Inventiones mathematicae (2023) 235:569–668
https://doi.org/10.1007/s00222-023-01228-y

Higher Siegel–Weil formula for unitary groups: the non-singular terms

Tony Feng[1] · **Zhiwei Yun**[2] · **Wei Zhang**[2]

Received: 2 January 2023 / Accepted: 3 November 2023 / Published online: 27 November 2023

Abstract
We construct special cycles on the moduli stack of hermitian shtukas. We prove an identity between (1) the r^{th} central derivative of non-singular Fourier coefficients of a normalized Siegel–Eisenstein series, and (2) the degree of special cycles of "virtual dimension 0" on the moduli stack of hermitian shtukas with r legs. This may be viewed as a function-field analogue of the Kudla-Rapoport Conjecture, that has the additional feature of encompassing all higher derivatives of the Eisenstein series.

Keywords Special cycle · Shtuka · Hitchin moduli · Springer theory · Arithmetic Siegel–Weil · Local density

Contents

✉ T. Feng
fengt@berkeley.edu

Z. Yun
zyun@mit.edu

W. Zhang
weizhang@mit.edu

[1] Department of Mathematics, University of California, 970 Evans Hall, Berkeley, CA 94720, USA

[2] Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA

1 Introduction

The classical Siegel–Weil formula ([22, 26]) relates the special values of Siegel–Eisenstein series on the symplectic group (resp. the unitary group) to theta functions, which are generating series of representation numbers of quadratic (resp. Hermitian) forms over number fields. In particular, by exploiting the factorization of the non-singular Fourier coefficients into a product of local terms, one arrives at Siegel's formula for representation numbers of global quadratic or Hermitian forms in terms of local representation densities.

In [11] Kudla began to study an arithmetic version of the Siegel–Weil formula and he discovered a relation between an "arithmetic theta function" — a generating series of arithmetic cycles on an integral model of a Shimura curve—and the first central derivative of a Siegel–Eisenstein series on Sp_4. In a series of papers, Kudla and Rapoport developed this paradigm by defining the non-singular terms of a generating series of special cycles on suitable integral models of Shimura varieties for $\mathrm{SO}(n-1,2)$ with $n \leq 4$ and for all $\mathrm{U}(n-1,1)$. Of particular relevance to our paper, in [12, 13] Kudla and Rapoport defined the sought-after special cycles on integral models of unitary Shimura varieties, now known as *Kudla–Rapoport cycles*, and conjectured a relationship to the non-singular Fourier coefficients of the central derivative of the Siegel–Eisenstein series. Their conjecture has been recently proved by Li and one of us [18]; we also refer to the introduction of [18] for a more detailed account of recent advances in some other related directions (see also [19] for the orthogonal analog). With the Kudla–Rapoport conjecture proved in [18] and its archimedean counterpart proved by Liu [20] and independently by Garcia and Sankaran [6] as some of the key ingredients, Li and Liu [16, 17] have recently proved an arithmetic Rallis inner product formula relating the height pairing of the generating series to the first derivative of L-functions for unitary groups, from which they deduced cases of Beilinson–Bloch conjecture for certain high rank motives.

In this paper we study a function field analogue of the arithmetic Siegel–Weil formula, for unitary groups. In particular, we will construct special cycles on the moduli space of hermitian shtukas with arbitrary number of legs. Then we formulate and prove the analogue of the Kudla-Rapoport conjecture for derivatives of *arbitrary order* at the center of the Siegel–Eisenstein series, relating the *non-singular* Fourier coefficients of such higher derivatives to the degrees of special cycles. We remark that the proofs here follow a completely different strategy than in [18].

In the sequel [4], we will construct the complete generating series of special cycles (including singular terms) and give evidence for their modularity.

1.1 Statement of main result

To formulate the result, let X be a smooth, proper and geometrically connected curve over $k = \mathbf{F}_q$ of characteristic $p \neq 2$, and $\nu\colon X' \to X$ be an étale double cover, with the non-trivial automorphism denoted $\sigma \in \mathrm{Aut}(X'/X)$. Let F be the function field of X and let F' be the ring of rational functions on X' (we allow $X' = X \coprod X$). In §6 we recall the definition of the moduli stack $\mathrm{Sht}^r_{U(n)}$ parametrizing rank n "Hermitian shtukas" with r legs. Roughly speaking it classifies chains of vector bundles with Hermitian structures

$$\mathcal{F}_0 \dashrightarrow \mathcal{F}_1 \dashrightarrow \cdots \dashrightarrow \mathcal{F}_r \cong {}^{\tau}\mathcal{F}_0 \tag{1.1}$$

related by elementary modifications. It admits a fibration $\mathrm{Sht}^r_{U(n)} \to (X')^r$, and will play the role of Shimura varieties in the function field context.

1.1.1 Special cycles

Drawing inspiration from the construction of Kudla-Rapoport cycles on unitary Shimura varieties [13], we introduce in §7 certain stacks $\mathcal{Z}^r_{\mathcal{E}}(a)$ over $\mathrm{Sht}^r_{U(n)}$ indexed by $\mathcal{E}$, a vector bundle of rank m with $1 \leq m \leq n$ on X', and a Hermitian map[1] $a\colon \mathcal{E} \to \sigma^*\mathcal{E}^\vee$ where $\mathcal{E}^\vee := \underline{\mathrm{Hom}}(\mathcal{E}, \omega_{X'})$ is the Serre dual of $\mathcal{E}$. They classify Hermitian shtukas as in (1.1) together with compatible maps $\mathcal{E} \to \mathcal{F}_i$ such that a is induced from the Hermitian form on $\mathcal{F}_0$.

If $\mathcal{E}$ is a line bundle on X', $\mathcal{Z}^r_{\mathcal{E}}(a)$ is an analogue of Kudla-Rapoport divisors although they have dimension r less than $\mathrm{Sht}^r_{U(n)}$. In general, $\mathcal{Z}^r_{\mathcal{E}}(a)$ are analogs of special cycles for function fields.

We will be particularly interested in the case $m = n$ and $a\colon \mathcal{E} \to \sigma^*\mathcal{E}^\vee$ is injective (by this we shall always mean as a map of coherent sheaves). In this case, the "virtual dimension" of $\mathcal{Z}^r_{\mathcal{E}}(a)$ is 0. However, as is already seen in the number field context [13], the literal dimension of $\mathcal{Z}^r_{\mathcal{E}}(a)$ is often significantly larger; this problem is exacerbated as r increases. Nevertheless, under the assumption that $a\colon \mathcal{E} \to \sigma^*\mathcal{E}^\vee$ is injective (as a map of coherent sheaves), we are able to construct an appropriate "virtual fundamental cycle" $[\mathcal{Z}^r_{\mathcal{E}}(a)] \in \mathrm{Ch}_0(\mathcal{Z}^r_{\mathcal{E}}(a))_{\mathbf{Q}}$. Interestingly, it turns out that there are some new difficulties present in this construction that do *not* appear in the Shimura variety setting. For a injective, it turns out that $\mathcal{Z}^r_{\mathcal{E}}(a)$ is proper over $\mathbf{F}_q$, so that $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ has a well-defined degree $\deg[\mathcal{Z}^r_{\mathcal{E}}(a)] \in \mathbf{Q}$.

1.1.2 The main result

Let $E(g, s, \Phi)$ be the Siegel–Eisenstein series for the standard split F'/F-skew-Hermitian space of dimension $2n$, with respect to the unramified standard section Φ. For a rank n vector bundle $\mathcal{E}$ on X' as above, $E(g, s, \Phi)$ admits a Fourier expansion

[1] A map of vector bundles of the form $a\colon \mathcal{E} \to \sigma^*\mathcal{E}^\vee$ is *Hermitian* if $\sigma^* a^\vee = a$.

with respect to $\mathcal{E}$ indexed by Hermitian maps $a\colon \mathcal{E} \to \sigma^*\mathcal{E}^\vee$. We let $\widetilde{E}_a(m(\mathcal{E}), s, \Phi)$ be the a^{th} Fourier coefficient multiplied by certain normalization factors, explained precisely in (12.1).

In our normalization, $s = 0$ is the center of the functional equation for $\widetilde{E}_a(m(\mathcal{E}), s, \Phi)$. Our main theorem relates the Taylor expansion at this central point to the degrees of special cycle classes.

Theorem 1.1 *Let $n \geq 1$ and $r \geq 0$. Let $\mathcal{E}$ be a rank n vector bundle on X' and $a\colon \mathcal{E} \to \sigma^*\mathcal{E}^\vee$ be an injective Hermitian map. Then we have*

$$\frac{1}{(\log q)^r}\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^r\Big|_{s=0}\left(q^{ds}\widetilde{E}_a(m(\mathcal{E}), s, \Phi)\right) = \deg[\mathcal{Z}^r_{\mathcal{E}}(a)], \tag{1.2}$$

where $d = -\deg(\mathcal{E}) + n\deg\omega_X = -\chi(X', \mathcal{E})$.

1.1.3 Initial comments on the proof

We stress that (1.2) holds for *all* r, regardless of the order of vanishing of $\widetilde{E}_a(m(\mathcal{E}), s, \Phi)$ at $s = 0$. This is a distinguishing novelty of Theorem 1.1 compared to all other works on the Seigel-Weil or arithmetic Siegel-Weil formula. The first results of this nature, giving motivic interpretations of Taylor coefficients of automorphic L-functions even "beyond the leading term", were proved in [28, 29] for PGL_2. Our results here are the first higher derivative formulas to be proved for groups of *arbitrary* rank. Our proof shares some common ingredients with these earlier works, but also has a number of interesting new ones. For example, a key discovery for us was a connection between the Fourier coefficients of Siegel–Eisenstein series and certain perverse sheaves arising from Springer theory. Another key realization was that the special cycles are governed by certain variants of the Hitchin fibration, whose geometry can also be described in terms of Springer theory. In particular, the geometry behind Theorem 1.1 is much more complicated than that in [28, 29] as soon as $n > 2$. An overview of the proof will be given in §1.2.

Another feature of the proof of Theorem 1.1 is that it is completely uniform in r, and in particular unites the "Siegel–Weil formula" and "arithmetic Siegel–Weil formula" in the same framework. For this reason, we propose to call (1.2) a *higher Siegel–Weil formula*. This formula will serve as the first step to establish a higher order derivative version of the aforementioned recent work of Li and Liu [16, 17] over function fields, which would give a geometric interpretation of higher derivatives of Langlands L-functions.

Remark 1.2 When $r = 0$, the coarse moduli space of $\mathrm{Sht}^r_{U(n)}$ is just the discrete set of points which form the domain of everywhere unramified automorphic forms for $U(n)$. In that case, Theorem 1.1 specializes to (the non-singular Fourier coefficients of) the classical Siegel–Weil formula, which can be found in [26].

One should imagine that when $r = 1$, $\mathrm{Sht}^r_{U(n)} \to X'$ is analogous to (the integral model of) a unitary Shimura variety. Now, under the technical assumptions of the present paper (namely the everywhere unramifiedness assumptions) this space is always empty, corresponding to the fact that the sign of the functional equation for

the Siegel–Eisenstein series is $+1$ (so that all odd order derivatives vanish). However, with a mild modification of the setup, the same methods may be used to prove variants of Theorem 1.1 in which the sign of the functional equation is -1. More precisely, in this paper we consider rank n vector bundles on X' with a Hermitian pairing valued in the canonical bundle $\omega_{X'} \cong \nu^* \omega_X$; if we replace ω_X here by a line bundle on X which is not a norm from X', then the sign of the functional equation is -1 when n is odd. The precise formulation is in [4, §9]. We mention also the work [25] over function fields, which should be thought of as being similar to the special case of Theorem 1.1 for $r = 1$ and $n = 1$.

When $r > 1$, no analogue of the spaces $\mathrm{Sht}^r_{U(n)}$ is presently known in the number field setting. Consequently, we do not know how to formulate an analogue of the main result for number fields.

1.1.4 Construction of virtual fundamental cycles

For a vector bundle $\mathcal{E}$ of rank m on X' and a Hermitian map $a\colon \mathcal{E} \to \sigma^*\mathcal{E}^\vee$, the dimension of $\mathcal{Z}^r_{\mathcal{E}}(a)$ differs from its "virtual dimension", which is $r(n-m)$. The situation gets worse if a is *singular* (i.e., not injective, in analogy to the terminology of [13]). For example, when $a = 0$, $\mathcal{Z}^r_{\mathcal{E}}(0)$ contains $\mathrm{Sht}^r_{U(n)}$ as a substack. It is a nontrivial task to define a cycle class $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ in the expected dimension $r(n-m)$.

Our companion paper [4] proposes two solutions to this problem, one using classical intersection theory and the other using derived algebraic geometry. There, we construct cycle classes $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ for all $\mathcal{E}$ of rank $\leq n$ and *possibly singular* $a\colon \mathcal{E} \to \sigma^*\mathcal{E}^\vee$. Moreover, we assemble them into generating series valued in the Chow groups of $\mathrm{Sht}^r_{U(n)}$ and conjecture it to be automorphic, in analogy to known results over number fields [2], which fall under the umbrella of the Kudla program.

In this paper, we use a more elementary method to define the 0-cycle $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ in the case $m = n$ and a injective. First, we prove that when $\mathcal{L}$ is a line bundle and $a\colon \mathcal{L} \to \sigma^*\mathcal{L}^\vee$ is an injective Hermitian map, $\mathcal{Z}^r_{\mathcal{L}}(a)$ has the expected dimension (cf. Proposition 7.9 and Remark 7.10). Next, when $\mathcal{E} = \oplus_{i=1}^n \mathcal{L}_i$ is a direct sum of line bundles, the class $[\mathcal{Z}^r_{\mathcal{E}}(a)] \in \mathrm{Ch}_0(\mathcal{Z}^r_{\mathcal{E}}(a))_{\mathbf{Q}}$ can be defined as (the restriction to $\mathcal{Z}^r_{\mathcal{E}}(a)$ of) the intersection product of $\mathcal{Z}^r_{\mathcal{L}_i}(a_{ii})$ for the diagonal entries a_{ii} of a; this is similar to the number field case. However, compared to the number field case, a new difficulty arises since $\mathcal{E}$ is not necessarily a direct sum of line bundles. We overcome this difficulty in §7.9 by introducing the notion of a *good framing* for $\mathcal{E}$ to reduce to the case of a sum of line bundles. A nontrivial task is to verify that the cycle class $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ is independent of the choice of the good framing, which occupies much of the sections §8–§10.

