \subset \mathbb{X}_*(T)$ be the coroot lattice. Let

$$W_{\text{aff}} := \mathbb{Z}\Phi^\vee \rtimes W; \qquad \widetilde{W} := \mathbb{X}_*(T) \rtimes W$$

be the *affine Weyl group* and the *extended affine Weyl group* of G. If G is almost simple, then W_{aff} is a Coxeter group with simple reflections $\Sigma_{\text{aff}} = \{s_0, s_1, \ldots, s_n\}$.

Let $(\cdot|\cdot)_{\text{can}}$ be the Killing form on $\mathbb{X}_*(T)$:

$$(x|y)_{\text{can}} := \sum_{\alpha \in \Phi} \langle \alpha, x\rangle \langle \alpha, y\rangle,$$

where $\Phi \subset \mathbb{X}^*(T)$ is the set of roots of G.

Let $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}$ be the Lie algebras of G, B, T respectively. Let $\mathfrak{c}$ be the GIT quotient $\mathfrak{g} /\!\!/ G = \mathfrak{t} /\!\!/ W = \operatorname{Spec} k[f_1, \ldots, f_n]$. The affine space $\mathfrak{c}$ inherits a weighted $\mathbb{G}_m$-action from $\mathfrak{t}$ such that f_i is homogeneous of degree $d_i \in \mathbb{Z}_{\geqslant 1}$.

Let $\mathfrak{c}^{\mathrm{rs}}$ be the regular semisimple locus of $\mathfrak{c}$, which is the complement of the discriminant divisor $\Delta \subset \mathfrak{c}$. For a stack $\mathfrak{X}$ or a morphism F over $\mathfrak{c}$ or $\mathfrak{c}/\mathbb{G}_m$, we use $\mathfrak{X}^{\mathrm{rs}}$ and F^{rs} to denote their restrictions to $\mathfrak{c}^{\mathrm{rs}}$ or $\mathfrak{c}^{\mathrm{rs}}/\mathbb{G}_m$.

2. The parabolic Hitchin fibration

In this section, we introduce the main players—the parabolic Hitchin moduli stack and the parabolic Hitchin fibration.

2.1. *The parabolic Hitchin moduli stack*

Let Bun_G be the moduli stack of G-torsors over the curve X. Let $\mathrm{Bun}_G^{\mathrm{par}}$ be the moduli stack of G-torsors on X with a B-reduction at a point. More precisely, for any scheme S, $\mathrm{Bun}_G^{\mathrm{par}}(S)$ is the groupoid of triples $(x, \mathcal{E}, \mathcal{E}_x^B)$ where

- $x : S \to X$ with graph $\Gamma(x)$;
- $\mathcal{E}$ is a G-torsor over $S \times X$;
- $\mathcal{E}_x^B$ is a B-reduction of $\mathcal{E}$ along $\Gamma(x)$.

Fix a divisor D on X such that $\deg(D) \geqslant 2g_X$ (g_X is the genus of X). We assume $D = 2D'$ for some other divisor D' on X; this assumption is only used to guarantee the existence of a global Kostant section (see [29, §4.2.4]).

Recall from [16] and [28, Definition 4.2.1] that the *Hitchin moduli stack* $\mathcal{M}^{\mathrm{Hit}} = \mathcal{M}^{\mathrm{Hit}}_{G,X,D}$ is the functor which sends a scheme S to the groupoid of *Hitchin pairs* $(\mathcal{E}, \varphi)$ where

- $\mathcal{E}$ is a G-torsor over $S \times X$;
- $\varphi \in \mathrm{H}^0(S \times X, \mathrm{Ad}(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D))$ is called a *Higgs field*.

Here $\mathrm{Ad}(\mathcal{E}) = \mathcal{E} \overset{G}{\times} \mathfrak{g}$ is the adjoint bundle associated to $\mathcal{E}$.

Definition 2.1.1. The *parabolic Hitchin moduli stack* $\mathcal{M}^{\mathrm{par}} = \mathcal{M}^{\mathrm{par}}_{G,X,D}$ is the functor which sends a scheme S to the groupoid of quadruples $(x, \mathcal{E}, \varphi, \mathcal{E}_x^B)$, where

- $x : S \to X$ with graph $\Gamma(x)$;
- $(\mathcal{E}, \varphi) \in \mathcal{M}^{\mathrm{Hit}}_{G,X,D}(S)$ is a Hitchin pair;
- $\mathcal{E}_x^B$ is a B-reduction of $\mathcal{E}$ along $\Gamma(x)$,

such that φ is *compatible* with $\mathcal{E}_x^B$, i.e.,

$$\varphi|_{\Gamma(x)} \in \mathrm{H}^0\big(\Gamma(x), \mathrm{Ad}\big(\mathcal{E}_x^B\big) \otimes_{\mathcal{O}_S} x^*\mathcal{O}_X(D)\big).$$

We will often write $\mathcal{M}^{\mathrm{par}}$ for $\mathcal{M}^{\mathrm{par}}_{G,X,D}$, and $\mathrm{Bun}^{\mathrm{par}}$ for $\mathrm{Bun}_G^{\mathrm{par}}$.

Forgetting the Higgs field φ gives a morphism $\mathcal{M}^{\mathrm{par}} \to \mathrm{Bun}_G^{\mathrm{par}}$. Forgetting the choice of the B-reduction gives a morphism

$$\pi_{\mathcal{M}} : \mathcal{M}^{\mathrm{par}} \to \mathcal{M}^{\mathrm{Hit}} \times X.$$

We give an alternative description of $\mathcal{M}^{\mathrm{par}}$. By [28, §4], the Hitchin moduli stack can be interpreted as classifying sections

$$X \to [\mathfrak{g}/G]_D := \rho_D \overset{\mathbb{G}_m}{\times} [\mathfrak{g}/G].$$

Here $[\mathfrak{g}/G]$ is the adjoint quotient stack of $\mathfrak{g}$ by G, ρ_D is the $\mathbb{G}_m$-torsor over X associated to the line bundle $\mathcal{O}_X(D)$ (for the meaning of the twisting $(-)_D$ in general, see Section 1.6.1). We have an evaluation morphism

$$\mathrm{ev}^{\mathrm{Hit}} : \mathcal{M}^{\mathrm{Hit}} \times X \to [\mathfrak{g}/G]_D.$$

Consider the D-twisted form of the *Grothendieck simultaneous resolution*:

$$\pi : [\mathfrak{b}/B]_D \to [\mathfrak{g}/G]_D.$$

It is easy to see that

Lemma 2.1.2. *The stack $\mathcal{M}^{\mathrm{par}}$ fits into a Cartesian square*

$$\begin{array}{ccc} \mathcal{M}^{\mathrm{par}} & \xrightarrow{\mathrm{ev}^{\mathrm{par}}} & [\mathfrak{b}/B]_D \\ \downarrow{\scriptstyle \pi_{\mathcal{M}}} & & \downarrow{\scriptstyle \pi} \\ \mathcal{M}^{\mathrm{Hit}} \times X & \xrightarrow{\mathrm{ev}^{\mathrm{Hit}}} & [\mathfrak{g}/G]_D \end{array} \tag{2.1}$$

2.2. The Hitchin base

Recall (see [16, §4], [28, Lemme 2.4]) that the Hitchin base space $\mathcal{A}^{\mathrm{Hit}}$ is the affine space of all sections of $X \to \mathfrak{c}_D$. Here the twisting $\mathfrak{c}_D := \rho_D \overset{\mathbb{G}_m}{\times} \mathfrak{c}$ uses the weighted $\mathbb{G}_m$-action on $\mathfrak{c}$ (see Section 1.6.3). Fixing homogeneous generators of $\mathrm{Sym}_k(\mathfrak{t}^*)^W$ of degrees $d_1, \ldots, d_n$, we can write

$$\mathcal{A}^{\mathrm{Hit}} = \bigoplus_{i=1}^{n} \mathrm{H}^0\big(X, \mathcal{O}_X(d_i D)\big).$$

The morphism $[\mathfrak{g}/G]_D \to \mathfrak{c}_D$ induces the *Hitchin fibration* (see [16, §4], [29, §4.2.3])

$$f^{\mathrm{Hit}} : \mathcal{M}^{\mathrm{Hit}} \to \mathcal{A}^{\mathrm{Hit}}.$$

Definition 2.2.1. The morphism

$$f^{\mathrm{par}} : \mathcal{M}^{\mathrm{par}} \to \mathcal{A}^{\mathrm{Hit}} \times X,$$
$$\big(x, \mathcal{E}, \varphi, \mathcal{E}_x^B\big) \mapsto \big(f^{\mathrm{Hit}}(\mathcal{E}, \varphi), x\big)$$

is called the *parabolic Hitchin fibration*. The geometric fibers of the morphism f^{par} are called *parabolic Hitchin fibers*.

Definition 2.2.2. The *universal cameral cover*, or the *enhanced Hitchin base* $\widetilde{\mathcal{A}}^{\mathrm{Hit}}$ is defined by the Cartesian diagram

$$\begin{array}{ccc} \widetilde{\mathcal{A}}^{\mathrm{Hit}} & \xrightarrow{\mathrm{ev}} & \mathfrak{t}_D \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q_{\mathfrak{t}}} \\ \mathcal{A}^{\mathrm{Hit}} \times X & \xrightarrow{\mathrm{ev}} & \mathfrak{c}_D \end{array}$$

For $a \in \mathcal{A}^{\mathrm{Hit}}(k)$, the preimage $X_a := q^{-1}(\{a\} \times X)$ is the *cameral curve* X_a defined by Faltings [11, §III] and Donagi [10, §4.2]. The projection $q_a : X_a \to X$ is a ramified W-cover.

The commutative diagram

$$\begin{array}{ccc} [\mathfrak{b}/B] & \longrightarrow & \mathfrak{t} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle /\!/W} \\ [\mathfrak{g}/G] & \xrightarrow{\chi} & \mathfrak{c} \end{array}$$

together with Lemma 2.1.2 gives a morphism

$$\tilde{f} : \mathcal{M}^{\mathrm{par}} \to \widetilde{\mathcal{A}}^{\mathrm{Hit}} \tag{2.2}$$

which we call the *enhanced parabolic Hitchin fibration*.

Lemma 2.2.3. *Recall that* $\deg(D) \geqslant 2g_X$. *Then* $\widetilde{\mathcal{A}}^{\mathrm{Hit}}$ *is smooth.*

Proof. The affine space bundle $\mathfrak{c}_D$ over X is non-canonically a direct sum of line bundles of the form $\mathcal{O}(eD)$, each having degree $\geqslant 2g_X$ since $e \geqslant 1$. Therefore the line bundle $\mathcal{O}(eD)$ is globally generated. Hence the evaluation map

$$\mathcal{A}^{\mathrm{Hit}} \times X = \mathrm{H}^0(X, \mathfrak{c}_D) \times X \to \mathfrak{c}_D$$

is a surjective bundle map over X, therefore it is smooth. By base change to $\mathfrak{t}_D$, we see that $\widetilde{\mathcal{A}}^{\mathrm{Hit}} \to \mathfrak{t}_D$ is smooth, hence $\widetilde{\mathcal{A}}^{\mathrm{Hit}}$ is smooth because $\mathfrak{t}_D$ is. □

We summarize the various stacks we considered into a commutative diagram

$$\begin{tikzcd}
\mathcal{M}^{\mathrm{par}} \arrow[r, "\nu_{\mathcal{M}}"] \arrow[dd, "\mathrm{ev}^{\mathrm{par}}"] & \mathcal{M}^{\mathrm{Hit}} \times_{\mathcal{A}^{\mathrm{Hit}}} \widetilde{\mathcal{A}}^{\mathrm{Hit}} \arrow[rr, "q_{\mathcal{M}}"] \arrow[dd] \arrow[dr] & & \mathcal{M}^{\mathrm{Hit}} \times X \arrow[dr, "f^{\mathrm{Hit}} \times \mathrm{id}_X"] \arrow[dd, "\mathrm{ev}^{\mathrm{Hit}}"] & \\
& & \widetilde{\mathcal{A}}^{\mathrm{Hit}} \arrow[rr, "q"] \arrow[dd, "\mathrm{ev}"] & & \mathcal{A}^{\mathrm{Hit}} \times X \arrow[dd, "\mathrm{ev}"] \\
[\mathfrak{b}/B]_D \arrow[r, "\nu"] & [\mathfrak{g}/G]_D \times_{\mathfrak{c}} \mathfrak{t} \arrow[rr, "q_{\mathfrak{g}}"] \arrow[dr] & & [\mathfrak{g}/G]_D \arrow[dr, "\chi"] & \\
& & \mathfrak{t}_D \arrow[rr, "q_{\mathfrak{t}}"] & & \mathfrak{c}_D
\end{tikzcd} \tag{2.3}$$

where the three squares formed by horizontal and vertical arrows are Cartesian.

Remark 2.2.4. The following open subsets of $\mathcal{A}^{\mathrm{Hit}}$ will be considered in the sequel:

$$\mathcal{A}^{\mathrm{Hit}} \supset \mathcal{A}^{\heartsuit} \supset \mathcal{A}^{\mathrm{ani}} = \mathcal{A}.$$

In [28, Definition 4.4], Ngô introduced the open subscheme $\mathcal{A}^{\heartsuit} \subset \mathcal{A}^{\mathrm{Hit}}$ consisting of those $a : X \to \mathfrak{c}_D$ which *generically* lie in the regular semisimple locus $\mathfrak{c}_D^{\mathrm{rs}} \subset \mathfrak{c}_D$. In [29, §6.1], Ngô introduced the open subset $\mathcal{A}^{\mathrm{ani}}$, which we recall in Definition 2.3.6. We will be mostly working over $\mathcal{A}$. The preimage of $\mathcal{A}$ in $\widetilde{\mathcal{A}}^{\mathrm{Hit}}$ is denoted by $\widetilde{\mathcal{A}}$.

Example 2.2.5. Let $G = \mathrm{GL}(n)$. The stack $\mathcal{M}^{\mathrm{par}}_{\mathrm{GL}(n)}$ classifies the data

$$\big(x; \mathcal{E}_0 \supset \mathcal{E}_1 \supset \cdots \supset \mathcal{E}_{n-1} \supset \mathcal{E}_n = \mathcal{E}_0(-x); \varphi\big)$$

where $\mathcal{E}_i$ are rank n vector bundles on X such that $\mathcal{E}_i/\mathcal{E}_{i+1}$ are skyscraper sheaves of length 1 supported at $x \in X$. The Higgs field φ in this case is a map of coherent sheaves

$$\varphi : \mathcal{E}_0 \to \mathcal{E}_0(D)$$

such that $\varphi(\mathcal{E}_i) \subset \mathcal{E}_i(D)$ for $i = 1, \dots, n-1$.

In this case, $\mathcal{A}^{\mathrm{Hit}}$ is the space of characteristic polynomials

$$\mathcal{A}^{\mathrm{Hit}} = \bigoplus_{i=1}^{n} \mathrm{H}^0\big(X, \mathcal{O}_X(iD)\big).$$

We follow Hitchin's original construction [16, §5.1] to describe $\mathcal{M}^{\mathrm{Hit}}_a$ in this case. For each $a = (a_1, \dots, a_n) \in \mathcal{A}^{\mathrm{Hit}}$, we can define the *spectral curve* $p_a : Y_a \to X$, here Y_a is an embedded curve in the total space of the line bundle $\mathcal{O}_X(D)$ defined by the equation

$$t^n + a_1 t^{n-1} + \cdots + a_n = 0.$$

Let $a \in \mathcal{A}^{\heartsuit}(k)$, then Y_a is a reduced curve. In this case, we have a natural isomorphism $\mathcal{M}_a^{\mathrm{Hit}} \cong \overline{\mathcal{P}\mathrm{ic}}(Y_a)$, the latter being the compactified Picard stack classifying torsion-free coherent sheaves on Y_a of generic rank 1. This isomorphism sends $\mathcal{F} \in \overline{\mathcal{P}\mathrm{ic}}(Y_a)$ to $(p_{a,*}\mathcal{F}, \varphi)$ where the Higgs field φ on $p_{a,*}\mathcal{F}$ comes from the action of a direct summand $\mathcal{O}_X(-D) \subset \mathcal{O}_{Y_a}$ on $\mathcal{F}$.

The reduced fiber of $q : \widetilde{\mathcal{A}}^{\mathrm{Hit}} \to \mathcal{A}^{\mathrm{Hit}} \times X$ over (a, x) classifies all orderings of $p_a^{-1}(x) \subset Y_a$.

For $x \in X$, the parabolic Hitchin fiber $\mathcal{M}_{a,x}^{\mathrm{par}}$ classifies the data

$$\mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots \supset \mathcal{F}_{n-1} \supset \mathcal{F}_n = \mathcal{F}_0(-x) \tag{2.4}$$

where $\mathcal{F}_i \in \overline{\mathcal{P}\mathrm{ic}}(Y_a)$ and each $\mathcal{F}_i/\mathcal{F}_{i+1}$ has length 1. If Y_a is étale over x, then $\mathcal{M}_{a,x}^{\mathrm{par}}$ classifies sheaves $\mathcal{F}_0 \in \overline{\mathcal{P}\mathrm{ic}}(Y_a)$ together with an ordering of the set $p_a^{-1}(x)$.

2.3. *Symmetries on parabolic Hitchin fibers*

We first review some facts about regular centralizers, following [29, §2]. Let $I_G \to \mathfrak{g}$ be the universal centralizer group scheme: for any $z \in \mathfrak{g}$, the fiber $I_{G,z}$ is the centralizer G_z of z in G. Let $\mathfrak{g}^{\mathrm{reg}}$ be the open subset of regular (i.e., centralizers have minimal dimension) elements in $\mathfrak{g}$. According to [29, Lemme 2.1.1], there is a smooth group scheme $J \to \mathfrak{c}$, the *group scheme of regular centralizers*, together with an $\mathrm{Ad}(G)$-equivariant homomorphism $j_G : \chi^* J =: J_{\mathfrak{g}} \to I_G$ which is an isomorphism over $\mathfrak{g}^{\mathrm{reg}}$ (here $\chi : \mathfrak{g} \to \mathfrak{c}$ is the adjoint quotient). In other words, the stack $\mathbb{B}J$ (over $\mathfrak{c}$) acts on the stack $[\mathfrak{g}/G]$, such that $[\mathfrak{g}^{\mathrm{reg}}/G]$ becomes a J-gerbe over $\mathfrak{c}$.

Let $J_{\mathfrak{b}}$ be the pull-back of J to $\mathfrak{b}$. Let I_B be the universal centralizer group scheme of the adjoint action of B on $\mathfrak{b}$. We claim that

Lemma 2.3.1. *There is a natural homomorphism $j_B : J_{\mathfrak{b}} \to I_B$ such that $j_G|_{\mathfrak{b}}$ factors as $J_{\mathfrak{b}} \xrightarrow{j_B} I_B \to I_G|_{\mathfrak{b}}$. Therefore, the stack $\mathbb{B}J$ acts on the stack $[\mathfrak{b}/B]$ over $\mathfrak{c}$.*

Proof. For $x \in \mathfrak{b}^{\mathrm{reg}} := \mathfrak{b} \cap \mathfrak{g}^{\mathrm{reg}}$, we have canonical identifications $J_x = I_{G,x} = I_{B,x}$ (cf. [29, Lemme 2.4.3]). This gives the desired map j_B over $\mathfrak{b}^{\mathrm{reg}}$. To extend j_B to the whole $\mathfrak{b}$, we can use the same argument as in [28, Proposition 3.2], because $J_{\mathfrak{b}}$ is smooth over $\mathfrak{b}$ and $\mathfrak{b} - \mathfrak{b}^{\mathrm{reg}}$ has codimension at least two. □

Let $J_{\mathfrak{t}}$ be the pull-back of J along $\mathfrak{t} \to \mathfrak{c}$. Recall the following fact:

Lemma 2.3.2. *(See [29, Proposition 2.4.2].) There is a W-equivariant homomorphism of group schemes over* $\mathfrak{t}$:

$$j_T : J_{\mathfrak{t}} \to T \times \mathfrak{t}$$

which is an isomorphism over $\mathfrak{t}^{\mathrm{rs}}$. Here the W-structure on $J_{\mathfrak{t}} = J \times_{\mathfrak{c}} \mathfrak{t}$ is given by its left action on $\mathfrak{t}$, while the W-structure on $T \times \mathfrak{t}$ is given by the diagonal left action.

Moreover, J carries a natural $\mathbb{G}_m$-action such that $J \to \mathfrak{c}$ is $\mathbb{G}_m$-equivariant, therefore it makes sense to twist J by the $\mathbb{G}_m$-torsor ρ_D over X and get $J_D \to \mathfrak{c}_D$ (see Section 1.6.2).

Recall from [29, §4.3.1] that we have a Picard stack $g : \mathcal{P} \to \mathcal{A}^{\mathrm{Hit}}$ whose fiber $\mathcal{P}_a$ over $a : S \to \mathcal{A}^{\mathrm{Hit}}$ (viewed as a morphism $a : S \times X \to \mathfrak{c}_D$) classifies $J_a := a^* J_D$-torsors on $S \times X$.

According to [28, Proposition 5.2], $\mathcal{P}$ is smooth over $\mathcal{A}^{\heartsuit}$. Since $\mathbb{B}J_D$ acts on $[\mathfrak{g}/G]_D$, $\mathcal{P}$ acts on $\mathcal{M}^{\mathrm{Hit}}$ preserving the base $\mathcal{A}^{\mathrm{Hit}}$.

There is an open dense substack $\mathcal{M}^{\mathrm{Hit,reg}} \subset \mathcal{M}^{\mathrm{Hit}}$ parametrizing those $(E, \varphi): X \to [\mathfrak{g}/G]_D$ which land entirely in $[\mathfrak{g}^{\mathrm{reg}}/G]_D$. Since $[\mathfrak{g}^{\mathrm{reg}}/G]_D$ is a $\mathbb{B}J_D$-gerbe over $\mathfrak{c}_D$, $\mathcal{M}^{\mathrm{Hit,reg}}$ is a $\mathcal{P}$-torsor over $\mathcal{A}^{\mathrm{Hit}}$.

Lemma 2.3.1 implies that $\mathbb{B}J_D$ acts on $[\mathfrak{b}/B]_D$, compatible with its action on $[\mathfrak{g}/G]_D$, and preserving the morphism $[\mathfrak{b}/B]_D \to \mathfrak{t}_D$. Using the moduli interpretations of $\mathcal{M}^{\mathrm{par}}$ and $\mathcal{P}$, we get

Lemma 2.3.3. *The Picard stack $\mathcal{P}$ acts on $\mathcal{M}^{\mathrm{par}}$. The action is compatible with its action on $\mathcal{M}^{\mathrm{Hit}}$ and preserves the enhanced parabolic Hitchin fibration $\tilde{f}: \mathcal{M}^{\mathrm{par}} \to \widetilde{\mathcal{A}}^{\mathrm{Hit}}$. In other words, if we let $\widetilde{\mathcal{P}} = \widetilde{\mathcal{A}}^{\mathrm{Hit}} \times_{\mathcal{A}^{\mathrm{Hit}}} \mathcal{P}$, viewed as a Picard stack over $\widetilde{\mathcal{A}}^{\mathrm{Hit}}$, then $\widetilde{\mathcal{P}}$ acts on $\mathcal{M}^{\mathrm{par}}$ over $\widetilde{\mathcal{A}}^{\mathrm{Hit}}$.*

We describe this action on the level of S-points. Let $(a, x) \in \mathcal{A}^{\mathrm{Hit}}(S) \times X(S)$ and $(x, \mathcal{E}, \varphi, \mathcal{E}_x^B) \in \mathcal{M}^{\mathrm{par}}(S)$ be a point over it. An S-point of $\mathcal{P}$ is the same as a J_a-torsor Q^J over $S \times X$. Then the effect of the Q^J-action is

$$Q^J \cdot \big(x, \mathcal{E}, \varphi, \mathcal{E}_x^B\big) = \big(x, Q^J \overset{J_a, j_G}{\times} (\mathcal{E}, \varphi), Q^J \overset{J_a, j_B}{\times} \mathcal{E}_x^B\big).$$

Example 2.3.4. We continue with Example 2.2.5 of $G = \mathrm{GL}(n)$. For $a \in \mathcal{A}^{\heartsuit}(k)$, $\mathcal{P}_a = \mathcal{P}\mathrm{ic}(Y_a)$ is the Picard stack of line bundles on Y_a. The action of $\mathcal{L} \in \mathcal{P}_a$ on $\mathcal{M}_{a,x}^{\mathrm{par}}$ is given by

$$\mathcal{L} \cdot (\mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots) = (\mathcal{L} \otimes \mathcal{F}_0 \supset \mathcal{L} \otimes \mathcal{F}_1 \supset \cdots)$$

where the tensor products are over $\mathcal{O}_{Y_a}$.

Remark 2.3.5. Although $\mathcal{P}_a$ still has an open orbit in $\mathcal{M}_{a,\tilde{x}}^{\mathrm{par}}$ (where $\tilde{x} \in X_a$) which is a torsor under $\mathcal{P}_a$, this open set may not be dense in $\mathcal{M}_{a,\tilde{x}}^{\mathrm{par}}$: there might be other irreducible components, as we will see in the example in Section 7.1.

Definition 2.3.6. (See [29, §4.10.5 and §6.1].) The *anisotropic* open subset $\mathcal{A} = \mathcal{A}^{\mathrm{ani}} \subset \mathcal{A}^{\heartsuit}$ is the open locus of $a \in \mathcal{A}^{\heartsuit}(k)$ where $\pi_0(\mathcal{P}_a)$ is finite.

The anisotropic locus $\mathcal{A}$ is nonempty if and only if G is semisimple. Over $\mathcal{A}$, $\mathcal{P}$ is Deligne–Mumford and of finite type.

2.4. Product formula

In this subsection, we will relate parabolic Hitchin fibers to products of affine Springer fibers, following the ideas in [28, §4.13], [28, Theorem 4.6] where the case of the usual Hitchin fibration was treated.

For each point $x \in X(k)$, denote the completed local ring of X at x and its field of fractions by $\widehat{\mathcal{O}}_x$ and $\widehat{F}_x$. Let $\gamma \in \mathfrak{g}(\widehat{F}_x)$. The Hitchin fibers have their local counterparts, the *affine Springer fiber* $M_x^{\mathrm{Hit}}(\gamma)$ in the affine Grassmannian ${}_{G,x} = G(\widehat{F}_x)/G(\widehat{\mathcal{O}}_x)$. For the definition of $M_x^{\mathrm{Hit}}(\gamma)$, see [29, Definition 3.2.2].

Similarly, the parabolic Hitchin fibers have their local counterparts $M_x^{\mathrm{par}}(\gamma)$ also called affine Springer fibers, which is a sub-ind-schemes of the affine flag variety $\mathrm{Fl}_{G,x} = G(\widehat{F}_x)/\mathbf{I}_x$ ($\mathbf{I}_x$ is the

Iwahori subgroup of $G(\widehat{\mathcal{O}}_x)$ corresponding to B). The functor $M_x^{\mathrm{par}}(\gamma)$ sends any scheme S to the set of isomorphism classes of quadruples $(\mathcal{E}, \varphi, \mathcal{E}_x^B, \alpha)$ where

- $\mathcal{E}$ is a G-torsor over $S \widehat{\times} \mathfrak{D}_x$;
- $\varphi \in \mathrm{H}^0(S \widehat{\times} \mathfrak{D}_x, \mathrm{Ad}(\mathcal{E}))$;
- $\mathcal{E}_x^B$ is a B-reduction along $S \times \{x\}$;
- α is an isomorphism $(\mathcal{E}, \varphi)|_{S \widehat{\times} \mathfrak{D}_x^\times} \cong (\mathcal{E}^{\mathrm{triv}}, \gamma)$, where $\mathcal{E}^{\mathrm{triv}}$ is the trivial G-torsor over $S \widehat{\times} \mathfrak{D}_x^\times$.

For $\gamma \in \mathfrak{g}^{\mathrm{rs}}(\widehat{F}_x)$, the reduced structures $M_x^{\mathrm{par,red}}(\gamma)$ and $M_x^{\mathrm{Hit,red}}(\gamma)$ are locally of finite type.

For $a \in \mathfrak{c}(\widehat{\mathcal{O}}_x)$, we also have the local counterpart $P_x(J_a)$ of the Picard stack $\mathcal{P}$ over $\mathcal{A}^{\mathrm{Hit}}$. The functor $P_x(J_a)$ sends any scheme S to the set of isomorphism classes of pairs (Q^J, τ) where

- Q^J is a J_a torsor over $S \widehat{\times} \mathfrak{D}_x$;
- τ a trivialization of Q^J over $S \widehat{\times} \mathfrak{D}_x^\times$.

If $\chi(\gamma) = a$, then the group ind-scheme $P_x(J_a)$ acts on $M_x^{\mathrm{Hit}}(\gamma)$ and $M_x^{\mathrm{par}}(\gamma)$.

Recall from [29, §4.2.4] that if $D = 2D'$, we have a global Kostant section

$$\epsilon : \mathcal{A}^{\mathrm{Hit}} \to \mathcal{M}^{\mathrm{Hit}}. \tag{2.5}$$

For $a \in \mathcal{A}^{\mathrm{Hit}}(k)$, consider the Hitchin pair $\epsilon(a) = (\mathcal{E}, \varphi)$. After trivializing $\mathcal{E}|_{\mathfrak{D}_x}$ and choosing an isomorphism $\widehat{\mathcal{O}}_x(D_x) \cong \widehat{\mathcal{O}}_x$, we can identify $(\mathcal{E}, \varphi)|_{\mathfrak{D}_x}$ with $(\mathcal{E}^{\mathrm{triv}}, \gamma_{a,x})$ for some element $\gamma_{a,x} \in \mathfrak{g}(\widehat{\mathcal{O}}_x)$ such that $\chi(\gamma) = a_x \in \mathfrak{c}(\widehat{\mathcal{O}}_x)$. Parallel to the product formula of Ngô in [29, Proposition 4.15.1], we have the following product formula, whose proof is identical with the case of Hitchin fibers in [29].

Proposition 2.4.1 *(Product formula). Let $(a, x) \in \mathcal{A}^{\heartsuit}(k) \times X(k)$ and let U_a be the dense open subset $a^{-1}(\mathfrak{c}_D^{\mathrm{rs}}) \subset X$. We have a* homeomorphism *of stacks*:

$$\mathcal{P}_a \overset{P_x^{\mathrm{red}}(J_a) \times P'}{\times} \bigl(M_x^{\mathrm{par,red}}(\gamma_{a,x}) \times M'\bigr) \to \mathcal{M}_{a,x}^{\mathrm{par}},$$

where

$$P' = \prod_{y \in X - U_a - \{x\}} P_y^{\mathrm{red}}(J_a);$$

$$M' = \prod_{y \in X - U_a - \{x\}} M_y^{\mathrm{Hit,red}}(\gamma_{a,y}).$$

Corollary 2.4.2. *For $a \in \mathcal{A}^{\heartsuit}(k)$ and $\tilde{x} \in X_a(k)$ with image $x \in X(k)$, then*:

$$\dim \mathcal{M}_{a,\tilde{x}}^{\mathrm{par}} = \dim \mathcal{M}_{a,x}^{\mathrm{par}} = \dim \mathcal{P}_a = \dim \mathcal{M}_a^{\mathrm{Hit}}.$$

Proof. By Kazhdan and Lusztig [20, §4, Corollary 2], we have equalities for the dimension of affine Springer fibers

$$\dim M_x^{\mathrm{par}}(\gamma) = \dim P_x(J_a) = \dim M_x^{\mathrm{Hit}}(\gamma)$$

where $\chi(\gamma) = a \in \mathfrak{c}(\widehat{\mathcal{O}}_x)$. Now the required statement follows from Proposition 2.4.1. □

2.5. Geometric properties of the parabolic Hitchin fibration

The results here are analogs of [29, Theorem 4.14.1 and Proposition 6.1.3] (which is based on Faltings's work [11]).