1.2 Method of proof

To summarize, we prove Theorem 1.1 by constructing two perverse sheaves that encode the two sides of (1.2) in the sense of sheaf-function correspondence, and then identifying these two perverse sheaves using a Hermitian variant of Springer theory, which labels these perverse sheaves by representations of the appropriate Weyl group. In this way, Theorem 1.1 is eventually unraveled into an elementary identity between representations of the Weyl group for type B/C.

On the geometric side, the connection between special cycles and Springer theory comes via the geometry of a moduli stack that resembles the Hitchin moduli space. On the other side, the connection between the Fourier coefficients of Siegel-Eisenstein series and Springer theory goes through local density formulas of Cho-Yamauchi.

Let us briefly explain the connection between the higher Siegel-Weil formula and the Hitchin moduli stack and Hermitian Springer theory, and defer details to the later paragraphs. The degree of the special cycle that appear on the right side of (1.2) is essentially an intersection number of cycles on $\mathrm{Sht}^r_{U(n)}$. The ambient space $\mathrm{Sht}^r_{U(n)}$ can itself be realized an intersection of a Hecke correspondence with the graph of a Frobenius endomorphism. We use this to "unfold" all the intersections, and then redo them in a different order, performing the *linear* intersections (i.e., those not involving the Frobenius map) first, and leaving the Frobenius semi-linear intersection till the last step (cf. (10.15) – (10.19)). In this process, a Hitchin-type moduli stack $\mathcal{M}_d$ appears naturally as we perform linear intersections (cf. (10.18)). The degree of the special cycle $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ can be expressed as a weighted counting of k-points on the fiber of a map $f_d : \mathcal{M}_d \to \mathcal{A}_d$ (analogue of Hitchin fibration) over the point $(\mathcal{E}, a) \in \mathcal{A}_d(k)$, where $(\mathcal{E}, a)$ are as in the statement of Theorem 1.1.

The cokernel $\mathcal{Q} = \mathrm{coker}(a)$ is a torsion sheaf on X' with a Hermitian structure inherited from a. This motivates the introduction of the moduli stack $\mathrm{Herm}_{2d}(X'/X)$ that parametrizes torsion coherent sheaves on X' of length $2d$ together with a Hermitian structure, so that $\mathcal{Q}$ is a k-point of $\mathrm{Herm}_{2d}(X'/X)$ (where $2d = \dim_k \Gamma(X', \mathcal{Q})$). We show that the fiber of $f_d : \mathcal{M}_d \to \mathcal{A}_d$ over $(\mathcal{E}, a)$ depends only on $\mathcal{Q} = \mathrm{coker}(a)$, therefore the degree of $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ depends only on the k-point $\mathcal{Q}$ of $\mathrm{Herm}_{2d}(X'/X)$.

On the other hand, the Eisenstein series side of (1.2) can be written as a product of local terms – representation density functions for Hermitian lattices. These density functions again only depend on the torsion sheaf $\mathcal{Q}$ together with its Hermitian structure, i.e., a k-point in $\mathrm{Herm}_{2d}(X'/X)$.

Therefore we reduce to proving that two quantities attached to a k-point in $\mathrm{Herm}_{2d}(X'/X)$ are equal. A key realization is that both quantities are of motivic nature: they come by the sheaf-to-function correspondence from two (graded, virtual) perverse sheaves on $\mathrm{Herm}_{2d}(X'/X)$. This is where Hermitian Springer theory enters. Classically, starting with a reductive Lie algebra $\mathfrak{g}$, Springer theory outputs a perverse sheaf $\mathrm{Spr}_{\mathfrak{g}}$ on $\mathfrak{g}$, defined as the direct image complex of the Grothendieck-Springer resolution $\pi_{\mathfrak{g}} : \widetilde{\mathfrak{g}} \to \mathfrak{g}$, together with an action of the Weyl group W. In our setting, $\mathrm{Herm}_{2d}(X'/X)$ will play the role of $\mathfrak{g}$. In §4, we construct a perverse sheaf $\mathrm{Spr}^{\mathrm{Herm}}_{2d}$ on $\mathrm{Herm}_{2d}(X'/X)$ together with an action of $W_d = (\mathbf{Z}/2\mathbf{Z})^d \rtimes S_d$ analogous to the Springer sheaf. If $\mathrm{Herm}_{2d}(X'/X)$ is replaced by $\mathrm{Coh}_d(X)$, the moduli of torsion coherent sheaves on X of length d, such a Springer sheaf was constructed by Laumon [15]. The Springer sheaf on $\mathrm{Coh}_d(X)$ (resp. $\mathrm{Herm}_{2d}(X'/X)$) can be viewed as a global version of the Springer sheaf for $\mathfrak{gl}_d$ (resp. $\mathfrak{o}_{2d}$). The perverse sheaves on $\mathrm{Herm}_{2d}(X'/X)$ that govern both sides of (1.2) will be constructed from direct summands of the Hermitian Springer sheaf $\mathrm{Spr}^{\mathrm{Herm}}_{2d}$.

Thus, the proof of Theorem 1.1 is completed in three steps:

(1) Construct a graded perverse sheaf on $\mathrm{Herm}_{2d}(X'/X)$

$$\mathcal{K}_d^{\mathrm{Eis}} = \bigoplus_{i=0}^{d} \mathcal{K}_{d,i}^{\mathrm{Eis}}$$

whose Frobenius trace at $\mathcal{Q}$ is related to the LHS of (1.2). More precisely,

$$\widetilde{E}_a(m(\mathcal{E}), s, \Phi) = \sum_{i=0}^{d} \mathrm{Tr}(\mathrm{Fr}_{\mathcal{Q}}, (\mathcal{K}_{d,i}^{\mathrm{Eis}})_{\mathcal{Q}}) q^{-2is}.$$

(2) Construct a graded perverse sheaf on $\mathrm{Herm}_{2d}(X'/X)$

$$\mathcal{K}_d^{\mathrm{Int}} = \bigoplus_{i=0}^{d} \mathcal{K}_{d,i}^{\mathrm{Int}}$$

whose Frobenius trace at $\mathcal{Q}$ is relate to the RHS of (1.2). More precisely,

$$\deg[\mathcal{Z}_{\mathcal{E}}^r(a)] = \sum_{i=0}^{d} \mathrm{Tr}(\mathrm{Fr}_{\mathcal{Q}}, (\mathcal{K}_{d,i}^{\mathrm{Int}})_{\mathcal{Q}}) \cdot (d-2i)^r. \tag{1.3}$$

(3) Prove that

$$\mathcal{K}_d^{\mathrm{Eis}} \cong \mathcal{K}_d^{\mathrm{Int}} \tag{1.4}$$

as graded perverse sheaves on $\mathrm{Herm}_{2d}(X'/X)$.

These three steps correspond to the three parts of the paper. We elaborate on the main ideas involved in each step.

1.2.1 Step (1)

After a standard procedure expressing the nonsingular Fourier coefficients of Eisenstein series in terms of local density of Hermitian lattices, we use the formula of Cho and Yamauchi [3] for these densities (more precisely, the unitary variant developed in [18]). We also need an extension of their formula in the split case (Theorem 2.3). The formula of Cho and Yamauchi depends only on the Hermitian torsion sheaf $\mathcal{Q} = \mathrm{coker}(a)$, which gives the hope that the local density, as a function on the set of Hermitian torsion sheaves, comes from a sheaf on $\mathrm{Herm}_{2d}(X'/X)$ via Grothendieck's sheaf-to-function dictionary. We do this by developing an analog of Springer theory over $\mathrm{Herm}_{2d}(X'/X)$ (§3-§4).

The key observation here is that the term in the Cho–Yamauchi formula resembles the Frobenius trace function for a certain linear combination of Springer sheaves for $\mathfrak{gl}_d$ or $\mathrm{Coh}_d(X)$, except for some signs. To match the signs exactly we consider an analogous linear combination of Springer sheaves on $\mathrm{Herm}_{2d}(X'/X)$, and we compare the Frobenius actions on the cohomology of Springer fibers over $\mathrm{Coh}_d(X)$ and over $\mathrm{Herm}_{2d}(X'/X)$, see §4.5 and §4.6.

1.2.2 Step (2)

This step consists of three substeps.

- First, we define special cycles for nonsingular a (§6-§7). When $\mathcal{E}$ is a direct sum of line bundles $\mathcal{L}_i$, we define, following Kudla and Rapoport, $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ as the intersection of cycle classes $[\mathcal{Z}^r_{\mathcal{L}_i}(a_{ii})]$, which, despite not being divisors in our setting, always have the "expected" dimension (more precisely, codimension r in $\mathrm{Sht}^r_{U(n)}$). The definition of $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ for general vector bundles $\mathcal{E}$ requires choosing a "good framing" on $\mathcal{E}$, i.e., an injective map from a direct sum of line bundles $\mathcal{E}' = \oplus_{i=1}^n \mathcal{L}_i \hookrightarrow \mathcal{E}$ satisfying certain conditions. In any case, the RHS of (1.2) is an intersection number of cycles on $\mathrm{Sht}^r_{U(n)}$.
- The well-definedness of $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ is proved in the second substep (§8-§10), which also gives a different definition of these cycle classes without any choices. The idea is similar to the one used in [28], namely by exchanging the order of intersection, we perform "linear intersections" first to form Hitchin-type moduli stacks (denoted $\mathcal{M}_d$, making sense over any base field), and in the last step we perform a shtuka-type construction by intersecting with the graph of Frobenius.
- In the last substep (§11) we use the Lefschetz trace formula to express the degree of $[\mathcal{Z}^r_{\mathcal{E}}(a)]$, formulated using the Hitchin-type moduli stack $\mathcal{M}_d$, as the trace of Frobenius composed with the r^{th} power of an endomorphism C on the direct image complex $Rf_*\overline{\mathbf{Q}}_\ell$ of the Hitchin map $f : \mathcal{M}_d \to \mathcal{A}_d$. Now, the "Hitchin base" $\mathcal{A}_d$ has a canonical smooth map to $\mathrm{Herm}_{2d}(X'/X)$, and it turns out that $Rf_*\overline{\mathbf{Q}}_\ell$ together with its endomorphism C descends through this map to a perverse sheaf $\mathcal{K}^{\mathrm{Int}}_d$ on $\mathrm{Herm}_{2d}(X'/X)$ with an endomorphism $\overline{C}$. The decomposition of $\mathcal{K}^{\mathrm{Int}}_d$ into graded pieces $\mathcal{K}^{\mathrm{Int}}_{d,i}$ is according to the eigenvalues of the $\overline{C}$-action, which are of the form $(d-2i)$. Combining these facts we get (1.3).

1.2.3 Step (3)

Both $\mathcal{K}^{\mathrm{Eis}}_d$ and $\mathcal{K}^{\mathrm{Int}}_d$ are linear combinations of isotypical summands of $\mathrm{Spr}^{\mathrm{Herm}}_{2d}$ under the action of W_d. The isomorphism (1.4) then comes from an isomorphism of two graded virtual representations of W_d, which we verify directly.

1.3 Notation

Throughout this paper, $k = \mathbf{F}_q$ is a finite field of odd characteristic p. Let $\ell \neq p$ be a prime. Let $\psi_0 : k \to \overline{\mathbf{Q}}_\ell^\times$ be a nontrivial character. For a stack $\mathcal{Y}$ over k, we write Fr or $\mathrm{Fr}_{\mathcal{Y}}$ for its q-power Frobenius endomorphism. We will use Frob or Frob_y for geometric Frobenius at an $\mathbf{F}_q$-point y.

Let X denote a smooth curve over k. With the exception of §3 and §4, X is assumed to be projective and geometrically connected. Let ω_X be the line bundle of 1-forms on X.

Let $F = k(X)$ denote the function field of X. Let $|X|$ be the set of closed points of X. For $v \in |X|$, let $\mathcal{O}_v$ be the completed local ring of X at v with fraction field F_v and residue field k_v. Let $\mathbb{A} = \mathbb{A}_F$ denote the ring of adèles of F, and $\widehat{\mathcal{O}} = \prod_{v\in|X|} \mathcal{O}_v$. Let

$\deg(v) = [k_v : k]$, and $q_v = q^{\deg(v)} = \#k_v$. A uniformizer of $\mathcal{O}_v$ is typically denoted ϖ_v. Let $|\cdot|_v : F_v^\times \to q_v^{\mathbf{Z}}$ be the absolute value such that $|\varpi_v|_v = q_v^{-1}$. Let $|\cdot|_F : \mathbb{A}_F^\times \to q^{\mathbf{Z}}$ be the absolute value that is $|\cdot|_v$ on $F_v^\times$.

Let X' be another smooth curve over k and $\nu : X' \to X$ be a finite map of degree 2 that is generically étale. We denote by σ the non-trivial automorphism of X' over X. With the exception of §4.1 and §4.2, ν is assumed to be étale. We emphasize that the case $X' = X \coprod X$ is allowed. Let F' be the ring of rational functions on X', which is either a quadratic extension of F or $F \times F$. We let k' be the ring of constants in F'. The notations $\omega_{X'}$, $|X'|$, $F'_{v'}$, $\mathcal{O}_{v'}$, $k_{v'}$, $\mathbb{A}_{F'}$, $|\cdot|_{v'}$, $|\cdot|_{F'}$, $q_{v'}$ and $\deg(v')$ (for $v' \in |X'|$) are defined similarly as their counterparts for X. Additionally, for $v \in |X|$, we use $\mathcal{O}'_v$ to denote the completion of $\mathcal{O}_{X'}$ along $\nu^{-1}(v)$, and define F'_v to be its total ring of fractions.

For a vector bundle $\mathcal{E}$ on X', let $\mathcal{E}^\vee = \underline{\mathrm{Hom}}(\mathcal{E}, \omega_{X'})$ be its Serre dual. For a torsion sheaf $\mathcal{T}$ on X', let $\mathcal{T}^\vee = \underline{\mathrm{Ext}}^1(\mathcal{T}, \omega_{X'})$.