Proposition 2.5.1. *Recall that* $\deg(D) \geqslant 2g_X$*, then we have*:

(1) $\mathcal{M}^{\mathrm{par}}|_{\mathcal{A}^\heartsuit} \to X$ *is smooth*;
(2) $\mathcal{M}^{\mathrm{par}}|_{\mathcal{A}^\heartsuit}$ *is a smooth and equidimensional algebraic stack of dimension* $\dim \mathcal{M}^{\mathrm{Hit}} + 1$;
(3) $\mathcal{M}^{\mathrm{par}}|_{\mathcal{A}}$ *is a smooth Deligne–Mumford stack.*

Proof. (1) We first do several steps of dévissage and then reduce to the proof of [29, Theorem 4.14.1]. The deformation-theoretic calculations below is the parabolic version of Biswas–Ramanan's calculation for the Hitchin moduli [4]. Let R be an artinian local k-algebra and $I \subset R$ a square-zero ideal. Let $R_0 = R/I$. Fix a point $x \in X(R)$ with image $x_0 \in X(R_0)$ and a point $m_0 = (x_0, \mathcal{E}_0, \varphi_0, \mathcal{E}^B_{x_0}) \in \mathcal{M}^{\mathrm{par}}(R_0)$ over $x_0 \in X(R_0)$. To establish the smoothness of $\mathcal{M}^{\mathrm{par}} \to X$, we need to lift this point to a point $(x, \mathcal{E}, \varphi, \mathcal{E}^B_x) \in \mathcal{M}^{\mathrm{par}}(R)$. In other words, we have to find the dotted arrows in the following diagrams

$$\begin{array}{ccc} X_{R_0} & \xrightarrow{m_0} & [\mathfrak{g}/G]_D \\ \big\downarrow & \overset{m}{\nearrow} & \big\downarrow \\ X_R & \xrightarrow{pr_X} & X \end{array} \qquad \begin{array}{ccc} \operatorname{Spec} R_0 & \xrightarrow{m_{x_0}} & [\mathfrak{b}/B]_D \\ \big\downarrow & \overset{m_x}{\nearrow} & \big\downarrow \\ \operatorname{Spec} R & \xrightarrow{x} & X \end{array}$$

making the following diagram commutative:

$$\begin{array}{ccc} \operatorname{Spec} R & \xrightarrow{m_x} & [\mathfrak{b}/B]_D \\ \big\downarrow{\scriptstyle (\mathrm{id},x)} & & \big\downarrow{\scriptstyle \pi} \\ X_R & \xrightarrow{m} & [\mathfrak{g}/G]_D \end{array} \tag{2.6}$$

For a morphism of stacks $\mathfrak{X} \to \mathfrak{Y}$, we write $L_{\mathfrak{X}/\mathfrak{Y}}$ for its cotangent complex. The infinitesimal deformations m of m_0 are controlled by the complex $\mathbf{R}\mathrm{Hom}_{X_{R_0}}(m_0^* L_{[\mathfrak{g}/G]_D/X}, I \otimes_k \mathcal{O}_X)$; the infinitesimal deformations of m_x of $m_{0,x}$ are controlled by the complex $\mathbf{R}\mathrm{Hom}_{R_0} \times (m_{x_0}^* L_{[\mathfrak{b}/B]_D/X}, I)$. The condition (2.6) implies that the infinitesimal liftings of $(x_0, \mathcal{E}_0, \varphi_0, \mathcal{E}^B_{x_0}) \in \mathcal{M}^{\mathrm{par}}(R_0)$ to $\mathcal{M}^{\mathrm{par}}(R)$ over x are controlled by the complex $\mathcal{K}_I$ which fits into an exact triangle

$$\mathcal{K}_I \to \mathbf{R}\mathrm{Hom}_{X_{R_0}}\big(m_0^* L_{[\mathfrak{g}/G]_D/X}, I \otimes_k \mathcal{O}_X\big) \oplus \mathbf{R}\mathrm{Hom}_{R_0}\big(m_{x_0}^* L_{[\mathfrak{b}/B]_D/X}, I\big)$$
$$\xrightarrow{(i^*,-\pi^*)} \mathbf{R}\mathrm{Hom}_{R_0}\big(i^* m_0^* L_{[\mathfrak{g}/G]_D/X}, I\big) \to \mathcal{K}_I[1]$$

where i is the embedding $\Gamma(x_0) \hookrightarrow X_{R_0}$ and π^* is induced by natural map between cotangent complexes $\pi^* L_{[\mathfrak{g}/G]_D/X} \to L_{[\mathfrak{b}/B]_D/X}$. According to the calculation in [29, §4.14], we have

$$m_0^* L_{[\mathfrak{g}/G]_D/X} = \big[\mathrm{Ad}(\mathcal{E}_0)^\vee \xrightarrow{\mathrm{Ad}(\varphi_0)} \mathrm{Ad}(\mathcal{E}_0)^\vee(D)\big];$$
$$m_{x_0}^* L_{[\mathfrak{b}/B]_D/X} = \big[\mathrm{Ad}\big(\mathcal{E}_{x_0}^B\big)^\vee \xrightarrow{\mathrm{Ad}(\varphi_0(x_0))} \mathrm{Ad}\big(\mathcal{E}_{x_0}^B\big)^\vee \otimes i^*\mathcal{O}_X(D)\big].$$

Here the two-term complexes sit in degrees -1 and 0. We see that

$$\mathcal{K}_I \cong \big[\mathcal{F} \otimes_{R_0} I \xrightarrow{\mathrm{Ad}(\varphi_0)} \mathcal{F}(D) \otimes_{R_0} I\big]$$

where $\mathcal{F} = \ker(\mathrm{Ad}(\mathcal{E}_0) \twoheadrightarrow i_*(i^*\mathrm{Ad}(\mathcal{E}_0)/\mathrm{Ad}(\mathcal{E}_{x_0}^B)))$. Since both the source and the target of the above surjection are flat over R_0, $\mathcal{F}$ is also flat over R_0. Also, as a subsheaf of the vector bundle $\mathrm{Ad}(\mathcal{E}_0)$ over X_{R_0}, $\mathcal{F}$ is locally free over $\mathcal{O}_X$.

The obstruction to the lifting lies in $\mathrm{H}^1(X_{R_0}, \mathcal{K}_I)$. By writing I as a quotient of a free R_0-module, we see that to prove $\mathrm{H}^1(X_{R_0}, \mathcal{K}_I) = 0$, it suffices to prove $\mathrm{H}^1(X_{R_0}, \mathcal{K}) = 0$ for $\mathcal{K} = \mathcal{K}_{R_0} = [\mathcal{F} \xrightarrow{\mathrm{Ad}(\varphi_0)} \mathcal{F}(D)]$.

Let $\mathfrak{m}$ be the maximal ideal of R_0. Consider the $\mathfrak{m}$-adic filtration of the complex $\mathcal{K}$. Since $\mathcal{F}$ is flat over R_0, the associated graded pieces $\mathrm{Gr}^n \mathcal{K}$ of this filtration is

$$\mathrm{Gr}^n \mathcal{K} = \mathfrak{m}^n/\mathfrak{m}^{n+1} \otimes \big[\mathcal{F}/\mathfrak{m}\mathcal{F} \xrightarrow{\mathrm{Ad}(\overline{\varphi_0})} \mathcal{F}(D)/\mathfrak{m}\mathcal{F}(D)\big] \tag{2.7}$$

where $\overline{\varphi_0}$ is the reduction of φ_0 mod $\mathfrak{m}$. Thus, to prove $\mathrm{H}^1(X_{R_0}, \mathcal{K}) = 0$, it suffices to prove $\mathrm{H}^1(X_{R_0}, \mathrm{Gr}^n \mathcal{K}) = 0$ for each n. By the expression (2.7), we eventually reduce to showing $\mathrm{H}^1(X_{R_0}, \mathcal{K}) = 0$ in the case R_0 is a field. In this case, as in the proof of [29, Theorem 4.14.1], using Serre duality, we reduce to showing that $\mathrm{H}^0(X, \mathcal{K}') = 0$ where

$$\mathcal{K}' = \ker\big(\mathcal{F}^\vee(-D) \otimes \omega_X \xrightarrow{\mathrm{Ad}(\varphi_0)} \mathcal{F}^\vee \otimes \omega_X\big).$$

Here $\mathcal{F}^\vee = \mathbf{R}\,\underline{\mathrm{Hom}}_X(\mathcal{F}, \mathcal{O}_X)$ is a subsheaf of $\mathrm{Ad}(\mathcal{E}_0)^\vee(x_0)$. Therefore

$$\mathcal{K}' \subset \mathcal{K}'' = \ker\big(\mathrm{Ad}(\mathcal{E}_0)^\vee(-D + x_0) \otimes \omega_X \xrightarrow{\mathrm{Ad}(\varphi_0)} \mathrm{Ad}(\mathcal{E}_0)^\vee(x_0) \otimes \omega_X\big).$$

In [29, Theorem 4.14.1], Ngô proves that $\mathrm{H}^0(X, \mathcal{K}'')$, as the obstruction to the lifting problem for the Hitchin moduli stack $\mathcal{M}^{\mathrm{Hit}}_{G,X,D-x_0}$, vanishes whenever $\deg(D - x_0) > 2g_X - 2$. In our case, $\deg(D) \geqslant 2g_X$ so this condition holds, therefore $\mathrm{H}^0(X, \mathcal{K}'') = 0$, hence $\mathrm{H}^0(X, \mathcal{K}') = 0$. This proves the vanishing of the obstruction group $\mathrm{H}^1(X_{R_0}, \mathcal{K}_I)$ in general, and completes the proof of smoothness of $\mathcal{M}^{\mathrm{par}} \to X$.

(2) The relative dimension of $\mathcal{M}^{\mathrm{par}} \to X$ at a k-point $(x_0, \mathcal{E}_0, \varphi_0, \mathcal{E}_{x_0}^B)$ is the Euler characteristic $\chi(X, \mathcal{K})$ of the complex $\mathrm{H}^*(X, \mathcal{K})$. Recall that $\mathcal{K}$ fits into the distinguished triangle $\mathcal{K} \to \widetilde{\mathcal{K}} \to \mathcal{Q} \to \mathcal{K}[1]$ where

$$\widetilde{\mathcal{K}}=\left[\mathrm{Ad}(\mathcal{E}_0)\xrightarrow{\mathrm{Ad}(\varphi_0)}\mathrm{Ad}(\mathcal{E}_0)(D)\right];$$

$$\mathcal{Q}=i_*\left[i^*\mathrm{Ad}(\mathcal{E}_0)/\mathrm{Ad}\left(\mathcal{E}^B_{x_0}\right)\xrightarrow{\mathrm{Ad}(\varphi_0)}i^*\mathrm{Ad}(\mathcal{E}_0)/\mathrm{Ad}\left(\mathcal{E}^B_{x_0}\right)\otimes i^*\mathcal{O}_X(D)\right].$$

It is clear that $\chi(X,\mathcal{Q})=0$. Therefore $\chi(X,\mathcal{K})=\chi(X,\widetilde{\mathcal{K}})$. But by the calculation in [29, §4.14], $\chi(X,\widetilde{\mathcal{K}})$ is the dimension of $\mathcal{M}^{\mathrm{Hit}}_{G,X,D}$ at the k-point $(\mathcal{E}_0,\varphi_0)$. Since $\mathcal{M}^{\mathrm{Hit}}$ is equidimensional by [29, Corollary 4.16.3], $\mathcal{M}^{\mathrm{par}}\to X$ is also equidimensional of relative dimension equal to $\dim\mathcal{M}^{\mathrm{Hit}}$.

(3) By [29, Proposition 6.1.3], $\mathcal{M}^{\mathrm{Hit}}|_{\mathcal{A}}$ is Deligne–Mumford. By Lemma 2.1.2, $\pi_{\mathcal{M}}:\mathcal{M}^{\mathrm{par}}\to\mathcal{M}^{\mathrm{Hit}}\times X$ is schematic and of finite type because the Grothendieck simultaneous resolution π is. Therefore $\mathcal{M}^{\mathrm{par}}|_{\mathcal{A}}$ is Deligne–Mumford. □

Corollary 2.5.2. *The parabolic Hitchin fibrations $f^{\mathrm{par}}:\mathcal{M}^{\mathrm{par}}|_{\mathcal{A}^\heartsuit}\to\mathcal{A}^\heartsuit\times X$ and $\tilde{f}:\mathcal{M}^{\mathrm{par}}|_{\mathcal{A}^\heartsuit}\to\widetilde{\mathcal{A}}^\heartsuit$ are flat. The fibers are local complete intersections. The restrictions of f^{par} and $\tilde{f}$ to $\mathcal{A}$ are proper.*

Proof. The source and the target of the maps f^{par} and $\tilde{f}$ are smooth and equidimensional. In both cases, the relative dimensions are equal to $\dim\mathcal{M}^{\mathrm{Hit}}-\dim\mathcal{A}^{\mathrm{Hit}}$. By Corollary 2.4.2, the dimension of the fibers of f^{par} and $\tilde{f}$ are also equal to the dimension of Hitchin fibers, which, in turn, equals to $\dim\mathcal{M}^{\mathrm{Hit}}-\dim\mathcal{A}^{\mathrm{Hit}}$ by [29, Proposition 4.16.4]. The fibers of f^{par} and $\tilde{f}$ are hence local complete intersections. By [27, Theorem 23.1], both maps are flat.

By [29, Proposition 6.1.3], $\mathcal{M}^{\mathrm{Hit}}|_{\mathcal{A}}$ is proper over $\mathcal{A}$. By Lemma 2.1.2, $\mathcal{M}^{\mathrm{par}}|_{\mathcal{A}}$ is proper over $\mathcal{M}^{\mathrm{Hit}}|_{\mathcal{A}}\times X$ and $\mathcal{M}^{\mathrm{Hit}}|_{\mathcal{A}}\times_{\mathcal{A}}\widetilde{\mathcal{A}}$ because they are obtained by base change from the proper maps $[\mathfrak{b}/B]_D\to[\mathfrak{g}/G]_D$ and $[\mathfrak{b}/B]_D\to[\mathfrak{g}/G]_D\times_{\mathfrak{c}_D}\mathfrak{t}_D$. Therefore $\mathcal{M}^{\mathrm{par}}|_{\mathcal{A}}$ is proper over $\mathcal{A}\times X$ and $\widetilde{\mathcal{A}}$. □

Remark 2.5.3. The parabolic Hitchin fibers $\mathcal{M}^{\mathrm{par}}_{a,x}$ are not reduced in general. For example, suppose $q_a:X_a\to X$ is ramified over x and $q_a^{-1}(x)$ consists of smooth points of X_a, then $\mathcal{M}^{\mathrm{par}}_{a,x}=\mathcal{M}^{\mathrm{Hit}}_a\times q_a^{-1}(x)$ is not reduced because $q_a^{-1}(x)$ is not.

2.6. Stratification by δ and codimension estimate

Recall that in [29, §3.7 and §4.9], Ngô introduced local and global Serre invariants δ.

The local Serre invariant [29, §3.7] assigns to every $(a,x)\in(\mathcal{A}^\heartsuit\times X)(k)$ the dimension of the corresponding affine Springer fiber $M^{\mathrm{Hit}}_x(\gamma_{a,x})$ (see Section 2.4), or, equivalently, the dimension of the local symmetry group $P_x(J_a)$. It defines an upper semi-continuous function

$$\delta:\mathcal{A}^\heartsuit\times X\to\mathbb{Z}_{\geqslant 0}.$$

For an integer $\delta\geqslant 0$, let $(\mathcal{A}^\heartsuit\times X)_\delta$ be the δ-level set.

The global invariant is the dimension of the affine part of the Picard stack $\mathcal{P}_a$. It defines an upper semi-continuous function

$$\delta:\mathcal{A}^\heartsuit\to\mathbb{Z}_{\geqslant 0}.$$

For an integer $\delta \geqslant 0$, let $\mathcal{A}^{\heartsuit}_{\delta}$ be the δ-level set of $\mathcal{A}^{\heartsuit}$. The local and global Serre invariants are related by

$$\delta(a) = \sum_{x \in X(k)} \delta(a, x).$$

The following two lemmas will only be used in the technical part of Section 3.2.

Lemma 2.6.1. *The open subset $(\mathcal{A}^{\heartsuit} \times X)_0$ is precisely the locus of $\mathcal{A}^{\heartsuit} \times X$ where $\nu_{\mathcal{M}} : \mathcal{M}^{\mathrm{par}}|_{\mathcal{A}^{\heartsuit}} \to \mathcal{M}^{\mathrm{Hit}} \times_{\mathcal{A}^{\heartsuit}} \widetilde{\mathcal{A}}^{\heartsuit}$ is an isomorphism.*

Proof. We need to check that for a geometric point $(a, x) \in \mathcal{A}^{\heartsuit}(k) \times X(k)$, $\nu_{\mathcal{M}}$ is an isomorphism over (a, x) if and only if $\delta(a, x) = 0$. By the left-most Cartesian square of the diagram (2.3), $\nu_{\mathcal{M}}$ is an isomorphism over a geometric point $(\mathcal{E}, \varphi, x) \in \mathcal{M}^{\mathrm{Hit}} \times X$ if and only if $\nu : [\mathfrak{b}/B] \to [\mathfrak{g}/G] \times_{\mathfrak{c}} \mathfrak{t}$ is an isomorphism over $\varphi(x) \in \mathfrak{g}$. By the result of Kostant (cf. [21]), this is equivalent to saying that $\varphi(x) \in \mathfrak{g}^{\mathrm{reg}}$, or that $\mathrm{ev}_x : \mathcal{M}^{\mathrm{Hit}}_a \to [\mathfrak{g}/G]_D$ lands in $[\mathfrak{g}^{\mathrm{reg}}/G]$. Let $\gamma = \gamma_{a,x} \in \mathfrak{g}(\widehat{\mathcal{O}}_x)$ be chosen as in Proposition 2.4.1. Recall from [29, §3.3] that we have an open sub-ind-scheme $M^{\mathrm{Hit,reg}}_x(\gamma) \subset M^{\mathrm{Hit}}_x(\gamma)$ defined in a similar way as $\mathcal{M}^{\mathrm{Hit,reg}}_a$, and which is a torsor under $P_x(J_a)$. The above discussion implies that $\nu_{\mathcal{M}}$ is an isomorphism over (a, x) if and only if $M^{\mathrm{Hit}}_x(\gamma) = M^{\mathrm{Hit,reg}}_x(\gamma)$. We have to show that this condition is equivalent to $\delta(a, x) = 0$.

If $\delta(a, x) = 0$, then $M^{\mathrm{Hit}}_x(\gamma) = M^{\mathrm{Hit,reg}}_x(\gamma)$ by [29, Corollary 3.7.2].

Conversely, suppose $M^{\mathrm{Hit}}_x(\gamma) = M^{\mathrm{Hit,reg}}_x(\gamma)$. According to [20] (see [29, Proposition 3.4.1]), there is a lattice subgroup $\Lambda \subset P_x(J_a)$ which acts freely on $M^{\mathrm{Hit}}_x(\gamma)$ such that $\Lambda \setminus M^{\mathrm{Hit,red}}_x(\gamma)$ is a projective variety. In this case, $\Lambda \setminus P^{\mathrm{red}}_x(J_a)$ is an affine group scheme (since it is a finite disjoint union of the affine group scheme denoted by $\mathcal{R}_x(a)$ in [29, Lemme 3.8.1]). Therefore the quotient $\Lambda \setminus M^{\mathrm{Hit,red}}_x(\gamma) = \Lambda \setminus M^{\mathrm{Hit,reg,red}}_x(\gamma)$ is both proper and affine since the latter is a torsor under $\Lambda \setminus P^{\mathrm{red}}_x(J_a)$. Hence they must be zero-dimensional, i.e., $\delta(a, x) = 0$. This completes the proof. □

Lemma 2.6.2. *The open subset $\mathcal{A}^{\heartsuit}_0$ of $\mathcal{A}^{\heartsuit}$ consists precisely of points $a \in \mathcal{A}^{\heartsuit}$ where $\mathcal{M}^{\mathrm{Hit,reg}}_a = \mathcal{M}^{\mathrm{Hit}}_a$.*

Proof. Since $\mathcal{M}^{\mathrm{Hit,reg}}_a$ is open in $\mathcal{M}^{\mathrm{Hit}}_a$, it is the whole of $\mathcal{M}^{\mathrm{Hit,reg}}_a$ if and only if its underlying topological space is the whole of $\mathcal{M}^{\mathrm{Hit}}_a$. By the product formula [29, Proposition 4.15.1], this is true if and only if $M^{\mathrm{Hit,reg}}_x(\gamma_{a,x}) = M^{\mathrm{Hit}}_x(\gamma_{a,x})$ for all $x \in X$. By the proof of Lemma 2.6.1, this is in turn equivalent to $\delta(a, x) = 0$ for all $x \in X$; i.e., $\delta(a) = 0$. □

We have the following codimension estimate:

Proposition 2.6.3 *(Consequence of [29, Proposition 5.7.1], which is based on [14]). For each $\delta_0 \geqslant 0$, there is an integer $d_0 > 0$ (depending on δ_0) such that whenever $\deg(D) > d_0$ and $0 < \delta \leqslant \delta_0$, we have*

$$\operatorname{codim}_{\mathcal{A}^{\heartsuit} \times X} \big(\mathcal{A}^{\heartsuit} \times X\big)_{\delta} \geqslant \delta + 1. \tag{2.8}$$

Proof. Let $\mathfrak{c}_{D,N}$ be the N-th jet bundle of the bundle $\mathfrak{c}_D$ over X: for $x \in X$, the fiber of $\mathfrak{c}_{D,N}$ over x is $\Gamma(\operatorname{Spec}\mathcal{O}_x/\mathfrak{m}_x^{N+1}, \mathfrak{c}_D)$ ($\mathfrak{m}_x$ is the maximal ideal of the local ring $\mathcal{O}_x$). We can define $\mathfrak{c}_{D,\infty}$ as the inverse limit of $\mathfrak{c}_{D,N}$ as $N \to \infty$. The local Serre invariant also gives a stratification $\bigsqcup \mathfrak{c}_{D,\infty,\delta}$ of $\mathfrak{c}_{D,\infty}$. According to [29, Proposition 5.7.1], when $\delta > 0$, the codimension of $\mathfrak{c}_{D,\infty,\delta}$ in $\mathfrak{c}_{D,\infty}$ is at least $\delta + 1$. For $\delta \leqslant \delta_0$, there is an $N = N(\delta_0) \geqslant 0$, such that $\mathfrak{c}_{D,\infty,\delta}$ is the preimage of a certain $\mathfrak{c}_{D,N,\delta} \subset \mathfrak{c}_{D,N}$ under the projection $\mathfrak{c}_{D,\infty} \to \mathfrak{c}_{D,N}$. Consider the evaluation map

$$\mathrm{ev}_N : \mathcal{A}^{\heartsuit} \times X \to \mathfrak{c}_{D,N}.$$

Since $\mathfrak{c}_D = \sum_{i=1}^n \mathcal{O}_X(d_i D)$, the map ev_N is surjective when $\deg(D) > d_0$, which depends on N, hence on δ_0. Therefore, for $0 < \delta \leqslant \delta_0$

$$\operatorname{codim}_{\mathcal{A}^{\heartsuit}\times X}\left(\mathcal{A}^{\heartsuit} \times X\right)_\delta = \operatorname{codim}_{\mathfrak{c}_{D,N}}(\mathfrak{c}_{D,N,\delta}) = \operatorname{codim}_{\mathfrak{c}_{D,\infty}}(\mathfrak{c}_{D,\infty,\delta}) \geqslant \delta + 1. \qquad \square$$

Remark 2.6.4. The main result in Section 3 will depend on this codimension estimate. For this purpose, there we will fix an open subset $(\mathcal{A}^{\heartsuit} \times X)' \subset \mathcal{A}^{\heartsuit} \times X$ on which the estimate

$$\operatorname{codim}_{(\mathcal{A}^{\heartsuit}\times X)'}\left(\mathcal{A}^{\heartsuit} \times X\right)'_\delta \geqslant \delta + 1$$

holds for any $\delta \in \mathbb{Z}_{\geqslant 0}$. According to Proposition 2.6.3, we can take $(\mathcal{A}^{\heartsuit} \times X)'$ to be the union of $(\mathcal{A}^{\heartsuit} \times X)_\delta$ for all $\delta \leqslant \delta_0$, as long as $\deg(D) \geqslant d_0$. In particular, we can take $(\mathcal{A}^{\heartsuit} \times X)'$ to be $\mathcal{A}' \times X$ for a suitable open subset $\mathcal{A}' \subset \mathcal{A}^{\heartsuit}$.

According to an unpublished argument of Ngô, when $\mathrm{char}(k) = 0$, one can take $(\mathcal{A}^{\heartsuit} \times X)' = \mathcal{A}^{\heartsuit} \times X$, i.e., the codimension estimate (2.8) always holds in this case.

3. The affine Weyl group action—the first construction

In this section, we prove the first main result: the affine Weyl group action on the parabolic Hitchin complex.

3.1. Hecke correspondences

We first recall the definition of the *Hecke correspondence* between $\mathrm{Bun}_G^{\mathrm{par}}$ and itself over X:

$$\begin{array}{ccccc} & & \mathcal{H}\mathrm{ecke}^{\mathrm{Bun}} & & \\ & \overleftarrow{b} \swarrow & & \searrow \overrightarrow{b} & \\ \mathrm{Bun}_G^{\mathrm{par}} & \longrightarrow & X & \longleftarrow & \mathrm{Bun}_G^{\mathrm{par}} \end{array}$$

For any scheme S, $\mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}(S)$ classifies tuples $(x, \mathcal{E}_1, \mathcal{E}_{1,x}^B, \mathcal{E}_2, \mathcal{E}_{2,x}^B, \alpha)$ where

- $(x, \mathcal{E}_i, \mathcal{E}_{i,x}^B) \in \mathrm{Bun}_G^{\mathrm{par}}(S)$ for $i = 1, 2$;
- $\alpha : \mathcal{E}_1|_{S\times X-\Gamma(x)} \xrightarrow{\sim} \mathcal{E}_2|_{S\times X-\Gamma(x)}$ is an isomorphism of G-torsors.

Definition 3.1.1. The *Hecke correspondence* $\mathcal{H}\mathrm{ecke}^{\mathrm{par}}$ is a self-correspondence of $\mathcal{M}^{\mathrm{par}}$ over $\mathcal{A}^{\mathrm{Hit}} \times X$:

$$\begin{array}{ccccc} & & \mathcal{H}\mathrm{ecke}^{\mathrm{par}} & & \\ & \swarrow^{\overleftarrow{h}} & & \searrow^{\overrightarrow{h}} & \\ \mathcal{M}^{\mathrm{par}} & \xrightarrow[f^{\mathrm{par}}]{} & \mathcal{A}^{\mathrm{Hit}} \times X & \xleftarrow[f^{\mathrm{par}}]{} & \mathcal{M}^{\mathrm{par}} \end{array}$$

For any scheme S, $\mathcal{H}\mathrm{ecke}^{\mathrm{par}}(S)$ classifies tuples $(x, \mathcal{E}_1, \varphi_1, \mathcal{E}^B_{1,x}, \mathcal{E}_2, \varphi_2, \mathcal{E}^B_{2,x}, \alpha)$ where

- $(x, \mathcal{E}_i, \varphi_i, \mathcal{E}^B_{i,x}) \in \mathcal{M}^{\mathrm{par}}(S)$ for $i = 1, 2$;
- α is an isomorphism of Hitchin pairs $(\mathcal{E}_1, \varphi_1)|_{S\times X-\Gamma(x)} \xrightarrow{\sim} (\mathcal{E}_2, \varphi_2)|_{S\times X-\Gamma(x)}$.

Comparing with the definition of $\mathcal{H}\mathrm{ecke}^{\mathrm{par}}$ in Definition 3.1.1, we have a commutative diagram of correspondences

$$\begin{array}{ccc} \mathcal{H}\mathrm{ecke}^{\mathrm{par}} & \xrightarrow{\beta} & \mathcal{H}\mathrm{ecke}^{\mathrm{Bun}} \\ \overleftarrow{h} \downarrow\downarrow \overrightarrow{h} & & \overleftarrow{b} \downarrow\downarrow \overrightarrow{b} \\ \mathcal{M}^{\mathrm{par}} & \longrightarrow & \mathrm{Bun}_G^{\mathrm{par}} \\ \downarrow f^{\mathrm{par}} & & \downarrow \\ \mathcal{A}^{\mathrm{Hit}} \times X & \longrightarrow & X \end{array}$$

Lemma 3.1.2. *The functor* $\mathcal{H}\mathrm{ecke}^{\mathrm{par}}$ *is representable by an ind-algebraic stack of ind-finite type, and the two projections* $\overleftarrow{h}, \overrightarrow{h} : \mathcal{H}\mathrm{ecke}^{\mathrm{par}} \to \mathcal{M}^{\mathrm{par}}$ *are ind-proper.*

Proof. From the definitions, we see that the fibers of $\overleftarrow{h}, \overrightarrow{h}$ are closed sub-ind-schemes of the fibers of $\overleftarrow{b}, \overrightarrow{b}$. Hence it suffices to check the same statement for $\mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}$.

Here we need some notation which will be introduced later in the article. In (5.3) (Section 5.2), we will define a stratification of $\mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}$ indexed by elements $\tilde{w} \in \widetilde{W}$. Let $\mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}_{\leqslant \tilde{w}}$ be the closure of $\mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}_{\tilde{w}}$. Then the projections

$$\overleftarrow{b}, \overrightarrow{b} : \mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}_{\leqslant \tilde{w}} \to \mathrm{Bun}_G^{\mathrm{par}}$$

are étale locally trivial bundles with fibers isomorphic to Schubert cycles in the affine flag variety Fl_G. In particular, $\mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}_{\leqslant \tilde{w}}$ is proper over $\mathrm{Bun}_G^{\mathrm{par}}$ for both projections. □

Let us describe the fibers of $\overleftarrow{h}$ and $\overrightarrow{h}$. Let $(x, \mathcal{E}, \varphi, \mathcal{E}^B_x) \in \mathcal{M}^{\mathrm{par}}(k)$. After trivializing $\mathcal{E}|_{\mathfrak{D}_x}$ and choosing an isomorphism $\widehat{\mathcal{O}}_x \cong \widehat{\mathcal{O}}_x(D)$, we get an isomorphism $\tau : (\mathcal{E}, \varphi)|_{\mathfrak{D}_x} \xrightarrow{\sim} (\mathcal{E}^{\mathrm{triv}}, \gamma_{a,x})$ for some $\gamma_{a,x} \in \mathfrak{g}(\widehat{\mathcal{O}}_x)$ such that $\chi(\gamma_{a,x}) = a \in \mathfrak{c}(\widehat{\mathcal{O}}_x)$.

Lemma 3.1.3. *The fibers of* $\overleftarrow{h}$ *and* $\overrightarrow{h}$ *over* $(x, \mathcal{E}, \varphi, \mathcal{E}^B_x) \in \mathcal{M}^{\mathrm{par}}(k)$ *are isomorphic to the affine Springer fiber* $M^{\mathrm{par}}_x(\gamma_{a,x})$.

Proof. We prove the statement for $\overrightarrow{h}$; the other one is similar. For any scheme S, we have a natural map

$$\begin{aligned}&\overrightarrow{h}^{-1}\big(x,\mathcal{E},\varphi,\mathcal{E}^B_x\big)(S)\to M^{\mathrm{par}}_x(\gamma_{a,x})(S),\\&\big(\mathcal{E}_1,\varphi_1,\mathcal{E}^B_{1,x},\alpha\big)\mapsto\big(\mathcal{E}_1|_{\mathfrak{D}_x},\varphi_1|_{\mathfrak{D}_x},\mathcal{E}^B_{1,x},\beta\big)\end{aligned}\tag{3.1}$$

where $\alpha:(\mathcal{E}_1,\varphi_1)|_{(X-\{x\})\times S}\xrightarrow{\sim}(\mathcal{E},\varphi)|_{(X-\{x\})\times S}$ and

$$\beta:(\mathcal{E}_1,\varphi_1)|_{\mathfrak{D}^\times_x\widehat{\times}S}\xrightarrow{\alpha}(\mathcal{E},\varphi)|_{\mathfrak{D}^\times_x\widehat{\times}S}\xrightarrow{\tau}\big(\mathcal{E}^{\mathrm{triv}},\gamma_{a,x}\big).$$

Now it is easy to see that (3.1) is injective; it is also surjective because any local modification of $(\mathcal{E}^{\mathrm{triv}},\gamma_{a,x})$ on $\mathfrak{D}^\times_x\widehat{\times}S$ can be glued with $(\mathcal{E},\varphi)|_{(X-\{x\})\times S}$ to get a Hitchin pair on X. Therefore (3.1) is an isomorphism for any S. □

Lemma 3.1.4. *Any finite type substack of* $\mathcal{H}\mathrm{ecke}^{\mathrm{par}}|_{(\mathcal{A}^\heartsuit\times X)'}$ *satisfies the condition* (G-2) *in Definition* A.6.1 *with respect to* $(\mathcal{A}^\heartsuit\times X)^{\mathrm{rs}}\subset(\mathcal{A}^\heartsuit\times X)'$.

Proof. First fix an integer $\delta>0$, let $\mathcal{M}^{\mathrm{par}}_\delta\subset\mathcal{M}^{\mathrm{par}}$ and $\mathcal{H}\mathrm{ecke}^{\mathrm{par}}_\delta\subset\mathcal{H}\mathrm{ecke}^{\mathrm{par}}$ be the preimages of $(\mathcal{A}^\heartsuit\times X)'_\delta$, the level set of the local Serre invariant. By Proposition 2.6.3, $\mathrm{codim}_{(\mathcal{A}^\heartsuit\times X)'}((\mathcal{A}^\heartsuit\times X)'_\delta)\geqslant\delta+1$. Since f^{par} is flat, we have $\mathrm{codim}_{\mathcal{M}^{\mathrm{par}}}(\mathcal{M}^{\mathrm{par}}_\delta)\geqslant\delta+1$.

By Lemma 3.1.3, the fibers of $\overleftarrow{h}:\mathcal{H}\mathrm{ecke}^{\mathrm{par}}_\delta\to\mathcal{M}^{\mathrm{par}}_\delta$ affine Springer fibers associated to elements $\gamma_{a,x}\in\mathfrak{g}(\mathcal{O}_x)$, which have dimensional $\delta(a,x)=\delta$. Therefore the relative dimension of $\overleftarrow{h}:\mathcal{H}\mathrm{ecke}^{\mathrm{par}}_\delta\to\mathcal{M}^{\mathrm{par}}_\delta$ is δ. Therefore we conclude $\dim\mathcal{H}\mathrm{ecke}^{\mathrm{par}}_\delta\leqslant\dim\mathcal{M}^{\mathrm{par}}-\delta-1+\delta=\dim\mathcal{M}^{\mathrm{par}}-1$.

Next we consider the locus $\delta=0$. Let $V=(\mathcal{A}^\heartsuit\times X)'_0-(\mathcal{A}^\heartsuit\times X)^{\mathrm{rs}}$ and $\mathcal{M}^{\mathrm{par}}_V\subset\mathcal{M}^{\mathrm{par}}$, $\mathcal{H}\mathrm{ecke}^{\mathrm{par}}_V\subset\mathcal{H}\mathrm{ecke}^{\mathrm{par}}$ be the preimages. Then obviously $\mathrm{codim}_{(\mathcal{A}\times X)'}(V)=\mathrm{codim}_{\mathcal{M}^{\mathrm{par}}}(\mathcal{M}^{\mathrm{par}}_V)\geqslant1$. Since the fibers of $\overleftarrow{h}:\mathcal{H}\mathrm{ecke}^{\mathrm{par}}_V\to\mathcal{M}^{\mathrm{par}}_V$ are zero-dimensional affine Springer fibers by Lemma 3.1.3, we still have $\dim\mathcal{H}\mathrm{ecke}^{\mathrm{par}}_V\leqslant\dim\mathcal{M}^{\mathrm{par}}-1$. □

3.2. Hecke correspondences over the nice locus

In this subsection, we determine the structure of the Hecke correspondence $\mathcal{H}\mathrm{ecke}^{\mathrm{par}}$ over the locus $(\mathcal{A}^\heartsuit\times X)_0$. For a stack $\mathfrak{X}$ or a morphism F over $\mathcal{A}^{\mathrm{Hit}}\times X$, we use $\mathfrak{X}^0$ and F^0 to denote their restrictions on the open subset $(\mathcal{A}^\heartsuit\times X)_0\subset\mathcal{A}^{\mathrm{Hit}}\times X$ (the locus where $\delta(a,x)=0$). Also, see Section 1.6.3 for the meaning of $(-)^{\mathrm{rs}}$. For example, we have $\widetilde{\mathcal{A}}^\heartsuit\supset\widetilde{\mathcal{A}}^0\supset\widetilde{\mathcal{A}}^{\mathrm{rs}}$ (here we abbreviate $\widetilde{\mathcal{A}}^{\mathrm{Hit},0}$ to $\widetilde{\mathcal{A}}^0$).

The goal is to prove

Proposition 3.2.1. *There exists a* right *action of* $\widetilde{W}$ *on* $\mathcal{M}^{\mathrm{par},0}$ *over* $(\mathcal{A}^\heartsuit\times X)_0$ *such that the reduced structure of* $\mathcal{H}\mathrm{ecke}^{\mathrm{par,rs}}$ *is the disjoint union of the graphs of this* $\widetilde{W}$*-action.*

Example 3.2.2. We describe the $\widetilde{W}$-action on $\mathcal{M}^{\mathrm{par},0}$ in the case $G=\mathrm{GL}(n)$. We continue with the notation in Example 2.2.5. Notice that $(a,x)\in(\mathcal{A}^\heartsuit\times X)_0$ if and only if the spectral curve Y_a is smooth at the points $p_a^{-1}(x)$. In this case, a chain of coherent sheaves as in (2.4) is determined by the sequence of points $\mathrm{Supp}(\mathcal{F}_i/\mathcal{F}_{i+1})\in p_a^{-1}(x)$. Therefore $\mathcal{M}^{\mathrm{par}}_{a,x}$ consists of $\mathcal{F}_0\in\overline{\mathcal{P}\mathrm{ic}}(Y_a)$ together with an ordering $(y_1,\dots,y_n)$ of the multi-set $p_a^{-1}(x)$. For $w\in W=S_n$, its action on

$\mathcal{M}^{\mathrm{par}}_{a,x}$ is the change of the ordering. For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n = \mathbb{X}_*(T)$, its action on $\mathcal{M}^{\mathrm{par}}_{a,x}$ is given by tensoring $\mathcal{F}_0$ with the line bundle $\mathcal{O}_{Y_a}(\lambda_1 y_1 + \cdots + \lambda_n y_n)$, leaving the ordering unchanged.

The proof of Proposition 3.2.1 will occupy the rest of the subsection, which the readers can safely skip if he only cares about groups of type A.

Consider the map

$$\mathcal{H}\mathrm{ecke}^{\mathrm{par}} \xrightarrow{\overleftrightarrow{h}} \mathcal{M}^{\mathrm{par}} \times_{\mathcal{A}^{\mathrm{Hit}} \times X} \mathcal{M}^{\mathrm{par}} \xrightarrow{(\tilde{f}, \tilde{f})} \widetilde{\mathcal{A}}^{\mathrm{Hit}} \times_{\mathcal{A}^{\mathrm{Hit}} \times X} \widetilde{\mathcal{A}}^{\mathrm{Hit}} \to \mathfrak{t}_D \times_{\mathfrak{c}_D} \mathfrak{t}_D. \tag{3.2}$$

For $w \in W$, let $\mathfrak{t}_{w,D} \subset \mathfrak{t}_D \times_{\mathfrak{c}_D} \mathfrak{t}_D$ be the graph of the *right* w-action on $\mathfrak{t}_D$, i.e., $\mathfrak{t}_{w,D}$ consists of points $(t, w^{-1}t)$. Let $\mathcal{H}\mathrm{ecke}^{\mathrm{par}}_{[w]} \subset \mathcal{H}\mathrm{ecke}^{\mathrm{par}}$ be the preimage of $\mathfrak{t}_{w,D}$ under the map (3.2). They are disjoint over $\mathcal{M}^{\mathrm{par,rs}}$.

Consider $\mathcal{H}\mathrm{ecke}^{\mathrm{St}} := \mathcal{M}^{\mathrm{par}} \times_{\mathcal{M}^{\mathrm{Hit}} \times X} \mathcal{M}^{\mathrm{par}}$ as a self-correspondence of $\mathcal{M}^{\mathrm{par}}$, where St stands for Steinberg. Then we have an embedding of correspondences $\mathcal{H}\mathrm{ecke}^{\mathrm{St}} \subset \mathcal{H}\mathrm{ecke}^{\mathrm{par}}$ by identifying $\mathcal{H}\mathrm{ecke}^{\mathrm{St}}$ with those Hecke modifications which only modifies the Borel reduction. Let $\mathcal{H}\mathrm{ecke}^{\mathrm{St}}_w := \mathcal{H}\mathrm{ecke}^{\mathrm{par}}_{[w]} \cap \mathcal{H}\mathrm{ecke}^{\mathrm{St}}$.

The following lemma is immediate from definition.

Lemma 3.2.3. *Over $(\mathcal{A}^{\heartsuit} \times X)_0$, we identify $\mathcal{M}^{\mathrm{par},0}$ with $\mathcal{M}^{\mathrm{Hit}} \times_{\mathcal{A}} \widetilde{\mathcal{A}}^0$ by Lemma* 2.6.1. *Then $\mathcal{H}\mathrm{ecke}^{\mathrm{St},0}_w$ is the graph of the right w-action on the second factor of $\mathcal{M}^{\mathrm{Hit}} \times_{\mathcal{A}^{\heartsuit}} \widetilde{\mathcal{A}}^0$.*

We define the *Beilinson–Drinfeld Grassmannian* Gr_J of the group scheme J over $\mathcal{A}^{\mathrm{Hit}} \times X$ as the functor which sends a scheme S to the set of isomorphism classes of quadruples (a, x, Q^J, τ) where $(a, x) \in \mathcal{A}^{\mathrm{Hit}}(S) \times X(S)$, Q^J is a J_a-torsor over $S \times X$ and τ a trivialization of Q^J over $S \times X - \Gamma(x)$. The fiber of Gr_J over $(a, x) \in (\mathcal{A}^{\mathrm{Hit}} \times X)(k)$ is canonically isomorphic to the local symmetry group $P_x(J_a)$ defined in Section 2.4.

Let $\widetilde{\mathrm{Gr}}_J$ be the pull back of Gr_J from $\mathcal{A}^{\mathrm{Hit}} \times X$ to $\widetilde{\mathcal{A}}^{\mathrm{Hit}}$. Since J is commutative, $\widetilde{\mathrm{Gr}}_J$ is naturally a group ind-scheme over $\widetilde{\mathcal{A}}^{\mathrm{Hit}}$. We have a homomorphism $\widetilde{\mathrm{Gr}}_J \to \widetilde{\mathcal{P}}$ of sheaves of groups over $\widetilde{\mathcal{A}}^{\mathrm{Hit}}$ by forgetting the trivialization. Since $\widetilde{\mathcal{P}}$ acts on $\mathcal{M}^{\mathrm{par}}$ over $\widetilde{\mathcal{A}}^{\mathrm{Hit}}$ by Lemma 2.3.3, we get an action of $\widetilde{\mathrm{Gr}}_J$ on $\mathcal{H}\mathrm{ecke}^{\mathrm{par}}$ by changing the second factor $(\mathcal{E}_2, \varphi_2, \mathcal{E}^B_{2,x})$ via the homomorphism $\widetilde{\mathrm{Gr}}_J \to \widetilde{\mathcal{P}}$. This (left) action preserves the maps $\overleftarrow{h}$ and $\tilde{f} \circ \overrightarrow{h}$. In particular, $\widetilde{\mathrm{Gr}}_J$ acts on $\mathcal{H}\mathrm{ecke}^{\mathrm{par}}_{[w]}$ for each $w \in W$.

Lemma 3.2.4. *Over $(\mathcal{A}^{\heartsuit} \times X)_0$, $\overleftarrow{h}^0_{[w]} : \mathcal{H}\mathrm{ecke}^{\mathrm{par},0}_{[w]} \to \mathcal{M}^{\mathrm{par},0}$ is a left $\widetilde{\mathrm{Gr}}^0_J$-torsor with a canonical trivialization given by the section $\mathcal{M}^{\mathrm{par},0} \overset{\overleftarrow{h}^{\mathrm{St},0}_w}{\cong} \mathcal{H}\mathrm{ecke}^{\mathrm{St},0}_w \hookrightarrow \mathcal{H}\mathrm{ecke}^{\mathrm{par},0}_{[w]}$.*

Proof. By Lemma 3.2.3, $\overleftarrow{h}^{\mathrm{St},0}_w$ is an isomorphism and hence $\mathcal{H}\mathrm{ecke}^{\mathrm{St},0}_w$ gives a section of $\overleftarrow{h}^0_{[w]}$. To prove the lemma, we only need to show that for any $m = (x, \mathcal{E}_1, \varphi_1, \mathcal{E}^B_{1,x}) \in \mathcal{M}^{\mathrm{par}}(S)$ over $(a, x) \in (\mathcal{A}^{\heartsuit} \times X)_0(S)$, the fiber $(\overleftarrow{h}_{[w]})^{-1}(m)$ is a $P_x(J_a)$-torsor (although $P_x(J_a)$ was only defined for geometric points (a, x) in Section 2.4, the definition makes sense for any S-point (a, x)).

Let $\tilde{f}(m) = (a, \tilde{x}_1)$ for some $\tilde{x}_1 \in X_a$ over x. For a point $m' = (m, \mathcal{E}_2, \varphi_2, \mathcal{E}^B_{2,x}, \alpha) \in (\overleftarrow{h}_{[w]})^{-1}(m)$, we have $\tilde{f}(m) = (a, \tilde{x}_2)$ where $\tilde{x}_2 = w^{-1} \cdot \tilde{x}_1$ because $m' \in \mathcal{H}\mathrm{ecke}^{\mathrm{par}}_{[w]}$. Such a point m' is completely determined by $(\mathcal{E}_2, \varphi_2, \alpha)$ because the choice of the Borel reduction $\mathcal{E}^B_{2,x}$ at x is fixed by $\tilde{x}_2$ (here we use the fact that $\delta_a(x) = 0$). By analogy with Lemma 3.1.3, we get a $P_x(J_a)$-equivariant isomorphism

$$(\overleftarrow{h}_{[w]})^{-1}(m) \cong M^{\mathrm{Hit}}_x(\gamma)$$

for some $\gamma \in \mathfrak{g}(\widehat{\mathcal{O}}_x)$ with $\chi(\gamma) = a \in \mathfrak{c}(\widehat{\mathcal{O}}_x)$. Consider the regular locus $M^{\mathrm{Hit,reg}}_x(\gamma) \subset M^{\mathrm{Hit}}_x(\gamma)$ [29, §3.3], which is a torsor under $P_x(J_a)$. Therefore it suffices to show that $M^{\mathrm{Hit,reg}}_x(\gamma) = M^{\mathrm{Hit}}_x(\gamma)$. Since $M^{\mathrm{Hit,reg}}_x(\gamma)$ is open in $M^{\mathrm{Hit}}_x(\gamma)$, we only need to check that they are equal over every geometric point of S. Hence we reduce to the case where S is the spectrum of an algebraically closed field. But in this case $\dim M^{\mathrm{Hit}}_x(\gamma) = \delta(a, x) = 0$. By [29, Corollary 3.7.2], we conclude that $M^{\mathrm{Hit}}_x(\gamma) = M^{\mathrm{Hit,reg}}_x(\gamma)$. This completes the proof of the lemma. □

Consider the Beilinson–Drinfeld Grassmannian Gr_T for the constant group scheme T over X. Let $\widetilde{\mathrm{Gr}}_T = \widetilde{\mathcal{A}}^{\mathrm{Hit}} \times_X \mathrm{Gr}_T$. The diagonal left action of W on $\widetilde{\mathcal{A}}^{\mathrm{Hit}} \times T$ gives a W-action on $\widetilde{\mathrm{Gr}}_T$. On the other hand, $\widetilde{\mathrm{Gr}}_J$ also carries a left W-action induced from the W-action on $\widetilde{\mathcal{A}}^{\mathrm{Hit}}$.

Lemma 3.2.5. *There is a W-equivariant isomorphism of group ind-schemes over $\widetilde{\mathcal{A}}^{\mathrm{rs}}$:*

$$j_{\mathrm{Gr}} : \widetilde{\mathrm{Gr}}^{\mathrm{rs}}_J \xrightarrow{\sim} \widetilde{\mathrm{Gr}}^{\mathrm{rs}}_T.$$

Proof. For $(a, \tilde{x}, Q^J, \tau) \in \widetilde{\mathrm{Gr}}_J(S)$ over $(a, \tilde{x}) \in \widetilde{\mathcal{A}}^{\mathrm{rs}}(S)$. The map j_T in Lemma 2.3.2 gives a homomorphism of group schemes $j_a : q^*_a J_a \to T \times X_a$. Let $Q^T = q^*_a Q^J \overset{q^*_a J_a}{\times} T$ be the induced T-torsor over X_a. Since $a(x) \in \mathfrak{c}^{\mathrm{rs}}$, $q^{-1}_a(\Gamma(x))$ is a disjoint union

$$q^{-1}_a\bigl(\Gamma(x)\bigr) = \bigsqcup_{w \in W} \Gamma(w\tilde{x}).$$

The trivialization τ gives a trivialization $q^*_a\tau$ of Q^T over $X_a - q^{-1}_a(\Gamma(x))$. We can glue the restriction of Q^T to the open set $X_a - \bigsqcup_{w \neq e} \Gamma(w\tilde{x})$ with the trivial T-torsor over the open set $X_a - \Gamma(\tilde{x})$ via the trivialization $q^*_a\tau$. This gives a new T-torsor Q^T_1 over X_a together with a tautological trivialization τ_1 of Q^T_1 on $X_a - \Gamma(\tilde{x})$. We define the morphism j_{Gr} by

$$j_{\mathrm{Gr}}\bigl(a, \tilde{x}, Q^J, \tau\bigr) = \bigl(a, \tilde{x}, Q^T_1, \tau_1\bigr) \in \widetilde{\mathrm{Gr}}_T(S).$$

The fact that j_{Gr} is a W-equivariant isomorphism follows from Lemma 2.3.2 that j_T is a W-equivariant isomorphism over $\mathfrak{t}^{\mathrm{rs}}$. □

It is well known that the reduced structure of T is the constant group scheme $\mathbb{X}_*(T)$ over X. Therefore, by the above lemma, the reduced structure of $\widetilde{\mathrm{Gr}}^{\mathrm{rs}}_J$ is the constant group scheme $\mathbb{X}_*(T)$ over $\widetilde{\mathcal{A}}^{\mathrm{rs}}$. In other words, for each $\lambda \in \mathbb{X}_*(T)$, we have a section $\tilde{s}_\lambda : \widetilde{\mathcal{A}}^{\mathrm{rs}} \to \widetilde{\mathrm{Gr}}^{\mathrm{rs}}_T$.

Lemma 3.2.6. *For each $\lambda \in \mathbb{X}_*(T)$, the section $\tilde{s}_\lambda : \widetilde{\mathcal{A}}^{\mathrm{rs}} \to \widetilde{\mathrm{Gr}}^{\mathrm{rs}}_J$ extends to a section $\tilde{s}_\lambda : \widetilde{\mathcal{A}}^0 \to \widetilde{\mathrm{Gr}}^0_J$.*

Proof. Let Z_λ be the scheme-theoretic closure of the image $s_\lambda(\widetilde{\mathcal{A}}^{\mathrm{rs}})$ in $\widetilde{\mathrm{Gr}}^0_J$. We only need to show that the projection induces an isomorphism $Z_\lambda \cong \widetilde{\mathcal{A}}^0$. Let $Z'_\lambda = \mathcal{M}^{\mathrm{par},0} \times_{\widetilde{\mathcal{A}}^0} Z_\lambda$ be the closed substack of $\mathcal{M}^{\mathrm{par},0} \times_{\widetilde{\mathcal{A}}^0} \widetilde{\mathrm{Gr}}^0_J$. Since $\mathcal{M}^{\mathrm{par},0}$ is faithfully flat over $\widetilde{\mathcal{A}}^0$ by Corollary 2.5.2, it suffices to show that $p' : Z'_\lambda \to \mathcal{M}^{\mathrm{par},0}$ is an isomorphism.

We first claim that p' is proper. In fact, by Lemma 3.2.4, $\mathcal{H}^0_{[e]}$ is isomorphic to the product $\mathcal{M}^{\mathrm{par},0} \times_{\widetilde{\mathcal{A}}^0} \widetilde{\mathrm{Gr}}^0_J$. Since $\mathcal{H}_{[e]}$ is the inductive limit of proper substacks over $\mathcal{M}^{\mathrm{par}}$ by Lemma 3.1.2 and Z'^{rs}_λ (hence its closure Z'_λ) is contained in one of these substacks, Z'_λ is also proper over $\mathcal{M}^{\mathrm{par},0}$.

Next, the fibers of p' are contained in $P_x(J_a)$, which have dimension equal to $\delta(a,x) = 0$. Since p' is proper, it is finite. Moreover, p' is an isomorphism over the dense open subset $\mathcal{M}^{\mathrm{par},\mathrm{rs}}$. We conclude that p' is an isomorphism because $\mathcal{M}^{\mathrm{par},0}$ is normal. This completes the proof. □

Proof of Proposition 3.2.1. From Lemma 3.2.6, we see that each $\lambda \in \mathbb{X}_*(T)$ gives a morphism

$$s_\lambda : \widetilde{\mathcal{A}}^0 \xrightarrow{\tilde{s}_\lambda} \widetilde{\mathrm{Gr}}_J \to \mathcal{P}$$

where the last arrow is the forgetful morphism (using the moduli meaning of $\widetilde{\mathrm{Gr}}_J$). Moreover, for any $w \in W$, we have

$$s_{w\lambda}(\tilde{a}) = s_\lambda\big(w^{-1}\tilde{a}\big) \tag{3.3}$$

for all $\tilde{a} \in \widetilde{\mathcal{A}}^0$. In fact, this follows from the W-equivariance of the isomorphism j_{Gr} in Lemma 3.2.5.

We first define the right $\widetilde{W}$-action. Note that $\mathcal{M}^{\mathrm{par},0} = \mathcal{M}^{\mathrm{Hit}} \times_{\mathcal{A}^{\mathrm{Hit}}} \widetilde{\mathcal{A}}^0$. For $(\lambda, w) \in \widetilde{W}$, we define its action on $(m, \tilde{x}) \in \mathcal{M}^{\mathrm{par},0}(S) = (\mathcal{M}^{\mathrm{Hit}} \times_{\mathcal{A}^{\mathrm{Hit}}} \widetilde{\mathcal{A}}^0)(S)$ by

$$(m, \tilde{a}) \cdot (\lambda, w) := \big(s_\lambda(\tilde{a})m, w^{-1}\tilde{a}\big) \tag{3.4}$$

where the action of s on m is given by the action of $\mathcal{P}_a$ on $\mathcal{M}^{\mathrm{Hit}}_a$ (a is the image of $\tilde{a}$ in $\mathcal{A}$). Using the relation (3.3), it is easy to check that (3.4) indeed gives a right action of $\widetilde{W}$: here we are using the fact that $\mathcal{P}$ is commutative.

Next, by Lemma 3.2.3, $\mathcal{H}\mathrm{ecke}^{\mathrm{St},0}_w$ is the graph of the right w-action. By Lemma 3.2.4, the reduced structure of $\mathcal{H}\mathrm{ecke}^{\mathrm{par},\mathrm{rs}}_{[w]}$ is the disjoint union of the $\mathbb{X}_*(T)$-translations of $\mathcal{H}\mathrm{ecke}^{\mathrm{St},\mathrm{rs}}_w$. In other words, the reduced structure of $\mathcal{H}\mathrm{ecke}^{\mathrm{par},\mathrm{rs}}_{[w]}$ is the disjoint union of the graphs of (λ, w) for $\lambda \in \mathbb{X}_*(T)$. This completes the proof. □

3.3. The affine Weyl group action

In this subsection, we restrict all relevant spaces over $\mathcal{A}^{\mathrm{Hit}}$ to $\mathcal{A}$ without changing notations. Recall $\widetilde{\mathcal{A}} \subset \widetilde{\mathcal{A}}^{\mathrm{Hit}}$ is the preimage of $\mathcal{A} \subset \mathcal{A}^{\mathrm{Hit}}$.

Definition 3.3.1. For each $\tilde{w} \in \widetilde{W}$, the *reduced Hecke correspondence* $\mathcal{H}_{\tilde{w}}$ indexed by $\tilde{w}$ is the closure (in $\mathcal{H}\mathrm{ecke}^{\mathrm{par}}$) of the graph of the right $\tilde{w}$-action constructed in Proposition 3.2.1.

Let $\mathcal{H}$ be the reduced structure of $\mathcal{H}\mathrm{ecke}^{\mathrm{par}}$, then $\mathcal{H}^{\mathrm{rs}}$ is the disjoint union of the graphs $\mathcal{H}^{\mathrm{rs}}_{\tilde{w}}$.

Definition 3.3.2. The direct image complex $f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell \in D^b(\mathcal{A} \times X)$ (resp. $\tilde{f}_*\overline{\mathbb{Q}}_\ell \in D^b(\widetilde{\mathcal{A}})$) of the constant sheaf under the parabolic Hitchin fibration (resp. enhanced parabolic Hitchin fibration) is called the *parabolic Hitchin complex* (resp. the *enhanced parabolic Hitchin complex*).

We apply the discussions in Appendix A.6 to the situation where $S = \mathcal{A} \times X$, $U = (\mathcal{A} \times X)^{\mathrm{rs}}$, $X = \mathcal{M}^{\mathrm{par}}$ and the reduced Hecke correspondences $C = \mathcal{H}_{\tilde{w}}$ for each $\tilde{w} \in \widetilde{W}$. By Definition 3.3.1, $\mathcal{H}_{\tilde{w}}$ is a graph-like correspondence with respect to U. By the discussion in Appendix A.6, we get a map

$$[\mathcal{H}_{\tilde{w}}]_\# : f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell \to f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell.$$

The first main theorem of this paper is

Theorem 3.3.3. *The assignment $\tilde{w} \mapsto [\mathcal{H}_{\tilde{w}}]_\#$ for $\tilde{w} \in \widetilde{W}$ gives a left action of $\widetilde{W}$ on the restriction $f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell|_{(\mathcal{A}\times X)'}$.*

Proof. By definition, the Hecke correspondence $\mathcal{H}\mathrm{ecke}^{\mathrm{par}}$ has a natural convolution structure $\mu : \mathcal{H}\mathrm{ecke}^{\mathrm{par}} * \mathcal{H}\mathrm{ecke}^{\mathrm{par}} \to \mathcal{H}\mathrm{ecke}^{\mathrm{par}}$ by forgetting the middle $\mathcal{M}^{\mathrm{par}}$, which is obviously associative. In Definition A.1.1 we define the vector space $\mathrm{Corr}(C; \mathcal{F}, \mathcal{G})$ of cohomological correspondences between two complexes of sheaves $\mathcal{F}, \mathcal{G}$. Let $\mathrm{Corr}(\mathcal{H}\mathrm{ecke}^{\mathrm{par}}; \overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_\ell)$ be the direct limit

$$\mathrm{Corr}\big(\mathcal{H}\mathrm{ecke}^{\mathrm{par}}; \overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_\ell\big) := \varinjlim \mathrm{Corr}\big(\mathcal{H}\mathrm{ecke}'; \overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_\ell\big)$$

where $\mathcal{H}\mathrm{ecke}'$ runs over substacks of $\mathcal{H}\mathrm{ecke}^{\mathrm{par}}$ which are of finite type over $\mathcal{M}^{\mathrm{par}}$ (via both projections). We will see in Lemma 3.1.2 that these $\mathcal{H}\mathrm{ecke}$'s exhaust $\mathcal{H}\mathrm{ecke}^{\mathrm{par}}$. Likewise we can define $\mathrm{Corr}(\mathcal{H}\mathrm{ecke}^{\mathrm{par,rs}}; \overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_\ell)$ as a direct limit. The discussions of Appendix A.7 can be applied to these direct limit situations, so that the groups $\mathrm{Corr}(\mathcal{H}\mathrm{ecke}^{\mathrm{par}}; \overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_\ell)$ and $\mathrm{Corr}(\mathcal{H}\mathrm{ecke}^{\mathrm{par,rs}}; \overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_\ell)$ have natural algebra structures given by convolutions.

By Proposition 3.2.1, $\mathcal{H}^{\mathrm{rs}}$ is the disjoint union of graphs of the right $\widetilde{W}$-action on $\mathcal{M}^{\mathrm{par,rs}}$, therefore we have an algebra homomorphism

$$\overline{\mathbb{Q}}_\ell[\widetilde{W}] \to \mathrm{H}^0\big(\mathcal{H}\mathrm{ecke}^{\mathrm{par,rs}}, \mathbb{D}_{\overleftarrow{h}}\big) \cong \mathrm{H}^0\big(\mathcal{H}^{\mathrm{rs}}, \mathbb{D}_{\overleftarrow{h}}\big) = \mathrm{Corr}\big(\mathcal{H}^{\mathrm{rs}}; \overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_\ell\big) \tag{3.5}$$

which sends $\tilde{w}$ to $[\mathcal{H}_{\tilde{w}}^{\mathrm{rs}}]$. Here $\overline{\mathbb{Q}}_\ell[\widetilde{W}]$ is the group algebra of $\widetilde{W}$.

By Lemma 3.1.4, any finite type substack of $\mathcal{H}\mathrm{ecke}^{\mathrm{par}}|_{(\mathcal{A}\times X)'}$ satisfies the condition (G-2) in Definition A.6.1 with respect to $U = (\mathcal{A} \times X)^{\mathrm{rs}}$. By Proposition A.7.2, the action of $\mathrm{Corr}(\mathcal{H}\mathrm{ecke}^{\mathrm{par}}|_{(\mathcal{A}\times X)'}; \overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_\ell)$ on $f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell$ factors through $\mathrm{Corr}(\mathcal{H}\mathrm{ecke}^{\mathrm{par,rs}}; \overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_\ell)$; by the homomorphism (3.5), we get an action of $\widetilde{W}$ on $f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell|_{(\mathcal{A}\times X)'}$ sending $\tilde{w}$ to $[\mathcal{H}_{\tilde{w}}]_\#$. The theorem is proved. □

Remark 3.3.4. The classical Springer action of W on $\pi_*\overline{\mathbb{Q}}_\ell$ (where $\pi : \tilde{\mathfrak{g}} \to \mathfrak{g}$) can also be constructed using cohomological correspondences. Let $\mathrm{St} = \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$ be the Steinberg variety. The smallness of π implies that St is a graph-like correspondence. Over $\mathfrak{g}^{\mathrm{rs}}$, $\mathrm{St}^{\mathrm{rs}}$ is the disjoint union of the graphs $\Gamma(w)$ of the W-action on $\tilde{\mathfrak{g}}^{\mathrm{rs}}$. Let St_w be the closure of $\Gamma(w)$ in St. In particular, the fundamental classes $[\mathrm{St}_w]_\#$ give the W-action on $\pi_*\overline{\mathbb{Q}}_\ell$.

We have a variant of the above theorem for enhanced parabolic Hitchin complexes. For each $\tilde{w} \in \widetilde{W}$ with image $w \in W$ under the projection $\widetilde{W} \to W$, the correspondence $\mathcal{H}_{\tilde{w}}$ can be viewed as a correspondence over $\widetilde{\mathcal{A}}$, which we denote by $\mathcal{H}^{\natural}_{\tilde{w}}$:

$$\begin{array}{ccccc} & & \mathcal{H}^{\natural}_{\tilde{w}} & & \\ & \swarrow & & \searrow & \\ \mathcal{M}^{\mathrm{par}} & \xrightarrow[R_w \circ \tilde{f}]{} & \widetilde{\mathcal{A}} & \xleftarrow[\tilde{f}]{} & \mathcal{M}^{\mathrm{par}} \end{array}$$

Here we have composed the left side $\tilde{f}$ with the right action of w on $\widetilde{\mathcal{A}}$ (denoted by R_w). Therefore $[\mathcal{H}^{\natural}_{\tilde{w}}]_{\#}$ induces an isomorphism

$$\big[\mathcal{H}^{\natural}_{\tilde{w}}\big]_{\#} : \tilde{f}_* \overline{\mathbb{Q}}_\ell \to R_{w,*} \tilde{f}_* \overline{\mathbb{Q}}_\ell. \tag{3.6}$$

Proposition 3.3.5. *The maps $[\mathcal{H}^{\natural}_{\tilde{w}}]_{\#}$ in* (3.6) *give a $\widetilde{W}$-equivariant structure on $\tilde{f}_* \overline{\mathbb{Q}}_\ell$, compatible with the right $\widetilde{W}$-action on $\widetilde{\mathcal{A}}$ through the quotient $\widetilde{W} \to W$.*

In particular, we get a $\mathbb{X}_*(T)$-action on $\tilde{f}_* \overline{\mathbb{Q}}_\ell$.

Finally, we check that the $\widetilde{W}$-action constructed above essentially commutes with Verdier duality.

Let $d = \dim \mathcal{M}^{\mathrm{par}}$. We fix a fundamental class of $\mathcal{M}^{\mathrm{par}}$, hence fixing an isomorphism $u = [\mathcal{M}^{\mathrm{par}}] : \overline{\mathbb{Q}}_{\ell, \mathcal{M}^{\mathrm{par}}}[d](d/2) \cong \mathbb{D}_{\mathcal{M}^{\mathrm{par}}}[d](d/2)$. This induces isomorphisms

$$v : f^{\mathrm{par}}_* \overline{\mathbb{Q}}_\ell[d](d/2) \xrightarrow{\sim} f^{\mathrm{par}}_* \mathbb{D}_{\mathcal{M}^{\mathrm{par}}}[d](d/2) = \mathbb{D}\big(f^{\mathrm{par}}_* \overline{\mathbb{Q}}_\ell[d](d/2)\big);$$
$$\tilde{v} : \tilde{f}_* \overline{\mathbb{Q}}_\ell[d](d/2) \xrightarrow{\sim} \tilde{f}_* \mathbb{D}_{\mathcal{M}^{\mathrm{par}}}[d](d/2) = \mathbb{D}\big(\tilde{f}_* \overline{\mathbb{Q}}_\ell[d](d/2)\big).$$

Proposition 3.3.6. *For any $\tilde{w} \in \widetilde{W}$, the following diagram is commutative when restricted to $(\mathcal{A} \times X)'$:*

$$\begin{array}{ccc} f^{\mathrm{par}}_* \overline{\mathbb{Q}}_\ell[d](d/2) & \xrightarrow{\quad \tilde{w} \quad} & f^{\mathrm{par}}_* \overline{\mathbb{Q}}_\ell[d](d/2) \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v} \\ \mathbb{D}(f^{\mathrm{par}}_* \overline{\mathbb{Q}}_\ell[d](d/2)) & \xrightarrow{\mathbb{D}(\tilde{w}^{-1})} & \mathbb{D}(f^{\mathrm{par}}_* \overline{\mathbb{Q}}_\ell[d](d/2)) \end{array} \tag{3.7}$$

Here the horizontal maps come from the $\widetilde{W}$-action constructed in Theorem 3.3.3. *Similar result holds for the $\mathbb{X}_*(T)$-action on $\tilde{f}_* \overline{\mathbb{Q}}_\ell[d](d/2)$.*

Proof. Let $\mathcal{F} = \overline{\mathbb{Q}}_{\ell, \mathcal{M}^{\mathrm{par}}}[d](d/2)$. By Lemma A.3.1, the map $\mathbb{D}(\tilde{w}^{-1})$ in diagram (3.7) is given by

$$\mathbb{D}\big([\mathcal{H}_{\tilde{w}^{-1}}]_{\#}\big) = \mathbb{D}\big([\mathcal{H}_{\tilde{w}^{-1}}]\big)_{\#}$$

where $\mathbb{D}([\mathcal{H}_{\tilde{w}^{-1}}])$ is the Verdier dual of the cohomological correspondence $[\mathcal{H}_{\tilde{w}^{-1}}] \in \mathrm{Corr}(\mathcal{H}_{\tilde{w}^{-1}}; \mathcal{F}, \mathcal{F})$. It is clear that $\mathcal{H}_{\tilde{w}^{-1}}$ and $\mathcal{H}_{w}^{\vee}$ coincide over $(\mathcal{A} \times X)^{\mathrm{rs}}$ since they are both the graph of the right $\tilde{w}^{-1}$-action on $\mathcal{M}^{\mathrm{par}}$. Therefore, taking closures in $\mathcal{H}\mathrm{ecke}^{\mathrm{par}}$, we get $\mathcal{H}_{\tilde{w}^{-1}} = \mathcal{H}_{\tilde{w}}^{\vee}$ as self-correspondences of $\mathcal{M}^{\mathrm{par}}$ over $\mathcal{A} \times X$. To prove the proposition, we only have to show that under the following two maps

$$\begin{array}{ccc}
[\mathcal{H}_{\tilde{w}}] \in \mathrm{Corr}(\mathcal{H}_{\tilde{w}}; \mathcal{F}, \mathcal{F}) & \xrightarrow{\mathrm{Corr}(\mathcal{H}_{\tilde{w}};u,u)} & \mathrm{Corr}(\mathcal{H}_{\tilde{w}}; \mathbb{D}\mathcal{F}, \mathbb{D}\mathcal{F}) \\
 & \nearrow_{\mathbb{D}} & \\
[\mathcal{H}_{\tilde{w}^{-1}}] \in \mathrm{Corr}(\mathcal{H}_{\tilde{w}^{-1}}; \mathcal{F}, \mathcal{F}) = \mathrm{Corr}(\mathcal{H}_{\tilde{w}}^{\vee}; \mathcal{F}, \mathcal{F}) & &
\end{array}$$

the elements $[\mathcal{H}_{\tilde{w}}]$ and $[\mathcal{H}_{\tilde{w}^{-1}}]$ have the same image in $\mathrm{Corr}(\mathcal{H}_{\tilde{w}}; \mathbb{D}\mathcal{F}, \mathbb{D}\mathcal{F})$. By Lemma 3.1.4, the correspondences involved are all graph-like. By Lemma A.6.2, it suffices to check the coincidence of the two images in $\mathrm{Corr}(\mathcal{H}_{\tilde{w}}^{\mathrm{rs}}; \mathbb{D}\mathcal{F}, \mathbb{D}\mathcal{F})$, which is obvious. □

4. Parahoric versions of the Hitchin moduli stack

In this section, we generalize the notion of Hitchin stacks to arbitrary parahoric level structures. Throughout this section, let $F = k((t))$ and $\mathcal{O}_F = k[\![t]\!]$. The group G over k determines a group scheme $\mathbf{G} = G \otimes_{\mathrm{Spec}\, k} \mathrm{Spec}\, \mathcal{O}_F$ over $\mathcal{O}_F$.