When X (hence X') is projective, let $\mathrm{Bun}_{\mathrm{GL}_n}$ (resp. $\mathrm{Bun}_{\mathrm{GL}'_n}$) be the moduli stack of rank n vector bundles over X (resp. X'). Let g be the genus of X and g' be the arithmetic genus of X'. Note that whenever ν is étale, we have $g' = 2g - 1$.

For an algebraic stack $\mathcal{Y}$, $\mathrm{Ch}(\mathcal{Y})$ denotes its *rationalized* Chow group and $D^b(\mathcal{Y}, \overline{\mathbf{Q}}_\ell)$ its bounded derived category of constructible $\overline{\mathbf{Q}}_\ell$-sheaves.

Part 1. The analytic side

2 Fourier coefficients of Eisenstein series

In this section we will define the Siegel-Eisenstein series featuring into our main theorem, and explain how to express their non-singular Fourier coefficients in terms of local density polynomials, which will be geometrized in later sections.

2.1 Siegel–Eisenstein series

For any one-dimensional F-vector space L, let $\mathrm{Herm}_n(F, L)$ be the F-vector space of F'/F-Hermitian forms $h : F'^n \times F'^n \to L \otimes_F F'$ (with respect to the involution $1 \otimes \sigma$ on $L \otimes_F F'$). For any F-algebra R, $\mathrm{Herm}_n(R, L) := \mathrm{Herm}_n(F, L) \otimes_F R$ is the set of $L \otimes_F R'$-valued R'/R-Hermitian forms on R'^n, where $R' = R \otimes_F F'$. When $L = F$ we write $\mathrm{Herm}_n(F) = \mathrm{Herm}_n(F, F)$ and $\mathrm{Herm}_n(R) = \mathrm{Herm}_n(F) \otimes_F R$ for any F-algebra R.

Let W be the standard split F'/F-skew-Hermitian space of dimension $2n$. Let $H_n = U(W)$. Write $\mathbb{A} := \mathbb{A}_F$ for the ring of adeles of F. Let $P_n(\mathbb{A}) = M_n(\mathbb{A})N_n(\mathbb{A})$ be the standard Siegel parabolic subgroup of $H_n(\mathbb{A})$, where

$$M_n(\mathbb{A}) = \left\{ m(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & {}^t\bar{\alpha}^{-1} \end{pmatrix} : \alpha \in \mathrm{GL}_n(\mathbb{A}_{F'}) \right\},$$

$$N_n(\mathbb{A}) = \left\{ n(\beta) = \begin{pmatrix} 1_n & \beta \\ 0 & 1_n \end{pmatrix} : \beta \in \mathrm{Herm}_n(\mathbb{A}_F) \right\}.$$

Let $\eta : \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times$ be the quadratic character associated to F'/F by class field theory. Fix $\chi : \mathbb{A}_{F'}^\times / F'^\times \to \mathbb{C}^\times$ a character such that $\chi|_{\mathbb{A}_F^\times} = \eta^n$. We may view χ as a character on $M_n(\mathbb{A})$ by $\chi(m(\alpha)) = \chi(\det(\alpha))$ and extend it to $P_n(\mathbb{A})$ trivially on $N_n(\mathbb{A})$. Define the *degenerate principal series* to be the unnormalized smooth induction

$$I_n(s, \chi) = \mathrm{Ind}_{P_n(\mathbb{A})}^{H_n(\mathbb{A})}(\chi \cdot |\cdot|_{F'}^{s+n/2}), \quad s \in \mathbb{C}.$$

For a standard section $\Phi(-, s) \in I_n(s, \chi)$, define the associated *Siegel–Eisenstein series*

$$E(g, s, \Phi) = \sum_{\gamma \in P_n(F) \backslash H_n(F)} \Phi(\gamma g, s), \quad g \in H_n(\mathbb{A}),$$

which converges for $\Re(s) \gg 0$ and admits meromorphic continuation to $s \in \mathbb{C}$. Notice that $E(g, s, \Phi)$ depends on the choice of χ.

Remark 2.1 In this paper, we will choose χ to be unramified everywhere. To see that such χ exists, observe that since $\mathbf{C}^\times$ is injective (in the category of abelian groups), it suffices to check that η^n is trivial on $\ker(\mathrm{Pic}(X) \to \mathrm{Pic}(X'))$. If X'/X is the trivial double cover or the double cover corresponding to $\mathbf{F}_{q^2}/\mathbf{F}_q$, then this kernel is trivial so the result is immediate. Otherwise, the cover is geometrically non-trivial. Since $\mathrm{char}(k) \neq 2$, the kernel consists of the 2-torsion line bundle whose class in $\mathrm{H}^1(X, \mu_2)$ agrees with $\eta \in H^1(X, \mathbf{Z}/2\mathbf{Z})$ under the isomorphism $\mu_2 \cong \mathbf{Z}/2\mathbf{Z}$. If n is even then there is nothing to check; if n is odd then the desired vanishing property amounts (when $\mathrm{char}(k) \neq 2$) to the alternating property of the cup product pairing $H^1(X_{\overline{\mathbf{F}}_q}, \mathbf{Z}/2\mathbf{Z}) \times \mathrm{H}^1(X_{\overline{\mathbf{F}}_q}, \mathbf{Z}/2\mathbf{Z}) \to \mathbf{Z}/2$, which follows from the graded commutativity of the cup product and the fact that the geometric $\mathbf{Z}_2$-cohomology of curves in characteristic $\neq 2$ is torsion-free.

As justified by Remark 2.1, we may choose χ to be everywhere unramified. Then $I_n(s, \chi)$ is unramified and we fix $\Phi(-, s) \in I_n(s, \chi)$ as the unique $K = H_n(\widehat{\mathcal{O}})$-invariant section normalized by

$$\Phi(1_{2n}, s) = 1.$$

Similarly we normalize $\Phi_v \in I_n(s, \chi_v)$ for every $v \in |X|$ and we then have a factorization $\Phi = \bigotimes_{v \in |X|} \Phi_v$.

2.2 Fourier expansion

Let ω_F be the generic fiber of the canonical bundle of X, and $\mathbb{A}_{\omega_F} = \mathbb{A} \otimes_F \omega_F$. The sum of the residues induces a pairing $\mathbb{A}_{\omega_F} \times \mathbb{A}_F \to k$ induces a pairing

$$\langle \cdot, \cdot \rangle : \mathrm{Herm}_n(\mathbb{A}, \omega_F) \times \mathrm{Herm}_n(\mathbb{A}) \to k$$

given by $\langle T, b \rangle = \mathrm{Res}(-\mathrm{Tr}(Tb))$. Composing this pairing with the fixed non-trivial additive character $\psi_0 : k \to \mathbb{C}^\times$ exhibits $\mathrm{Herm}_n(\mathbb{A}, \omega_F)$ as the Pontryagin

dual of $\mathrm{Herm}_n(\mathbb{A})$. Moreover, it exhibits $\mathrm{Herm}_n(F,\omega_F)$ as the Pontryagin dual of $\mathrm{Herm}_n(F)\backslash\mathrm{Herm}_n(\mathbb{A}) = N_n(F)\backslash N_n(\mathbb{A})$. The global residue pairing is the sum of local residue pairings $\langle\cdot,\cdot\rangle_v : \mathrm{Herm}_n(F_v,\omega_{F_v})\times\mathrm{Herm}_n(F_v)\to k$ defined by $\langle T,b\rangle_v = \mathrm{tr}_{k_v/k}\,\mathrm{Res}_v(-\mathrm{Tr}(Tb))$.

We have a Fourier expansion

$$E(g,s,\Phi) = \sum_{T\in\mathrm{Herm}_n(F,\omega_F)} E_T(g,s,\Phi),$$

where

$$E_T(g,s,\Phi) = \int_{N_n(F)\backslash N_n(\mathbb{A})} E(n(b)g,s,\Phi)\psi_0(\langle T,b\rangle)\,\mathrm{d}n(b),$$

and the Haar measure $\mathrm{d}n(b)$ is normalized such that $N_n(F)\backslash N_n(\mathbb{A})$ has volume 1. For any $\alpha\in M_n(\mathbb{A})$ we have

$$E_T(m(\alpha)g,s,\Phi) = \chi(\det(\bar\alpha))^{-1}|\det(\alpha)|_{F'}^{-s+n/2}E_{{}^t\overline{\alpha}T\alpha}(g,s,\Phi). \tag{2.1}$$

Suppose T is *nonsingular*, meaning that for one (equivalently, any) choice of trivialization of ω_F it has non-vanishing determinant, for a factorizable $\Phi = \bigotimes_{v\in|X|}\Phi_v$ we have a factorization of the Fourier coefficient into a product (cf. [11, §4])

$$E_T(g,s,\Phi) = |\omega_X|_F^{-n^2/2}\prod_v W_{T,v}(g_v,s,\Phi_v), \tag{2.2}$$

where the *local (generalized) Whittaker function* is defined by

$$W_{T,v}(g_v,s,\Phi_v) = \int_{N_n(F_v)}\Phi_v(w_n^{-1}n(b)g_v,s)\psi_0(\langle T,b\rangle_v)\,\mathrm{d}_v n(b),$$
$$w_n = \begin{pmatrix} 0 & 1_n\\ -1_n & 0\end{pmatrix}$$

and has analytic continuation to $s\in\mathbb{C}$. Here the local Haar measure $\mathrm{d}_v n(b)$ is the one such that the volume of $N_n(\mathcal{O}_v)$ is 1. The factor $|\omega_X|_F^{-n^2/2}$ is the ratio between the global measure $\mathrm{d}n$ and the product of the local measures $\prod_v \mathrm{d}_v n$.

Note that for $\alpha\in M_n(F_v)$,

$$W_{T,v}(m(\alpha),s,\Phi_v) = \chi(\det(\bar\alpha))^{-1}|\det(\alpha)|_{F_v'}^{-s+n/2}W_{{}^t\bar\alpha T\alpha,v}(1,s,\Phi_v). \tag{2.3}$$

We define the *regular* part of the Eisenstein series to be

$$E^{\mathrm{reg}}(g,s,\Phi) = \sum_{\substack{T\in\mathrm{Herm}_n(F,\omega_F)\\ \mathrm{rank}\,T=n}} E_T(g,s,\Phi). \tag{2.4}$$

2.3 Local densities for Hermitian lattices

The local density for Hermitian lattices in the non-split case has been studied in [18, §3] following the strategy of Cho–Yamauchi [3]. Here we recall the result of [18] and extend the results to the split case.

From now on until §2.5, let F be a non-archimedean local field of characteristic not equal to 2 (but possibly with residue characteristic 2). Let F' be either an unramified quadratic field extension or the split quadratic F-algebra $F' = F \times F$. Denote by $\mathcal{O}_F$ (resp. $\mathcal{O}_{F'}$) the ring of integers in F (resp. F'). In the split case we have $\mathcal{O}_{F'} = \mathcal{O}_F \times \mathcal{O}_F$. Let $\eta = \eta_{F'/F} : F^\times \to \{\pm 1\}$ be the quadratic character attached to F'/F by class field theory. Let ϖ be a uniformizer of F, k the residue field, $q = \#k$.

Let L, M be two Hermitian $\mathcal{O}_{F'}$-lattices. In the split case, the datum of a *Hermitian $\mathcal{O}_{F'}$-lattice* L is a pair (L_1, L_2) of $\mathcal{O}_F$-lattices together with an $\mathcal{O}_F$-bilinear pairing

$$(\cdot, \cdot) : L_1 \times L_2 \to \mathcal{O}_F$$

that is perfect after base change to F. We will define $L^\vee = (L_1^\vee, L_2^\vee)$ where $L_1^\vee = \{x \in L_1 \otimes_{\mathcal{O}_F} F : (x, L_2) \subset \mathcal{O}_F\}$ and similarly for $L_2^\vee$.

Let $\mathrm{Rep}_{M,L}$ be the *scheme of integral representations of M by L*, an $\mathcal{O}_F$-scheme such that for any $\mathcal{O}_F$-algebra R,

$$\mathrm{Rep}_{M,L}(R) = \mathrm{Herm}(L \otimes_{\mathcal{O}_F} R, M \otimes_{\mathcal{O}_F} R),$$

where Herm denotes the set of Hermitian R-module homomorphisms. In the split case, if we write L and M in terms of pairs (L_1, L_2) and (M_1, M_2) with their $\mathcal{O}_F$-bilinear pairings, then a Hermitian module homomorphism consists of a pair of R-linear maps $\phi_i : L_i \otimes_{\mathcal{O}_F} R \to M_i \otimes_{\mathcal{O}_F} R$ preserving the base change to R of the $\mathcal{O}_F$-bilinear pairings.

The *local density* of integral representations of M by L is defined to be

$$\mathrm{Den}(M, L) := \lim_{N \to +\infty} \frac{\#\mathrm{Rep}_{M,L}(\mathcal{O}_F/\varpi^N)}{q^{N \cdot \dim(\mathrm{Rep}_{M,L})_F}}.$$

Note that if L, M have $\mathcal{O}_{F'}$-rank n, m respectively and the generic fiber $(\mathrm{Rep}_{M,L})_F \neq \varnothing$, then $n \leq m$ and

$$\dim(\mathrm{Rep}_{M,L})_F = \dim \mathrm{U}_m - \dim \mathrm{U}_{m-n} = n \cdot (2m - n).$$

2.4 Cho–Yamauchi formula for local density

Definition 2.2 For $a \in \mathbf{Z}_{\geq 0}$ we define a polynomial of degree a

$$\mathfrak{m}(a; T) := \prod_{i=0}^{a-1} (1 - (\eta(\varpi)q)^i T) \in \mathbf{Z}[T].$$

Note that $\mathfrak{m}(a;T)$ depends on F'/F.

In both the non-split and the split cases, for a finite torsion $\mathcal{O}_F$-module $\mathcal{T}$ we define

$$\begin{aligned}\ell(\mathcal{T}) &:= \text{ length of } \mathcal{T} \text{ as an } \mathcal{O}_F\text{-module};\\ t(\mathcal{T}) &:= \dim_k(\mathcal{T}\otimes_{\mathcal{O}_F} k).\end{aligned} \tag{2.5}$$

For an $\mathcal{O}_{F'}$-Hermitian lattice L, we define its type

$$t(L) := t(L^\vee/L)$$

where we view the finite torsion $\mathcal{O}_{F'}$-module $L^\vee/L$ as an $\mathcal{O}_F$-module.

When F'/F is non-split, for a finite torsion $\mathcal{O}_{F'}$-module $\mathcal{T}$ we define

$$\begin{aligned}\ell'(\mathcal{T}) &:= \text{length of } \mathcal{T} \text{ as an } \mathcal{O}_{F'}\text{-module};\\ t'(\mathcal{T}) &:= \dim_{k'}(\mathcal{T}\otimes_{\mathcal{O}_{F'}} k').\end{aligned}$$

Then we have

$$\ell(\mathcal{T}) = 2\ell'(\mathcal{T}), \quad t(\mathcal{T}) = 2t'(\mathcal{T}). \tag{2.6}$$

When $F' = F\times F$ is split, for a finite torsion $\mathcal{O}_{F'}$-module $\mathcal{T}$ we may define $\ell'(\mathcal{T})$ and $t'(\mathcal{T})$ by (2.6). Moreover, for $\mathcal{O}_{F'}$-Hermitian lattices $L = (L_1, L_2)$ and $L' = (L_1', L_2')$ such that $L\subset L'$ (meaning that $L_1\subset L_1'$ and $L_2\subset L_2'$), we have

$$\ell(L'/L) = \ell(L_1'/L_1) + \ell(L_2'/L_2)$$

and

$$t'(L^\vee/L) = t(L_2^\vee/L_1) = t(L_1^\vee/L_2).$$

In both the split and non-split case, we define

$$t'(L) = t'(L^\vee/L).$$

We have the following analog of Cho–Yamauchi formula [3].

Theorem 2.3 *Let $j\geq 0$ be an integer. Let $\langle 1\rangle^j$ be the self-dual Hermitian $\mathcal{O}_{F'}$-lattice of rank j with Hermitian form given the identity matrix $\mathbf{1}_j$. Let L be a Hermitian $\mathcal{O}_{F'}$-lattice of rank n.*

(1) *We have*

$$\mathrm{Den}(\langle 1\rangle^{n+j}, \langle 1\rangle^n) = \prod_{i=1}^{n}(1-(\eta(\varpi)q)^{-i}T)\Big|_{T=(\eta(\varpi)q)^{-j}}.$$

(2) *There is a (unique) polynomial* $\mathrm{Den}(T,L)\in\mathbb{Z}[T]$, *called (normalized)* local Siegel series *of L, such that for all $j\geq 0$,*

$$\mathrm{Den}((\eta(\varpi)q)^{-j}, L) = \frac{\mathrm{Den}(\langle 1\rangle^{n+j}, L)}{\mathrm{Den}(\langle 1\rangle^{n+j}, \langle 1\rangle^n)}.$$

(3) *We have*

$$\mathrm{Den}(T, L) = \sum_{L \subset L' \subset L'^{\vee} \subset L^{\vee}} T^{2\ell'(L'/L)} \mathfrak{m}(t'(L'); T). \tag{2.7}$$

Here the sum is over $\mathcal{O}_{F'}$-lattices L' (in the F'-Hermitian space spanned by L) containing L on which the Hermitian form is integral.

Proof The non-split case is proved in [18, Thm. 3.5.1] and here we indicate the necessary change in the split case. Now suppose $F' = F \times F$ and hence $k' = k \times k$. Let $L_k = L \otimes_{\mathcal{O}_F} k$ and $\langle 1 \rangle_k^m = \langle 1 \rangle^m \otimes_{\mathcal{O}_F} k$, which are free k'-modules with the induced k'/k-Hermitian forms. In particular, $\langle 1 \rangle_k^m$ is non-degenerate and the radical of $L_k = L \otimes_{\mathcal{O}_F} k$ has k'-rank equal to $t'(L) = t(L_1^{\vee}/L_1) = t(L_2^{\vee}/L_2)$. Let $\mathrm{Isom}_{\langle 1 \rangle_k^m, L_k}$ be the k-scheme of "isometric embeddings" from L_k to $\langle 1 \rangle_k^m$, i.e., *injective* k'-linear maps from L_k to $\langle 1 \rangle_k^m$ preserving the Hermitian forms.

Similar to the orthogonal case [3, §3.3], we have

$$\mathrm{Den}(\langle 1 \rangle^m, L) = q^{-\dim \mathrm{Rep}(\langle 1 \rangle^m, L)_F} \sum_{L \subset L' \subset L'^{\vee}} \#(L'/L)^{-(m-n)} \# \mathrm{Isom}_{\langle 1 \rangle_k^m, L_k}(k),$$

where $\dim \mathrm{Rep}(\langle 1 \rangle^m, L)_F = m^2 - (m-n)^2 = 2mn - n^2$.

It remains to show that

$$\# \mathrm{Isom}_{\langle 1 \rangle_k^m, L_k}(k) = q^{m^2-(m-n)^2} \cdot \prod_{i=0}^{n+a-1} (1 - q^{i-m}) \tag{2.8}$$

where $a = t'(L)$ is the k'-rank of the radical of L_k. Note that up-to-isomorphism, L_k is determined by its rank and the rank of its radical. Let $U_{n-a,a}$ be a k'/k-Hermitian space of rank n with radical of rank a. Let $V_m = U_{m,0}$ be a (non-degenerate) k'/k-Hermitian space of dimension $m \geq n$. Then it is easy to see that $U_{n-a,a} \simeq U_{n-a,0} \oplus U_{0,a}$ and

$$\# \mathrm{Isom}_{V_m, U_{n-a,a}}(k) = \# \mathrm{Isom}_{V_m, U_{n-a,0}}(k) \cdot \# \mathrm{Isom}_{V_{m-(n-a)}, U_{0,a}}(k). \tag{2.9}$$

By (2.9) (note that $\# \mathrm{Isom}_{\langle 1 \rangle_k^m, L_k}(k) = \# \mathrm{Isom}_{V_m, U_{n-a,a}}(k)$), it suffices to show (2.8) in the two extreme cases: $a = 0$ and $a = n$.

First we consider the case $a = n$. Then, to give an isometric embedding from $U = U_{0,n} = k'^n$ to $V = U_{m,0} = k'^m$ is equivalent to give an injective k-linear map $\phi : k^n \to k^m$ and then an injective k-linear map $\varphi : k^n \to \mathrm{Im}(\phi)^{\perp} \subset k^m$. Therefore, denoting by $\mathrm{Hom}_k^*(k^n, k^m)$ the set of injective k-linear maps $\phi : k^n \to k^m$, we have

$$\begin{aligned} \# \mathrm{Isom}_{V_m, U_{0,n}}(k) =& \# \mathrm{Hom}_k^*(k^n, k^m) \cdot \# \mathrm{Hom}_k^*(k^n, k^{m-n}) \\ =& q^{mn} \prod_{i=0}^{n-1} (1 - q^{i-m}) \cdot q^{(m-n)n} \prod_{i=0}^{n-1} (1 - q^{i-m+n}) \end{aligned}$$

$$=q^{2mn-n^2}\prod_{i=0}^{2n-1}(1-q^{i-m}).$$

It remains to consider the case $a=0$. Then a similar argument shows

$$\begin{aligned}\#\mathrm{Isom}_{V_m,U_{n,0}}(k)=&\#\mathrm{Hom}_k^*(k^n,k^m)\cdot\#\mathrm{Hom}_k(k^n,k^{m-n})\\ =&q^{mn}\prod_{i=0}^{n-1}(1-q^{i-m})\cdot q^{(m-n)n}\\ =&q^{2mn-n^2}\prod_{i=0}^{n-1}(1-q^{i-m}).\end{aligned}$$

This completes the proof. □

Remark 2.4 By Theorem 2.3, the polynomial $\mathrm{Den}(T,L)$ depends only on the induced Hermitian form on the torsion module $L^\vee/L$. Indeed, for a Hermitian torsion module[2] Q we define $\mathrm{Den}(T,Q)$ by the formula

$$\mathrm{Den}(T,Q)=\sum_{Q'\subset(Q')^\vee\subset Q}T^{2\ell'(Q')}\mathfrak{m}(t'(Q');T).$$

Then by (2.7), we have $\mathrm{Den}(T,L)=\mathrm{Den}(T,L^\vee/L)$.

Remark 2.5 In the split case, write $L=(L_1,L_2)$ and $L'=(L_1',L_2')$. Then the formula reads

$$\mathrm{Den}(T,L)=\sum_{L_1\subset L_1'\subset L_2'^\vee\subset L_2^\vee}T^{\ell(L_1'/L_1)+\ell(L_2'/L_2)}\mathfrak{m}(t({L_2'}^\vee/L_1');T).$$

Remark 2.6 The local Siegel series satisfies a functional equation

$$\mathrm{Den}(T,L)=(\eta(\varpi)T)^{\ell'(L^\vee/L)}\cdot\mathrm{Den}\left(\frac{1}{T},L\right).$$

A proof in the inert case can be found in [8, Theorem 5.3]. By Theorem, 2.3 the constant term of $\mathrm{Den}(T,L)$ is 1. It follows that the degree of the polynomial $\mathrm{Den}(T,L)$ is equal to $\ell'(L^\vee/L)$. We will not use this fact in this paper. See Corollary 11.14 for the (global) geometric analog.

2.5 Relation with local Whittaker functions

We continue to let F be a local field. Define the local L-function

$$\mathscr{L}_{n,F'/F}(s):=\prod_{i=1}^{n}L(i+2s,\eta^i)=\prod_{i=1}^{n}\frac{1}{1-\eta^i(\varpi)q^{-i-2s}}.$$

[2] By this we mean a torsion $\mathcal{O}_{F'}$-module with an $\mathcal{O}_{F'}/\mathcal{O}_F$-Hermitian form.

Lemma 2.7 *Let L be a Hermitian $\mathcal{O}_{F'}$-lattice of rank n. Let $T = ((x_i, x_j))_{1\le i,j\le n}$ be the fundamental matrix of an $\mathcal{O}_{F'}$-basis $\{x_1, \dots, x_n\}$ of L, an $n \times n$ Hermitian matrix over F. Let θ be a generator of $\omega_{\mathcal{O}_F}$ so that $T\theta \in \mathrm{Herm}_n(F, \omega_F)$. Then*

$$W_{T\theta}(1, s, \Phi) = \mathscr{L}_{n,F'/F}(s)^{-1}\,\mathrm{Den}(q^{-2s}, L).$$

Here Φ is the local unramified section normalized by $\Phi(1_{2n}, s) = 1$.

Proof Note that by Theorem 2.3

$$\mathscr{L}_{n,F'/F}(j) = \mathrm{Den}(\langle 1\rangle^{n+2j}, \langle 1\rangle^n)^{-1}.$$

It is known that $W_{T\theta}(1, s, \Phi)$ is a rational function in q^s. Therefore the formula is equivalent to

$$W_{T\theta}(1, j, \Phi) = \mathrm{Den}(\langle 1\rangle^{n+2j}, L).$$

for all integer $j \ge 0$. In the non-split case this is essentially [13, Prop. 10.1] (cf. [18, §3.3]), which can be easily modified to the split case. We note that $W_{T\theta}$ is the same as W_T in *loc. cit.*. □

2.6 Fourier coefficients revisited

Now we return to the global situation. We need the following global L-function to normalize the Eisenstein series

$$\mathscr{L}_n(s) = \prod_{i=1}^{n} L(i + 2s, \eta^i).$$

We now consider the restriction of the regular part $E^{\mathrm{reg}}(\cdot, s, \Phi)$ (as a function in $g \in H_n(\mathbb{A})$, cf. (2.4)) to the Levi subgroup $M_n(\mathbb{A})$. Since the restriction is left $M_n(F)$-invariant and right K-invariant, it descends to a function on

$$M_n(F)\backslash M_n(\mathbb{A})/M_n(\widehat{\mathcal{O}}) \simeq \mathrm{Bun}_{M_n}(k) \simeq \mathrm{Bun}_{\mathrm{GL}'_n}(k),$$

via the canonical identifications. From now on we will freely switch between $g = m(\alpha) \in M_n(\mathbb{A})$ and the corresponding element $\mathcal{E} \in \mathrm{Bun}_{\mathrm{GL}'_n}(k)$ and we will write

$$E^{\mathrm{reg}}(m(\mathcal{E}), s, \Phi) = E^{\mathrm{reg}}(m(\alpha), s, \Phi).$$

Note that the absolute value on $\mathbb{A}_{F'}^{\times}$ is normalized such that $|\det(\alpha)|_{F'} = q^{\deg(\mathcal{E})}$. By abuse of notation we also view χ as a function on $\mathrm{Bun}_{\mathrm{GL}'_1}(k)$.

Recall that $\mathcal{E}^\vee = \underline{\mathrm{Hom}}_{\mathcal{O}_{X'}}(\mathcal{E}, \omega_{X'})$ denotes the Serre dual of $\mathcal{E}$. Consider a rational Hermitian map $a : \mathcal{E} \dashrightarrow \sigma^*\mathcal{E}^\vee$ (i.e., a is a map defined over the generic point of X', such that $\sigma^* a = a$). Given a pair $(\mathcal{E}, a)$ as above, we shall define the Fourier coefficient

$$E_a(m(\mathcal{E}), s, \Phi)$$

as follows. For any generic trivialization $\tau : \mathcal{E}_{F'} \xrightarrow{\sim} (F')^n$, the pair $(\mathcal{E}, \tau)$ gives a point $\alpha = \alpha(\mathcal{E}, \tau) \in M_n(\mathbb{A})/M_n(\widehat{\mathcal{O}})$ such that $\mathcal{E}$ is glued from $(F')^n$ and the lattices $\alpha_v \mathcal{O}_{F'_v}^n$. Under τ, the restriction of a at the generic point gives an ω_F-valued Hermitian form on $(F')^n$ which we denote by $T = T(a, \tau)$. Then we define

$$E_a(m(\mathcal{E}), s, \Phi) := E_{T(a,\tau)}(m(\alpha(\mathcal{E}, \tau)), s, \Phi). \tag{2.10}$$

If we change τ to $\gamma\tau$ for some $\gamma \in M_n(F) = \mathrm{GL}_n(F')$, then $\alpha(\mathcal{E}, \gamma\tau) = \gamma\alpha(\mathcal{E}, \tau)$ and $T(a, \gamma\tau) = {}^t\overline{\gamma}^{-1} T(a, \tau)\gamma^{-1}$. By (2.1), we have

$$E_{T(a,\gamma\tau)}(m(\alpha(\mathcal{E}, \gamma\tau)), s, \Phi) = E_{T(a,\tau)}(m(\alpha(\mathcal{E}, \tau)), s, \Phi)$$

for all $\gamma \in M_n(F)$. Therefore $E_a(m(\mathcal{E}), s, \Phi)$ is well-defined.