4.1. Parahoric subgroups

Parahoric subgroups are local notions. In order to make sense of them over a global curve, we have to deal with parahoric subgroups in a “Virasoro-equivariant” way. For this, we need to consider local coordinates on the curve X.

We follow [13, 2.1.2] in the following discussion. Let $\mathrm{Aut}_{\mathcal{O}}$ be the pro-algebraic group of automorphisms of the topological ring $\mathcal{O}_F$. More precisely, for any k-algebra R, $\mathrm{Aut}_{\mathcal{O}}(R)$ is the set of R-linear continuous automorphisms of the topological ring $R[\![t]\!] = R \widehat{\otimes}_k \mathcal{O}_F$ (with t-adic topology).

There is a canonical $\mathrm{Aut}_{\mathcal{O}}$-torsor $\mathrm{Coor}(X)$ over X, called the *space of local coordinates of X*. For any k-algebra R, the set $\mathrm{Coor}(X)(R)$ consists of pairs (x, α) where $x \in X(R)$ and α is an R-linear continuous isomorphism $\alpha : R[\![t]\!] \xrightarrow{\sim} \widehat{\mathcal{O}}_x$ (see Section 1.6.2 for notations). The group $\mathrm{Aut}_{\mathcal{O}}$ acts on $\mathrm{Coor}(X)$ on the right by changing the isomorphism α. The following fact is well known.

Lemma 4.1.1. *The group scheme* $\mathrm{Aut}_{\mathcal{O}}$ *acts on both the formal scheme* $\mathrm{Spf}\, \mathcal{O}_F$ *and the scheme* $\mathrm{Coor}(X)$. *Then the contracted product*

$$\mathrm{Coor}(X) \overset{\mathrm{Aut}_{\mathcal{O}}}{\times} \mathrm{Spf}\, \mathcal{O}_F \cong \widehat{X^2}_{\Delta},$$

where the RHS is the formal completion of X^2 along the diagonal $\Delta(X) \subset X^2$, viewed as a formal X-scheme via the projection to the first factor.

The pro-algebraic group $\mathrm{Aut}_{\mathcal{O}}$ has the Levi quotient $\mathbb{G}_m$ given by

$$\mathrm{Aut}_{\mathcal{O}} \ni \sigma \mapsto \sigma(t)/t \mod t \in \mathbb{G}_m.$$

We call this quotient $\mathbb{G}_m$ the *rotation torus* $\mathbb{G}_m^{\mathrm{rot}}$.

From Lemma 4.1.1, we immediately get

Corollary 4.1.2. *There is an isomorphism of $\mathbb{G}_m$-torsors on X:*

$$\mathrm{Coor}(X) \overset{\mathrm{Aut}_{\mathcal{O}}}{\times} \mathbb{G}_m^{\mathrm{rot}} \xrightarrow{\sim} \rho_{\omega_X}.$$

(*See Section* 1.6.2 *for notations such as* ρ_{ω_X}.)

Let $\mathfrak{B}(G,F)$ be the (semisimple) Bruhat–Tits building of $G(F)$. The group $\mathrm{Aut}(\mathcal{O}_F)$ acts on $\mathfrak{B}(G,F)$ in a simplicial way, hence acts on the set of parahoric subgroups of $G(F)$. The fixed Borel subgroup $B \subset G$ determines an Iwahori subgroup $\mathbf{I} \subset \mathbf{G}$. Any parahoric subgroup $\mathbf{P}$ containing $\mathbf{I}$ is called a *standard parahoric subgroup* of $G(F)$.

Let $\mathbf{P} \subset G(F)$ be a parahoric subgroup, corresponding to a facet $\mathfrak{F}_{\mathbf{P}}$ in $\mathfrak{B}(G,F)$. By Bruhat–Tits theory, $\mathbf{P}$ determines a smooth group scheme $\mathcal{G}_{\mathbf{P}}$ over $\mathrm{Spec}\,\mathcal{O}_F$ with generic fiber $G \otimes_k F$ and such that $\mathcal{G}_{\mathbf{P}}(\mathcal{O}_F) = \mathbf{P}$ (see [31, 3.4.1]).

Let $\mathfrak{g}_{\mathbf{P}}$ be the Lie algebra of $\mathcal{G}_{\mathbf{P}}$, which is a free $\mathcal{O}_F$-module of rank $\dim_k G$. Let $L_{\mathbf{P}}$ be the Levi quotient of the special fiber of $\mathcal{G}_{\mathbf{P}}$, which is a connected reductive group over k. Let $\widetilde{\mathbf{P}}$ be the stabilizer of $\mathfrak{F}_{\mathbf{P}}$ under $G(F)$ (equivalently, $\widetilde{\mathbf{P}}$ is the normalizer of $\mathbf{P}$ in $G(F)$). Let $\Omega_{\mathbf{P}} = \widetilde{\mathbf{P}}/\mathbf{P}$, a finite group.

Let $G((t))$ be the group ind-scheme over k whose R-points are $G(R \widehat{\otimes}_k F) = G(R((t)))$. We call $G((t))$ the *loop group* of G. Similarly, let $G_{\mathbf{P}}$ be the group schemes over k whose R-points are $\mathcal{G}_{\mathbf{P}}(R \widehat{\otimes}_k \mathcal{O}_F) = \mathcal{G}_{\mathbf{P}}(R[\![t]\!])$. For $\mathbf{P} = \mathbf{G}$, we write $G[\![t]\!]$ instead of $G_{\mathbf{G}}$.

If $\mathbf{P}$ is a parahoric subgroup stabilized by $\mathrm{Aut}(\mathcal{O}_F)$, then $\mathrm{Aut}(\mathcal{O}_F)$ naturally acts on the group scheme $\mathcal{G}_{\mathbf{P}}$, lifting its action on $\mathcal{O}_F$. Therefore, the pro-algebraic group $\mathrm{Aut}_{\mathcal{O}}$ acts on the group ind-scheme $G((t))$ and the group scheme $G_{\mathbf{P}}$, hence also on $L_{\mathbf{P}}$. We form the twisted product

$$\underline{L}_{\mathbf{P}} := \mathrm{Coor}(X) \overset{\mathrm{Aut}_{\mathcal{O}}}{\times} L_{\mathbf{P}} \tag{4.1}$$

which is a reductive group scheme over X with geometric fibers isomorphic to $L_{\mathbf{P}}$. Let $\mathfrak{l}_{\mathbf{P}}$ be the Lie algebra of $L_{\mathbf{P}}$, and let $\underline{\mathfrak{l}}_{\mathbf{P}}$ be the Lie algebra of $\underline{L}_{\mathbf{P}}$, which is the vector bundle $\mathrm{Coor}(X) \overset{\mathrm{Aut}_{\mathcal{O}}}{\times} \mathfrak{l}_{\mathbf{P}}$ over X.

Let $\mathbf{P}$ be a standard parahoric subgroup. The Borel B gives a Borel subgroup $B_{\mathbf{I}}^{\mathbf{P}} \subset L_{\mathbf{P}}$ whose quotient torus is canonically isomorphic to T. Let $W_{\mathbf{P}}$ be the Weyl group of $L_{\mathbf{P}}$ determined by $B_{\mathbf{I}}^{\mathbf{P}}$ and T. Then $W_{\mathbf{P}}$ is naturally a subgroup of $\widetilde{W}$. In fact, any maximal torus in B gives an apartment $\mathfrak{A}$ in $\mathfrak{B}(G,F)$, on which $\widetilde{W}$ acts by affine transformations. The Weyl group $W_{\mathbf{P}}$ can be identified with the subgroup of $\widetilde{W}$ which fixes $\mathfrak{F}_{\mathbf{P}}$ pointwise. The resulting subgroup $W_{\mathbf{P}} \subset \widetilde{W}$ is independent of the choice of the maximal torus in B.

4.2. Bundles with parahoric level structures

Definition 4.2.1. Let $\widetilde{\mathrm{Bun}}_\infty : \{k\text{-algebras}\} \to \{\text{Groupoids}\}$ be the fpqc sheaf associated to the following presheaf $\widetilde{\mathrm{Bun}}^{pre}_\infty$: for any k-algebra R, $\widetilde{\mathrm{Bun}}^{pre}_\infty(R)$ is the groupoid of quadruples $(x, \alpha, \mathcal{E}, \tau_x)$ where

- $x \in X(R)$ with graph $\Gamma(x) \subset X_R$;
- $\alpha : R[\![t]\!] \xrightarrow{\sim} \widehat{\mathcal{O}}_x$ a local coordinate;
- $\mathcal{E}$ is a G-torsor over X_R;
- $\tau_x : G \times \mathfrak{D}_x \xrightarrow{\sim} \mathcal{E}|_{\mathfrak{D}_x}$ is a trivialization of the restriction of $\mathcal{E}$ to $\mathfrak{D}_x = \operatorname{Spec} \widehat{\mathcal{O}}_x$.

In other words, $\widetilde{\mathrm{Bun}}_\infty$ parametrizes G-bundles on X with a full level structure at a point of X, and a choice of a local coordinate at that point. This moduli space is also considered in [3, §2.8.3].

Construction 4.2.2. (Compare [3, §2.8.4].) Consider the semi-direct product $G((t)) \rtimes \mathrm{Aut}_{\mathcal{O}}$ formed using the action of $\mathrm{Aut}_{\mathcal{O}}$ on $G((t))$ defined in Section 4.1. We claim that this group ind-scheme naturally acts on $\widetilde{\mathrm{Bun}}_\infty$ from the right. In fact, for any k-algebra R, $g \in G(R((t)))$, $\sigma \in \mathrm{Aut}(R[\![t]\!])$ and $(x, \alpha, \mathcal{E}, \tau_x) \in \widetilde{\mathrm{Bun}}_\infty(R)$, let

$$R_{g,\sigma}(x, \alpha, \mathcal{E}, \tau_x) = \big(x, \alpha \circ \sigma, \mathcal{E}^g, \tau_x^g\big).$$

Let us explain the notations. By a variant of the main result of [2] (using Tannakian formalism to reduce to the case of vector bundles, see [12, Corollary 1.3.3]), to give a G-torsor on X_R is the same as to give G-torsors on $X_R - \Gamma(x)$ and on $\mathfrak{D}_x$ respectively, together with a G-isomorphism between their restrictions to $\mathfrak{D}_x^\times$. Now let $\mathcal{E}^g$ be the G-torsor on X_R obtained by gluing $\mathcal{E}|_{X_R - \Gamma(x)}$ with the trivial G-torsor $G \times \mathfrak{D}_x$ via the isomorphism

$$G \times \mathfrak{D}_x^\times \xrightarrow{\alpha_*^{-1} g \alpha_*} G \times \mathfrak{D}_x^\times \xrightarrow{\tau_x} \mathcal{E}|_{\mathfrak{D}_x^\times}.$$

Here the first arrow is the transport of the left multiplication by g on $G \times \operatorname{Spec} R((t))$ to $G \times \mathfrak{D}_x^\times$ via the local coordinate α. The trivialization τ_x^g is tautologically given by the construction of $\mathcal{E}^g$.

Definition 4.2.3. Let $\mathbf{P}$ be a standard parahoric subgroup of $G(F)$. The moduli stack $\mathrm{Bun}_{\mathbf{P}}$ of *G-bundles over X with parahoric level structures of type* $\mathbf{P}$ is the fpqc sheaf associated to the quotient presheaf $R \mapsto \widetilde{\mathrm{Bun}}_\infty(R)/(G_{\mathbf{P}} \rtimes \mathrm{Aut}_{\mathcal{O}})(R)$.

We will use the notation $(x, \mathcal{E}, \tau_x \bmod \mathbf{P})$ to denote the point in $\mathrm{Bun}_{\mathbf{P}}$ which is the image of $(x, \alpha, \mathcal{E}, \tau_x) \in \widetilde{\mathrm{Bun}}_\infty(R)$.

In the special cases $\mathbf{P} = \mathbf{G} = G(\mathcal{O}_F)$, we have

Lemma 4.2.4. *There is a canonical isomorphism* $\mathrm{Bun}_{\mathbf{G}} \cong \mathrm{Bun}_G \times X$*, where* Bun_G *is the usual moduli stack of G-torsors over X.*

Proof. We need to show that the forgetful morphism $\widetilde{\mathrm{Bun}}_\infty \to \mathrm{Bun}_G \times X$ is a $G[\![t]\!] \rtimes \mathrm{Aut}_{\mathcal{O}}$-torsor. The only not-so-obvious part is the essential surjectivity, i.e., for any k-algebra R and any $(x, \mathcal{E}) \in X(R) \times \mathrm{Bun}_G(R)$, we have to find a trivialization of $\mathcal{E}|_{\mathfrak{D}_x}$ locally in the flat or étale topology of $\mathrm{Spec}\, R$. Since $\mathcal{E}|_{\Gamma(x)}$ is a G-torsor, by definition, there is an étale covering $\mathrm{Spec}\, R' \to \mathrm{Spec}\, R$ which trivializes $\mathcal{E}|_{\Gamma(x)}$, i.e., there is a section $\tau'_0 : \mathrm{Spec}\, R' \to \mathcal{E}|_{\mathfrak{D}_x}$ over $\mathrm{Spec}\, R' \to \mathrm{Spec}\, R = \Gamma(x) \subset \mathfrak{D}_x$. Since $\mathcal{E}|_{\mathfrak{D}_x}$ is smooth over $\mathfrak{D}_x$, the section τ'_0 extends to a section $\tau' : \mathrm{Spec}(R' \otimes_R \widehat{\mathcal{O}}_x) \to \mathcal{E}|_{\mathfrak{D}_x}$. In other words, after pulling back to the étale covering $\mathrm{Spec}\, R' \to \mathrm{Spec}\, R$, $\mathcal{E}|_{\mathfrak{D}_x}$ can be trivialized. □

If $\mathbf{P} \subset G(\mathcal{O}_F)$ (i.e., $\mathbf{P}$ corresponds to a parabolic subgroup $P \subset G$), then $\mathrm{Bun}_{\mathbf{P}}$ is the moduli stack of G-torsors on X with a parabolic reduction of type P at a point of X. In particular, if $\mathbf{P} = \mathbf{I}$, the standard Iwahori subgroup, $\mathrm{Bun}_{\mathbf{I}}$ is what we denoted by $\mathrm{Bun}_G^{\mathrm{par}}$ in Section 2.1.

Lemma 4.2.5. *There is a right action of $\Omega_{\mathbf{P}} = \widetilde{\mathbf{P}}/\mathbf{P}$ on* $\mathrm{Bun}_{\mathbf{P}}$.

Proof. Let $g \in \widetilde{\mathbf{P}}$. Let $\widetilde{\mathrm{Bun}}_{\mathbf{P}} = \widetilde{\mathrm{Bun}}_\infty / G_{\mathbf{P}}$, which is an $\mathrm{Aut}_{\mathcal{O}}$-torsor over $\mathrm{Bun}_{\mathbf{P}}$. Since $g^{-1}\mathbf{P}g = \mathbf{P}$, the right action of g on $\widetilde{\mathrm{Bun}}_\infty$ (see Construction 4.2.2) descends to an isomorphism $R_g : \widetilde{\mathrm{Bun}}_{\mathbf{P}} \xrightarrow{\sim} \widetilde{\mathrm{Bun}}_{\mathbf{Q}}$. Moreover, for any $\sigma \in \mathrm{Aut}(R[\![t]\!])$, we have a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathrm{Bun}}_{\mathbf{P}}(R) & \xrightarrow{R_g} & \widetilde{\mathrm{Bun}}_{\mathbf{P}}(R) \\ \downarrow{\cdot\sigma} & & \downarrow{\cdot\sigma} \\ \widetilde{\mathrm{Bun}}_{\mathbf{P}}(R) & \xrightarrow{R_{\sigma(g)}} & \widetilde{\mathrm{Bun}}_{\mathbf{P}}(R) \end{array} \tag{4.2}$$

Since $\mathbf{P}$ is stable under σ, $\sigma(g)g^{-1}$ normalizes $\mathbf{P}$, hence $\sigma(g)g^{-1} \in \widetilde{\mathbf{P}}(R)$. The assignment $\sigma \mapsto \sigma(g)g^{-1} \bmod \mathbf{P}$ gives a morphism from the connected group $\mathrm{Aut}_{\mathcal{O}}$ to the discrete group $\widetilde{\mathbf{P}}/\mathbf{P}$, which must be trivial. Therefore $\sigma(g)g^{-1} \in \mathbf{P}(R)$, hence acts trivially on $\widetilde{\mathrm{Bun}}_{\mathbf{P}}$. Therefore $R_{\sigma(g)} = R_g \circ R_{\sigma(g)g^{-1}} = R_g$. Using diagram (4.2), we conclude that $R_g : \widetilde{\mathrm{Bun}}_{\mathbf{P}}(R) \xrightarrow{\sim} \widetilde{\mathrm{Bun}}_{\mathbf{P}}(R)$ is equivariant under $\mathrm{Aut}_{\mathcal{O}}(R)$, hence descends to an isomorphism

$$R_g : \mathrm{Bun}_{\mathbf{P}} \xrightarrow{\sim} \mathrm{Bun}_{\mathbf{P}}.$$

It is clear that R_g only depends on the coset $[g] = g\mathbf{P} \in \Omega_{\mathbf{P}}$, hence giving a right action of $\Omega_{\mathbf{P}}$ on $\mathrm{Bun}_{\mathbf{P}}$. □

Suppose $\mathbf{P} \subset \mathbf{Q}$ are standard parahoric subgroups, then we have a forgetful morphism

$$\mathrm{For}_{\mathbf{P}}^{\mathbf{Q}} : \mathrm{Bun}_{\mathbf{P}} \to \mathrm{Bun}_{\mathbf{Q}} \tag{4.3}$$

whose fibers are isomorphic to $G_{\mathbf{Q}}/G_{\mathbf{P}}$, a partial flag variety of the reductive group $L_{\mathbf{Q}}$. In particular, $\mathrm{For}_{\mathbf{P}}^{\mathbf{Q}}$ is representable, proper, smooth and surjective.

Corollary 4.2.6. *For any standard parahoric subgroup $\mathbf{P} \subset G(F)$, the stack* $\mathrm{Bun}_{\mathbf{P}}$ *is an algebraic stack locally of finite type.*

Proof. Since $\mathrm{For}_{\mathbf{I}}^{\mathbf{G}} : \mathrm{Bun}_{\mathbf{I}} \to \mathrm{Bun}_{\mathbf{G}} \cong \mathrm{Bun}_G \times X$ is representable and of finite type, and Bun_G is an algebraic stack locally of finite type, $\mathrm{Bun}_{\mathbf{I}}$ is also algebraic and locally of finite type. On the other hand, $\mathrm{For}_{\mathbf{I}}^{\mathbf{P}} : \mathrm{Bun}_{\mathbf{I}} \to \mathrm{Bun}_{\mathbf{P}}$ is representable, smooth and surjective, hence $\mathrm{Bun}_{\mathbf{P}}$ is also algebraic and locally of finite type. □

4.3. The parahoric Hitchin fibrations

In this subsection, we define parahoric analogs of $\mathcal{M}^{\mathrm{par}}$ and f^{par} considered in [34, §3.1]. These are analogs of the partial Grothendieck resolutions in the classical Springer theory, cf. [5].

Let $\mathbf{P}$ be a standard parahoric subgroup of $G(F)$.

Construction 4.3.1 *(The Higgs fields).* For any k-algebra R and $(x, \alpha, \mathcal{E}, \tau_x) \in \widetilde{\mathrm{Bun}}_\infty(R)$, consider the composition

$$j_* j^* \mathrm{Ad}(\mathcal{E}) \to \mathrm{Ad}(\mathcal{E}) \otimes \widehat{\mathcal{O}}_x^{\mathrm{punc}} \xrightarrow{\tau_x^{-1}} \mathfrak{g} \otimes_k \widehat{\mathcal{O}}_x^{\mathrm{punc}} \xrightarrow{\alpha^{-1}} \mathfrak{g} \otimes_k R((t)) \tag{4.4}$$

where $j : X_R - \Gamma(x) \hookrightarrow X_R$ is the inclusion and the first arrow is the natural embedding. Let $\mathrm{Ad}_{\mathbf{P}}(\mathcal{E})$ be the preimage of $\mathfrak{g}_{\mathbf{P}} \otimes_k R \subset \mathfrak{g} \otimes_k R((t))$ under the injection (4.4). Sheafifying this procedure, the assignment $(x, \mathcal{E}, \tau_x \bmod \mathbf{P}) \mapsto \mathrm{Ad}_{\mathbf{P}}(\mathcal{E})$ gives a quasi-coherent sheaf $\mathrm{Ad}_{\mathbf{P}}$ on $\widetilde{\mathrm{Bun}}_\infty \times X$.

Lemma 4.3.2.

(1) *The quasi-coherent sheaf* $\mathrm{Ad}_{\mathbf{P}}$ *descends to* $\mathrm{Bun}_{\mathbf{P}} \times X$*, and is in fact coherent*;
(2) $\mathrm{Ad}_{\mathbf{P}}$ *admits a natural* $\Omega_{\mathbf{P}}$*-equivariant structure* (*with respect to the* $\Omega_{\mathbf{P}}$*-action on* $\mathrm{Bun}_{\mathbf{P}}$ *in Lemma* 4.2.5).

Proof. (1) Since $\mathfrak{g}_{\mathbf{P}}$ is stable under $G_{\mathbf{P}} \rtimes \mathrm{Aut}(\mathcal{O}_F)$, the subsheaf $\mathrm{Ad}_{\mathbf{P}}(\mathcal{E}) \subset j_* j^* \mathrm{Ad}(\mathcal{E})$ only depends on the image of $(x, \alpha, \mathcal{E}, \tau_x)$ in $\mathrm{Bun}_{\mathbf{P}}(R)$.

Fix any k-algebra R and $(x, \mathcal{E}, \tau_x \bmod \mathbf{P}) \in \mathrm{Bun}_{\mathbf{P}}(R)$, we want to show that $\mathrm{Ad}_{\mathbf{P}}(\mathcal{E})$ is a coherent sheaf on X_R. For $\mathbf{P} = \mathbf{I}$, we rewrite this point as $(x, \mathcal{E}, \mathcal{E}_x^B) \in \mathrm{Bun}_{\mathbf{I}}(R)$ ($\mathcal{E}_x^B$ is a B-reduction of $\mathcal{E}_x$), we have an exact sequence

$$0 \to \mathrm{Ad}_{\mathbf{I}}(\mathcal{E}) \to \mathrm{Ad}(\mathcal{E}) \to i_*\big(\mathrm{Ad}(\mathcal{E}_x)/\mathrm{Ad}\big(\mathcal{E}_x^B\big)\big) \to 0$$

where $i : \Gamma(x) \hookrightarrow X_R$ is the closed inclusion. Since the middle and final terms of the above exact sequence are coherent, $\mathrm{Ad}_{\mathbf{I}}(\mathcal{E})$ is also coherent.

In general, since $\mathbf{P} \supset \mathbf{I}$, we have an embedding $\mathrm{Ad}_{\mathbf{I}}(\mathcal{E}) \hookrightarrow \mathrm{Ad}_{\mathbf{P}}(\mathcal{E})$ whose cokernel is again a finite R-module supported on $\Gamma(x)$, hence $\mathrm{Ad}_{\mathbf{P}}(\mathcal{E})$ is also a coherent sheaf on X_R.

(2) Similar to the proof of Lemma 4.2.5. □

Definition 4.3.3. Fix a divisor D on X with $\deg(D) \geqslant 2g_X$. The *parahoric Hitchin moduli stack* $\mathcal{M}_{\mathbf{P}}$ *of type* $\mathbf{P}$ is the fpqc sheaf which associates to every k-algebra R the groupoid of pairs (ξ, φ) where

- $\xi = (x, \mathcal{E}, \tau_x \bmod \mathbf{P}) \in \mathrm{Bun}_{\mathbf{P}}(R)$;
- $\varphi \in \mathrm{H}^0(X_R, \mathrm{Ad}_{\mathbf{P}}(\mathcal{E}) \otimes \mathcal{O}_X(D))$.

Corollary 4.3.4 *(of Lemma 4.3.2(2)). The group $\Omega_{\mathbf{P}}$ acts on $\mathcal{M}_{\mathbf{P}}$ on the right, lifting its right action on* $\mathrm{Bun}_{\mathbf{P}}$.

Lemma 4.3.5. *The stack $\mathcal{M}_{\mathbf{P}}$ is an algebraic stack locally of finite type.*

Proof. Consider the forgetful morphism $\mathcal{M}_{\mathbf{P}} \to \mathrm{Bun}_{\mathbf{P}}$. The fiber of this morphism over a point $(x, \mathcal{E}, \tau_x \bmod \mathbf{P}) \in \mathrm{Bun}_{\mathbf{P}}(R)$ is the finite R-module $\mathrm{H}^0(X_R, \mathrm{Ad}_{\mathbf{P}}(\mathcal{E})(D))$ (the finiteness follows from the coherence of $\mathrm{Ad}_{\mathbf{P}}(\mathcal{E})$ as proved in Lemma 4.3.2(1) and the properness of X). Therefore, the forgetful morphism $\mathcal{M}_{\mathbf{P}} \to \mathrm{Bun}_{\mathbf{P}}$ is representable and of finite type. By Corollary 4.2.6, $\mathrm{Bun}_{\mathbf{P}}$ is algebraic and locally of finite type, hence so is $\mathcal{M}_{\mathbf{P}}$. □

Construction 4.3.6. We claim that there is a natural morphism

$$\mathrm{ev}_{\mathbf{P}} : \mathcal{M}_{\mathbf{P}} \to [\underline{\mathfrak{l}}_{\mathbf{P}}/\underline{L}_{\mathbf{P}}]_D \tag{4.5}$$

of "evaluating the Higgs fields at the point of the $\mathbf{P}$-level structure". For the notation $\underline{\mathfrak{l}}_{\mathbf{P}}$, $\underline{L}_{\mathbf{P}}$, see (4.1). In fact, to construct $\mathrm{ev}_{\mathbf{P}}$, it suffices to construct a $G_{\mathbf{P}} \rtimes \mathrm{Aut}_{\mathcal{O}}$-equivariant morphism

$$\widetilde{\mathrm{ev}}_{\mathbf{P}} : \widetilde{\mathrm{Bun}}_{\infty} \times_{\mathrm{Bun}_{\mathbf{P}}} \mathcal{M}_{\mathbf{P}} \to \mathfrak{l}_{\mathbf{P}} \overset{\mathbb{G}_m}{\times} \rho_D.$$

Here the $G_{\mathbf{P}} \rtimes \mathrm{Aut}_{\mathcal{O}}$-action on the LHS is on the $\widetilde{\mathrm{Bun}}_{\infty}$-factor, and the action on the RHS factors through the $L_{\mathbf{P}} \rtimes \mathrm{Aut}_{\mathcal{O}}$-action on $\mathfrak{l}_{\mathbf{P}}$.

For any k-algebra R and $(x, \alpha, \mathcal{E}, \tau_x, \varphi) \in (\widetilde{\mathrm{Bun}}_{\infty} \times_{\mathrm{Bun}_{\mathbf{P}}} \mathcal{M}_{\mathbf{P}})(R)$, by the definition of $\mathrm{Ad}_{\mathbf{P}}(\mathcal{E})$, the maps in (4.4) give

$$\mathrm{Ad}_{\mathbf{P}}(\mathcal{E}) \to \mathfrak{g}_{\mathbf{P}} \otimes_k R \twoheadrightarrow \mathfrak{l}_{\mathbf{P}} \otimes_k R.$$

Twisting by $\mathcal{O}_X(D)$, we get

$$\widetilde{\mathrm{ev}}_{\mathbf{P},x} : \mathrm{H}^0\big(X_R, \mathrm{Ad}_{\mathbf{P}}(\mathcal{E})(D)\big) \to \mathfrak{l}_{\mathbf{P}} \otimes_k x^*\mathcal{O}_X(D).$$

The assignment $(x, \alpha, \mathcal{E}, \tau_x, \varphi) \mapsto \widetilde{\mathrm{ev}}_{\mathbf{P},x}(\varphi)$ gives the desired morphism $\widetilde{\mathrm{ev}}_{\mathbf{P}}$. It is easy to check that $\widetilde{\mathrm{ev}}_{\mathbf{P}}$ is equivariant under $G_{\mathbf{P}} \rtimes \mathrm{Aut}_{\mathcal{O}}$, hence giving the desired morphism $\mathrm{ev}_{\mathbf{P}}$ in (4.5).

For two standard parahoric subgroups $\mathbf{P} \subset \mathbf{Q}$, there is a unique parabolic subgroup $B^{\mathbf{Q}}_{\mathbf{P}} \subset L_{\mathbf{Q}}$, such that $\mathbf{P}$ is the inverse image of $B^{\mathbf{Q}}_{\mathbf{P}}$ under the natural quotient $\mathbf{Q} \twoheadrightarrow L_{\mathbf{Q}}$. There is a canonical $\mathrm{Aut}_{\mathcal{O}}$-action on $B^{\mathbf{Q}}_{\mathbf{P}}$ making the embedding $B^{\mathbf{Q}}_{\mathbf{P}} \hookrightarrow L_{\mathbf{Q}}$ equivariant under $\mathrm{Aut}_{\mathcal{O}}$. Let $\mathfrak{b}^{\mathbf{Q}}_{\mathbf{P}}$ be the Lie algebra of $B^{\mathbf{Q}}_{\mathbf{P}}$ and let $\underline{B}^{\mathbf{Q}}_{\mathbf{P}}, \underline{\mathfrak{b}}^{\mathbf{Q}}_{\mathbf{P}}$ be the group scheme and Lie algebra over X obtained by applying $\mathrm{Coor}(X) \overset{\mathrm{Aut}_{\mathcal{O}}}{\times} (-)$. The same construction as in Construction 4.3.6 gives the *relative evaluation map*

$$\mathrm{ev}^{\mathbf{Q}}_{\mathbf{P}} : \mathcal{M}_{\mathbf{P}} \to \big[\underline{\mathfrak{b}}^{\mathbf{Q}}_{\mathbf{P}}/\underline{B}^{\mathbf{Q}}_{\mathbf{P}}\big]_D.$$

Similar to the morphism $\mathrm{For}^{\mathbf{Q}}_{\mathbf{P}} : \mathrm{Bun}_{\mathbf{P}} \to \mathrm{Bun}_{\mathbf{Q}}$ in (4.3), there is a morphism

$$\widetilde{\mathrm{For}}^{\mathbf{Q}}_{\mathbf{P}} : \mathcal{M}_{\mathbf{P}} \to \mathcal{M}_{\mathbf{Q}}$$

lifting $\mathrm{For}_{\mathbf{P}}^{\mathbf{Q}}$. It is easy to show that the following diagram is Cartesian

$$\begin{array}{ccc} \mathcal{M}_{\mathbf{P}} & \xrightarrow{\mathrm{ev}_{\mathbf{P}}^{\mathbf{Q}}} & [\underline{\mathfrak{b}}_{\mathbf{P}}^{\mathbf{Q}}/\underline{B}_{\mathbf{P}}^{\mathbf{Q}}]_D \\ \downarrow{\scriptstyle \widetilde{\mathrm{For}}_{\mathbf{P}}^{\mathbf{Q}}} & & \downarrow{\scriptstyle \pi_{\mathbf{P}}^{\mathbf{Q}}} \\ \mathcal{M}_{\mathbf{Q}} & \xrightarrow{\mathrm{ev}_{\mathbf{Q}}} & [\underline{\mathfrak{l}}_{\mathbf{Q}}/\underline{L}_{\mathbf{Q}}]_D \end{array} \tag{4.6}$$

For any k-algebra R and $(x, \mathcal{E}, \tau_x \bmod \mathbf{P}) \in \mathrm{Bun}_{\mathbf{P}}(R)$, the natural map of taking invariants $\mathrm{Ad}(\mathcal{E})(D) \to \mathfrak{c}_D$ gives a map

$$\chi_{\mathbf{P},\mathcal{E}} : \mathrm{H}^0\big(X_R, \mathrm{Ad}_{\mathbf{P}}(\mathcal{E})(D)\big) \hookrightarrow \mathrm{H}^0\big(X_R - \Gamma(x), \mathrm{Ad}(\mathcal{E})(D)\big) \\ \to \mathrm{H}^0\big(X_R - \Gamma(x), \mathfrak{c}_D\big).$$

Lemma 4.3.7. *The image of the map $\chi_{\mathbf{P},\mathcal{E}}$ lands in $\mathrm{H}^0(X_R, \mathfrak{c}_D)$, hence giving a morphism*

$$f_{\mathbf{P}} : \mathcal{M}_{\mathbf{P}} \to \mathcal{A}^{\mathrm{Hit}} \times X.$$

Proof. The statement obviously holds for $\mathbf{P} = \mathbf{I}$. In general, we may assume $\mathbf{I} \subset \mathbf{P}$. By diagram (4.6), for any point $(\xi, \varphi) \in \mathcal{M}_{\mathbf{P}}(R)$, after passing to a fpqc base change of R, there is always a point $(\tilde{\xi}, \tilde{\varphi}) \in \mathcal{M}^{\mathrm{par}}(R)$ mapping to it under $\widetilde{\mathrm{For}}_{\mathbf{I}}^{\mathbf{Q}}$. Since $\chi_{\mathbf{P},\mathcal{E}}(\varphi) = \chi_{\mathbf{I},\mathcal{E}}(\tilde{\varphi})$, we conclude that $\chi_{\mathbf{P},\mathcal{E}}(\varphi) \in \mathrm{H}^0(X_R, \mathfrak{c}_D)$. □

Definition 4.3.8. The morphism $f_{\mathbf{P}} : \mathcal{M}_{\mathbf{P}} \to \mathcal{A}^{\mathrm{Hit}} \times X$ in Lemma 4.3.7 is called the *parahoric Hitchin fibration of type* $\mathbf{P}$.