Now suppose $a : \mathcal{E} \hookrightarrow \sigma^*\mathcal{E}^\vee$ is an injective Hermitian map. Let $(\mathcal{E}_v, a)$ denote the Hermitian $\mathcal{O}'_v$-lattice (valued in $\omega_{\mathcal{O}_{F_v}} := \omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_{F_v}$) induced by a at $v \in |X|$. Choosing a generator of the free $\mathcal{O}_{F_v}$-module $\omega_{\mathcal{O}_{F_v}}$ of rank one, we obtain a Hermitian lattice $\mathcal{E}_v$ (valued in $\mathcal{O}_{F_v}$) and hence the density polynomial $\mathrm{Den}(T, \mathcal{E}_v)$ defined by (2.7) relative to F'_v/F_v. We define the density polynomial for $(\mathcal{E}_v, a)$ as

$$\mathrm{Den}_v(T, (\mathcal{E}_v, a)) := \mathrm{Den}(T, \mathcal{E}_v). \tag{2.11}$$

It is easy to see that the result is independent of the choice of the generator of $\omega_{\mathcal{O}_{F_v}}$. We then define

$$\mathrm{Den}(q^{-2s}, (\mathcal{E}, a)) = \prod_{v \in |X|} \mathrm{Den}_v(q_v^{-2s}, (\mathcal{E}_v, a)),$$

Note that the degree of $\mathrm{Den}(q^{-2s}, (\mathcal{E}, a))$ (as a polynomial of q^{-s}) is

$$\deg(\sigma^*\mathcal{E}^\vee) - \deg(\mathcal{E}) = -2\deg(\mathcal{E}) + 2n \deg \omega_X.$$

Theorem 2.8 *Let $\mathcal{E}$ be a vector bundle over X' of rank n. Then*

$$E^{\mathrm{reg}}(m(\mathcal{E}), s, \Phi) = \sum_{a : \mathcal{E} \hookrightarrow \sigma^*\mathcal{E}^\vee} E_a(m(\mathcal{E}), s, \Phi) \tag{2.12}$$

where the sum runs over all injective Hermitian maps $a : \mathcal{E} \to \sigma^*\mathcal{E}^\vee$. Moreover, we have

$$\begin{aligned} &E_a(m(\mathcal{E}), s, \Phi) \\ &\quad = \chi(\det(\mathcal{E}))q^{-\deg(\mathcal{E})(s-n/2)-\frac{1}{2}n^2 \deg \omega_X} \mathscr{L}_n(s)^{-1} \mathrm{Den}(q^{-2s}, (\mathcal{E}, a)). \end{aligned} \tag{2.13}$$

Proof From the definitions it is clear that

$$E^{\mathrm{reg}}(m(\mathcal{E}), s, \Phi) = \sum_{a : \mathcal{E} \dashrightarrow \sigma^*\mathcal{E}^\vee} E_a(m(\mathcal{E}), s, \Phi)$$

where a runs over *rational* Hermitian maps $\mathcal{E} \dashrightarrow \sigma^*\mathcal{E}^\vee$ that are generically nonsingular.

Now let $a : \mathcal{E} \dashrightarrow \sigma^*\mathcal{E}^\vee$ be such a rational nonsingular Hermitian map. We continue with the convention defining (2.10)

$$E_a(m(\mathcal{E}), s, \Phi) = E_T(m(\alpha), s, \Phi). \tag{2.14}$$

By (2.2) and (2.3), and noting that the character χ is trivial on the norm of $\mathbb{A}_{F'}^\times$, we have

$$E_T(m(\alpha), s, \Phi) = \chi(\det(\alpha))|\alpha|_{F'}^{-s+n/2}|\omega_X|^{-\frac{1}{2}n^2} \prod_{v\in|X|} W_{T_v}(1, s, \Phi_v). \tag{2.15}$$

If the (ω_{F_v}-valued) Hermitian form T_v does not have integral entries, then $W_{T_v}(1, s, \Phi_v) = 0$ (since Φ is invariant under $N_n(\mathcal{O}_v)$). Therefore $E_T(m(\alpha), s, \Phi)$ is nonzero only when T_v is integral for all v, i.e., a is an everywhere regular Hermitian map $\mathcal{E} \hookrightarrow \sigma^*\mathcal{E}^\vee$. This proves (2.12).

For such $a : \mathcal{E} \hookrightarrow \sigma^*\mathcal{E}^\vee$, by Lemma 2.7, the right side of (2.15) is

$$\chi(\det(\alpha))|\det(\alpha)|_{F'}^{-s+n/2}|\omega_X|^{-\frac{1}{2}n^2} \prod_{v\in|X|} \frac{1}{\mathscr{L}_{n,F'_v/F_v}(s)} \mathrm{Den}(q_v^{-2s}, \mathcal{E}_v), \tag{2.16}$$

where, by choosing a generator of $\omega_{\mathcal{O}_{F_v}}$, the $\mathcal{O}_{F'_v}$-module $\mathcal{E}_v$ is endowed with the Hermitian form (valued in $\mathcal{O}_{F_v}$) induced by the ω_X-valued Hermitian form a.

By $\chi(\det(\alpha)) = \chi(\det(\mathcal{E}))$, $|\det(\alpha)|_{F'} = q^{\deg(\mathcal{E})}$, and (2.11), we obtain

$$\mathrm{Den}(q^{-2s}, (\mathcal{E}, a)) = \prod_{v\in|X|} \mathrm{Den}(q_v^{-2s}, (\mathcal{E}_v, a_v)) = \prod_{v\in|X|} \mathrm{Den}(q_v^{-2s}, \mathcal{E}_v).$$

Combining these facts with (2.14), (2.15) and (2.16), we get (2.13). □

3 Springer theory for torsion coherent sheaves

In this section we review the construction of the Springer sheaf on the moduli stack of torsion coherent sheaves on a curve following Laumon [15]. We also compute the Frobenius trace function of a particular summand of the Springer sheaf called the Steinberg sheaf.

In this section let X be any smooth (not necessarily projective or connected) curve over $k = \mathbf{F}_q$. For $d \in \mathbf{N}$, let X_d be the d^{th} symmetric power of X.

3.1 Local geometry of Coh_d

Let $\mathrm{Coh}_d = \mathrm{Coh}_d(X)$ be the moduli stack of torsion coherent sheaves on X of length d. For any k-scheme S, $\mathrm{Coh}_d(S)$ is the groupoid of coherent sheaves on $X \times S$ whose pushforward to S is locally free of rank d.

Let $s_d^{\mathrm{Coh}} : \mathrm{Coh}_d \to X_d$ be the support map. Recall that for any k-scheme S, $[\mathfrak{gl}_d/\mathrm{GL}_d](S)$ is the groupoid of (V, T) where V is a vector bundle of rank d on S and T is an endomorphism of V. When $X = \mathbf{A}^1$, we have a canonical isomorphism

$$\mathrm{Coh}_d(\mathbf{A}^1) \cong [\mathfrak{gl}_d/\mathrm{GL}_d]$$

given as follows. For $\mathcal{Q} \in \mathrm{Coh}_d(\mathbf{A}^1)(S)$, $\Gamma(\mathbf{A}^1_S, \mathcal{Q})$ is a locally free rank d $\mathcal{O}_S$-module equipped with an endomorphism given by the affine coordinate t for $\mathbf{A}^1$, giving an S-point of $[\mathfrak{gl}_d/\mathrm{GL}_d]$; conversely, given an object $(V, T) \in [\mathfrak{gl}_d/\mathrm{GL}_d]$ we may view V as an $\mathcal{O}_S[t]$-module $\mathcal{Q}$ (viewed as a coherent sheaf on $\mathbf{A}^1_S$) with t acting as T.

Let $U \subset X_{\overline{k}}$ be open and $f : U \to \mathbf{A}^1_{\overline{k}}$ be an étale map. Such a pair (U, f) is called an *étale chart* for $X_{\overline{k}}$. It induces a map $f_d^{\mathrm{Coh}} : \mathrm{Coh}_d(U) \to \mathrm{Coh}_d(\mathbf{A}^1_{\overline{k}})$ sending $\mathcal{Q}$ to $f_*\mathcal{Q}$ which is compatible with the symmetric power $f_d : U_d \to (\mathbf{A}^1_{\overline{k}})_d$ under s_d^{Coh}. Let $\mathfrak{D}_{d,\mathbf{A}^1} \subset (\mathbf{A}^1_{\overline{k}})_d$ and $\mathfrak{D}_{d,U} \subset U_d$ be the discriminant divisors, i.e., they parametrize divisors with multiplicities. Clearly $\mathfrak{D}_{d,U} \subset f_d^{-1}(\mathfrak{D}_{d,\mathbf{A}^1})$, therefore we may write $f_d^{-1}(\mathfrak{D}_{d,\mathbf{A}^1}) = \mathfrak{D}_{d,U} + \mathfrak{R}_{d,f}$ as Cartier divisors on U_d.

Lemma 3.1 *Let D be an effective divisor of degree d on U. Then $D \in U_d \backslash \mathfrak{R}_{d,f}$ if and only if for all pairs of distinct points x, y in the support of D, $f(x) \neq f(y)$.*

Proof Let $\pi_d : U^d \to U_d$ be the quotient map by the symmetric group S_d. We compute the divisor $\pi_d^{-1}(\mathfrak{R}_{d,f})$. Consider $U \times_{\mathbf{A}^1_{\overline{k}}} U$. Since U is étale over $\mathbf{A}^1_{\overline{k}}$, $\Delta(U) \subset U \times_{\mathbf{A}^1} U$ is open and closed, hence we can write $U \times_{\mathbf{A}^1_{\overline{k}}} U = \Delta(U) \coprod \mathfrak{R}$. For geometric points $x, y \in U$, $(x, y) \in \mathfrak{R}$ if and only if $x \neq y$ and $f(x) = f(y)$.

For $1 \leq i < j \leq d$, let $p_{ij} : U^d \to U \times U$ be the projection to the ith and jth coordinates. Let

$$\widetilde{\Delta}_{ij} = p_{ij}^{-1}(U \times_{\mathbf{A}^1} U) = \Delta_{ij} \coprod \mathfrak{R}_{ij}, \text{ where } \Delta_{ij} = p_{ij}^{-1}(\Delta(U)), \mathfrak{R}_{ij} = p_{ij}^{-1}(\mathfrak{R}).$$

By definition, $\pi_d^{-1}(\mathfrak{D}_{d,\mathbf{A}^1}) = \sum_{1 \leq i < j \leq d} \widetilde{\Delta}_{ij}$, $\pi_d^{-1}(\mathfrak{D}_{d,U}) = \sum_{1 \leq i < j \leq d} \Delta_{ij}$ as divisors on U^d. Therefore

$$\pi_d^{-1}(\mathfrak{R}_{d,f}) = \sum_{1 \leq i < j \leq d} \mathfrak{R}_{ij}. \tag{3.1}$$

From this we see, if $D = x_1 + x_2 + \cdots + x_d$, where $x_i \in U(\overline{k})$, then $D \notin \mathfrak{R}_{d,f}$ if and only if $(x_1, \ldots, x_d) \notin \pi_d^{-1}(\mathfrak{R}_{d,f})$. By (3.1), the latter happens if and only if for all $1 \leq i < j \leq d$, $(x_i, x_j) \notin \mathfrak{R}$, i.e., either $x_i = x_j$ or $f(x_i) \neq f(x_j)$. □

Let $\mathrm{Coh}_d(U)^f \subset \mathrm{Coh}_d(U)$ be the preimage of $U_d \backslash \mathfrak{R}_{d,f}$. Then $\mathrm{Coh}_d(U)^f$ is an open substack of $\mathrm{Coh}_d(X)_{\overline{k}} = \mathrm{Coh}_d(X_{\overline{k}})$.

The following lemma shows that $\mathrm{Coh}_d(X)$ is étale locally isomorphic to $\mathrm{Coh}_d(\mathbf{A}^1) \cong [\mathfrak{gl}_d/\mathrm{GL}_d]$.

Lemma 3.2 (1) *For any étale chart (U, f) of $X_{\overline{k}}$, the map $f_d^{\mathrm{Coh}} : \mathrm{Coh}_d(U) \to \mathrm{Coh}_d(\mathbf{A}^1)_{\overline{k}}$ is étale when restricted to $\mathrm{Coh}_d(U)^f$.*

(2) *The stack $\mathrm{Coh}_d(X)_{\overline{k}}$ is covered by the substacks $\mathrm{Coh}_d(U)^f$ for various étale charts (U, f) of $X_{\overline{k}}$.*

Proof (1) For any $\mathcal{Q} \in \mathrm{Coh}_d(U)^f(\overline{k})$, the tangent map of f_d^{Coh} at $\mathcal{Q}$ is $\mathrm{Ext}^*_U(\mathcal{Q},\mathcal{Q}) \to \mathrm{Ext}^*_{\mathbf{A}^1}(f_*\mathcal{Q}, f_*\mathcal{Q})$. By Lemma 3.1, different points in the support of $\mathcal{Q}$ map to different points in $\mathbf{A}^1$, the above map is the direct sum of $\tau_z : \mathrm{Ext}^*_{\mathcal{O}_{U,z}}(\mathcal{Q}_z,\mathcal{Q}_z) \to \mathrm{Ext}^*_{\mathcal{O}_{\mathbf{A}^1,f(z)}}(\mathcal{Q}_z,\mathcal{Q}_z)$ over $z \in \mathrm{supp}(\mathcal{Q})$. Since f is étale at each such z, τ_z are isomorphisms, and hence f_d^{Coh} is étale at $\mathcal{Q}$ by the Jacobian criterion.

(2) For every point $\mathcal{Q} \in \mathrm{Coh}_d(X)(\overline{k})$ we will construct an étale chart (U, f) such that $\mathcal{Q} \in \mathrm{Coh}_d(U)^f(\overline{k})$. Let $Z \subset X(\overline{k})$ be the support of $\mathcal{Q}$. For $z \in Z$, let $\mathcal{O}_z$ be the completed local ring of $X_{\overline{k}}$ at z with a uniformizer ϖ_z. The map of sheaves $r : \mathcal{O}_{X_{\overline{k}}} \to \oplus_{x\in Z}\mathcal{O}_z/\varpi_z^2$ is surjective. Let $c : Z \to \overline{k}$ be any injective map of sets. Then there exists an open neighborhood U_1 of Z and $f \in \mathcal{O}(U_1)$ such that $r(f) = (c(z) + \varpi_z)_{z\in Z}$. Viewing f as a map $f : U_1 \to \mathbf{A}^1_{\overline{k}}$, it is then étale at Z, hence étale in an open neighborhood $U \subset U_1$ of Z, i.e., (U, f) is an étale chart. Since $\{f(z) = c(z)\}_{z\in Z}$ are distinct points in $\mathbf{A}^1_{\overline{k}}$, we see that $\mathcal{Q} \in \mathrm{Coh}_d(U)^f_{\overline{k}}$ by Lemma 3.1. □

3.2 Springer theory for Coh_d

Let $\widetilde{\mathrm{Coh}}_d(X)$ be the moduli stack classifying a full flag of torsion sheaves on X

$$0 \subset \mathcal{Q}_1 \subset \mathcal{Q}_2 \subset \cdots \subset \mathcal{Q}_d = \mathcal{Q}$$

where $\mathcal{Q}_j$ has length j. Let

$$\pi^{\mathrm{Coh}}_{d,X} : \widetilde{\mathrm{Coh}}_d(X) \to \mathrm{Coh}_d(X)$$

be the forgetful map recording only $\mathcal{Q} = \mathcal{Q}_d$.

Lemma 3.3 (Laumon [15, Theorem 3.3.1]) *The stacks $\widetilde{\mathrm{Coh}}_d(X)$ and $\mathrm{Coh}_d(X)$ are smooth of dimension zero, and the map $\pi^{\mathrm{Coh}}_{d,X}$ is proper and small.*

Proof It is enough to check the same statements after base change to $\overline{k}$. We give a quick alternative proof using Lemma 3.2: for an étale chart (U, f) (over $\overline{k}$), we have a diagram in which both squares are Cartesian:

$$\begin{array}{ccccc}
\widetilde{\mathrm{Coh}}_d(X)_{\overline{k}} & \hookleftarrow & \widetilde{\mathrm{Coh}}_d(U)^f & \longrightarrow & \widetilde{\mathrm{Coh}}_d(\mathbf{A}^1)_{\overline{k}} \\
\downarrow{\scriptstyle \pi^{\mathrm{Coh}}_{d,X}} & & \downarrow{\scriptstyle \pi^{\mathrm{Coh}}_{d,U}} & & \downarrow{\scriptstyle \pi^{\mathrm{Coh}}_{d,\mathbf{A}^1}} \\
\mathrm{Coh}_d(X)_{\overline{k}} & \hookleftarrow & \mathrm{Coh}_d(U)^f & \xrightarrow{f_d^{\mathrm{Coh}}} & \mathrm{Coh}_d(\mathbf{A}^1)_{\overline{k}}
\end{array}$$

Here $\widetilde{\mathrm{Coh}}_d(U)^f$ is the preimage of $\mathrm{Coh}_d(U)^f$ in $\widetilde{\mathrm{Coh}}_d(U)$. Since the horizontal maps are étale and the $\mathrm{Coh}_d(U)^f$ cover $\mathrm{Coh}_d(X)_{\overline{k}}$ by Lemma 3.2, the desired properties of $\pi^{\mathrm{Coh}}_{d,X}$ follow from the same properties of $\pi^{\mathrm{Coh}}_{d,\mathbf{A}^1}$, which is the Grothendieck alteration $\pi_{\mathfrak{gl}_d} : [\widetilde{\mathfrak{gl}}_d/\mathrm{GL}_d] \to [\mathfrak{gl}_d/\mathrm{GL}_d]$. □

Let $X_d^\circ \subset X_d$ be the open subset of multiplicity-free divisors (i.e., the complement of $\mathfrak{D}_{d,X}$), and let $\mathrm{Coh}_d(X)^\circ$ (resp. $\widetilde{\mathrm{Coh}}_d(X)^\circ$) be its preimage under s_d^{Coh} (resp. under $s_d^{\mathrm{Coh}} \circ \pi_d^{\mathrm{Coh}}$). Then $\widetilde{\mathrm{Coh}}_d(X)^\circ \to \mathrm{Coh}_d(X)^\circ$ is an S_d-torsor.

Corollary 3.4 (Laumon [15, p.320]) *The complex*

$$\mathrm{Spr}_d := R\pi_{d,X*}^{\mathrm{Coh}} \overline{\mathbf{Q}}_\ell \in D^b(\mathrm{Coh}_d(X), \overline{\mathbf{Q}}_\ell)$$

is a perverse sheaf on $\mathrm{Coh}_d(X)$ *that is the middle extension from its restriction to* $\mathrm{Coh}_d(X)^\circ$. *In particular, the natural* S_d*-action on* $\mathrm{Spr}_d|_{\mathrm{Coh}_d(X)^\circ}$ *extends to the whole* Spr_d.

3.3 Springer fibers

Let $\mathcal{Q} \in \mathrm{Coh}_d(X)(\overline{k})$ with image D in $X_d(\overline{k})$, an effective divisor of degree d. Let $Z = (\mathrm{supp}\, D)(\overline{k})$. Let $\Sigma(Z)$ be the set of maps $y : \{1, 2, \ldots, d\} \to Z$ such that $\sum_{i=1}^d y(i) = D$. Let $\mathcal{B}_\mathcal{Q}$ be the fiber of π_d^{Coh} over $\mathcal{Q}$. Then $\mathcal{B}_\mathcal{Q}$ classifies complete flags of subsheaves $0 \subset \mathcal{Q}_1 \subset \mathcal{Q}_2 \subset \cdots \subset \mathcal{Q}_{d-1} \subset \mathcal{Q}$. We write $\mathrm{H}^*(-) := \mathrm{H}^*(-; \mathbf{Q}_\ell)$ for ℓ-adic cohomology (regarded as a graded $\mathbf{Q}_\ell$-vector space). By Corollary 3.4, $\mathrm{H}^*(\mathcal{B}_\mathcal{Q}) = (\mathrm{Spr}_d)_\mathcal{Q}$ carries an action of S_d.

For $y \in \Sigma(Z)$, let $\mathcal{B}_\mathcal{Q}(y)$ be the open and closed subscheme of $\mathcal{B}_\mathcal{Q}$ defined by the condition $\mathrm{supp}\, \mathcal{Q}_i/\mathcal{Q}_{i-1} = y(i)$. Then $\mathcal{B}_\mathcal{Q}$ is the disjoint union of $\mathcal{B}_\mathcal{Q}(y)$ for $y \in \Sigma(Z)$. Hence

$$\mathrm{H}^*(\mathcal{B}_\mathcal{Q}) \cong \bigoplus_{y \in \Sigma(Z)} \mathrm{H}^*(\mathcal{B}_\mathcal{Q}(y)).$$

There is an action of S_d on $\Sigma(Z)$ by precomposing.

Lemma 3.5 *The action of* $w \in S_d$ *on* $\mathrm{H}^*(\mathcal{B}_\mathcal{Q})$ *sends* $\mathrm{H}^*(\mathcal{B}_\mathcal{Q}(y))$ *to* $\mathrm{H}^*(\mathcal{B}_\mathcal{Q}(y \circ w^{-1}))$, *for all* $y \in \Sigma(Z)$.

Proof It suffices to check the statement for each simple reflection s_i switching i and $i+1$ ($1 \le i \le d-1$). Let $\widetilde{\mathrm{Coh}}_d^i(X)$ be the moduli stack classifying chains of torsion coherent sheaves $0 \subset \mathcal{Q}_1 \subset \cdots \subset \mathcal{Q}_{i-1} \subset \mathcal{Q}_{i+1} \subset \cdots \subset \mathcal{Q}_d$ with $\mathcal{Q}_i$ missing. Then we have a factorization

$$\pi_d^{\mathrm{Coh}} : \widetilde{\mathrm{Coh}}_d(X) \xrightarrow{\rho_i} \widetilde{\mathrm{Coh}}_d^i(X) \xrightarrow{\pi_i} \mathrm{Coh}_d(X).$$

The map ρ_i is an étale double cover over the open dense locus $\widetilde{\mathrm{Coh}}_d^{i,\heartsuit}(X)$ where $\mathcal{Q}_{i+1}/\mathcal{Q}_{i-1}$ (which has length 2) is supported at two distinct points. The map ρ_i is small by Lemma 3.3, and $R\rho_{i*}\overline{\mathbf{Q}}_\ell$ carries an involution $\widetilde{s}_i$, which induces an involution $\widetilde{s}_i$ on $R\pi_{i*}R\rho_{i*}\overline{\mathbf{Q}}_\ell \cong \mathrm{Spr}_d$. This action coincides with the action of s_i over $\mathrm{Coh}_d(X)^\circ$, hence coincides with s_i everywhere.

Let $\mathcal{B}_\mathcal{Q}^i = \pi_i^{-1}(\mathcal{Q})$. By considering the support of the successive quotients, we have a decomposition of $\mathcal{B}_\mathcal{Q}^i$ by the orbit set $\Sigma(Z)/\langle s_i \rangle$. When $y \in \Sigma(Z)$ satisfies $y \neq$

$y \circ s_i$, the s_i-orbit $\eta = \{y, y \circ s_i\}$ gives an open and closed substack $\mathcal{B}^i_{\mathcal{Q}}(\eta) \subset \mathcal{B}^i_{\mathcal{Q}}$, such that $\rho_i^{-1}(\mathcal{B}^i_{\mathcal{Q}}(\eta)) = \mathcal{B}_{\mathcal{Q}}(y) \coprod \mathcal{B}_{\mathcal{Q}}(y \circ s_i)$, and $\mathcal{B}^i_{\mathcal{Q}}(\eta) \subset \widetilde{\mathrm{Coh}}^{i,\heartsuit}_{2d}$. Therefore in this case the action of $\widetilde{s_i}$ on $\mathrm{H}^*(\rho_i^{-1}(\mathcal{B}^i_{\mathcal{Q}}(\eta)))$ comes from the involution on $\mathcal{B}_{\mathcal{Q}}(y) \coprod \mathcal{B}_{\mathcal{Q}}(y \circ s_i)$ that interchanges the two components. Since $\widetilde{s_i} = s_i$, this proves the statement for s_i and y such that $y \neq y \circ s_i$. For $y = y \circ s_i$ the statement is vacuous. This finishes the proof. □

Let $\mathcal{Q}_x$ be the direct summand of $\mathcal{Q}$ supported at $x \in Z$. Let $d_x = \dim_{\overline{k}} \mathcal{Q}_x$. Then for any $y \in \Sigma(Z)$, there is a canonical isomorphism over $\overline{k}$

$$\beta_y : \mathcal{B}_{\mathcal{Q}}(y) \cong \prod_{x \in Z} \mathcal{B}_{\mathcal{Q}_x} \tag{3.2}$$

sending $(\mathcal{Q}_i) \in \mathcal{B}_{\mathcal{Q}}(y)$ to the full flag of $\mathcal{Q}_x$ given by taking the summands of $\mathcal{Q}_i$ supported at x.

The proof above implies the following statement that we record for future reference.

Lemma 3.6 *Let $y, y' \in \Sigma(Z)$ and let $w \in S_d$ be such that $y \circ w^{-1} = y'$. Assume that w has minimal length (in terms of the simple reflections $s_1, \ldots, s_{d-1}$) among such elements (such w is unique). Then the Springer action $w : \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}(y)) \to \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}(y'))$ is induced by the composition of the canonical isomorphisms $\beta_{y,y'} := \beta_{y'}^{-1} \circ \beta_y : \mathcal{B}_{\mathcal{Q}}(y) \xrightarrow{\sim} \mathcal{B}_{\mathcal{Q}}(y')$. In particular, w sends the fundamental class of $\mathcal{B}_{\mathcal{Q}}(y)$ to the fundamental class of $\mathcal{B}_{\mathcal{Q}}(y')$.*

Proof Let $w^{-1} = s_{i_1} \cdots s_{i_N}$ be a reduced word for w^{-1}. Let $y_j = y s_{i_1} \cdots s_{i_j}$, $1 \leq j \leq N$. Let $y_0 = y$, and $y' = y_N$. Since w has minimal length among $w' \in S_d$ such that $y \circ w'^{-1} = y'$, for each $1 \leq j \leq N$, $y_{j-1} \neq y_j$ for otherwise one could delete s_{i_j} to shorten w. Since $y_j = y_{j-1} \circ s_{i_j} \neq y_{j-1}$, the proof of Lemma 3.5 shows that the Springer action of $s_{i_j} : \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}(y_{j-1})) \to \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}(y_j))$ is induced by the canonical isomorphism $\sigma_j = \beta_{y_j}^{-1} \circ \beta_{y_{j-1}} : \mathcal{B}_{\mathcal{Q}}(y_{j-1}) \xrightarrow{\sim} \mathcal{B}_{\mathcal{Q}}(y_j)$. The action $w : \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}(y)) \to \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}(y'))$, being the composition $\sigma_N \circ \cdots \circ \sigma_1$, is then equal to $\beta_{y'}^{-1} \circ \beta_y : \mathcal{B}_{\mathcal{Q}}(y) \xrightarrow{\sim} \mathcal{B}_{\mathcal{Q}}(y')$. □

Corollary 3.7 *Let $y \in \Sigma(Z)$ and $S_y \cong \prod_{x \in Z} S_{d_x}$ be the stabilizer of y under S_d. There is an isomorphism of graded S_d-representations*

$$\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}) \cong \mathrm{Ind}_{S_y}^{S_d} \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}(y)) \cong \mathrm{Ind}_{S_y}^{S_d} \left(\bigotimes_{x \in Z} \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}_x}) \right).$$

Here on the right side, each factor S_{d_x} of S_y acts on the tensor factor indexed by x (for $x \in Z$) via the Springer action in Corollary 3.4 on $(\mathrm{Spr}_{d_x})_{\mathcal{Q}_x}$.