Parallel to Proposition 2.5.1, we have

Proposition 4.3.9. *Recall* $\deg(D) \geqslant 2g_X$. *Then we have*:

(1) *The morphism $\mathcal{M}_{\mathbf{P}}|_{\mathcal{A}^{\heartsuit}} \to X$ is smooth*;
(2) *The stack $\mathcal{M}_{\mathbf{P}}|_{\mathcal{A}}$ is Deligne–Mumford*;
(3) *The morphism $f_{\mathbf{P}}^{\mathrm{ani}} : \mathcal{M}_{\mathbf{P}}|_{\mathcal{A}} \to \mathcal{A} \times X$ is flat and proper.*

Example 4.3.10. Let $G = \mathrm{GL}(n)$. The standard parahoric subgroups are in 1–1 correspondence with sequences of integers

$$\underline{i} = (0 \leqslant i_0 < \cdots < i_m < n), \quad m \geqslant 0.$$

For each such sequence $\underline{i}$, let $\mathbf{P}_{\underline{i}}$ be the corresponding parahoric subgroup. Then $\mathrm{Bun}_{\mathbf{P}_{\underline{i}}}$ classifies

$$\big(x, \mathcal{E}_{i_0} \supset \mathcal{E}_{i_1} \supset \cdots \supset \mathcal{E}_{i_m} \supset \mathcal{E}_{i_0}(-x)\big)$$

where $x \in X$, $\mathcal{E}_{i_j}$ are vector bundles of rank n on X such that $\mathcal{E}_{i_0}/\mathcal{E}_{i_j}$ has length $i_j - i_0$ for $j = 0, 1, \ldots, m$.

The Hitchin base is again

$$\mathcal{A}^{\mathrm{Hit}} = \bigoplus_{i=1}^{n} \mathrm{H}^0\big(X, \mathcal{O}_X(iD)\big).$$

For $a = (a_1, \dots, a_n) \in \mathcal{A}^{\heartsuit}(k)$ (where $a_i \in \mathrm{H}^0(X, \mathcal{O}_X(iD))$), define the spectral curve Y_a as in Example 2.2.5. Fix a point $x \in X$. Then the parahoric Hitchin fiber $\mathcal{M}_{\mathbf{P},a,x}$ classifies the data

$$\big(\mathcal{F}_{i_0} \supset \mathcal{F}_{i_1} \supset \cdots \supset \mathcal{F}_{i_m} \supset \mathcal{F}_{i_0}(-x)\big)$$

where $\mathcal{F}_{i_j} \in \overline{\mathcal{P}\mathrm{ic}}(Y_a)$ such that $\mathcal{F}_{i_0}/\mathcal{F}_{i_j}$ has length $i_0 - i_j$ for $j = 0, 1, \dots, m$.

5. The affine Weyl group action—the second construction

In this section, we give the second construction of the $\widetilde{W}$-action on $f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell$ using the Coxeter presentation of $\widetilde{W}$. This construction is valid over all of $\mathcal{A} \times X$. We will restrict all relevant spaces and morphisms over $\mathcal{A}^{\mathrm{Hit}}$ to $\mathcal{A}$ without changing notations.

5.1. The construction

Let $\Omega = \widetilde{W}/W_{\mathrm{aff}} = \mathbb{X}_*(T)/\mathbb{Z}\Phi^\vee$. Then Ω can be identified with the subgroup $\Omega_{\mathbf{I}} \subset \widetilde{W}$, the stabilizer of the alcove corresponding to the standard Iwahori $\mathbf{I}$ in the building of $G((t))$ (for more details, see Section 4.1). Therefore we can write $\widetilde{W}$ as a semi-direct product $\widetilde{W} = W_{\mathrm{aff}} \rtimes \Omega_{\mathbf{I}}$.

Construction 5.1.1. We will define the action of a set of generators of $\widetilde{W}$ on the parabolic Hitchin complex $f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell$.

- For each affine simple reflection $s_i \in \Sigma_{\mathrm{aff}}$, consider the Cartesian diagram (4.6) for $\mathbf{P} = \mathbf{I}$ and $\mathbf{Q} = \mathbf{P}_i$, the standard parahoric subgroup whose Lie algebra $\mathfrak{g}_{\mathbf{P}_i}$ is spanned by $\mathfrak{g}_{\mathbf{I}}$ and the root space of $-\alpha_i$. We will abbreviate $L_{\mathbf{P}_i}, \mathfrak{l}_{\mathbf{P}_i}, B_{\mathbf{I}}^{\mathbf{P}_i}, \mathfrak{b}_{\mathbf{I}}^{\mathbf{P}_i}$, etc. by $L_i, \mathfrak{l}_i, B^i, \mathfrak{b}^i$, etc. The morphism $\pi_{\mathbf{I}}^{\mathbf{P}_i} : [\underline{\mathfrak{b}}^i/\underline{B}^i]_D \to [\underline{\mathfrak{l}}_i/\underline{L}_i]_D$ fits into the following Cartesian diagram

$$\begin{array}{ccccc} \mathcal{M}^{\mathrm{par}} & \xrightarrow{\mathrm{ev}_{\mathbf{I}}^{\mathbf{P}_i}} & [\underline{\mathfrak{b}}^i/\underline{B}^i]_D & \xrightarrow{\tilde{\beta}_i} & [\tilde{\mathfrak{l}}_i/L_i^\natural] \\ \downarrow{\scriptstyle \mathrm{For}_{\mathbf{I}}^{\mathbf{P}_i}} & & \downarrow{\scriptstyle \pi_{\mathbf{I}}^{\mathbf{P}_i}} & & \downarrow{\scriptstyle \pi^i} \\ \mathcal{M}_{\mathbf{P}_i} & \xrightarrow{\mathrm{ev}_{\mathbf{P}_i}} & [\underline{\mathfrak{l}}_i/\underline{L}_i]_D & \xrightarrow{\beta_i} & [\mathfrak{l}_i/L_i^\natural] \end{array} \tag{5.1}$$

  We explain the notation. Here $\tilde{\mathfrak{l}}_i$ is the Grothendieck simultaneous resolution of $\mathfrak{l}_i$. The action of $\mathrm{Aut}_{\mathcal{O}}$ on L_i necessarily factors through a finite dimensional quotient Q. We may assume that Q surjects to $\mathbb{G}_m^{\mathrm{rot}}$. The group $L_i^\natural$ in the diagram (5.1) is defined as

$$L_i^\natural = (L_i \rtimes Q) \times \mathbb{G}_m.$$

The conjugation action of L_i on L_i and the action of $\mathrm{Aut}_{\mathcal{O}}$ on L_i gives an action of $L_i \rtimes Q$ on L_i, and hence on $\mathfrak{l}_i, \tilde{\mathfrak{l}}_i$. Hence $L_i^{\natural}$ acts on $\mathfrak{l}_i$ and $\tilde{\mathfrak{l}}_i$ with $\mathbb{G}_m$ acting by dilation.
The Springer action for $\mathfrak{l}_i$ is an action of $W_{\mathbf{P}_i} = \{1, s_i\}$ on the complex $\pi^i_*\overline{\mathbb{Q}}_\ell \in D^b([\mathfrak{l}_i/L_i^{\natural}])$. By proper base change, the complex

$$\pi^{\mathbf{P}_i}_{\mathbf{I},*}\overline{\mathbb{Q}}_\ell = \beta_i^*\pi^i_*\overline{\mathbb{Q}}_\ell$$

carries a natural s_i-action. Therefore the complex $f^{\mathrm{par}}_*\overline{\mathbb{Q}}_\ell = f_{\mathbf{P}_i,*}\pi^{\mathbf{P}_i}_{\mathbf{I}}\overline{\mathbb{Q}}_\ell$ carries an s_i-action.

- The $\Omega_{\mathbf{I}}$-action on $f^{\mathrm{par}}_*\overline{\mathbb{Q}}_\ell$ is given by the action of $\Omega_{\mathbf{I}}$ on $\mathcal{M}^{\mathrm{par}}$ in Corollary 4.3.4. For $\omega \in \Omega$, $R_\omega : \mathcal{M}^{\mathrm{par}} \to \mathcal{M}^{\mathrm{par}}$ is the right action, and ω acts on $f^{\mathrm{par}}_*\overline{\mathbb{Q}}_\ell$ via the pull-back R^*_ω.

Theorem 5.1.2. *The action of the affine simple reflections Σ_{aff} and $\Omega_{\mathbf{I}}$ given in Construction* 5.1.1 *generates an action of $\widetilde{W}$ on $f^{\mathrm{par}}_*\overline{\mathbb{Q}}_\ell$.*

Proof. We first check that the actions of Σ_{aff} extend to a W_{aff}-action on $f^{\mathrm{par}}_*\overline{\mathbb{Q}}_\ell$. For each parahoric $\mathbf{P}$, we can apply the similar argument of Construction 5.1.1 to get a $W_{\mathbf{P}}$-action on $f^{\mathrm{par}}_*\overline{\mathbb{Q}}_\ell = f_{\mathbf{P},*}\,\mathrm{ev}^*_{\mathbf{P}}\,\pi^{\mathbf{P}}_{\mathbf{I},*}\overline{\mathbb{Q}}_\ell$, induced from the classical Springer action on $\pi^{\mathbf{P}}_{\mathbf{I},*}\overline{\mathbb{Q}}_\ell$. This construction is compatible with inclusions $\mathbf{P} \subset \mathbf{Q}$ of parahoric subgroups (using the Cartesian diagram (4.6) for $\mathbf{I} \subset \mathbf{P} \subset \mathbf{Q}$). Since the braid relations among the s_i's and the relations $s_i^2 = 1$ are all contained in certain $W_{\mathbf{P}}$, we conclude that the actions of s_i extend to the W_{aff}-action.

We then check the commutation relations between Σ_{aff} and $\Omega_{\mathbf{I}}$. The conjugation action of $\omega \in \Omega_{\mathbf{I}}$ on W_{aff} is given by permuting the affine simple reflections: $s_i \mapsto s_{\omega(i)}$. Similar argument as in Lemma 4.2.5 and Corollary 4.3.4 gives a canonical isomorphism $\mathcal{M}_{\mathbf{P}_{\omega(i)}} \xrightarrow{\sim} \mathcal{M}_{\mathbf{P}_i}$, which we also denote by R_ω. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{M}^{\mathrm{par}} & \xrightarrow{R_\omega} & \mathcal{M}^{\mathrm{par}} \\ \downarrow{\scriptstyle \mathrm{For}^{\mathbf{P}_{\omega(i)}}_{\mathbf{I}}} & & \downarrow{\scriptstyle \mathrm{For}^{\mathbf{P}_i}_{\mathbf{I}}} \\ \mathcal{M}_{\mathbf{P}_{\omega(i)}} & \xrightarrow{R_\omega} & \mathcal{M}_{\mathbf{P}_i} \end{array}$$

From this we deduce $R^*_\omega s_i R^*_{\omega^{-1}} = s_{\omega(i)}$ on $f^{\mathrm{par}}_*\overline{\mathbb{Q}}_\ell$. This completes the proof. □

5.2. Comparison of two constructions

The goal of this subsection is to prove

Proposition 5.2.1. *The $\widetilde{W}$-action on $f^{\mathrm{par}}_*\overline{\mathbb{Q}}_\ell$ defined in Theorem* 5.1.2*, when restricted to $(\mathcal{A} \times X)'$, coincides with the action defined in Theorem* 3.3.3.

We first make some remarks on the Hecke correspondence. Using the identification $\mathrm{Bun}^{\mathrm{par}}_G = \widetilde{\mathrm{Bun}}_\infty/G_{\mathbf{I}} \rtimes \mathrm{Aut}_{\mathcal{O}}$ in Section 4.2, we write

$$\mathcal{H}\mathrm{ecke}^{\mathrm{Bun}} = \widetilde{\mathrm{Bun}}_\infty \overset{G_{\mathbf{I}} \rtimes \mathrm{Aut}_{\mathcal{O}}}{\times} \big(G((t)) \rtimes \mathrm{Aut}_{\mathcal{O}} / G_{\mathbf{I}} \rtimes \mathrm{Aut}_{\mathcal{O}}\big) \tag{5.2}$$

with the two projections given by

$$\overleftarrow{b}(\xi,\tilde{g})=\xi \bmod G_{\mathbf{I}}\rtimes \mathrm{Aut}_{\mathcal{O}};$$
$$\overrightarrow{b}(\xi,\tilde{g})=R_{\tilde{g}}(\xi) \bmod G_{\mathbf{I}}\rtimes \mathrm{Aut}_{\mathcal{O}}$$

for $\xi\in\widetilde{\mathrm{Bun}}_\infty(S)$ and $\tilde{g}\in(G((t))\rtimes\mathrm{Aut}_{\mathcal{O}})(S)/(G_{\mathbf{I}}\rtimes\mathrm{Aut}_{\mathcal{O}})(S)$. The Bruhat decomposition $G((t))=\bigsqcup_{\tilde{w}\in\widetilde{W}}G_{\mathbf{I}}\tilde{w}G_{\mathbf{I}}$ gives a stratification

$$\mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}=\bigsqcup_{\tilde{w}\in\widetilde{W}}\mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}_{\tilde{w}},$$

where

$$\mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}_{\tilde{w}}=\widetilde{\mathrm{Bun}}_\infty\overset{G_{\mathbf{I}}\rtimes\mathrm{Aut}_{\mathcal{O}}}{\times}\big((G_{\mathbf{I}}\rtimes\mathrm{Aut}_{\mathcal{O}})\tilde{w}(G_{\mathbf{I}}\rtimes\mathrm{Aut}_{\mathcal{O}})/G_{\mathbf{I}}\rtimes\mathrm{Aut}_{\mathcal{O}}\big). \tag{5.3}$$

The Bruhat order on $\widetilde{W}$ coincides with the partial order induced by the closure relation among the strata $\mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}_{\tilde{w}}$.

Recall from Definition 3.3.1 that we have reduced Hecke correspondences $\mathcal{H}_{\tilde{w}}$ indexed by $\tilde{w}\in\widetilde{W}$.

Lemma 5.2.2. *The image of $\mathcal{H}^{\mathrm{rs}}_{\tilde{w}}$ in $\mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}$ is contained in $\mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}_{\tilde{w}}$.*

Proof. It suffices to check this statement on the geometric points. Fix $(a,x)\in(\mathcal{A}\times X)^{\mathrm{rs}}(k)$. Using the product formula Proposition 2.4.1, $\mathcal{M}^{\mathrm{par}}_{a,x}$ is homeomorphic to

$$\mathcal{P}_a\overset{P^{\mathrm{red}}_x(J_a)\times P'}{\times}\big(M^{\mathrm{par,red}}_x(\gamma)\times M'\big), \tag{5.4}$$

where M' and P' are the products of local terms over $y\in X-\{x\}$, and $\gamma\in\mathfrak{g}(\widehat{\mathcal{O}}_x)$ lifting $a(x)\in\mathfrak{c}(\widehat{\mathcal{O}}_x)$. Since $a(x)$ has regular semisimple reduction in $\mathfrak{c}$, we can conjugate γ by $G(\widehat{\mathcal{O}}_x)$ so that $\gamma\in\mathfrak{t}(\widehat{\mathcal{O}}_x)$ (here $\mathfrak{t}\subset\mathfrak{b}$ is a Cartan subalgebra). The choice of γ gives a point $\tilde{x}\in q_a^{-1}(x)$. In particular, we can use $\tilde{x}$ to get an isomorphism $P_x(J_a)=T(\widehat{F}_x)/T(\widehat{\mathcal{O}}_x)$.

Fix a uniformizing parameter $t\in\widehat{\mathcal{O}}_x$. Now the reduced structure of $M^{\mathrm{par}}_x(\gamma)\subset\mathrm{Fl}_{G,x}$ consists of points $\tilde{w}_1\mathbf{I}_x/\mathbf{I}_x=t^{\lambda_1}w_1\mathbf{I}_x/\mathbf{I}_x$ for $\tilde{w}_1=(\lambda_1,w_1)\in\widetilde{W}$. Using the definition of the right $\widetilde{W}$-action in Proposition 3.2.1, one easily checks that the action of $\tilde{w}=(\lambda,w)\in\widetilde{W}$ on $\mathcal{M}^{\mathrm{par}}_{a,x}$, under the product formula (5.4), is trivial on M' and sends $\tilde{w}_1\mathbf{I}_x/\mathbf{I}_x\in M^{\mathrm{par}}_x(\gamma)$ to $\tilde{w}_1\tilde{w}\mathbf{I}_x/\mathbf{I}_x$. Since the pair $(\tilde{w}_1\mathbf{I}_x/\mathbf{I}_x,\tilde{w}_1\tilde{w}\mathbf{I}_x/\mathbf{I}_x)\in\mathrm{Fl}_{G,x}\times\mathrm{Fl}_{G,x}$ is in relative position $\tilde{w}$, the image of $\mathcal{H}^{\mathrm{rs}}_{\tilde{w}}$ in $\mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}$ is contained in $\mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}_{\tilde{w}}$. □

Lemma 5.2.3. *Let $\omega\in\Omega_{\mathbf{I}}$. The reduced Hecke correspondence $\mathcal{H}_\omega$ for $\mathcal{M}^{\mathrm{par}}$ is the graph of the automorphism $R_\omega:\mathcal{M}^{\mathrm{par}}\to\mathcal{M}^{\mathrm{par}}$ (see Corollary 4.3.4). In particular, the action of $\omega\in\Omega_{\mathbf{I}}\subset\widetilde{W}$ on $f^{\mathrm{par}}_*\overline{\mathbb{Q}}_\ell$ defined in Theorem 3.3.3 is the same as R^*_ω.*

Proof. We first show that $\mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}_\omega$ is the graph of the automorphism $R_\omega:\mathrm{Bun}^{\mathrm{par}}_G\to\mathrm{Bun}^{\mathrm{par}}_G$ defined in Lemma 4.2.5. In fact, this follows from the description (5.3) of $\mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}_\omega$, and the fact that the Schubert cell $G_{\mathbf{I}}\omega G_{\mathbf{I}}/G_{\mathbf{I}}$ consists of one point for any $\omega\in\Omega_{\mathbf{I}}$.

By the construction of the $\Omega_{\mathbf{I}}$-action on $\mathcal{M}^{\mathrm{par}}$, for any $m\in\mathcal{M}^{\mathrm{par}}(R)$ with image $x\in X(R)$, the Hitchin pairs on $X_R-\Gamma(x)$ given by m and $R_\omega(m)$ are canonically identified. Therefore, the graph $\Gamma(R_\omega)$ of R_ω naturally embeds into $\mathcal{H}\mathrm{ecke}^{\mathrm{par}}$. By the discussion above, the image of $\Gamma(R_\omega)$ in $\mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}$ lies in $\mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}_\omega$ under the map $\beta : \mathcal{H}\mathrm{ecke}^{\mathrm{par}} \to \mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}$:

$$\Gamma(R_\omega) \subset \beta^{-1}\big(\mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}_\omega\big)^{\mathrm{red}}.$$

On the other hand, by Lemma 5.2.2, the reduced structure of $\beta^{-1}(\mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}_\omega)^{\mathrm{rs}}$ is $\mathcal{H}^{\mathrm{rs}}_\omega$, hence $\Gamma(R_\omega)^{\mathrm{rs}} \subset \mathcal{H}^{\mathrm{rs}}_\omega$. Since both $\Gamma(R_\omega)^{\mathrm{rs}}$ and $\mathcal{H}^{\mathrm{rs}}_\omega$ are graphs, we must have $\Gamma(R_\omega)^{\mathrm{rs}} = \mathcal{H}^{\mathrm{rs}}_\omega$. Taking closures, we get $\Gamma(R_\omega) = \mathcal{H}_\omega$. □

Let $\mathbf{P} \subset G(F)$ be a standard parahoric subgroup of $G(F)$. By the Cartesian diagram (4.6), there is a right $W_{\mathbf{P}}$-action on $\mathcal{M}^{\mathrm{par,rs}}$ making $\mathcal{M}^{\mathrm{par,rs}} \to \mathcal{M}^{\mathrm{rs}}_{\mathbf{P}}$ a $W_{\mathbf{P}}$-torsor (because $[\underline{\mathfrak{b}}/\underline{B}]_D \to [\underline{\mathfrak{l}}/\underline{L}]_D$ is a $W_{\mathbf{P}}$-torsor over the regular semisimple locus).

Lemma 5.2.4. *The $W_{\mathbf{P}}$-action on $\mathcal{M}^{\mathrm{par,rs}}$ defined in Proposition* 3.2.1 *and the $W_{\mathbf{P}}$-action defined by the diagram* (4.6) *coincide.*

Proof. For two points $(x, \mathcal{E}_j, \varphi_j, \mathcal{E}^B_{x,j}) \in \mathcal{M}^{\mathrm{par}}(R)$ $(j = 1, 2)$ with the same image in $\mathcal{M}_{\mathbf{P}}$, their Higgs fields on $X - \Gamma(x)$ are identified. Therefore, we have a canonical embedding $\gamma_{\mathbf{P}} : \mathcal{M}^{\mathrm{par}} \times_{\mathcal{M}_{\mathbf{P}}} \mathcal{M}^{\mathrm{par}} \hookrightarrow \mathcal{H}\mathrm{ecke}^{\mathrm{par}}$. Similarly, we have an embedding $\delta_{\mathbf{P}} : \mathrm{Bun}^{\mathrm{par}} \times_{\mathrm{Bun}_{\mathbf{P}}} \mathrm{Bun}^{\mathrm{par}} \hookrightarrow \mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}$ making the following diagram commutative:

$$\begin{array}{ccc}
\mathcal{M}^{\mathrm{par}} \times_{\mathcal{M}_{\mathbf{P}}} \mathcal{M}^{\mathrm{par}} & \xrightarrow{\gamma_{\mathbf{P}}} & \mathcal{H}\mathrm{ecke}^{\mathrm{par}} \\
\downarrow & & \downarrow{\scriptstyle\beta} \\
\mathrm{Bun}^{\mathrm{par}} \times_{\mathrm{Bun}_{\mathbf{P}}} \mathrm{Bun}^{\mathrm{par}} & \xrightarrow{\delta_{\mathbf{P}}} & \mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}
\end{array}$$

Using the description of $\mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}$ in (5.2), we may identify the image of $\delta_{\mathbf{P}}$ with the union $\bigsqcup_{\tilde{w} \in W_{\mathbf{P}}} \mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}_{\tilde{w}}$. Therefore, the image of $\gamma_{\mathbf{P}}$ lies in $\beta^{-1}(\bigsqcup_{\tilde{w} \in W_{\mathbf{P}}} \mathcal{H}\mathrm{ecke}^{\mathrm{Bun}}_{\tilde{w}})$. By Lemma 5.2.2(1), the restriction of $\gamma_{\mathbf{P}}$ to $\mathcal{M}^{\mathrm{par,rs}} \times_{\mathcal{M}_{\mathbf{P}}} \mathcal{M}^{\mathrm{par,rs}}$ (which is reduced) has image in $\bigsqcup_{\tilde{w} \in W_{\mathbf{P}}} \mathcal{H}^{\mathrm{rs}}_{\tilde{w}}$.

For $\mathbf{P} = \mathbf{P}_i$ corresponding to an affine simple reflection s_i, $\mathcal{M}^{\mathrm{par,rs}} \times_{\mathcal{M}_{\mathbf{P}_i}} \mathcal{M}^{\mathrm{par,rs}}$ is the disjoint union of the diagonal and the graph $\Gamma(s_i)$ of the action of s_i given by the Cartesian diagram (4.6). Therefore, $\Gamma(s_i)$ must map to $\mathcal{H}^{\mathrm{rs}}_{s_i}$ under $\gamma_{\mathbf{P}_i}$. This proves the lemma for $\mathbf{P} = \mathbf{P}_i$. The general case follows because $W_{\mathbf{P}}$ is generated by affine simple reflections. □

Proof of Proposition 5.2.1. We only need to check that generators $\Sigma_{\mathrm{aff}} \cup \Omega_{\mathbf{I}}$ of $\widetilde{W}$ give the same action on $f^{\mathrm{par}}_* \overline{\mathbb{Q}}_\ell|_{\mathcal{A}' \times X}$ under the two constructions. For $\omega \in \Omega_{\mathbf{I}}$, this follows from Lemma 5.2.3.

For an affine simple reflection s_i, let St^-_i be the non-diagonal component of the Steinberg variety $\mathrm{St}_i = \tilde{\mathfrak{l}}_i \times_{\mathfrak{l}_i} \tilde{\mathfrak{l}}_i$. Consider the commutative diagram of correspondences

$$\begin{array}{ccccc}
\mathcal{H}\mathrm{ecke}^{\mathrm{par}} & \longleftarrow & C_i & \xrightarrow{\tilde{\epsilon}} & [\mathrm{St}^-_i/L^\natural_i] \\
 & & \overleftarrow{c_i}\downarrow\;\downarrow\overrightarrow{c_i} & & \overleftarrow{\mathrm{st}_i}\downarrow\;\downarrow\overrightarrow{\mathrm{st}_i} \\
 & & \mathcal{M}^{\mathrm{par}} & \xrightarrow{\epsilon} & [\tilde{\mathfrak{l}}_i/L^\natural_i]
\end{array} \tag{5.5}$$

where all the squares are Cartesian (which then defines $C_i \subset \mathcal{M}^{\mathrm{par}} \times_{\mathcal{M}_{\mathbf{P}_i}} \mathcal{M}^{\mathrm{par}} \subset \mathcal{H}\mathrm{ecke}^{\mathrm{par}}$). Classical Springer action (see Remark 3.3.4) of s_i on $\pi^i_*\overline{\mathbb{Q}}_\ell$ is given by $[\mathrm{St}_i^-/L_i^\natural]_\#$. By Lemma A.4.1, the s_i-action on $f^{\mathrm{par}}_*\overline{\mathbb{Q}}_\ell$ in Construction 5.1.1 is give by $(\tilde{\epsilon}^*[\mathrm{St}_i^-/L_i^\natural])_\#$, where $\tilde{\epsilon}^*[\mathrm{St}_i^-/L_i^\natural] \in \mathrm{Corr}(C_i; \overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_\ell)$. As finite-type substacks of $\mathcal{H}\mathrm{ecke}^{\mathrm{par}}|_{(\mathcal{A}\times X)'}$, $\mathcal{C}_i|_{(\mathcal{A}\times X)'}$ and $\mathcal{H}_{s_i}|_{(\mathcal{A}\times X)'}$ satisfy (G-2) in Definition A.6.1 with respect to $(\mathcal{A}\times X)^{\mathrm{rs}} \subset (\mathcal{A}\times X)'$ by Lemma 3.1.4. By Lemma 5.2.4, $\tilde{\epsilon}^*[\mathrm{St}_i^-/L_i^\natural]$ coincides with both $[C_i]$ and $[\mathcal{H}_{s_i}]$ over $(\mathcal{A}\times X)^{\mathrm{rs}}$. By Lemma A.6.2, we conclude that $(\tilde{\epsilon}^*[\mathrm{St}_i^-/L_i^\natural])_\# = [\mathcal{H}_{s_i}]_\#$ on $f^{\mathrm{par}}_*\overline{\mathbb{Q}}_\ell|_{(\mathcal{A}\times X)'}$. □

6. The graded DAHA action

In this section, we will state and prove the second main result of the paper: the graded double affine Hecke algebra action on the parabolic Hitchin complex. Throughout this section, *we assume that G is almost simple*.

6.1. The graded DAHA and its action

Define the *extended Cartan torus* to be

$$\widetilde{T} := \mathbb{G}_m^{\mathrm{cen}} \times T \times \mathbb{G}_m^{\mathrm{rot}}, \tag{6.1}$$

where $\mathbb{G}_m^{\mathrm{cen}}$ and $\mathbb{G}_m^{\mathrm{rot}}$ are one-dimensional tori. We denote the canonical generators of the $\mathbb{Z}$-lattices $\mathbb{X}_*(\mathbb{G}_m^{\mathrm{cen}}), \mathbb{X}^*(\mathbb{G}_m^{\mathrm{cen}}), \mathbb{X}_*(\mathbb{G}_m^{\mathrm{rot}})$ and $\mathbb{X}^*(\mathbb{G}_m^{\mathrm{rot}})$ by $K_{\mathrm{can}}, \Lambda_{\mathrm{can}}, d$ and δ. We will see in Section 6.2 that $\widetilde{T}$ is the Cartan torus of the affine Kac–Moody group $\mathcal{G}$ of G, and $\widetilde{W}$ is the Weyl group of $\mathcal{G}$. Hence $\widetilde{W}$ acts $\widetilde{T}$. The induced action on $\mathbb{X}^*(\widetilde{T})$ and $\mathbb{X}_*(\widetilde{T})$ are denoted by $\eta \mapsto {}^{\tilde{w}}\eta$ and $\xi \mapsto {}^{\tilde{w}}\xi$. The following description of the action is standard.

Lemma 6.1.1.

(1) *$W \subset \widetilde{W}$ fixes $K_{\mathrm{can}}, d, \Lambda_{\mathrm{can}}$ and δ, and acts in the usual way on $\mathbb{X}_*(T)$ and $\mathbb{X}^*(T)$;*
(2) *$\lambda \in \mathbb{X}_*(T)$ acts on $\eta \in \mathbb{X}_*(\widetilde{T})$ and $\xi \in \mathbb{X}^*(\widetilde{T})$ by*

$$ {}^{\lambda}\eta = \eta - \langle\delta, \eta\rangle\lambda + \left((\eta|\lambda)_{\mathrm{can}} - \frac{1}{2}(\lambda|\lambda)_{\mathrm{can}}\langle\delta, \eta\rangle\right)K_{\mathrm{can}};$$

$$ {}^{\lambda}\xi = \xi - \langle\xi, K_{\mathrm{can}}\rangle\lambda^* + \left(\langle\xi, \lambda\rangle - \frac{1}{2}(\lambda|\lambda)_{\mathrm{can}}\langle\xi, K_{\mathrm{can}}\rangle\right)\delta,$$

here $\lambda \mapsto \lambda^*$ is the isomorphism $\mathbb{X}_*(T)_{\mathbb{Q}} \xrightarrow{\sim} \mathbb{X}^*(T)_{\mathbb{Q}}$ induced by the form $(\cdot|\cdot)_{\mathrm{can}}$ (see Section 1.6.3).

Definition 6.1.2. The *graded double affine Hecke algebra* (or *graded DAHA* for short) is an evenly graded $\overline{\mathbb{Q}}_\ell$-algebra $\mathbb{H}$ which, as a vector space, is the tensor product

$$\mathbb{H} = \overline{\mathbb{Q}}_\ell[\widetilde{W}] \otimes \mathrm{Sym}_{\overline{\mathbb{Q}}_\ell}\left(\mathbb{X}^*(\widetilde{T})_{\overline{\mathbb{Q}}_\ell}\right) \otimes \overline{\mathbb{Q}}_\ell[u].$$

Here $\overline{\mathbb{Q}}_\ell[\widetilde{W}]$ is the group ring of $\widetilde{W}$, and $\overline{\mathbb{Q}}_\ell[u]$ is a polynomial algebra in the indeterminate u. The grading on $\mathbb{H}$ is given by

- $\deg(\tilde{w}) = 0$ for $\tilde{w} \in \widetilde{W}$;
- $\deg(u) = \deg(\xi) = 2$ for $\xi \in \mathbb{X}^*(\widetilde{T})$.

The algebra structure on $\mathbb{H}$ is determined by

(DH-1) $\overline{\mathbb{Q}}_\ell[\widetilde{W}]$, $\mathrm{Sym}_{\overline{\mathbb{Q}}_\ell}(\mathbb{X}^*(\widetilde{T})_{\overline{\mathbb{Q}}_\ell})$ and $\overline{\mathbb{Q}}_\ell[u]$ are subalgebras of $\mathbb{H}$;
(DH-2) u is in the center of $\mathbb{H}$;
(DH-3) For any simple reflection $s_i \in \Sigma_{\mathrm{aff}}$ (corresponding to a simple root α_i) and $\xi \in \mathbb{X}^*(\widetilde{T})$, we have

$$s_i\xi - {}^{s_i}\xi s_i = \langle \xi, \alpha_i^\vee \rangle u;$$

(DH-4) For any $\omega \in \Omega_{\mathbf{I}} \subset \widetilde{W}$ and $\xi \in \mathbb{X}^*(\widetilde{T})$, we have

$$\omega\xi = {}^{\omega}\xi\omega.$$

Remark 6.1.3. The graded DAHA, also known as the trigonometric degeneration of the usual DAHA, was introduced in Cherednik's book [7, §2.12.3]. If we replace $\widetilde{W}$ by the finite Weyl group W, and $\widetilde{T}$ by T, the corresponding algebra is the equal-parameter case of the "graded affine Hecke algebras" considered by Lusztig in [25].

Construction 6.1.4. We define the action of the generators of $\mathbb{H}$.

- The action of $\widetilde{W}$ has been constructed in Theorem 5.1.2.
- The action of $\xi \in \mathbb{X}^*(T)$. There is a tautological T-torsor over $\mathrm{Bun}_G^{\mathrm{par}}$ (at the point $(x, \mathcal{E}, \mathcal{E}_x^B)$, the fiber of this T-torsor is $\mathcal{E}_x^B/N$). Hence, for each $\xi \in \mathbb{X}^*(T)$, we have a line bundle $\mathcal{L}(\xi)$ on $\mathrm{Bun}_G^{\mathrm{par}}$ or on $\mathcal{M}^{\mathrm{par}}$ by pull-back. We view the Chern class $c_1(\mathcal{L}(\xi)) \in \mathrm{H}^2(\mathcal{M}^{\mathrm{par}})(1)$ as a map $\overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_\ell[2](1)$ in $D^b(\mathcal{M}^{\mathrm{par}})$. Define the action of ξ on $f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell$ to be the cup product with the Chern class of $\mathcal{L}(\xi)$:

$$\xi = f_*^{\mathrm{par}}\big(c_1\big(\mathcal{L}(\xi)\big)\big) : f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell \to f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell[2](1).$$

- Similar as above, the actions of $\delta \in \mathbb{X}^*(\mathbb{G}_m^{\mathrm{rot}})$, $\Lambda_{\mathrm{can}} \in \mathbb{X}^*(\mathbb{G}_m^{\mathrm{cen}})$ and u are given by cup product with the pull-backs of the Chern classes $c_1(\omega_X)$, $c_1(\omega_{\mathrm{Bun}})$ and $c_1(\mathcal{O}_X(D))$. Here ω_{Bun} and ω_X are the canonical bundles of Bun_G and X respectively.