Proof By Lemma 3.5, $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}(y \circ w^{-1})) = w\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}(y))$ for $w \in S_d$. In particular, $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}(y))$ is stable under S_y, and $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}) \cong \mathrm{Ind}_{S_y}^{S_d} \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}(y))$. By (3.2) and the Künneth formula, we have $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}(y)) \cong \otimes_{x \in Z} \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}_x})$.

It remains to check that the action of S_y on $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}(y))$ (as the restriction of the S_d-action on $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}})$) is the same as the tensor product of the Springer action of S_{d_x} on $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}_x})$. Since the action of S_d on $\Sigma(Z)$ is transitive, it suffices to check this statement for a particular $y \in \Sigma(Z)$.

Order points in Z as $x_1, \ldots, x_r$. Let $y_0 \in \Sigma(Z)$ be the unique increasing function, i.e. such that if $i < j$ then the index of $y_0(i)$ is less than or equal to the index of $y_0(j)$. Let $d_i = d_{x_i}$. Let $\delta = (\delta_i)_{1 \le i \le r}$ be the increasing sequence $\delta_i = d_1 + \cdots + d_i$. Let $\mathrm{Coh}_\delta(X)$ be the moduli stack of partial chains of torsion coherent sheaves $0 \subset \mathcal{Q}_{\delta_1} \subset \cdots \subset \mathcal{Q}_{\delta_{r-1}} \subset \mathcal{Q}_{\delta_r} = \mathcal{Q}$ such that $\mathcal{Q}_{\delta_i}$ has length δ_i. The map π_d^{Coh} then factorizes as

$$\widetilde{\mathrm{Coh}}_d(X) \xrightarrow{\pi_\delta} \mathrm{Coh}_\delta(X) \xrightarrow{\nu_\delta} \mathrm{Coh}_d(X).$$

We have a Cartesian diagram

$$\begin{array}{ccc} \widetilde{\mathrm{Coh}}_d(X) & \xrightarrow{\widetilde{c}} & \prod_{i=1}^r \widetilde{\mathrm{Coh}}_{d_i} \\ \downarrow{\scriptstyle \pi_\delta} & & \downarrow{\scriptstyle \prod \pi_{d_i}^{\mathrm{Coh}}} \\ \mathrm{Coh}_\delta(X) & \xrightarrow{c} & \prod_{i=1}^r \mathrm{Coh}_{d_i}(X) \end{array} \tag{3.3}$$

where c sends $(\mathcal{Q}_{\delta_i})$ to $(\mathcal{Q}_{\delta_i}/\mathcal{Q}_{\delta_{i-1}})$. By proper base change we have $R\pi_{\delta *}\overline{\mathbf{Q}}_\ell \cong c^*(\boxtimes_{i=1}^r \mathrm{Spr}_{d_i})$, and the latter carries the Springer action of $S_{d_1} \times \cdots \times S_{d_r} = S_{y_0}$ (pulled back along c). Pushing forward along ν_δ, this induces an action of S_{y_0} on $R\nu_{\delta *} R\pi_{\delta *}\overline{\mathbf{Q}}_\ell = \mathrm{Spr}_d$. This action coincides with the restriction of the action of S_d because both actions come from deck transformations over $\mathrm{Coh}_d(X)^\circ$.

Now $\nu_\delta^{-1}(\mathcal{Q})$ contains the point $\mathcal{Q}^\dagger \in \mathrm{Coh}_\delta(X)$ where $\mathrm{supp}\, \mathcal{Q}_{\delta_i}/\mathcal{Q}_{\delta_{i-1}} = \{x_i\}$ for $1 \le i \le r$. This is an isolated point in $\nu_\delta^{-1}(\mathcal{Q})$, and $\mathcal{B}_{\mathcal{Q}}(y_0) = \pi_\delta^{-1}(\mathcal{Q}^\dagger)$. Moreover, the isomorphism (3.2) is the one given by taking the Cartesian diagram (3.3) and restricting to $\mathcal{Q}^\dagger \in \mathrm{Coh}_\delta(X)$. The above discussion shows that the action of $S_{y_0} \subset S_d$ on $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}(y_0)) \subset \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}})$ is the same as the Springer action of $\prod_i S_{d_i}$ on $\otimes_i \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}_{x_i}})$ via the isomorphism (3.2). □

3.4 The Steinberg sheaf

Let $\mathrm{St}_d \in D^b(\mathrm{Coh}_d(X), \overline{\mathbf{Q}}_\ell)$ be the direct summand of Spr_d where S_d acts through the sign representation. We will describe its Frobenius trace function below. The result is well-known but we include a self-contained proof.

We call $\mathcal{Q} \in \mathrm{Coh}_d(X)(\overline{k})$ *semisimple* if it is a direct sum of skyscraper sheaves at closed points.

Proposition 3.8 (1) *If $\mathcal{Q} \in \mathrm{Coh}_d(X)(\overline{k})$ is not semisimple, then the stalk of St_d at $\mathcal{Q}$ is zero.*

(2) *Let $\mathcal{Q} = \oplus_{v \in |X|} k_v^{\oplus d_v} \in \mathrm{Coh}_d(X)(k)$ be semisimple. Then the stalk of St_d at $\mathcal{Q}$ is 1-dimensional, and the geometric Frobenius* Frob *acts on the stalk $\mathrm{St}_{d,\mathcal{Q}}$ by the*

scalar

$$\varepsilon(\mathcal{Q}) \prod_{v \in \operatorname{supp} \mathcal{Q}} q_v^{d_v(d_v-1)/2}$$

where $\varepsilon(\mathcal{Q}) \in \{\pm 1\}$ is the sign of Frobenius permuting the geometric points in the support of $\mathcal{Q}$ counted with multiplicities (as a multi-set of cardinality d).

Proof Let $\mathcal{Q} \in \mathrm{Coh}_d(X)(\overline{k})$. Let $Z \subset X(\overline{k})$ be the geometric points in the support of $\mathcal{Q}$ and $y \in \Sigma(Z)$. By Corollary 3.7 and Frobenius reciprocity,

$$\begin{aligned} \mathrm{St}_{d,\mathcal{Q}} &\cong \mathrm{Hom}_{S_d}(\mathrm{sgn}, \mathrm{Ind}_{S_y}^{S_d}(\otimes \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}_x}))) \cong \mathrm{Hom}_{S_y}(\mathrm{sgn}, \otimes \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}_x})) \\ &\cong \otimes_{x \in Z} \mathrm{Hom}_{S_{d_x}}(\mathrm{sgn}, \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}_x})) \cong \otimes_{x \in Z} \mathrm{St}_{d_x,\mathcal{Q}_x}. \end{aligned} \tag{3.4}$$

(1) By the above factorization of $\mathrm{St}_{d,\mathcal{Q}}$, it suffices to show that if $\mathcal{Q}_x$ is not semisimple, then $\mathrm{St}_{d_x,\mathcal{Q}_x} = 0$. By Lemma 3.2 we may reduce to the case $X = \mathbf{A}^1$ and $\mathcal{Q}$ is concentrated at $x = 0$. In this case Spr is the usual Springer sheaf on $[\mathfrak{gl}_d/\mathrm{GL}_d]$, and $\mathcal{Q}$ corresponds to a nilpotent element $e \in \mathcal{N}_d \subset \mathfrak{gl}_d$ (here $\mathcal{N}_d$ is the nilpotent cone in $\mathfrak{gl}_d$). It is well-known that $\mathrm{St}_d|_{\mathcal{N}_d} \cong \delta_0[-d(d-1)]$ where δ_0 is the skyscraper sheaf at $0 \in \mathcal{N}_d$. Indeed, by [1, §3.4, Corollary (b)], for any nonzero nilpotent element $e \in \mathcal{N}_d$, the sign representation of S_d does not appear in $\mathrm{H}^*(\mathcal{B}_e)$ ($\mathcal{B}_e$ is the Springer fiber for e). For $e = 0$, the sign representation of S_d only appears in the top degree $\mathrm{H}^{d(d-1)}(\mathcal{B}_e)$, which is one-dimensional. This implies that $\mathrm{St}_d|_{\mathcal{N}} \cong \delta_0[-d(d-1)]$. In particular, $\mathrm{St}_{d,e} = 0$ for all nilpotent $e \neq 0$.

(2) Let $\mathcal{Q} \in \mathrm{Coh}_d(X)(k)$ be semisimple. Let $|Z|$ be the set of closed points in the support of $\mathcal{Q}$. The above discussion shows that $\mathrm{St}_{d_x,\mathcal{Q}_x} \cong \mathrm{H}^{top}(\mathcal{B}_{\mathcal{Q}_x}) = \mathrm{H}^{d_x(d_x-1)}(\mathrm{Fl}_{d_x})$ where Fl_{d_x} is the flag variety for GL_{d_x}. By (3.4), $\mathrm{St}_{d,\mathcal{Q}}$ is 1-dimensional and is in the top degree cohomology of $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}})$. Let

$$N = \dim \mathcal{B}_{\mathcal{Q}} = \sum_{x \in Z} d_x(d_x-1)/2 = \sum_{v \in |Z|} \deg(v) d_v(d_v-1)/2$$

(here $d_v = d_x$ for any $x|v$). Let $0 \neq \xi \in \mathrm{St}_{d,\mathcal{Q}} \subset \oplus_{y \in \Sigma(Z)} \mathrm{H}^{2N}(\mathcal{B}_{\mathcal{Q}}(y))$. Let $\mathrm{Fr} : \mathcal{B}_{\mathcal{Q}} \to \mathcal{B}_{\mathcal{Q}}$ be the Frobenius morphism. We need to show that $\mathrm{Fr}^* \xi = \varepsilon(\mathcal{Q}) q^N \xi$.

For $y \in \Sigma(Z)$, let $\eta_y \in \mathrm{H}^{2N}(\mathcal{B}_{\mathcal{Q}}(y))$ be the fundamental class of $\mathcal{B}_{\mathcal{Q}}(y)$. Then Fr sends $\mathcal{B}_{\mathcal{Q}}(y)$ onto $\mathcal{B}_{\mathcal{Q}}(\mathrm{Fr}(y))$ (here $\mathrm{Fr}(y)$ means post-composing y with the Frobenius permutation on Z), and hence $\mathrm{Fr}^* \eta_{\mathrm{Fr}(y)} = q^N \eta_y$. On the other hand, let $w \in S_d$ be the minimal length element such that $\mathrm{Fr}(y) = y \circ w^{-1}$. By Lemma 3.6, the Springer action of w satisfies $w\eta_y = \eta_{\mathrm{Fr}(y)}$. Write $\xi = (\xi_y)_{y \in \Sigma(Z)}$ where $\xi_y = c_y \eta_y$ for some $c_y \in \overline{\mathbf{Q}}_\ell^\times$. Since $w\xi = \mathrm{sgn}(w)\xi$, we see that $w\xi_y = \mathrm{sgn}(w)\xi_{\mathrm{Fr}(y)}$. Since $w\eta_y = \eta_{\mathrm{Fr}(y)}$, we have $c_y = \mathrm{sgn}(w) c_{\mathrm{Fr}(y)}$. Therefore

$$\begin{aligned} (\mathrm{Fr}^* \xi)_y &= \mathrm{Fr}^*(\xi_{\mathrm{Fr}(y)}) = c_{\mathrm{Fr}(y)} \mathrm{Fr}^* \eta_{\mathrm{Fr}(y)} = q^N c_{\mathrm{Fr}(y)} \eta_y = \mathrm{sgn}(w) q^N c_y \eta_y \\ &= \mathrm{sgn}(w) q^N \xi_y. \end{aligned}$$

Note that, for any choice of y and w above, $\mathrm{sgn}(w)$ is equal to the sign of the Frobenius permutation of the multiset $\{y(i)\}_{1\le i\le d}$, which is $\varepsilon(\mathcal{Q})$. This implies $\mathrm{Fr}^*\xi = \varepsilon(\mathcal{Q})q^N\xi$ as desired. □

4 Springer theory for Hermitian torsion sheaves

In this section we extend the construction in §3 to the case of Hermitian torsion sheaves. The main output is a perverse sheaf $\mathrm{Spr}_{2d}^{\mathrm{Herm}}$ on the moduli stack of Hermitian torsion sheaves with an action of $W_d := (\mathbf{Z}/2\mathbf{Z})^d \rtimes S_d$. We will compare the stalks and Frobenius trace functions of $\mathrm{Spr}_{2d}^{\mathrm{Herm}}$ with those of Spr_d.

As in §3, X is a smooth curve over k (not necessarily projective or connected). Recall from §1.3 that $\nu : X' \to X$ is a finite map of degree 2 that is assumed to be *generically étale* (and X' is smooth over k). We develop the Hermitian Springer theory in this generality. Starting from §4.3 we will assume ν to be étale, which is the case needed for proving the main theorem. Let $\sigma \in \mathrm{Gal}(X'/X)$ be the nontrivial involution.

4.1 Local geometry of Herm_d

We say that a map of torsion coherent sheaves $a : \mathcal{Q} \to \sigma^*\mathcal{Q}^\vee$ is *Hermitian* if $\sigma^*a^\vee = a$. Let $d \in \mathbf{N}$. Let

$$\mathrm{Herm}_d(X'/X), \text{ or simply } \mathrm{Herm}_d$$

be the moduli stack of pairs $(\mathcal{Q}, h)$ where $\mathcal{Q}$ is a torsion coherent sheaf on X' of length d, and h is a Hermitian isomorphism $\mathcal{Q} \xrightarrow{\sim} \sigma^*\mathcal{Q}^\vee := \sigma^*\underline{\mathrm{Ext}}^1(\mathcal{Q}, \omega_{X'})$.

We offer two other ways to think about a Hermitian torsion sheaf $(\mathcal{Q}, h)$. For a torsion sheaf $\mathcal{Q}$ on X' of length d, the datum of a Hermitian isomorphism $h : \mathcal{Q} \xrightarrow{\sim} \sigma^*\mathcal{Q}^\vee$ is equivalent to either:

(1) a symmetric k-bilinear nondegenerate pairing

$$(\cdot,\cdot) : V \times V \to k$$

on $V = \Gamma(X', \mathcal{Q})$ satisfying $(fv_1, v_2) = (v_1, \sigma^*(f)v_2)$ for any function f on X' regular near the support of $\mathcal{Q}$, or

(2) an $\mathcal{O}_{X'}$-sesquilinear nondegenerate pairing

$$\langle\cdot,\cdot\rangle : \mathcal{Q} \times \mathcal{Q} \to \omega_{F'}/\omega_{X'}$$

satisfying $\langle v_1, v_2\rangle = \sigma^*\langle v_2, v_1\rangle$. Here $\omega_{F'}$ is the constant (and quasi-coherent) sheaf on X' whose local sections are the rational 1-forms on X'.