Example 6.1.5. We describe the line bundles $\mathcal{L}(\xi)$ for $G = \mathrm{GL}(n)$, following the notation in Example 2.2.5. In this case $\mathbb{X}^*(T)$ is naturally identified with $\mathbb{Z}^n$, which canonical basis λ_i, $i = 1, \dots, n$. The line bundle $\mathcal{L}(\lambda_i)$ on $\mathrm{Bun}_n^{\mathrm{par}}$ assigns to each $(x; \mathcal{E}_0 \supset \mathcal{E}_1 \supset \cdots \supset \mathcal{E}_n = \mathcal{E}_0(-x))$ the line $\mathcal{E}_{n-i}/\mathcal{E}_{n-i+1}$. The canonical bundle ω_{Bun} of Bun_n assigns to each vector bundle $\mathcal{E}$ the line $\det \mathbf{R}\Gamma(X, \underline{\mathrm{End}}(\mathcal{E}))$.

The second main theorem of the paper is:

Theorem 6.1.6. *The actions of $\widetilde{W}$, u and $\mathbb{X}^*(\widetilde{T})$ on $f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell$ given in Construction* 6.1.4 *extend to an action of $\mathbb{H}$ on $f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell$. More precisely, we have a graded algebra homomorphism*

$$\mathbb{H} \to \bigoplus_{i\in\mathbb{Z}} \mathrm{End}^{2i}_{\mathcal{A}\times X}\big(f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell\big)(i)$$

such that the image of the elements in $\widetilde{W}$, $\mathbb{X}^*(T)$ and the elements δ, Λ_{can}, u are the same as the ones given in Construction 6.1.4.

The proof will occupy the subsequent subsections till Section 6.5.

For any geometric point $(a,x)\in\mathcal{A}\times X$, we can specialize the above theorem to the action of $\mathbb{H}$ on the stalk of $f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell$ at (a,x), i.e., $\mathrm{H}^*(\mathcal{M}^{\mathrm{par}}_{a,x})$.

Corollary 6.1.7. *For any geometric point $(a,x)\in\mathcal{A}\times X$, Construction* 6.1.4 *gives an action of $\mathbb{H}/(\delta,u)$ on $\mathrm{H}^*(\mathcal{M}^{\mathrm{par}}_{a,x})$. In other words, the actions of $\bar{\xi}\in\mathbb{X}^*(\mathbb{G}_m^{\mathrm{cen}}\times T)$ and $\tilde{w}\in\widetilde{W}$ satisfy the following simple relation*:

$$\tilde{w}\bar{\xi} = {}^{\tilde{w}}\bar{\xi}\tilde{w}.$$

Here $\bar{\xi}\mapsto {}^{\tilde{w}}\bar{\xi}$ is the action of $\tilde{w}$ on $\mathbb{X}^*(\mathbb{G}_m^{\mathrm{cen}}\times T)=\mathbb{X}^*(\widetilde{T})/\mathbb{X}^*(\mathbb{G}_m^{\mathrm{rot}})$.

Proof. Since the restrictions of $\mathcal{O}_X(D)$ and ω_X to $\mathcal{M}^{\mathrm{par}}_{a,x}$ are trivial, the actions of u and δ on $\mathrm{H}^*(\mathcal{M}^{\mathrm{par}}_{a,x})$ are zero. □

6.2. Remarks on the Kac–Moody group

The general reference for the constructions below is [1, §4] where the case $G=\mathrm{SL}_n$ was treated.

6.2.1. The determinant line bundle

For any k-algebra R and a projective R-module M, we use $\det(M)$ to denote the top wedge power of M. We define the *determinant line bundle* $\mathcal{L}_{\mathrm{can}}$ on $G((t))$. For any $R[\![t]\!]$-submodule Ξ of $\mathfrak{g}\otimes_k R((t))$ which is commensurable with the standard $R[\![t]\!]$-submodule $\Xi_0:=\mathfrak{g}\otimes_k R[\![t]\!]$ (i.e., $t^N\Xi_0\subset\Xi\subset t^{-N}\Xi_0$ for some $N\in\mathbb{Z}_{\geqslant 0}$ and $t^{-N}\Xi_0/\Xi$ and $\Xi/t^N\Xi_0$ are both projective R-modules), define the *relative determinant line* of Ξ with respect to Ξ_0 to be:

$$\det(\Xi:\Xi_0)=\big(\det(\Xi/\Xi\cap\Xi_0)\big)\otimes_R\big(\det(\Xi_0/\Xi\cap\Xi_0)\big)^{\otimes -1}.$$

Any $g\in G(R((t)))$ acts on $\mathfrak{g}\otimes_k R((t))$ by the adjoint representation. The functor $\mathcal{L}_{\mathrm{can}}$ then sends g to the invertible R-module $\det(\mathrm{Ad}(g)\Xi_0:\Xi_0)$. Since $\mathrm{Ad}(g)\Xi_0$ only depends on the coset $gG(R[\![t]\!])$, the line bundle $\mathcal{L}_{\mathrm{can}}$ descends to the affine Grassmannian Gr_G.

6.2.2. The completed Kac–Moody group

Let $\widehat{G((t))}=\rho_{\mathcal{L}_{\mathrm{can}}}\to G((t))$ be the total space of the $\mathbb{G}_m$-torsor associated to the line bundle $\mathcal{L}_{\mathrm{can}}$. The set $\widehat{G((t))}(R)$ consists of pairs (g,γ) where $g\in G(R((t)))$ and γ is an R-linear isomorphism $R \xrightarrow{\sim} \det(\mathrm{Ad}(g)\Xi_0 : \Xi_0)$. There is a natural group structure on $\widehat{G((t))}$ making it a central extension

$$1 \to \mathbb{G}_m^{\mathrm{cen}} \to \widehat{G\big((t)\big)} \to G\big((t)\big) \to 1, \tag{6.2}$$

where the one-dimensional torus $\mathbb{G}_m^{\mathrm{cen}}$ is the fiber of $\widehat{G((t))}$ over $1 \in G((t))$. Since $\mathcal{L}_{\mathrm{can}}$ descends to Gr_G, the central extension (6.2) can be canonically trivialized over $G[\![t]\!] \subset G((t))$.

The action of $\mathrm{Aut}_{\mathcal{O}}$ on $G((t))$ lifts to an action on $\widehat{G((t))}$, hence we can form the semi-direct product

$$\mathcal{G} := \widehat{G\big((t)\big)} \rtimes \mathrm{Aut}_{\mathcal{O}}. \tag{6.3}$$

We call this object the (*completed*) *Kac–Moody group* associated to the loop group $G((t))$.

Let $\mathbf{I}^u \subset \mathbf{I}$ be the unipotent radical and $G_{\mathbf{I}}^u \subset G_{\mathbf{I}}$ be the corresponding pro-unipotent group. Let $\mathrm{Aut}_{\mathcal{O}}^u \subset \mathrm{Aut}_{\mathcal{O}}$ be the pro-unipotent radical. Consider the subgroups

$$\mathcal{G}_{\mathbf{I}} := \mathbb{G}_m^{\mathrm{cen}} \times G_{\mathbf{I}} \rtimes \mathrm{Aut}_{\mathcal{O}} \subset \mathcal{G};$$
$$\mathcal{G}_{\mathbf{I}}^u := G_{\mathbf{I}}^u \rtimes \mathrm{Aut}_{\mathcal{O}}^u \subset \mathcal{G}_{\mathbf{I}}.$$

The extended Cartan torus $\widetilde{T}$ defined in (6.1) can be canonically identified with $\mathcal{G}_{\mathbf{I}}/\mathcal{G}_{\mathbf{I}}^u$. Any section ι of the quotient $B \to T$ gives a section $\iota : \widetilde{T} \to \mathcal{G}_{\mathbf{I}}$. The extended affine Weyl group $\widetilde{W}$ can be identified with the Weyl group of the pair $(\mathcal{G}, \iota(\widetilde{T}))$, and this identification is independent of ι.

Let $\theta \in \Phi$ be highest root and $\theta^\vee \in \Phi^\vee$ be the corresponding coroot. Let ρ be half of the sum of the positive roots in Φ. Let $h^\vee$ be the dual Coxeter number of $\mathfrak{g}$, which is one plus the sum of coefficients of $\theta^\vee$ written as a linear combination of simple coroots.

Lemma 6.2.3. $\frac{1}{2}(\theta^\vee|\theta^\vee)_{\mathrm{can}} = 2(\langle\rho, \theta^\vee\rangle + 1) = 2h^\vee$.

Proof. For any positive root $\alpha \neq \theta$, we have $\langle\alpha, \theta^\vee\rangle = 0$ or 1 (see [6, Chapter VI, 1.8, Proposition 25(iv)]). Hence $\langle\alpha, \theta^\vee\rangle^2 = \langle\alpha, \theta^\vee\rangle$ for $\alpha \in \Phi^+ - \{\theta\}$. Therefore

$$\begin{aligned}\frac{1}{2}\big(\theta^\vee\big|\theta^\vee\big)_{\mathrm{can}} &= \sum_{\alpha\in\Phi^+}\big\langle\alpha, \theta^\vee\big\rangle^2 = \big\langle\theta, \theta^\vee\big\rangle^2 + \sum_{\alpha\in\Phi^+-\{\theta\}}\big\langle\alpha, \theta^\vee\big\rangle \\ &= 4 + \big\langle 2\rho - \theta, \theta^\vee\big\rangle = 2\big(\big\langle\rho, \theta^\vee\big\rangle + 1\big).\end{aligned}$$

Since $\langle\rho, \alpha_i^\vee\rangle = 1$ for every simple coroot $\alpha_i^\vee \in \Phi$, we get $\langle\rho, \theta^\vee\rangle + 1 = h^\vee$. $\square$

Remark 6.2.4. Let

$$K := 2h^\vee K_{\mathrm{can}}; \qquad \Lambda_0 := \frac{1}{2h^\vee}\Lambda_{\mathrm{can}}.$$

We see from Lemma 6.1.1 that our definitions of K, Λ_0, d and δ are consistent with the notation for Kac–Moody algebras in [17, 6.5]. The simple roots of the complete Kac–Moody group $\mathcal{G}$ are

$\{\alpha_0 = \delta - \theta, \alpha_1, \dots, \alpha_n\} \subset \mathbb{X}^*(T \times \mathbb{G}_m^{\mathrm{rot}})$; the simple coroots are $\{\alpha_0^\vee = K - \theta^\vee, \alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \mathbb{X}_*(\mathbb{G}_m^{\mathrm{cen}} \times T)$.

6.3. *Line bundles on* $\mathrm{Bun}_G^{\mathrm{par}}$

In this subsection, we give a uniform construction of the action of the degree two elements $\mathbb{X}^*(\widetilde{T}) \subset \mathbb{H}$ on $f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell$.

Let ω_{Bun} be the canonical bundle of Bun_G. Since the tangent complex at a point $\mathcal{E} \in \mathrm{Bun}_G(R)$ is $\mathbf{R}\Gamma(X_R, \mathrm{Ad}(\mathcal{E}))[1]$, the value of the canonical bundle ω_{Bun} at the point $\mathcal{E}$ is the invertible R-module $\det \mathbf{R}\Gamma(X_R, \mathrm{Ad}(\mathcal{E}))$ (see [3, 2.2.1], [23]). Let $\widehat{\mathrm{Bun}}_\infty \to \widetilde{\mathrm{Bun}}_\infty$ be the total space of the $\mathbb{G}_m$-torsor associated to the pull-back of ω_{Bun}. More concretely, for any k-algebra R, $\widehat{\mathrm{Bun}}_\infty(R)$ classifies tuples $(x, \alpha, \mathcal{E}, \tau_x, \epsilon)$ where $(x, \alpha, \mathcal{E}, \tau_x) \in \widetilde{\mathrm{Bun}}_\infty(R)$ and ϵ is an R-linear isomorphism $R \xrightarrow{\sim} \det \mathbf{R}\Gamma(X_R, \mathrm{Ad}(\mathcal{E}))$.

Construction 6.3.1. (Compare [3, §2.8.5].) There is a natural action of $\mathcal{G}$ on $\widehat{\mathrm{Bun}}_\infty$, lifting the action of $G((t)) \rtimes \mathrm{Aut}_{\mathcal{O}}$ on $\widetilde{\mathrm{Bun}}_\infty$ in Construction 4.2.2. In fact, for $(x, \alpha, \mathcal{E}, \tau_x) \in \widetilde{\mathrm{Bun}}_\infty(R)$ and $g \in G(R((t)))$, the G-torsor $\mathcal{E}^g$ is obtained by gluing the trivial G-torsor on $\mathfrak{D}_x \cong \mathrm{Spec}\, R[\![t]\!]$ (using α) with $\mathcal{E}|_{X_R - \Gamma(x)}$ via the identification $\tau_x \circ g$. Hence $\mathrm{Ad}(\mathcal{E}^g)$ is obtained by gluing $\mathfrak{g}(\widehat{\mathcal{O}}_x) \cong \mathfrak{g} \otimes_k R[\![t]\!] = \Xi_0$ with $\mathrm{Ad}(\mathcal{E})|_{X_R - \Gamma(x)}$ via the identification $\mathrm{Ad}(\tau_x) \circ \mathrm{Ad}(g)$. In other words, $\mathrm{Ad}(\mathcal{E}^g)$ is obtained by gluing $\mathrm{Ad}(g)\Xi_0$ with $\mathrm{Ad}(\mathcal{E})|_{X_R - \Gamma(x)}$ via $\mathrm{Ad}(\tau_x)$. Thus we have a canonical isomorphism

$$\bigl(\det \mathbf{R}\Gamma\bigl(X_R, \mathrm{Ad}\bigl(\mathcal{E}^g\bigr)\bigr)\bigr) \otimes_R \bigl(\det \mathbf{R}\Gamma\bigl(X_R, \mathrm{Ad}(\mathcal{E})\bigr)\bigr)^{\otimes -1} \cong \det\bigl(\mathrm{Ad}(g)\Xi_0 : \Xi_0\bigr).$$

Therefore, for trivializations $\epsilon : R \xrightarrow{\sim} \det \mathbf{R}\Gamma(X_R, \mathrm{Ad}(\mathcal{E}))$ and $\gamma : R \xrightarrow{\sim} \det(\mathrm{Ad}(g)\Xi_0 : \Xi_0)$, $\epsilon \otimes \gamma$ defines a trivialization of $\det \mathbf{R}\Gamma(X_R, \mathrm{Ad}(\mathcal{E}^g))$. We then define the action of $\hat{g} = (g, \gamma, \sigma) \in (\widehat{G((t))} \rtimes \mathrm{Aut}_{\mathcal{O}})(R) = \mathcal{G}(R)$ on $(x, \alpha, \mathcal{E}, \tau_x, \epsilon) \in \widehat{\mathrm{Bun}}_\infty(R)$ by

$$R_{\hat{g}}(x, \alpha, \mathcal{E}, \tau_x, \epsilon) = \bigl(x, \alpha \circ \sigma, \mathcal{E}^g, \tau_x^g, \epsilon \otimes \gamma\bigr).$$

Construction 6.3.2. We define a natural $\widetilde{T}$-torsor $\mathcal{L}^{\widetilde{T}}$ on $\mathrm{Bun}_G^{\mathrm{par}}$, hence line bundles $\mathcal{L}(\xi)$ for $\xi \in \mathbb{X}^*(\widetilde{T})$. In fact, take $\mathcal{L}^{\widetilde{T}} = \widehat{\mathrm{Bun}}_\infty / \mathcal{G}_{\mathbf{I}}^u$ (as a fpqc sheaf). The right translation of $\mathcal{G}_{\mathbf{I}}$ on $\widehat{\mathrm{Bun}}_\infty$ descends to a right action of $\widetilde{T} = \mathcal{G}_{\mathbf{I}}/\mathcal{G}_{\mathbf{I}}^u$ on $\mathcal{L}^{\widetilde{T}}$, and realizes the natural projection $\mathcal{L}^{\widetilde{T}} \to \mathrm{Bun}_G^{\mathrm{par}}$ as a $\widetilde{T}$-torsor.

It is easy to identify the line bundles $\mathcal{L}(\xi)$ for $\xi \in \mathbb{X}^*(\widetilde{T})$ (compare [22, the map in Theorem 1.1]):

Lemma 6.3.3.

(1) *$\mathcal{L}(\Lambda_{\mathrm{can}})$ is the pull-back of ω_{Bun} via the forgetful morphism* $\mathrm{Bun}_G^{\mathrm{par}} \to \mathrm{Bun}_G$;
(2) *For $\xi \in \mathbb{X}^*(T)$, the value of the line bundle $\mathcal{L}(\xi)$ at a point $(x, \mathcal{E}, \mathcal{E}_x^B) \in \mathrm{Bun}_G^{\mathrm{par}}(R)$ is the invertible R-module associated to the B-torsor $\mathcal{E}_x^B$ over $\Gamma(x) \cong \mathrm{Spec}\, R$ and the character* $B \to T \xrightarrow{\xi} \mathbb{G}_m$;
(3) *$\mathcal{L}(\delta)$ is isomorphic to the pull-back of ω_X via the morphism* $\mathrm{Bun}_G^{\mathrm{par}} \to X$ (*cf. Lemma* 4.1.2).

Remark 6.3.4. Comparing Lemma 6.3.3 with Construction 6.1.4, we see that the action of any $\xi \in \widetilde{T}$ on $f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell$ is given by

$$\xi = f_*^{\mathrm{par}}\Big(\bigcup c_1\big(\mathcal{L}(\xi)\big)\Big) : f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell \to f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell[2](1),$$

where $\mathcal{L}(\xi)$ are the line bundles in Construction 6.3.2.

6.4. Simple reflections—a calculation in $\mathfrak{sl}_2$

In this subsection, we check the condition (DH-3) for $\xi \in \mathbb{X}^*(T \times \mathbb{G}_m^{\mathrm{rot}})$. We use the notation in Construction 5.1.1.

The natural projection $B^i \to T$ extends to the projection $B^i \rtimes Q \to T \times \mathbb{G}_m^{\mathrm{rot}}$. Therefore we have a morphism $[\tilde{\mathfrak{l}}_i/L_i^\natural] = [\mathfrak{b}^i/(B^i \rtimes Q \times \mathbb{G}_m)] \to \mathbb{B}(T \times \mathbb{G}_m^{\mathrm{rot}})$, which gives a $T \times \mathbb{G}_m^{\mathrm{rot}}$-torsor on $[\tilde{\mathfrak{l}}_i/L_i^\natural]$. The associated line bundles on $[\tilde{\mathfrak{l}}_i/L_i^\natural]$ are denoted by $\mathcal{N}(\xi)$, for $\xi \in \mathbb{X}^*(T \times \mathbb{G}_m^{\mathrm{rot}})$.

Recall $\mathrm{St}_i = \tilde{\mathfrak{l}}_i \times_{\mathfrak{l}_i} \tilde{\mathfrak{l}}_i$ is the Steinberg variety for $\mathfrak{l}_i$. Let $\mathrm{St}_i = \mathrm{St}_i^+ \cup \mathrm{St}_i^-$ be the decomposition into two irreducible components, where St_i^+ is the diagonal copy of $\tilde{\mathfrak{l}}_i$, and St_i^- is the non-diagonal component. We have a correspondence diagram

$$\begin{array}{ccccc} & & [\mathrm{St}_i/L_i^\natural] & & \\ & \overleftarrow{\mathrm{st}_i} \swarrow & & \searrow \overrightarrow{\mathrm{st}_i} & \\ [\tilde{\mathfrak{l}}_i/L_i^\natural] & \xrightarrow{\pi^i} & [\mathfrak{l}_i/L_i^\natural] & \xleftarrow{\pi^i} & [\tilde{\mathfrak{l}}_i/L_i^\natural] \end{array}$$

Lemma 6.4.1.

(1) *For* $\xi \in \mathbb{X}^*(T \times \mathbb{G}_m^{\mathrm{rot}})$*, the action of* $s_i\xi - {}^{s_i}\xi s_i - \langle \xi, \alpha_i^\vee\rangle u$ *on* $f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell = f_{\mathbf{P}_i,*}\beta_i^*\pi_*^i\overline{\mathbb{Q}}_\ell$ *is induced from a map* $\sigma : \pi_*^i\overline{\mathbb{Q}}_\ell \to \pi_*^i\overline{\mathbb{Q}}_\ell[2](1)$. *In other words,*

$$s_i\xi - {}^{s_i}\xi s_i - \langle \xi, \alpha_i^\vee\rangle u = f_{\mathbf{P}_i,*}\beta_i^*\sigma \in \mathrm{End}^2\big(f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell\big)(1).$$

(2) *The map* σ *is induced by the following element in* $\mathrm{Corr}([\mathrm{St}_i/L_i^\natural]; \overline{\mathbb{Q}}_\ell[2](1), \overline{\mathbb{Q}}_\ell)$

$$\big[\mathrm{St}_i^-/L_i^\natural\big] \cup \big(\overrightarrow{\mathrm{st}_i}^* c_1\big(\mathcal{N}(\xi)\big) - \overleftarrow{\mathrm{st}_i}^* c_1\big(\mathcal{N}\big({}^{s_i}\xi\big)\big)\big) - \big[\mathrm{St}_i^+/L_i^\natural\big] \cup \langle \xi, \alpha_i^\vee\rangle v.$$

Here $v \in \mathrm{H}^2([\mathrm{St}_i/L_i^\natural])(1)$ *is the image of the generator of* $\mathrm{H}^2(\mathbb{B}\mathbb{G}_m)(1)$ (*for the* $\mathbb{G}_m$ *factor in* $L_i^\natural$).

Proof. By Construction 5.1.1, the s_i-action on $f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell$ is induced from that on $\pi_*^i\overline{\mathbb{Q}}_\ell$. Recall the construction of the Springer action via Steinberg varieties (see Remark 3.3.4): the action of s_i on $\pi_*^i\overline{\mathbb{Q}}_\ell$ is given by $[\mathrm{St}_i^-/L_i^\natural] \in \mathrm{Corr}([\mathrm{St}_i/L_i^\natural]; \overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_\ell)$.

On the other hand, consider the evaluation map $\epsilon : \mathcal{M}^{\mathrm{par}} \to [\tilde{\mathfrak{l}}_i/L_i^\natural]$ in (5.1). We have $\mathcal{L}(\xi) = \epsilon^*\mathcal{N}(\xi)$ for $\xi \in \mathbb{X}^*(T \times \mathbb{G}_m^{\mathrm{rot}})$, and $c_1(\mathcal{O}_X(D)) = \epsilon^* v$. The rest of the lemma follows from Lemma A.5.2 about the cup product action on cohomological correspondences. □

Thus the condition (DH-3) for $\xi \in \mathbb{X}^*(T \times \mathbb{G}_m^{\mathrm{rot}})$ reduces to

Proposition 6.4.2. *For each $\xi \in \mathbb{X}^*(T \times \mathbb{G}_m^{\mathrm{rot}})$, we have*

$$\big[\mathrm{St}_i^-/L_i^\natural\big] \cup \big(\overrightarrow{\mathrm{st}_i}^* c_1\big(\mathcal{N}(\xi)\big) - \overleftarrow{\mathrm{st}_i}^* c_1\big(\mathcal{N}\big({}^{s_i}\xi\big)\big)\big) = \big[\mathrm{St}_i^+/L_i^\natural\big] \cup \big\langle \xi, \alpha_i^\vee \big\rangle v \tag{6.4}$$

as elements in $\mathrm{Corr}([\mathrm{St}_i/L_i^\natural]; \overline{\mathbb{Q}}_\ell[2](1), \overline{\mathbb{Q}}_\ell)$.

Proof. We decompose $\mathbb{X}^*(T \times \mathbb{G}_m^{\mathrm{rot}})_\mathbb{Q} = \mathbb{X}^*(T \times \mathbb{G}_m^{\mathrm{rot}}) \otimes_\mathbb{Z} \mathbb{Q}$ into ± 1-eigenspaces of the reflection s_i:

$$\mathbb{X}^*\big(T \times \mathbb{G}_m^{\mathrm{rot}}\big)_\mathbb{Q} = \mathbb{X}^*\big(T \times \mathbb{G}_m^{\mathrm{rot}}\big)_\mathbb{Q}^{s_i} \oplus \mathbb{Q}\alpha_i,$$

where α_i spans the (-1)-eigenspace of s_i.

To prove (6.4), it suffices to prove it for $\xi \in \mathbb{X}^*(T \times \mathbb{G}_m^{\mathrm{rot}})^{s_i}$ and $\xi = \alpha_i$ separately. In the first case, taking Chern class induces an isomorphism

$$c_1 : \mathbb{X}^*\big(T \times \mathbb{G}_m^{\mathrm{rot}}\big)_{\overline{\mathbb{Q}}_\ell}^{s_i} \xrightarrow{\sim} \mathrm{H}^2\big(\mathbb{B}(L_i \rtimes Q)\big)(1).$$

Hence $c_1(\mathcal{N}(\xi))$ lies in the image of the pull-back map

$$\pi^{i,*} : \mathrm{H}^2\big(\mathbb{B}L_i^\natural\big)(1) \to \mathrm{H}^2\big(\big[\mathfrak{l}_i/L_i^\natural\big]\big)(1) \to \mathrm{H}^2\big(\big[\tilde{\mathfrak{l}}_i/L_i^\natural\big]\big)(1).$$

Since $\pi^i \circ \overleftarrow{\mathrm{st}_i} = \pi^i \circ \overrightarrow{\mathrm{st}_i}$, we conclude that

$$\overrightarrow{\mathrm{st}_i}^* c_1\big(\mathcal{N}(\xi)\big) = \overleftarrow{\mathrm{st}_i}^* c_1\big(\mathcal{N}(\xi)\big) = \overleftarrow{\mathrm{st}_i}^* c_1\big(\mathcal{N}\big({}^{s_i}\xi\big)\big).$$

Therefore, the LHS of (6.4) is zero. On the other hand, since ${}^{s_i}\xi = \xi$, we have $\langle \xi, \alpha_i^\vee \rangle = 0$, hence the RHS of (6.4) is also zero. This proves the identity (6.4) in the case $\xi \in \mathbb{X}^*(T \times \mathbb{G}_m^{\mathrm{rot}})^{s_i}$.

Finally we treat the case $\xi = \alpha_i$. Since $L_i \rtimes Q$ is connected, the action of $L_i \rtimes Q$ on L_i factors through a homomorphism $L_i \rtimes Q \to L_i^{\mathrm{ad}}$, where L_i^{ad} is the adjoint form of L_i (isomorphic to PGL(2)). Let $\mathbb{P}_i^1 = L_i/B^i = L_i \rtimes Q/B^i \rtimes Q = L_i^{\mathrm{ad}}/B^{\mathrm{ad},i}$ be the flag variety of L_i or L_i^{ad}. The pull-back

$$\mathrm{H}^2_{L_i^{\mathrm{ad}}}\big(\mathbb{P}_i^1\big)(1) \to \mathrm{H}^2_{L_i \rtimes Q}\big(\mathbb{P}_i^1\big)(1) = \mathrm{H}^2\big(\mathbb{B}\big(B^i \rtimes Q\big)\big)(1) = \mathbb{X}^*\big(T \times \mathbb{G}_m^{\mathrm{rot}}\big)_{\overline{\mathbb{Q}}_\ell}$$

has image $\overline{\mathbb{Q}}_\ell\alpha_i$, and the line bundle $\mathcal{N}(\alpha_i)$ on $\tilde{\mathfrak{l}}_i$ is the pull-back of the canonical bundle $\omega_{\mathbb{P}_i^1}$ on $\mathbb{P}_i^1$. We can therefore only consider the $L_i^{\mathrm{ad}} \times \mathbb{G}_m$-action on St_i. The equality (6.4) then reduces to the following identity in the $L_i^{\mathrm{ad}} \times \mathbb{G}_m$-equivariant Borel–Moore homology group $\mathrm{H}_{2d-2}^{\mathrm{BM},L_i^{\mathrm{ad}}\times\mathbb{G}_m}(\mathrm{St}_i)(1)$ ($d = \dim \mathrm{St}_i$):

$$h^{-,*}c_1(\omega_{\mathbb{P}_i^1\times\mathbb{P}_i^1}) \cup \big[\mathrm{St}_i^-\big] = 2v \cup \big[\mathrm{St}_i^+\big], \tag{6.5}$$

where h^- is the $L_i^{\mathrm{ad}} \times \mathbb{G}_m$-equivariant morphism $h^- : \mathrm{St}_i^- \to \mathbb{P}_i^1 \times \mathbb{P}_i^1$. We will show that both sides of (6.5) are equal to the cycle class $2[\mathrm{St}^{\mathrm{nil}}] \in \mathrm{H}_{2d-2}^{\mathrm{BM},L_i^{\mathrm{ad}}\times\mathbb{G}_m}(\mathrm{St}_i)(1)$, where $\mathrm{St}^{\mathrm{nil}}$ is the preimage of the nilpotent cone in $\mathfrak{l}_i$ under $\mathrm{St}_i \to \mathfrak{l}_i$.

On one hand, $\mathrm{St}^{\mathrm{nil}}$ is the preimage of the diagonal $\Delta(\mathbb{P}^1_i) \subset \mathbb{P}^1_i \times \mathbb{P}^1_i$ under h^-. Let $\mathcal{I}_\Delta$ be the ideal sheaf of the diagonal $\Delta(\mathbb{P}^1_i)$, viewed as an L_i^{ad}-equivariant line bundle on $\mathbb{P}^1_i \times \mathbb{P}^1_i$. We claim that

$$\mathcal{I}_\Delta^{\otimes 2} \cong \omega_{\mathbb{P}^1_i \times \mathbb{P}^1_i} \in \mathrm{Pic}_{L_i^{\mathrm{ad}}}\big(\mathbb{P}^1_i \times \mathbb{P}^1_i\big). \tag{6.6}$$

In fact, since L_i^{ad} does not admit nontrivial characters, we have $\mathrm{Pic}_{L_i^{\mathrm{ad}}}(\mathbb{P}^1_i \times \mathbb{P}^1_i) \cong \mathbb{Z} \oplus \mathbb{Z}$. Then (6.6) follows by comparing the degrees along the rulings.

Since the Poincaré dual of $c_1(\mathcal{I}_\Delta)$ is the cycle class $[\Delta(\mathbb{P}^1_i)] \in \mathrm{H}_2^{\mathrm{BM},L_i^{\mathrm{ad}}}(\mathbb{P}^1_i \times \mathbb{P}^1_i)(1)$, we get from (6.6) that

$$c_1(\omega_{\mathbb{P}^1_i \times \mathbb{P}^1_i}) \cup \big[\mathbb{P}^1_i \times \mathbb{P}^1_i\big] = 2\big[\Delta\big(\mathbb{P}^1_i\big)\big]. \tag{6.7}$$

Since h^- is smooth (St_i^- is in fact a vector bundle over $\mathbb{P}^1_i \times \mathbb{P}^1_i$), we can pull-back (6.7) along h^- to get

$$h^{-,*}c_1(\omega_{\mathbb{P}^1_i \times \mathbb{P}^1_i}) \cup \big[\mathrm{St}_i^-\big] = 2\big[\mathrm{St}_i^{\mathrm{nil}}\big] \in \mathrm{H}_{2d-2}^{\mathrm{BM},L_i^{\mathrm{ad}} \times \mathbb{G}_m}\big(\mathrm{St}_i^-\big)(1). \tag{6.8}$$

On the other hand, consider the projection $\tau : \tilde{\mathfrak{l}}_i \to \mathfrak{t} \to \mathfrak{t}^{\mathrm{ad}}$ ($\mathfrak{t}^{\mathrm{ad}}$ is the universal Cartan for L_i^{ad}), then $\mathrm{St}_i^{\mathrm{nil}} = \tau^{-1}(0)$. The class $[0] \in \mathrm{H}_0^{\mathrm{BM},\mathbb{G}_m}(\mathfrak{t}^{\mathrm{ad}})(1)$ is the Poincaré dual of v ($\mathbb{G}_m$ acts on the affine line $\mathfrak{t}^{\mathrm{ad}}$ by dilation). Since τ is $L_i^{\mathrm{ad}} \times \mathbb{G}_m$-equivariant and L_i^{ad} acts trivially on $\mathfrak{t}^{\mathrm{ad}}$, we conclude that

$$\big[\mathrm{St}_i^{\mathrm{nil}}\big] = v \cup \big[\mathrm{St}_i^+\big] \in \mathrm{H}_{2d-2}^{\mathrm{BM},L_i^{\mathrm{ad}} \times \mathbb{G}_m}\big(\mathrm{St}_i^+\big)(1). \tag{6.9}$$

If we view both (6.8) and (6.9) as identities in $\mathrm{H}_{2d-2}^{\mathrm{BM},L_i^{\mathrm{ad}} \times \mathbb{G}_m}(\mathrm{St}_i)(1)$, we get the identity (6.5). This completes the proof. □

6.5. Completion of the proof

We need to check that the four conditions in Definition 6.1.2 hold for the actions defined in Construction 6.1.4. The condition (DH-1) obvious holds.

The condition (DH-2) is also easy to check. In fact, the s_i-action is constructed by pulling back the Springer action on $\pi^i_*\overline{\mathbb{Q}}_\ell$ via diagram (5.1). Since the Springer action on $\pi^i_*\overline{\mathbb{Q}}_\ell$ is given by a self-correspondence of $[\tilde{\mathfrak{l}}_i/L_i^\natural]$ over $[\mathfrak{l}_i/L_i^\natural]$, it commutes with cupping with any class in $\mathrm{H}^*([\mathfrak{l}_i/L_i^\natural])$. Therefore, the s_i-action on $f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell$ commutes with cupping with any class in $\mathrm{H}^*(\mathcal{M}_{\mathbf{P}_i})$, in particular u. Similarly, the action of $\Omega_{\mathbf{I}}$ on $f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell$ is given by self-correspondences of $\mathcal{M}^{\mathrm{par}}$ over $\mathcal{A} \times X$, it commutes with cupping with any class from $\mathrm{H}^*(X)$, in particular u. Moreover, the action of $\xi \in \mathbb{X}^*(\widetilde{T})$ is defined as cupping with $c_1(\mathcal{L}(\xi)) \in \mathrm{H}^2(\mathcal{M}^{\mathrm{par}})(1)$, which certainly commutes with the action of u. Therefore the actions of the generators of $\mathbb{H}$ all commute with the action of u. This verifies (DH-2).

Now we verify the condition (DH-4) in Definition 6.1.2. This is implied by the following lemma.

Lemma 6.5.1. *For each $\omega \in \Omega_{\mathbf{I}}$ and $\xi \in \mathbb{X}^*(\widetilde{T})$, there is an isomorphism of line bundles on* $\mathrm{Bun}_G^{\mathrm{par}}$:

$$R_\omega^* \mathcal{L}(\xi) \cong \mathcal{L}\big({}^{\omega}\xi\big).$$

Proof. Recall from Construction 6.3.2 that the right action of $\widetilde{T}$ on $\mathcal{L}^{\widetilde{T}}$ comes from the right action of $\mathcal{G}_{\mathbf{I}}$ on $\widehat{\mathrm{Bun}}_\infty$. On the other hand, the right action of $\Omega_{\mathbf{I}}$ on $\mathrm{Bun}_G^{\mathrm{par}}$ comes from the right action of $\mathcal{G}_{\widetilde{\mathbf{I}}}$ on $\widehat{\mathrm{Bun}}_\infty$ (see Lemma 4.2.5). For any $\hat{g} \in \mathcal{G}_{\mathbf{I}}$ and $\hat{\omega} \in \mathcal{G}_{\widetilde{\mathbf{I}}}$, we have

$$R_{\mathrm{Ad}(\hat{\omega}^{-1})\hat{g}} \circ R_{\hat{\omega}} = R_{\hat{g}\hat{\omega}} = R_{\hat{\omega}} \circ R_{\hat{g}}.$$

Taking the quotient by $\mathcal{G}_{\mathbf{I}}^u$, we get an equality of actions on $\mathcal{L}^{\widetilde{T}} = \widehat{\mathrm{Bun}}_\infty/\mathcal{G}_{\mathbf{I}}^u$:

$$R_{\mathrm{Ad}(\omega^{-1})g} \circ R_\omega = R_\omega \circ R_g, \quad \text{for } \omega \in \Omega_{\mathbf{I}},\ g \in \widetilde{T}.$$

Therefore the $\widetilde{T}$-torsor $R_\omega^* \mathcal{L}^{\widetilde{T}}$ on $\mathrm{Bun}_G^{\mathrm{par}}$ is the $\mathrm{Ad}(\omega)$-twist of $\mathcal{L}^{\widetilde{T}}$. □

We have verified (DH-3) for $\xi \in \mathbb{X}^*(T \times \mathbb{G}_m^{\mathrm{rot}})$ in Section 6.4. It remains is to verify (DH-3) for $\xi = \Lambda_{\mathrm{can}}$. For each standard parahoric subgroup $\mathbf{P} \subset G(F)$, we define a line bundle $\mathcal{L}_{\mathbf{P},\mathrm{can}}$ on $\mathrm{Bun}_{\mathbf{P}}$ which to every point $(x, \mathcal{E}, \tau_x \bmod \mathbf{P}) \in \mathrm{Bun}_{\mathbf{P}}(R)$ assigns the invertible R-module $\det \mathbf{R}\Gamma(X_R, \mathrm{Ad}_{\mathbf{P}}(\mathcal{E}))$.

Lemma 6.5.2. *For each standard parahoric subgroup $\mathbf{P} \subset G(F)$, we have*

$$\mathcal{L}_{\mathbf{I},\mathrm{can}} \otimes \mathcal{L}(-2\rho_{\mathbf{P}}) \cong \mathrm{For}_{\mathbf{I}}^{\mathbf{P},*} \mathcal{L}_{\mathbf{P},\mathrm{can}} \in \mathrm{Pic}\big(\mathrm{Bun}_G^{\mathrm{par}}\big). \tag{6.10}$$

Here $2\rho_{\mathbf{P}}$ is the sum of positive roots in $L_{\mathbf{P}}$ (with respect to the Borel $B_{\mathbf{I}}^{\mathbf{P}}$).

Proof. For $(x, \alpha, \mathcal{E}, \tau_x \bmod \mathbf{I}) \in \widetilde{\mathrm{Bun}}_{\mathbf{I}}(R)$, we have an exact sequence of vector bundles on X_R

$$0 \to \mathrm{Ad}_{\mathbf{I}}(\mathcal{E}) \to \mathrm{Ad}_{\mathbf{P}}(\mathcal{E}) \to i_* \mathcal{Q}(\mathcal{E}) \to 0 \tag{6.11}$$

where $\mathcal{Q}(\mathcal{E})$ is a coherent sheaf supported on $\Gamma(x)$. The assignment $(\mathcal{E}, \tau_x \bmod \mathbf{I}) \mapsto \mathcal{Q}(\mathcal{E})$ gives a coherent sheaf $\mathcal{Q}$ on $\mathrm{Bun}^{\mathrm{par}}$. Via the local coordinate α and the full level structure τ_x, we can identify $\mathcal{Q}(\mathcal{E})$ with $(\mathfrak{g}_{\mathbf{P}}/\mathfrak{g}_{\mathbf{I}}) \otimes_k R$. Hence $\mathcal{Q}$ can be viewed as the vector bundle

$$\mathcal{Q} \cong \widetilde{\mathrm{Bun}}_\infty \overset{G_{\mathbf{I}} \rtimes \mathrm{Aut}_{\mathcal{O}}}{\times} (\mathfrak{g}_{\mathbf{P}}/\mathfrak{g}_{\mathbf{I}})$$

over $\mathrm{Bun}^{\mathrm{par}}$. Taking the determinant, we get

$$\det \mathcal{Q} \cong \widetilde{\mathrm{Bun}}_\infty \overset{G_{\mathbf{I}} \rtimes \mathrm{Aut}_{\mathcal{O}}}{\times} \det(\mathfrak{g}_{\mathbf{P}}/\mathfrak{g}_{\mathbf{I}}).$$

Since $G_{\mathbf{I}} \rtimes \mathrm{Aut}_{\mathcal{O}}$ acts on $\det(\mathfrak{g}_{\mathbf{P}}/\mathfrak{g}_{\mathbf{I}})$ via the character $G_{\mathbf{I}} \rtimes \mathrm{Aut}_{\mathcal{O}} \to T \times \mathbb{G}_m^{\mathrm{rot}} \xrightarrow{-2\rho_{\mathbf{P}}} \mathbb{G}_m$, we conclude that $\det \mathcal{Q} \cong \mathcal{L}(-2\rho_{\mathbf{P}})$. Taking the determinant of the exact sequence (6.11), we get the isomorphism (6.10). □

Corollary 6.5.3. *For each $i = 0, \dots, n$, there is an isomorphism of line bundles on* $\mathrm{Bun}_G^{\mathrm{par}}$:

$$\mathrm{For}_{\mathbf{I}}^{\mathbf{P}_i,*}\mathcal{L}_{\mathbf{P}_i,\mathrm{can}} \cong \mathrm{For}_{\mathbf{I}}^{\mathbf{G},*}\omega_{\mathrm{Bun}} \otimes \mathcal{L}(2\rho - \alpha_i)$$

where 2ρ is the sum of positive roots in G.

By Construction 5.1.1 (see the diagram (5.1)), the action of s_i on $f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell$ commutes with cupping with any cohomology class pulled back from $\mathcal{M}_{\mathbf{P}_i}$. In particular, it commutes with $\bigcup c_1(\mathcal{L}_{\mathbf{P}_i,\mathrm{can}})$. Using Corollary 6.5.3, we conclude that

$$s_i(\Lambda_{\mathrm{can}} + 2\rho - \alpha_i) = (\Lambda_{\mathrm{can}} + 2\rho - \alpha_i)s_i \in \mathrm{Hom}_{\mathcal{A}\times X}\big(f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell, f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell[2](1)\big). \tag{6.12}$$

Observe that for $i = 1, \dots, n$, we have

$$\big\langle \Lambda_{\mathrm{can}} + 2\rho - \alpha_i, \alpha_i^\vee\big\rangle = 2\big\langle\rho, \alpha_i^\vee\big\rangle - 2 = 0.$$

For $i = 0$, we have

$$\begin{aligned}\big\langle \Lambda_{\mathrm{can}} + 2\rho - \alpha_0, \alpha_0^\vee\big\rangle &= \big\langle \Lambda_{\mathrm{can}} + 2\rho - \delta + \theta, K - \theta^\vee\big\rangle \\ &= \langle \Lambda_{\mathrm{can}}, K\rangle - 2\big\langle\rho, \theta^\vee\big\rangle - 2 = 2h^\vee - 2h^\vee = 0.\end{aligned}$$

Here we have used the fact that $\langle \Lambda_{\mathrm{can}}, K\rangle = 2h^\vee$ (see Remark 6.2.4) and $h^\vee = \langle\rho, \theta^\vee\rangle + 1$ (see Lemma 6.2.3). In any case, we have $\langle \Lambda_{\mathrm{can}} + 2\rho - \alpha_i, \alpha_i^\vee\rangle = 0$ for $i = 0, \dots, n$. This, together with (6.12) implies that (DH-3) holds for s_i and $\xi = \Lambda_{\mathrm{can}} + 2\rho - \alpha_i$. Since we have already proved (DH-3) for s_i and $2\rho - \alpha_i \in \mathbb{X}^*(T \times \mathbb{G}_m^{\mathrm{rot}})$ in Section 6.4, we can subtract this relation from the one for $\xi = \Lambda_{\mathrm{can}} + 2\rho - \alpha_i$, and conclude that (DH-3) holds for s_i and $\xi = \Lambda_{\mathrm{can}}$. This completes the proof of (DH-3), and hence the proof of Theorem 6.1.6.

6.6. Variants of the main results

We first state a parahoric version of Theorem 6.1.6. Let $\mathbf{1}_{W_{\mathbf{P}}} \in \overline{\mathbb{Q}}_\ell[\widetilde{W}]$ be the characteristic function of $W_{\mathbf{P}}$, then $\frac{1}{\#W_{\mathbf{P}}}\mathbf{1}_{W_{\mathbf{P}}}$ is an idempotent in $\mathbb{Q}[\widetilde{W}]$. Let $\mathbb{H}_{\mathbf{P}} = \mathbf{1}_{W_{\mathbf{P}}}\mathbb{H}\mathbf{1}_{W_{\mathbf{P}}}$.

Theorem 6.6.1. *There is a graded algebra homomorphism*:

$$\mathbb{H}_{\mathbf{P}} \to \bigoplus_{i\in\mathbb{Z}} \mathrm{End}^{2i}_{\mathcal{A}\times X}(f_{\mathbf{P},*}\overline{\mathbb{Q}}_\ell)(i).$$

Sketch of proof. By Theorem 5.1.2, we have a $\widetilde{W}$-action on $f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell$. Since $W_{\mathbf{P}} \subset \widetilde{W}$ is finite, it makes sense to extract the subcomplex $(f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell)^{W_{\mathbf{P}}}$ of $W_{\mathbf{P}}$-invariants. Recall the diagram (4.6) and Construction 5.1.1. The $W_{\mathbf{P}}$-action on $f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell$ is induced from that on $\pi_{\mathbf{P},*}^{\mathbf{I}}\overline{\mathbb{Q}}_\ell \in D^b([\mathfrak{p}/\underline{L}_{\mathbf{P}}]_D)$ via proper base change. Classical Springer theory for $\pi_{\mathbf{P}}^{\mathbf{I}}$ implies that $(\pi_{\mathbf{P},*}^{\mathbf{I}}\overline{\mathbb{Q}}_\ell)^{W_{\mathbf{P}}}$ is the constant sheaf on $[\mathfrak{p}/\underline{L}_{\mathbf{P}}]_D$. Therefore

$$f_{\mathbf{P},*}\overline{\mathbb{Q}}_\ell = f_{\mathbf{P},*}\big(\mathrm{ev}_{\mathbf{P}}^*\big(\pi_{\mathbf{P},*}^{\mathbf{I}}\overline{\mathbb{Q}}_\ell\big)^{W_{\mathbf{P}}}\big) \cong \big(f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell\big)^{W_{\mathbf{P}}}.$$

The theorem easily follows from this isomorphism. □

Next, we state a version of Theorem 3.3.3 over an *enhanced* Hitchin base. Let $\mathfrak{c}_{\mathbf{P}} = \mathfrak{l}_{\mathbf{P}} /\!\!/ L_{\mathbf{P}} = \mathfrak{t} /\!\!/ W_{\mathbf{P}}$ be the adjoint quotient of $\mathfrak{l}_{\mathbf{P}}$. The projection $\widetilde{W} \to W$ restricted to the subgroup $W_{\mathbf{P}} \subset \widetilde{W}$ induces an injection $W_{\mathbf{P}} \hookrightarrow W$. Therefore we have finite flat morphisms

$$\mathfrak{t} \xrightarrow{q_{\mathbf{I}}^{\mathbf{P}}} \mathfrak{c}_{\mathbf{P}} \xrightarrow{q_{\mathbf{P}}} \mathfrak{c}.$$

Definition 6.6.2. The *enhanced Hitchin base* $\widetilde{\mathcal{A}}_{\mathbf{P}}$ of type $\mathbf{P}$ is defined by the following Cartesian square

$$\begin{array}{ccc} \widetilde{\mathcal{A}}_{\mathbf{P}} & \longrightarrow & \mathfrak{c}_{\mathbf{P},D} \\ \downarrow & & \downarrow{\scriptstyle q_{\mathbf{P}}} \\ \mathcal{A}^{\mathrm{Hit}} \times X & \xrightarrow{\mathrm{ev}} & \mathfrak{c}_D \end{array}$$

where "ev" is the evaluation map. Note that $\widetilde{\mathcal{A}}_{\mathbf{I}} = \widetilde{\mathcal{A}}^{\mathrm{Hit}}$ defined in Definition 2.2.2, and $\widetilde{\mathcal{A}}_{\mathbf{G}} = \mathcal{A}^{\mathrm{Hit}} \times X$. We fix an open subset $(\mathcal{A} \times X)' \subset \mathcal{A} \times X$ as in Remark 2.6.4, and let $\widetilde{\mathcal{A}}'_{\mathbf{P}}$ be the preimage of $(\mathcal{A} \times X)'$ in $\widetilde{\mathcal{A}}_{\mathbf{P}}$.

Recall from (4.1) that the twisted forms $\underline{L}_{\mathbf{P}}$ and $\underline{\mathfrak{l}}_{\mathbf{P}}$ of $L_{\mathbf{P}}$ and $\mathfrak{l}_{\mathbf{P}}$ are formed using the $\mathrm{Aut}_{\mathcal{O}}$-torsor $\mathrm{Coor}(X)$. Since $\mathrm{Aut}_{\mathcal{O}}$ is connected, its action on $L_{\mathbf{P}}$ factors through the adjoint action of $L_{\mathbf{P}}^{\mathrm{ad}}$ on $L_{\mathbf{P}}$. Therefore the adjoint quotient $[\mathfrak{l}_{\mathbf{P}}/L_{\mathbf{P}}] \to \mathfrak{c}_{\mathbf{P}}$ is automatically $\mathrm{Aut}_{\mathcal{O}}$-invariant, inducing a morphism

$$\chi_{\mathbf{P}} : [\underline{\mathfrak{l}}_{\mathbf{P}}/\underline{L}_{\mathbf{P}}] \to \mathfrak{c}_{\mathbf{P}}.$$

Using the morphism $\mathcal{M}_{\mathbf{P}} \xrightarrow{\mathrm{ev}_{\mathbf{P}}} [\underline{\mathfrak{l}}_{\mathbf{P}}/\underline{L}_{\mathbf{P}}]_D \xrightarrow{\chi_{\mathbf{P}}} \mathfrak{c}_{\mathbf{P},D}$, we get the *enhanced parahoric Hitchin fibration of type* $\mathbf{P}$:

$$\tilde{f}_{\mathbf{P}} : \mathcal{M}_{\mathbf{P}} \xrightarrow{(\chi_{\mathbf{P}} \circ \mathrm{ev}_{\mathbf{P}}, f_{\mathbf{P}})} \mathfrak{c}_{\mathbf{P},D} \times_{\mathfrak{c}_D} \left(\mathcal{A}^{\mathrm{Hit}} \times X\right) = \widetilde{\mathcal{A}}_{\mathbf{P}}. \tag{6.13}$$

Construction 6.6.3. For a standard parahoric subgroup $\mathbf{P}$, we can similarly define a Hecke correspondence $\mathcal{H}\mathrm{ecke}_{\mathbf{P}}$ of $\mathcal{M}_{\mathbf{P}}$ as in Definition 3.1.1. We have a natural morphism $h_{\mathbf{P}} : \mathcal{H}\mathrm{ecke}^{\mathrm{par}} \to \mathcal{H}\mathrm{ecke}_{\mathbf{P}}$.

For $\lambda \in \mathbb{X}_*(T)$, let $|\lambda|_{\mathbf{P}}$ denote its $W_{\mathbf{P}}$-orbit in $\mathbb{X}_*(T)$. Let $\mathcal{H}_{|\lambda|_{\mathbf{P}}}$ be the reduced image of $\mathcal{H}_\lambda$ under $h_{\mathbf{P}}$ (which only depends on the $W_{\mathbf{P}}$-orbit of λ). The same argument as in Lemma 3.1.4 shows that $\mathcal{H}_{|\lambda|_{\mathbf{P}}}$ is a graph-like self-correspondence of $\mathcal{M}_{\mathbf{P}}$ over the enhanced Hitchin base $\widetilde{\mathcal{A}}'_{\mathbf{P}}$.

Proposition 6.6.4. *There is a unique algebra homomorphism*

$$\overline{\mathbb{Q}}_\ell\big[\mathbb{X}_*(T)\big]^{W_{\mathbf{P}}} \to \mathrm{End}_{\widetilde{\mathcal{A}}'_{\mathbf{P}}}\big(\tilde{f}_{\mathbf{P},*}\overline{\mathbb{Q}}_\ell|_{\widetilde{\mathcal{A}}'_{\mathbf{P}}}\big),$$

such that $\mathrm{Av}_{\mathbf{P}}(\lambda) := \sum_{\lambda' \in |\lambda|_{\mathbf{P}}} \lambda'$ *acts by* $[\mathcal{H}_{|\lambda|_{\mathbf{P}}}]_\#$ *for any* $\lambda \in \mathbb{X}_*(T)$.

The proof is similar to that of Theorem 3.3.3. When $\mathbf{P} = \mathbf{I}$, we recover the $\mathbb{X}_*(T)$-action on $\tilde{f}_*\overline{\mathbb{Q}}_\ell$ constructed in Proposition 3.3.5. When $\mathbf{P} = \mathbf{G}$, we get a $\overline{\mathbb{Q}}_\ell[\mathbb{X}_*(T)]^W$-action on the complex $f_*^{\mathrm{Hit}}\overline{\mathbb{Q}}_\ell \boxtimes \overline{\mathbb{Q}}_{\ell,X} \in D^b((\mathcal{A} \times X)')$. This is an algebro-geometric construction of the *'t Hooft operators* considered by Kapustin and Witten [18].

7. A sample calculation

The goal of this section is to calculate the global Springer representation in the first nontrivial example.

7.1. Description of the parabolic Hitchin fiber

Throughout this section, we specialize to the case $X = \mathbb{P}^1$, $\mathcal{O}_X(D) \cong \mathcal{O}(2)$ and $G = \mathrm{SL}(2)$.

The Hitchin base $\mathcal{A}^{\mathrm{Hit}} = \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(4))$ parametrizes degree 4 homogeneous polynomials $a(\xi,\eta) = \sum_{i=0}^4 a_i \xi^i \eta^{4-i}$. For $a \in \mathcal{A}^{\mathrm{Hit}}$, the cameral curve X_a coincides with the spectral curve Y_a: $t^2 = a(\xi,\eta)$, which carries an involution $\tau : (\xi,\eta,t) \mapsto (\xi,\eta,-t)$. Let $p_a : Y_a \to X$ be the projection. We have $a \in \mathcal{A}(k)$ if and only if Y_a is irreducible.

For an anisotropic $a \in \mathcal{A}(k)$, the Hitchin fiber $\mathcal{M}_a^{\mathrm{Hit}}$ is:

$$\mathcal{M}_a^{\mathrm{Hit}} = \left\{(\mathcal{F},\alpha) \,\middle|\, \mathcal{F} \in \overline{\mathcal{P}\mathrm{ic}}(Y_a),\ \alpha : \det(p_{a,*}\mathcal{F}) \xrightarrow{\sim} \mathcal{O}_X\right\}.$$

For the stack $\overline{\mathcal{P}\mathrm{ic}}(Y_a)$ see Example 2.2.5. For any $(\mathcal{F},\alpha) \in \mathcal{M}_a^{\mathrm{Hit}}$, $\mathcal{E} = p_{a,*}\mathcal{F}$ is a rank 2 vector bundle on X with trivial determinant, therefore $\chi(Y_a,\mathcal{F}) = \chi(X,\mathcal{E}) = 2$, hence $\mathcal{F} \in \overline{\mathcal{P}\mathrm{ic}}^2(Y_a)$. Since $\chi(Y_a,\mathcal{O}_{Y_a}) = \chi(X,\mathcal{O}_X) + \chi(X,\mathcal{O}_X(-2)) = 0$, Y_a is an irreducible curve of arithmetic genus 1. The degree -1 Abel–Jacobi map

$$\begin{aligned} \mathrm{AJ} : Y_a &\to \overline{\mathrm{Pic}}^{-1}(Y_a) \\ y &\mapsto \mathcal{I}_y \quad \text{(the ideal sheaf of } y) \end{aligned} \tag{7.1}$$

is an isomorphism, here $\overline{\mathrm{Pic}}^{-1}(Y_a)$ is the coarse moduli space of $\overline{\mathcal{P}\mathrm{ic}}^{-1}(Y_a)$. Moreover, $\mathcal{M}_a^{\mathrm{Hit}} \cong \overline{\mathrm{Pic}}^2(Y_a)/\mu_2$ where the center $\mu_2 \subset \mathrm{SL}(2)$ acts trivially on $\overline{\mathrm{Pic}}^2(Y_a)$. Via the Abel–Jacobi map (7.1), $\mathcal{M}_a^{\mathrm{Hit}}$ is non-canonically isomorphic to $Y_a \times \mathbb{B}\mu_2$ (we have to choose an isomorphism $\overline{\mathrm{Pic}}^2(Y_a) \cong \overline{\mathrm{Pic}}^{-1}(Y_a)$, which is non-canonical).

The Picard stack $\mathcal{P}_a$ acting on $\mathcal{M}_a^{\mathrm{Hit}}$ is the Prym variety

$$\mathcal{P}\mathrm{ic}(Y_a)^{\tau=-1} = \left\{(\mathcal{L},\iota) \,\middle|\, \mathcal{L} \in \mathcal{P}\mathrm{ic}^0(Y_a),\ \iota : \mathcal{L} \xrightarrow{\sim} \tau^*\mathcal{L}^{\otimes -1} \text{ such that } \iota = \tau^*\left(\iota^{\otimes -1}\right)\right\}.$$

Since Y_a is an irreducible curve of arithmetic genus 1, $\mathcal{P}_a \cong \mathrm{Pic}^0(Y_a) \times \mathbb{B}\mu_2$.

The parabolic Hitchin fiber $\mathcal{M}_{a,x}^{\mathrm{par}}$ is

$$\mathcal{M}_{a,x}^{\mathrm{par}} = \left\{(\mathcal{F}_0,\mathcal{F}_1,\alpha) \,\middle|\, (\mathcal{F}_0,\alpha) \in \mathcal{M}_a^{\mathrm{Hit}},\ \mathcal{F}_0 \supsetneq \mathcal{F}_1 \supsetneq \mathcal{F}_0(-x)\right\}.$$

We have two forgetful morphisms:

$$\begin{aligned} \rho_0 &: \mathcal{M}_{a,x}^{\mathrm{par}} \to \overline{\mathcal{P}\mathrm{ic}}^2(Y_a), \\ \rho_1 &: \mathcal{M}_{a,x}^{\mathrm{par}} \to \overline{\mathcal{P}\mathrm{ic}}^1(Y_a) \end{aligned}$$

sending $(\mathcal{F}_0, \mathcal{F}_1, \alpha)$ to $\mathcal{F}_0$ and $\mathcal{F}_1$ respectively. As in the case of $\mathcal{M}_a^{\mathrm{Hit}}$, $\mathcal{M}_{a,x}^{\mathrm{par}}$ is the quotient of its coarse moduli schemes by the trivial action of μ_2.

For $x \in X$, let $v_a(x)$ be the order of vanishing of the polynomial a at x. Then the local Serre invariant is

$$\delta(a, x) = \big[v_a(x)/2\big].$$

Note that $(a, x) \in (\mathcal{A}^{\heartsuit} \times X)^{\mathrm{rs}}$ means $v_a(x) = 0$ while $(a, x) \in (\mathcal{A}^{\heartsuit} \times X)_0$ means $v_a(x) = 0$ or 1.

For each partition $\underline{p}$ of 4, let $\mathcal{A}_{\underline{p}}$ be the locus where the multiplicities of the roots of $a(\xi, \eta) = 0$ are given by $\underline{p}$. We have

$$\delta(a) = \sum_i [p_i/2], \quad \text{if } a \in \mathcal{A}_{\underline{p}}, \ \underline{p} = (p_1, p_2, \ldots).$$

Let us analyze the anisotropic parabolic Hitchin fibers on each stratum:

- $\underline{p} = (1, 1, 1, 1)$. Then Y_a is a smooth curve of genus one; $\mathcal{M}_{a,x}^{\mathrm{par}} = \mathcal{M}_a^{\mathrm{Hit}} \times p_a^{-1}(x)$ which is non-canonically isomorphic to $Y_a \times p_a^{-1}(x) \times \mathbb{B}\mu_2$.
- $\underline{p} = (2, 1, 1)$. Then Y_a is a nodal curve of arithmetic genus 1. Let $\pi : \mathbb{P}^1 \to Y_a$ be the normalization. Then the node of $\mathcal{M}_a^{\mathrm{Hit}}$ (recall the coarse moduli space of $\mathcal{M}_a^{\mathrm{Hit}}$ is isomorphic to Y_a, which has a node) corresponds to $\mathcal{F} = \pi_*\mathcal{O}_{\mathbb{P}^1}(1)$. If $v_a(x) = 0$ or 1, then $\mathcal{M}_{a,x}^{\mathrm{par}} = \mathcal{M}_a^{\mathrm{Hit}} \times p_a^{-1}(x)$, which is isomorphic to $Y_a \times p_a^{-1}(x) \times \mathbb{B}\mu_2$. If $v_a(x) = 2$, i.e., x is the projection of the node, then the reduced structure of $\mathcal{M}_{a,x}^{\mathrm{par}}$ (ignore the μ_2-action) consists of two $\mathbb{P}^1$'s meeting transversally at two points: one component (call it C_1) corresponds to $\mathcal{F}_0 = \pi_*\mathcal{O}_{\mathbb{P}^1}(1)$ and varying $\mathcal{F}_1$; the other component (call it C_0) corresponds to $\mathcal{F}_1 = \pi_*\mathcal{O}_{\mathbb{P}^1}$ and varying $\mathcal{F}_0$. The generic point of each component C_0 or C_1 is non-reduced of length 2.
- $\underline{p} = (3, 1)$. The situation is similar to the above case, except that Y_a is a cuspidal curve of arithmetic genus 1. If $v_a(x) = 3$, i.e., x is the projection of the cusp, then the reduced structure of $\mathcal{M}_{a,x}^{\mathrm{par}}$ (ignore the μ_2-action as well) consists of two $\mathbb{P}^1$'s tangent to each other at one point.

7.2. The DAHA action for a "subregular" parabolic Hitchin fiber

Now we concentrate on $\mathcal{M}_{a,x}^{\mathrm{par}}$ for $a \in \mathcal{A}_{(2,1,1)}$ and x the projection of the node of Y_a. In this subsection, we compute the action of $\widetilde{W}$ on its cohomology. We ignore the Tate twists. We restrict $\mathcal{M}^{\mathrm{par}}$ to the anisotropic locus $\mathcal{A} \times X$ without changing notation. We also ignore the stack issue from now on because the finite automorphism group μ_2 does not affect the $\overline{\mathbb{Q}}_\ell$-cohomology. Hence we will work with $\overline{\mathrm{Pic}}$ rather than $\overline{\mathcal{P}\mathrm{ic}}$.

We first fix a basis for $\mathrm{H}_2(\mathcal{M}_{a,x}^{\mathrm{par}})$. Recall from the end of the last subsection that the reduced structure of $\mathcal{M}_{a,x}^{\mathrm{par}}$ consists of two components C_0 and C_1, each isomorphic to $\mathbb{P}^1$, we get a basis $\{[C_0], [C_1]\}$ for $\mathrm{H}_2(\mathcal{M}_{a,x}^{\mathrm{par}})$. The dual basis in $\mathrm{H}^*(\mathcal{M}_{a,x}^{\mathrm{par}})$ is denoted by $\{\zeta_0, \zeta_1\}$.

Lemma 7.2.1. *The homology class $[C_0] + [C_1] \in \mathrm{H}_2(\mathcal{M}_{a,x}^{\mathrm{par}})$ is invariant under the $\widetilde{W}$-action.*

Proof. By proper base change, for sufficiently small étale neighborhood U of (a,x) in $\mathcal{A}\times X$, there is a $\widetilde{W}$-equivariant surjection:

$$i^*_{a,x}:\mathrm{H}^*\big(\mathcal{M}^{\mathrm{par}}_U\big)\twoheadrightarrow\mathrm{H}^*\big(\mathcal{M}^{\mathrm{par}}_{a,x}\big).$$

Therefore, it suffices to show that for any $\zeta\in\mathrm{H}^2(\mathcal{M}^{\mathrm{par}}_U)$ and $\tilde{w}\in\widetilde{W}$, we have $\langle i^*_{a,x}\tilde{w}\zeta,[C_0]+[C_1]\rangle=\langle i^*_{a,x}\zeta,[C_0]+[C_1]\rangle$. Equivalently, we have to show that

$$\big\langle\tilde{w}\zeta-\zeta,i_{a,x,*}\big([C_0]+[C_1]\big)\big\rangle=0. \tag{7.2}$$

By the description at the end of last subsection, the scheme-theoretic fiber $\mathcal{M}^{\mathrm{par}}_{a,x}$ is the union of C_0 and C_1, each with a non-reduced structure of length 2. Therefore, the cycle class $[\mathcal{M}^{\mathrm{par}}_{a,x}]$ in $\mathcal{M}^{\mathrm{par}}_U$ is $2[C_0]+2[C_1]$, which is algebraically equivalent to any other fiber $\mathcal{M}^{\mathrm{par}}_{a',x'}$. In particular, we can pick $(a',x')\in U^{\mathrm{rs}}$. Then

$$\begin{aligned}\big\langle\tilde{w}\zeta-\zeta,i_{a,x,*}\big([C_0]+[C_1]\big)\big\rangle&=\frac{1}{2}\big\langle\tilde{w}\zeta-\zeta,i_{a',x',*}\big[\mathcal{M}^{\mathrm{par}}_{a',x'}\big]\big\rangle\\&=\frac{1}{2}\big\langle i^*_{a',x'}(\tilde{w}\zeta-\zeta),\big[\mathcal{M}^{\mathrm{par}}_{a',x'}\big]\big\rangle.\end{aligned} \tag{7.3}$$

For generic $(a',x')\in U^{\mathrm{rs}}$, the $\widetilde{W}$-action on $\mathrm{H}^2(\mathcal{M}^{\mathrm{par}}_{a',x'})$ is given by automorphisms of $\mathcal{M}^{\mathrm{par}}_{a',x'}$ (see Proposition 3.2.1 and Definition 3.3.1), hence the functional $[\mathcal{M}^{\mathrm{par}}_{a',x'}]:\mathrm{H}^2(\mathcal{M}^{\mathrm{par}}_{a',x'})\to\overline{\mathbb{Q}}_\ell$ is $\widetilde{W}$-invariant. Therefore, the quantity in (7.3) is zero, hence (7.2) holds. □

Now we compute the action of the simple reflection $s_1\in W$ on $\mathrm{H}^*(\mathcal{M}^{\mathrm{par}}_{a,x})$. We restrict the Cartesian diagram (2.1) to (a,x), and trivialize the line bundle $\mathcal{O}_X(2)$ near x. Since $a(x)=0$, any Hitchin pair in $\mathcal{M}^{\mathrm{Hit}}_a$ has nilpotent Higgs field at x. We get a Cartesian diagram

$$\begin{array}{ccc}\mathcal{M}^{\mathrm{par}}_{a,x} & \xrightarrow{\mathrm{ev}^{\mathrm{par}}_x} & [\widetilde{\mathcal{N}}/G]\\ \downarrow{\scriptstyle\pi_{\mathcal{M}}} & & \downarrow{\scriptstyle\pi}\\ \mathcal{M}^{\mathrm{Hit}}_a & \xrightarrow{\mathrm{ev}_x} & [\mathcal{N}/G]\end{array}$$

Here $\mathcal{N}$ is the nilpotent cone of G, and $\widetilde{\mathcal{N}}$ is the scheme-theoretic preimage of $\mathcal{N}$ in $\tilde{\mathfrak{g}}$. We have an exact sequence of shifted perverse sheaves in $D^b([\mathfrak{g}/G])$:

$$0\to\overline{\mathbb{Q}}_\ell\to\pi_*\overline{\mathbb{Q}}_\ell\to i_{0,*}\overline{\mathbb{Q}}_\ell[-2]\to 0 \tag{7.4}$$

where $i_0:\{0\}\hookrightarrow\mathcal{N}$ is the inclusion. By the Springer theory for G we know that s_1 acts on the constant sheaf $\overline{\mathbb{Q}}_\ell$ trivially and acts on $i_*\overline{\mathbb{Q}}_\ell[-2]$ by -1. Pulling back (7.4) to $\mathcal{M}^{\mathrm{Hit}}_a$ via ev^*_x, we get a distinguished triangle in $D^b(\mathcal{M}^{\mathrm{Hit}}_a)$:

$$\overline{\mathbb{Q}}_\ell\to\pi_{\mathcal{M},*}\overline{\mathbb{Q}}_\ell\to i_*\overline{\mathbb{Q}}_\ell[-2]\to \tag{7.5}$$

where i is the inclusion of the node into the nodal curve $\mathcal{M}_a^{\mathrm{Hit}}$. Therefore, all classes in the image of $\pi_{\mathcal{M}}^* : \mathrm{H}^*(\mathcal{M}_a^{\mathrm{Hit}}) \to \mathrm{H}^*(\mathcal{M}_{a,x}^{\mathrm{par}})$ are invariant under s_i. These include the whole of $\mathrm{H}^0(\mathcal{M}_{a,x}^{\mathrm{par}})$ and $\mathrm{H}^1(\mathcal{M}_{a,x}^{\mathrm{par}})$ and the class $\zeta_0 \in \mathrm{H}^2(\mathcal{M}_{a,x}^{\mathrm{par}})$. Furthermore, (7.5) induces an exact sequence

$$0 \to \mathrm{H}^2(\mathcal{M}_a^{\mathrm{Hit}}) \to \mathrm{H}^2(\mathcal{M}_{a,x}^{\mathrm{par}}) \to \overline{\mathbb{Q}}_\ell \to 0.$$

The discussion above shows that s_i acts trivially on $\mathrm{H}^2(\mathcal{M}_a^{\mathrm{Hit}}) = \overline{\mathbb{Q}}_\ell \zeta_0$ and acts by -1 on the quotient $\overline{\mathbb{Q}}_\ell$. Therefore the matrix of s_i on $\mathrm{H}^2(\mathcal{M}_{a,x}^{\mathrm{par}})$ under the basis $\{\zeta_{0,\zeta_1}\}$ takes the form $s_1 = \begin{pmatrix} 1 & c \\ 0 & -1 \end{pmatrix}$ for some $c \in \overline{\mathbb{Q}}_\ell$. By Lemma 7.2.1, s_1 fixes $[C_0] + [C_1]$, hence annihilates $\zeta_0 - \zeta_1 \in \mathrm{H}^2(\mathcal{M}_{a,x}^{\mathrm{par}})$, we conclude that $c = 2$.

The action of s_0 can be similarly calculated, with the roles of C_0 and C_1 interchanged. In summary,

Proposition 7.2.2. *The group $\widetilde{W}$ acts trivially on $\mathrm{H}^0(\mathcal{M}_{a,x}^{\mathrm{par}})$ and $\mathrm{H}^1(\mathcal{M}_{a,x}^{\mathrm{par}})$. Under the basis $\{\zeta_0, \zeta_1\}$ of $\mathrm{H}^2(\mathcal{M}_{a,x}^{\mathrm{par}})$ dual to $\{[C_0], [C_1]\}$, the elements $s_1, s_0 \in \widetilde{W}$ and $\alpha^\vee = s_0 s_1 \in \mathbb{X}_*(T)$ act as matrices*

$$s_1 = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}; \qquad s_0 = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}; \qquad \alpha^\vee = \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix}.$$

In particular, the action of the lattice part $\mathbb{X}_*(T) \subset \widetilde{W}$ is unipotent, but not the identity. The action of $\widetilde{W}$ on $\mathrm{H}^2(\mathcal{M}_{a,x}^{\mathrm{par}})$ contains the trivial representation spanned by $\zeta_0 - \zeta_1$, but is not completely reducible.

Finally we describe the line bundles $\mathcal{L}(\xi)$ on $\mathcal{M}_{a,x}^{\mathrm{par}}$ for $\xi \in \mathbb{X}^*(\mathbb{G}_m^{\mathrm{cen}} \times T)$, following the recipe in Lemma 6.3.3. Let $\alpha \in \mathbb{X}^*(T)$ be the simple root, then $\mathcal{L}(\alpha)$ is $\mathcal{O}(-2)$ when restricted to C_0, and $\mathcal{O}(2)$ when restricted to C_1. The line bundle $\mathcal{L}(\Lambda_{\mathrm{can}})$ is $\mathcal{O}(-3)$ when restricted to C_0, and is trivial on C_1.

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Appendix A. Generalities on cohomological correspondences

In this appendix, we review the formalism of cohomological correspondences. In Appendices A.1–A.5, the results are standard and we omit the proofs. In Appendix A.6, We introduce a nice class of correspondences called *graph-like correspondences*, which will be used to construct the global Springer action.

The spaces in the following discussions can be any algebraic stack whenever the sheaf-theoretic operations make sense. In practice, we will apply the formalism to either Deligne–Mumford stacks or global quotients $[X/G]$ where X is a scheme and G is a group scheme.

We sometimes put a label over an arrow to describe the nature of the map. The label "b.c." means proper base change; "ad." means adjunction; "$*! \to !*$" means the natural transformation $\phi^* f^! \to f_1^! \psi^*$ (adjoint to the proper base change) associated to the following Cartesian diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\phi} & X \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} \\ Y_1 & \xrightarrow{\psi} & Y \end{array}$$

A.1. Cohomological correspondences

We recall the general formalism of cohomological correspondences, following [15]. Consider the correspondence diagram

$$\begin{array}{ccccc} & & C & & \\ & \swarrow^{\overleftarrow{c}} & & \searrow^{\overrightarrow{c}} & \\ X & \xrightarrow[f]{} & S & \xleftarrow[g]{} & Y \end{array} \tag{A.1}$$

where all maps are representable and of finite type. *We always assume that* $\overleftrightarrow{c} = (\overleftarrow{c}, \overrightarrow{c}) : C \to X \times_S Y$ *is proper*. Let $\overleftarrow{p}, \overrightarrow{p}$ be the projections from $X \times_S Y$ to X and Y respectively.

Definition A.1.1. A *cohomological correspondence between a complex* $\mathcal{F} \in D^b(X)$ *and a complex* $\mathcal{G} \in D^b(Y)$ *with support on* C is an element in

$$\mathrm{Corr}(C; \mathcal{F}, \mathcal{G}) := \mathrm{Hom}_C\big(\overrightarrow{c}^* \mathcal{G}, \overleftarrow{c}^! \mathcal{F}\big).$$

Such a cohomological correspondence ζ induces a morphism $\zeta_\# : g_! \mathcal{G} \to f_* \mathcal{F}$, which is defined by the following procedure:

$$(-)_\# : \mathrm{Corr}(C; \mathcal{F}, \mathcal{G}) \xrightarrow{\alpha} \mathrm{Corr}(X \times_S Y; \mathcal{F}, \mathcal{G}) \xrightarrow{\beta} \mathrm{Hom}_S\big(g_! \mathcal{G}, f^* \mathcal{F}\big),$$

where α is the composition

$$\begin{aligned} \mathrm{Corr}(C; \mathcal{F}, \mathcal{G}) &\xrightarrow{\overleftrightarrow{c}_*} \mathrm{Hom}_{X \times_S Y}\big(\overleftrightarrow{c}_* \overrightarrow{c}^* \mathcal{G}, \overleftrightarrow{c}_* \overleftarrow{c}^! \mathcal{F}\big) \\ &= \mathrm{Hom}_{X \times_S Y}\big(\overleftrightarrow{c}_* \overleftrightarrow{c}^* \overrightarrow{p}^* \mathcal{G}, \overleftrightarrow{c}_! \overleftrightarrow{c}^! \overleftarrow{p}^! \mathcal{F}\big) (\overleftrightarrow{c} \text{ is proper}) \\ &\xrightarrow{\text{ad.}} \mathrm{Corr}(X \times_S Y; \mathcal{F}, \mathcal{G}) \end{aligned}$$

and β is the composition

$$\mathrm{Corr}(X \times_S Y; \mathcal{F}, \mathcal{G}) \overset{\text{ad.}}{=} \mathrm{Hom}_S\big(\mathcal{G}, \overrightarrow{p}_* \overleftarrow{p}^! \mathcal{F}\big) \overset{\text{b.c.}}{\cong} \mathrm{Hom}_S\big(\mathcal{G}, g^! f_* \mathcal{F}\big) \overset{\text{ad.}}{=} \mathrm{Hom}_S(g_! \mathcal{G}, f_* \mathcal{F}).$$

The morphism α is a special case of the *push-forward* of cohomological correspondences: Suppose $\gamma : C \to C'$ is a proper map of correspondences between X and Y over S, then we can define

$$\gamma_* : \mathrm{Corr}(C; \mathcal{F}, \mathcal{G}) \to \mathrm{Corr}\big(C'; \mathcal{F}, \mathcal{G}\big)$$

in the same way as we defined α. It follows directly from the definition that

Lemma A.1.2. *For any $\zeta \in \mathrm{Corr}(C; \mathcal{F}, \mathcal{G})$, we have*

$$(\gamma_*\zeta)_\# = \zeta_\# \in \mathrm{Hom}_S(g_!\mathcal{G}, f_*\mathcal{F}).$$

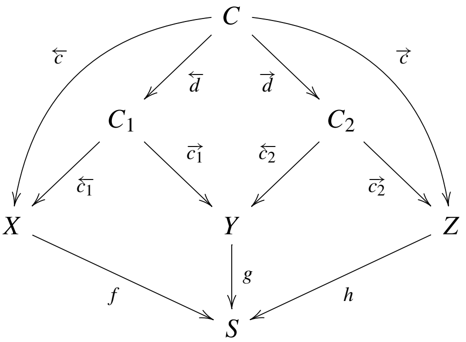
We will mainly be interested in the special case where $\mathcal{F}$ and $\mathcal{G}$ are the constant sheaves (in degree 0) on X and Y. In this case, we have

$$\mathrm{Corr}(C; \overline{\mathbb{Q}}_{\ell,X}, \overline{\mathbb{Q}}_{\ell,Y}) = \mathrm{Hom}_C\big(\overrightarrow{c}^*\overline{\mathbb{Q}}_{\ell,Y}, \overleftarrow{c}^!\overline{\mathbb{Q}}_{\ell,X}\big) = H^0(C, \mathbb{D}_{\overleftarrow{c}}).$$

Here $\mathbb{D}_{\overleftarrow{c}}$ means the dualizing complex relative to the morphism $\overleftarrow{c}$.

A.2. Composition of correspondences

Let C_1 be a correspondence between X and Y over S, and C_2 be a correspondence between Y and Z over S. *Assume that Y is proper over S*. The *composition* $C = C_1 * C_2$ of C_1 and C_2 is defined to be $C_1 \times_Y C_2$, viewed as a correspondence between X and Z over S:



(A.2)

Note that the properness of Y/S ensures the properness of C over $X \times_S Z$.

Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be complexes on X, Y and Z respectively. We define the *convolution product*

$$\circ_Y : \mathrm{Corr}(C_1; \mathcal{F}, \mathcal{G}) \otimes \mathrm{Corr}(C_2; \mathcal{G}, \mathcal{H}) \to \mathrm{Corr}(C; \mathcal{F}, \mathcal{H})$$

as follows. Let $\zeta_1 \in \mathrm{Corr}(C_1; \mathcal{F}, \mathcal{G})$ and $\zeta_2 \in \mathrm{Corr}(C_2; \mathcal{G}, \mathcal{H})$, then $\zeta_1 \circ_Y \zeta_2$ is

$$\overrightarrow{c}^*\mathcal{H} = \overrightarrow{d}^*\overrightarrow{c_2}^*\mathcal{H} \xrightarrow{\zeta_2} \overrightarrow{d}^*\overleftarrow{c_2}^!\mathcal{G} \xrightarrow{*!\to!*} \overleftarrow{d}^!\overrightarrow{c_1}^*\mathcal{G} \xrightarrow{\zeta_1} \overleftarrow{d}^!\overleftarrow{c_1}^!\mathcal{F} = \overleftarrow{c}^!\mathcal{F}.$$

Lemma A.2.1. *(See [15, §5.2].) For $\zeta_1 \in \mathrm{Corr}(C_1; \mathcal{F}, \mathcal{G})$, $\zeta_2 \in \mathrm{Corr}(C_2; \mathcal{G}, \mathcal{H})$, we have*

$$\zeta_{1,\#} \circ \zeta_{2,\#} = (\zeta_1 \circ_Y \zeta_2)_\# : h_! \mathcal{H} \to f_* \mathcal{F}.$$

Consider the correspondences C_i between X_i and X_{i+1}, $i = 1, 2, 3$. Assume X_2, X_3 are proper over S. It follows from the definition of convolution that:

Lemma A.2.2. *The convolution product is associative. More precisely, for $\mathcal{F}_i \in D^b(X_i), i = 1, \dots, 4$ and $\zeta_i \in \mathrm{Corr}(C_i; \mathcal{F}_i, \mathcal{F}_{i+1})$, $i = 1, 2, 3$, we have*

$$(\zeta_1 \circ_{X_2} \zeta_2) \circ_{X_3} \zeta_3 = \zeta_1 \circ_{X_2} (\zeta_2 \circ_{X_3} \zeta_3).$$

A.3. Verdier duality and correspondences

Consider a correspondence diagram as in (A.1). The *transposition* $C^\vee$ of the correspondence C is the same stack C with two projections:

$$\overleftarrow{c^\vee} = \overrightarrow{c} : C \to Y; \qquad \overrightarrow{c^\vee} = \overleftarrow{c} : C \to X.$$

The Verdier duality functor gives an isomorphism

$$\mathbb{D}(-) : \mathrm{Corr}(C; \mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{Corr}\big(C^\vee; \mathbb{D}\mathcal{G}, \mathbb{D}\mathcal{F}\big)$$

which sends the map $\zeta : \overrightarrow{c}^* \mathcal{G} \to \overleftarrow{c}^! \mathcal{F}$ to its Verdier dual

$$\mathbb{D}\zeta : \overrightarrow{c^\vee}^* \mathbb{D}\mathcal{F} = \mathbb{D}\big(\overleftarrow{c}^! \mathcal{F}\big) \to \mathbb{D}\big(\overrightarrow{c}^* \mathcal{G}\big) = \overleftarrow{c^\vee}^! \mathbb{D}\mathcal{G}.$$

On the other hand, the Verdier duality functor also gives an isomorphism

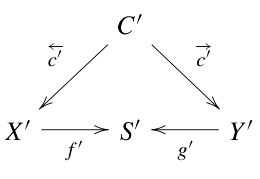
$$\mathbb{D}(-) : \mathrm{Hom}_S(g_! \mathcal{G}, f_* \mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_S(f_! \mathbb{D}\mathcal{F}, g_* \mathbb{D}\mathcal{G}).$$

Lemma A.3.1. *(See [15, diagram 5.1.7].) For any $\zeta \in \mathrm{Corr}(C; \mathcal{F}, \mathcal{G})$, we have*

$$\mathbb{D}(\zeta_\#) = (\mathbb{D}\zeta)_\# \in \mathrm{Hom}_S(f_! \mathbb{D}\mathcal{F}, g_* \mathbb{D}\mathcal{G}).$$

A.4. Pull-back of correspondences

Suppose we have another correspondence diagram



and a morphisms $\gamma : C' \to C, \phi : X' \to X, \psi : Y' \to Y$ such that the following diagrams are Cartesian

$$\begin{array}{ccc} C' & \xrightarrow{\gamma} & C \\ {\scriptstyle \overleftarrow{c'}}\downarrow & & \downarrow{\scriptstyle \overleftarrow{c}} \\ X' & \xrightarrow{\phi} & X \end{array} \qquad \begin{array}{ccc} C' & \xrightarrow{\gamma} & C \\ {\scriptstyle \overrightarrow{c'}}\downarrow & & \downarrow{\scriptstyle \overrightarrow{c}} \\ Y' & \xrightarrow{\psi} & Y \end{array}$$

We define the *pull-back*:

$$\gamma^* : \mathrm{Corr}(C; \mathcal{F}, \mathcal{G}) \to \mathrm{Corr}\big(C'; \phi^*\mathcal{F}, \psi^*\mathcal{G}\big)$$

as follows: for $\zeta \in \mathrm{Corr}(C; \mathcal{F}, \mathcal{G})$, $\gamma^*\zeta$ is defined as the composition

$$\overrightarrow{c'}^*\psi^*\mathcal{G} = \gamma^*\overrightarrow{c}^*\mathcal{G} \xrightarrow{\gamma^*\zeta} \gamma^*\overleftarrow{c}^!\mathcal{F} \xrightarrow{*!\to !*} \overleftarrow{c'}^!\phi^*\mathcal{F}.$$

Lemma A.4.1. *Let $\beta : S' \to S$ be a morphism. Suppose γ, ϕ, ψ are base changes of β, and that f is proper, then we have a commutative diagram*

$$\begin{array}{ccccc} \mathrm{Corr}(C; \mathcal{F}, \mathcal{G}) & \xrightarrow{(-)_\#} & \mathrm{Hom}_S(g_!\mathcal{G}, f_*\mathcal{F}) & & \\ \downarrow{\scriptstyle \gamma^*} & & \downarrow{\scriptstyle \beta^*} & & \\ \mathrm{Corr}(C'; \phi^*\mathcal{F}, \psi^*\mathcal{G}) & \xrightarrow{(-)_\#} & \mathrm{Hom}_{S'}(\beta^*g_!\mathcal{G}, \beta^*f_*\mathcal{F}) & \overset{\text{b.c.}}{=\!=\!=} & \mathrm{Hom}_{S'}(g'_!\psi^*\mathcal{G}, f'_*\phi^*\mathcal{F}) \end{array}$$

A.5. Cup product and correspondences

For each $i \in \mathbb{Z}$, let

$$\mathrm{Corr}^i(C; \mathcal{F}, \mathcal{G}) = \mathrm{Corr}\big(C; \mathcal{F}[i], \mathcal{G}\big),$$
$$\mathrm{Corr}^*(C; \mathcal{F}, \mathcal{G}) = \bigoplus_i \mathrm{Corr}^i(C; \mathcal{F}, \mathcal{G}).$$

We have a left action of $\mathrm{H}^*(X)$ and a right action of $\mathrm{H}^*(Y)$ on $\mathrm{Corr}^*(C; \mathcal{F}, \mathcal{G})$. More precisely, for $\alpha \in \mathrm{H}^j(X)$, $\beta \in \mathrm{H}^j(Y)$ and $\zeta \in \mathrm{Corr}^i(C; \mathcal{F}, \mathcal{G})$, we define $\alpha \cdot \zeta, \zeta \cdot \beta \in \mathrm{Corr}^{i+j}(C; \mathcal{F}, \mathcal{G})$ to be

$$\alpha \cdot \zeta : \overrightarrow{c}^*\mathcal{G} \xrightarrow{\zeta} \overleftarrow{c}^!\mathcal{F}[i] \xrightarrow{\overleftarrow{c}^!(\cup\alpha)} \overleftarrow{c}^!\mathcal{F}[i+j];$$
$$\zeta \cdot \beta : \overrightarrow{c}^*\mathcal{G} \xrightarrow{\overrightarrow{c}^*(\cup\beta)} \overrightarrow{c}^*\mathcal{G}[j] \xrightarrow{\zeta} \overleftarrow{c}^!\mathcal{F}[i+j].$$

The following lemma is obvious.

Lemma A.5.1. *For $\alpha \in \mathrm{H}^j(X)$, $\beta \in \mathrm{H}^j(Y)$ and $\zeta \in \mathrm{Corr}^i(C; \mathcal{F}, \mathcal{G})$, we have*

$$(\alpha \cdot \zeta)_\# = f_*(\cup\alpha) \circ \zeta_\#; \qquad (\zeta \cdot \beta)_\# = \zeta_\# \circ g_!(\cup\beta).$$

On the other hand, $\mathrm{H}^*(C)$ acts on $\mathrm{Corr}^*(C;\mathcal{F},\mathcal{G})=\mathrm{Ext}^*_C(\overrightarrow{c}^*\mathcal{G},\overleftarrow{c}^!\mathcal{F})$ by cup product, which we denote simply by $\cup$.

Lemma A.5.2. *Let $\alpha\in\mathrm{H}^*(X)$, $\beta\in\mathrm{H}^*(Y)$ and $\zeta\in\mathrm{Corr}^*(C;\mathcal{F},\mathcal{G})$, then we have*

$$\alpha\cdot\zeta=\zeta\cup\big(\overleftarrow{c}^*\alpha\big);\qquad \zeta\cdot\beta=\zeta\cup\big(\overrightarrow{c}^*\beta\big).$$

If we have a base change diagram of correspondences induced from $S'\to S$ as in Appendix A.4, then the pull-back map

$$\gamma^*:\mathrm{Corr}^*(C;\mathcal{F},\mathcal{G})\to\mathrm{Corr}^*\big(C';\phi^*\mathcal{F},\psi^*\mathcal{G}\big)$$

commutes with cup products $\cdot$ and $\cup$ defined above.

A.6. Integration along a graph-like correspondence

In the sequel, we assume X to be smooth and equidimensional of dimension d.

Definition A.6.1. Let $U\subset S$ be an open subscheme. A correspondence C between X and Y over S is said to be *left graph-like with respect to U* if it satisfies the following conditions:

(G-1) *The projection $\overleftarrow{c}:C_U\to X_U$ is étale.*
(G-2) $\dim C_U\leqslant d$ *and the image of $C-C_U\to X\times_S Y$ has dimension $<d$.*

Similarly, C is said to be *right graph-like with respect to U* if $\overrightarrow{c}:C_U\to Y_U$ is étale and (G-2) is satisfied; C is said to be *graph-like with respect to U* if it is both left and right graph-like.

Note that the condition $\dim C_U\leqslant d$ in (G-2) is implied by (G-1); we leave it there because sometimes we want to refer to (G-2) alone without assuming (G-1).

Lemma A.6.2. *Suppose C is a correspondence between X and Y over S satisfying* (G-2) *with respect to $U\subset S$. Let $\zeta,\zeta'\in\mathrm{Corr}(C;\overline{\mathbb{Q}}_{\ell,X},\overline{\mathbb{Q}}_{\ell,Y})$. If $\zeta|_U=\zeta'|_U\in\mathrm{Corr}(C_U;\overline{\mathbb{Q}}_{\ell,X_U},\overline{\mathbb{Q}}_{\ell,Y_U})$, then $\zeta_\#=\zeta'_\#\in\mathrm{Hom}_S(g_!\overline{\mathbb{Q}}_{\ell,Y},f_*\overline{\mathbb{Q}}_{\ell,X})$.*

Proof. Let Z be the image of $\overleftrightarrow{c}$ and $\overleftarrow{z}:Z\to X,\overrightarrow{z}:Z\to Y$ be the projections. Under the above assumptions, after choosing a fundamental class of X, we can identify $\overline{\mathbb{Q}}_{\ell,X}$ with $\mathbb{D}_X[-2d](-d)$, hence identify $\mathbb{D}_{\overleftarrow{z}}$ with $\mathbb{D}_Z[-2d](-d)$. Similar remark applies to Z_U. Consider the restriction map

$$j^*:\mathrm{H}^{-2d}(Z,\mathbb{D}_Z)(-d)=\mathrm{H}^{\mathrm{BM}}_{2d}(Z)(-d)\to\mathrm{H}^{\mathrm{BM}}_{2d}(Z_U)(-d)=\mathrm{H}^{-2d}(Z_U,\mathbb{D}_{Z_U})(-d).$$

The source and target have bases consisting of fundamental classes of d-dimensional irreducible components of Z and Z_U. By condition (G-2), the d-dimensional irreducible components of Z and Z_U are naturally in bijection. Therefore j^* is an isomorphism, and the restriction map

$$\mathrm{Corr}(Z;\overline{\mathbb{Q}}_{\ell,X},\overline{\mathbb{Q}}_{\ell,Y})=\mathrm{H}^0(Z,\mathbb{D}_{\overleftarrow{z}})\to\mathrm{H}^0(Z_U,\mathbb{D}_{\overleftarrow{z}})=\mathrm{Corr}(Z_U;\overline{\mathbb{Q}}_{\ell,X_U},\overline{\mathbb{Q}}_{\ell,Y_U})$$

is also an isomorphism. Since $\zeta|_U = \zeta'|_U \in \mathrm{Corr}(C_U; \overline{\mathbb{Q}}_{\ell,X_U}, \overline{\mathbb{Q}}_{\ell,Y_U})$, hence $(\overleftrightarrow{c}_*\zeta)|_U = (\overleftrightarrow{c}_*\zeta')|_U \in \mathrm{Corr}(Z_U; \overline{\mathbb{Q}}_{\ell,X_U}, \overline{\mathbb{Q}}_{\ell,Y_U})$, therefore $\overleftrightarrow{c}_*\zeta = \overleftrightarrow{c}_*\zeta' \in \mathrm{Corr}(Z; \overline{\mathbb{Q}}_{\ell,X}, \overline{\mathbb{Q}}_{\ell,Y})$. It remains to apply Lemma A.1.2 to $\overleftrightarrow{c} : C \to Z$. □

Example A.6.3. Let $\phi : X \to Y$ be a morphism over S and $\Gamma(\phi) \subset X \times_S Y$ be its graph, then $\Gamma(\phi)$ is a left graph-like correspondence between X and Y. The relative fundamental class $[\Gamma(\phi)] \in \mathrm{H}^0(\Gamma(\phi), \mathbb{D}_{\Gamma(\phi)/X}) = \mathrm{H}^0(\Gamma(\phi))$ is the class of the constant function 1. Then $[\Gamma(\phi)]_\#$ takes the form

$$\left[\Gamma(\phi)\right]_\# : g_!\overline{\mathbb{Q}}_{\ell,Y} \to g_*\overline{\mathbb{Q}}_{\ell,Y} \xrightarrow{\phi^*} f_*\overline{\mathbb{Q}}_{\ell,X}.$$

Let C be a correspondence between X and Y over S satisfying (G-2) with respect to some $U \subset S$. Consider the fundamental class $[C_U] \in \mathrm{H}^{\mathrm{BM}}_{2d}(C)(-d)$, defined as the sum of the fundamental classes of the closures of d-dimensional irreducible components of C_U. Using the fundamental class of X, we can identify $\overline{\mathbb{Q}}_{\ell,X}$ with $\mathbb{D}_X[-2d](-d)$, and get an isomorphism $\mathbb{D}_{\overleftarrow{c}} \cong \mathbb{D}_C[-2d](-d)$. Therefore $[C_U]$ can be viewed as a class in $\mathrm{H}^0(C, \mathbb{D}_{\overleftarrow{c}}) = \mathrm{Corr}(C; \overline{\mathbb{Q}}_{\ell,X}, \overline{\mathbb{Q}}_{\ell,Y})$. We claim that the induced map $[C_U]_\# : g_!\overline{\mathbb{Q}}_{\ell,Y} \to f_*\overline{\mathbb{Q}}_{\ell,X}$ is independent of U. In fact, if C also satisfies the condition (G-2) with respect to another $V \subset S$, then it again satisfies (G-2) with respect to $U \cap V$. Since $[C_U]$ and $[C_V]$ both restrict to $[C_{U\cap V}]$ in $\mathrm{Corr}(C_{U\cap V}; \overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_\ell)$, Lemma A.6.2 implies that $[C_U]_\# = [C_V]_\#$. Therefore it is unambiguous to write

$$[C]_\# : g_!\overline{\mathbb{Q}}_{\ell,Y} \to f_*\overline{\mathbb{Q}}_{\ell,X},$$

which is the sheaf-theoretic analog of *integration along the correspondence* C.

Now we study the composition of such integrations. We use the notation in the diagram (A.2). Let X, Y be smooth, equidimensional and Y be proper over S.

Proposition A.6.4. *Assume C_2 is left graph-like and $C_1, C = C_1 * C_2$ satisfy* (G-2) *with respect to $U \subset S$, then*

$$[C_1]_\# \circ [C_2]_\# = [C]_\# : h_!\overline{\mathbb{Q}}_{\ell,Z} \to f_*\overline{\mathbb{Q}}_{\ell,X}.$$

Similarly, if we assume C_1 is right graph-like and C_2, $C = C_1 * C_2$ satisfy (G-2) *with respect to $U \subset S$, the same conclusion holds.*

Proof. We prove the first statement. The proof follows from a sequence of dévissages. By Lemma A.2.1, it suffices to prove

$$[C_1] \circ_Y [C_2] = [C] \in \mathrm{Corr}(C; \overline{\mathbb{Q}}_{\ell,X}, \overline{\mathbb{Q}}_{\ell,Z}).$$

By property (G-2) and Lemma A.6.2, it suffices to prove

$$[C_{1,U}] \circ_{Y_U} [C_{2,U}] = [C_U] \in \mathrm{H}^0(C_U, \mathbb{D}_{\overleftarrow{c}}). \tag{A.3}$$

Therefore we have reduced to the case where $\overleftarrow{c_2}$, and hence $\overleftarrow{d}$ are étale. In this case, we can identify $\mathbb{D}_{\overleftarrow{c_2}}$ with $\overline{\mathbb{Q}}_{\ell,C_2}$, and the convolution product becomes

$$\mathrm{H}^0(C_1, \mathbb{D}_{\overleftarrow{c_1}}) \otimes \mathrm{H}^0(C_2) \to \mathrm{H}^0(C, \mathbb{D}_{\overleftarrow{c}}).$$

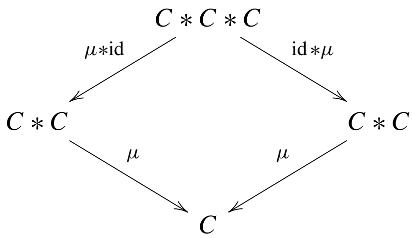
Moreover, $[C_2]$ becomes the class of constant function 1 in $\mathrm{H}^0(C_2)$. Therefore, convolution with $[C_2]$ becomes the pull-back along the étale morphism $\overleftarrow{d}$:

$$\overleftarrow{d}^* : \mathrm{H}^0(C_1, \mathbb{D}_{\overleftarrow{C_1}}) \to \mathrm{H}^0(C, \mathbb{D}_{\overleftarrow{C}}).$$

It is obvious that $\overleftarrow{d}^*[C_1] = [C]$. Therefore (A.3) is proved. □

A.7. The convolution algebra

Assume X is smooth of equidimension d and $f : X \to S$ is proper. Let C be a self correspondence of X over S satisfying (G-2). Assume we have a morphism $\mu : C * C \to C$ as correspondences which is associative, i.e., the following diagram is commutative:



Then the convolution gives a multiplication on $\mathrm{Corr}(C; \overline{\mathbb{Q}}_{\ell,X}, \overline{\mathbb{Q}}_{\ell,X})$:

$$\begin{aligned} \circ : \mathrm{Corr}(C; \overline{\mathbb{Q}}_{\ell,X}, \overline{\mathbb{Q}}_{\ell,X}) \otimes \mathrm{Corr}(C; \overline{\mathbb{Q}}_{\ell,X}, \overline{\mathbb{Q}}_{\ell,X}) &\xrightarrow{\circ_X} \mathrm{Corr}(C * C; \overline{\mathbb{Q}}_{\ell,X}, \overline{\mathbb{Q}}_{\ell,X}) \\ &\xrightarrow{\mu_*} \mathrm{Corr}(C; \overline{\mathbb{Q}}_{\ell,X}, \overline{\mathbb{Q}}_{\ell,X}), \end{aligned}$$

which is associative by Lemma A.2.2. Therefore $\mathrm{Corr}(C; \overline{\mathbb{Q}}_{\ell,X}, \overline{\mathbb{Q}}_{\ell,X})$ acquires a (non-unital) algebra structure. Restricting to C_U, $\mathrm{Corr}(C_U; \overline{\mathbb{Q}}_{\ell,X_U}, \overline{\mathbb{Q}}_{\ell,X_U})$ is also a (non-unital) algebra.

Remark A.7.1. We have a map

$$\mathrm{End}_S(X) \to \mathrm{Corr}(X \times_S X; \overline{\mathbb{Q}}_{\ell,X}, \overline{\mathbb{Q}}_{\ell,X})$$

which sends a morphism $\phi : X \to X$ to the fundamental class of its graph. By Example A.6.3, this is an *anti*-homomorphism of monoids. Here the monoid structures are given by the composition of morphisms on the LHS and the convolution $\circ$ on the RHS.

Proposition A.7.2.

(1) *The map*

$$(-)_\# : \mathrm{Corr}(C; \overline{\mathbb{Q}}_{\ell,X}, \overline{\mathbb{Q}}_{\ell,X}) \to \mathrm{End}_S(f_*\overline{\mathbb{Q}}_{\ell,X}) \tag{A.4}$$

is an algebra homomorphism.

(2) *The map* (A.4) *factors through the restriction map*

$$j^*: \mathrm{Corr}(C; \overline{\mathbb{Q}}_{\ell,X}, \overline{\mathbb{Q}}_{\ell,X}) \to \mathrm{Corr}(C_U; \overline{\mathbb{Q}}_{\ell,X_U}, \overline{\mathbb{Q}}_{\ell,X_U})$$

so that we also have an algebra homomorphism

$$(-)_{U,\#}: \mathrm{H}^0(C_U, \mathbb{D}_{\widetilde{C}}) \to \mathrm{End}_S(f_*\overline{\mathbb{Q}}_{\ell,X}). \tag{A.5}$$

Proof. (1) follows from Lemma A.2.1; (2) The factorization follows from Lemma A.6.2. Using the identification $\overline{\mathbb{Q}}_{\ell,X} \cong \mathbb{D}_X[-2d](-d)$, we can identify j^* with the restriction map

$$\mathrm{H}^{\mathrm{BM}}_{2d}(C)(-d) \to \mathrm{H}^{\mathrm{BM}}_{2d}(C_U)(-d),$$

which is surjective. Therefore the map (A.5) is also an algebra homomorphism because (A.4) is. □

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