For example, to pass from (2) to (1), form cohomology and apply the trace map $\mathrm{H}^0(\mathcal{Q}) \to k$. To pass from h to the pairing in (1), observe that $\underline{\mathrm{Ext}}^1(\mathcal{Q}, \omega_{X'}) \cong \underline{\mathrm{Hom}}(\mathcal{Q}, \omega_{F'}/\omega_{X'})$ by the long exact sequence associated to $\omega_{X'} \to \omega_{F'} \to \omega_{F'}/\omega_{X'}$. Therefore, h is equivalent to a sesquilinear pairing $\mathcal{Q} \times \mathcal{Q} \to \omega_{F'}/\omega_{X'}$, which upon

taking global sections and applying the residue map $\mathrm{H}^0(\omega_{F'}/\omega_{X'}) \to k$ gives the pairing in (1).

We refer to h, or any of the above equivalent data, as a *Hermitian structure* on $\mathcal{Q}$.

We have the support map

$$s_d^{\mathrm{Herm}} : \mathrm{Herm}_d(X'/X) \to (X'_d)^\sigma.$$

Note that we have an isomorphism $(X'_{2d})^\sigma \cong X_d$, sending a σ-invariant divisor on X' to its descent on X. so we will also allow ourselves to view the support map as $s_{2d}^{\mathrm{Herm}} : \mathrm{Herm}_{2d}(X'/X) \to X_d$.

Remark 4.1 When ν is étale and d is odd, $(X'_d)^\sigma = \varnothing$ hence $\mathrm{Herm}_d(X'/X) = \varnothing$.

In general, when ν is ramified over the points $R \subset X(\overline{k})$, $(X'_d)^\sigma$ has a decomposition into open and closed subschemes according to the parity of the multiplicities of the divisor at each point $x \in R$.

Let $\mathbf{A}^1_{\sqrt{t}} \to \mathbf{A}^1_t$ be the square map of affine lines.

Lemma 4.2 *There is a canonical isomorphism*

$$\mathrm{Herm}_d(\mathbf{A}^1_{\sqrt{t}}/\mathbf{A}^1_t) \cong [\mathfrak{o}_d/\mathrm{O}_d].$$

Here O_d *denotes the orthogonal group on a* d*-dimensional nondegenerate quadratic space over* k *and* $\mathfrak{o}_d$ *is its Lie algebra* (*the stack* $[\mathfrak{o}_d/\mathrm{O}_d]$ *is independent of the quadratic form*).

Proof We give the map $\mathrm{Herm}_d(\mathbf{A}^1_{\sqrt{t}}/\mathbf{A}^1_t) \to [\mathfrak{o}_d/\mathrm{O}_d]$ on S-points. For an S-point $(\mathcal{Q}, h)$ of $\mathrm{Herm}_d(\mathbf{A}^1_{\sqrt{t}}/\mathbf{A}^1_t)$, $V = \Gamma(\mathbf{A}^1_S, \mathcal{Q})$ is a locally free $\mathcal{O}_S$-module of rank d with a nondegenerate symmetric self-duality $(\cdot,\cdot)$, i.e., an O_d-torsor over S. Moreover the action of $\sqrt{t}$ on V satisfies $(\sqrt{t}v_1, v_2) = -(v_1, \sqrt{t}v_2)$ since $\sigma^*\sqrt{t} = -\sqrt{t}$. Therefore $\sqrt{t}$ gives a section of the adjoint bundle of V. It is easy to check this map is an equivalence of groupoids $\mathrm{Herm}_d(\mathbf{A}^1_{\sqrt{t}}/\mathbf{A}^1_t)(S) \xrightarrow{\sim} [\mathfrak{o}_d/\mathrm{O}_d](S)$. □

An *σ-equivariant étale chart* of $X'_{\overline{k}}$ is a pair (U, f), where $U \subset X_{\overline{k}}$ is an open subset (with preimage $U' \subset X'_{\overline{k}}$) and a regular function $f : U' \to \mathbf{A}^1_{\sqrt{t},\overline{k}}$ that is an étale map satisfying $\sigma^* f = -f$. Note that if ν is étale, the image of f has to lie in $\mathbf{A}^1_{\sqrt{t},\overline{k}} \backslash \{0\}$.

A σ-equivariant étale chart (U, f) of $X'_{\overline{k}}$ induces a map

$$f_d^{\mathrm{Herm}} : \mathrm{Herm}_d(U'/U) \to \mathrm{Herm}_d(\mathbf{A}^1_{\sqrt{t}}/\mathbf{A}^1_t)_{\overline{k}}$$

by sending $\mathcal{Q}$ to $f_*\mathcal{Q}$. Let $\mathrm{Herm}_d(U'/U)^f$ be the preimages of $(U'_d)^\sigma \backslash \mathfrak{R}^\sigma_{d,f}$ under the support maps (here $\mathfrak{R}_{d,f} \subset U'_d$ is defined using the map $f : U' \to \mathbf{A}^1_{\sqrt{t},\overline{k}}$; see §3.1).

We have an analog of Lemma 3.2 in the Hermitian setting.

Lemma 4.3 (1) *Let (U, f) be a σ-equivariant étale chart for $X'_{\overline{k}}$. Then the map f_d^{Herm} is étale when restricted to $\mathrm{Herm}_d(U'/U)^f$.*

(2) *Assume ν is ramified at most one point (over $\overline{k}$). Then the stack $\mathrm{Herm}_d(X'/X)_{\overline{k}}$ is covered by $\mathrm{Herm}_d(U'/U)^f$ for various σ-equivariant étale charts (U, f) of $X'_{\overline{k}}$. In particular, $\mathrm{Herm}_d(X'/X)$ is étale locally isomorphic to $[\mathfrak{o}_d/\mathrm{O}_d]$.*

(3) *In general, $\mathrm{Herm}_d(X'/X)$ is smooth of dimension 0.*

Proof (1) is similar to that of Lemma 3.2(1).

(2) We only need to construct for $(\mathcal{Q}, h) \in \mathrm{Herm}_d(X'/X)(\overline{k})$, a σ-equivariant étale chart (U, f) such that $(\mathcal{Q}, h) \in \mathrm{Herm}_d(U'/U)^f$. Let Z' be its support in X'; since the Hermitian property of $(\mathcal{Q}, h)$ implies that Z' is stable under σ, hence is the preimage of some $Z \subset X(\overline{k})$ under ν. Let $\mathcal{L} = (\nu_* \mathcal{O}_{X'_{\overline{k}}})^{\sigma=-1}$, a line bundle over X. Then the map $r = (r_z)_{z\in Z} : \mathcal{L} \to \oplus_{z\in Z} \mathcal{L}_z/\varpi_z^2$ is surjective. Let $Z_0 \subset Z$ be the points over which ν is étale (so $Z - Z_0$ is empty or has one point). For each $z \in Z_0$, upon choosing $z' \in Z'$ over z, we may identify $\mathcal{L}_z$ with $\mathcal{O}_z = \overline{k}[[\varpi_z]]$; changing z' to $\sigma(z')$ changes the identification by a sign. If $z \in Z - Z_0$, then $\mathcal{L}_z \cong \sqrt{\varpi_z}\overline{k}[[\varpi_z]]$. Choose a map $c : Z_0 \to \overline{k}^\times$ such that $c(z)^2$ are distinct for $z \in Z$. Let f be a section of $\mathcal{L}$ over some open neighborhood $U_1 \subset X_{\overline{k}}$ of Z such that $r_z(f) = c(z) + \varpi_z$ for $z \in Z_0$ under one of the two identifications $\mathcal{L}_z \cong \mathcal{O}_z$, and $r_z(f) \equiv \sqrt{\varpi_z} \mod \varpi_z$ for $z \in Z - Z_0$. Then f restricts to an étale map $U' = \nu^{-1}(U) \to \mathbf{A}^1_{\sqrt{t},\overline{k}}$ for some open neighborhood U of Z in U_1. The definition of $\mathcal{L}$ implies $\sigma^*(f) = -f$. Now $\{f(z')|z' \in Z'\}$ is the union of $\{c(z), -c(z)|z \in Z_0\}$ and possibly $\{0\}$ if $Z - Z_0$ is nonempty, which are all distinct points in $\mathbf{A}^1_{\sqrt{t},\overline{k}}$ by construction. We conclude that $(\mathcal{Q}, h) \in \mathrm{Herm}_d(U'/U)^f$.

(3) Let $R \subset X_{\overline{k}}$ be the ramification locus of ν. The case $|R| \le 1$ is treated in (2), so we may assume $|R| \ge 2$. For $x \in R$, let $Y_x = X\backslash(R\backslash\{x\})$ and let $Y'_x = \nu^{-1}(Y_x)$. For any function $\delta : R \to \mathbf{Z}_{\ge 0}$ such that $\sum_{x\in R} \delta(x) = d$ we have a map $\mathfrak{Y}_\delta := \prod_{x\in R} (Y'_{x,d(x)})^\sigma \to (X'_d)^\sigma$ by adding divisors. Let $\mathfrak{Y}_\delta^\heartsuit \subset \mathfrak{Y}_\delta$ be the open locus where the divisors indexed by different $x \in R$ are disjoint. It is clear that $\mathfrak{Y}_\delta^\heartsuit \to (X'_d)^\sigma$ is étale and for varying δ their images cover $(X'_d)^\sigma$. To prove the statement it suffices to show that the base change $\mathrm{Herm}_d(X'/X)|_{\mathfrak{Y}_\delta^\heartsuit}$ is smooth of dimension 0 for each δ. Observe that $\mathrm{Herm}_d(X'/X)|_{\mathfrak{Y}_\delta^\heartsuit}$ is isomorphic to the restriction of the product $\prod_{x\in R} \mathrm{Herm}_{\delta(x)}(Y'_x/Y_x)$ to $\mathfrak{Y}_\delta^\heartsuit$. Since $\nu|_{Y_x} : Y'_x \to Y_x$ is ramified at one point, by (2) $\mathrm{Herm}_{\delta(x)}(Y'_x/Y_x)$ is smooth of dimension 0. Therefore $\mathrm{Herm}_d(X'/X)|_{\mathfrak{Y}_\delta^\heartsuit} \cong \prod_{x\in R} \mathrm{Herm}_{\delta(x)}(Y'_x/Y_x)|_{\mathfrak{Y}_\delta^\heartsuit}$ is smooth of dimension 0. □

Remark 4.4 There is an obvious notion of skew-Hermitian torsion sheaves. Let $\mathrm{SkHm}_d(X'/X)$ be the moduli stack of skew-Hermitian torsion sheaves on (X', σ) of length d. Then d is even if $\mathrm{SkHm}_d(X'/X) \neq \varnothing$. The skew-Hermitian analog of Lemma 4.3 says that $\mathrm{SkHm}_d(X'/X)$ is étale locally isomorphic to $[\mathfrak{sp}_d/\mathrm{Sp}_d]$, at least when ν is étale.

4.2 The Hermitian Springer sheaf

Let $\widetilde{\mathrm{Herm}}_d(X'/X)$ be the moduli stack classifying $(\mathcal{Q}, h) \in \mathrm{Herm}_d(X'/X)$ together with a full flag

$$0 \subset \mathcal{Q}_1 \subset \cdots \subset \mathcal{Q}_i \subset \cdots \subset \mathcal{Q}_{d-1} = \mathcal{Q}_1^{\perp} \subset \mathcal{Q}_d = \mathcal{Q},$$

where $\mathcal{Q}_i$ has length i and $\mathcal{Q}_{d-i} = \mathcal{Q}_i^{\perp}$ (the orthogonal of $\mathcal{Q}_i$ under the Hermitian pairing $\mathcal{Q} \times \mathcal{Q} \to \omega_{F'}/\omega_{X'}$). Let

$$\pi_d^{\mathrm{Herm}} : \widetilde{\mathrm{Herm}}_d(X'/X) \to \mathrm{Herm}_d(X'/X)$$

be the forgetful map. Let $\mathrm{Herm}_d(X'/X)^{\circ} \subset \mathrm{Herm}_d(X'/X)$ be the preimage of the multiplicity-free part $X_d'^{\circ}$ under the support map s_d^{Herm}.

We recall the Grothendieck alteration for the full orthogonal group $\mathrm{O}(V, Q)$ for some vector space V of dimension d over k and a nondegenerate quadratic form Q on V. Let $\mathrm{Fl}(V, Q)$ be the flag variety that parametrizes full isotropic flags $V_\bullet = (V_1 \subset \cdots \subset V_d = V)$ in V. Note that when d is even, this is different from the flag variety of $\mathrm{SO}(V, Q)$ but rather a double cover of it because there are two choices for $V_{d/2}$ given the rest of members of a flag. Let $\mathfrak{o}(V, Q)$ be the Lie algebra of $\mathrm{O}(V, Q)$. Let $\widetilde{\mathfrak{o}}(V, Q)$ be the moduli space of pairs $(A, V_\bullet) \in \mathfrak{o}(V, Q) \times \mathrm{Fl}(V, Q)$ such that $AV_i \subset V_i$ for all i. The Grothendieck alteration for $\mathrm{O}(V, Q)$ is the $\mathrm{O}(V, Q)$-equivariant map $\widetilde{\mathfrak{o}}(V, Q) \to \mathfrak{o}(V, Q)$ forgetting the flag. The quotient stacks $[\mathfrak{o}(V, Q)/\mathrm{O}(V, Q)]$ and $[\widetilde{\mathfrak{o}}(V, Q)/\mathrm{O}(V, Q)]$ are canonical; they are independent of the quadratic form Q and only depend on $d = \dim V$. Therefore we also write the Grothendieck alteration as $\pi_{\mathrm{O}_d} : [\widetilde{\mathfrak{o}}_d/\mathrm{O}_d] \to [\mathfrak{o}_d/\mathrm{O}_d]$.

Proposition 4.5 (1) *If v is ramified at most one point, then the map π_d^{Herm} is étale locally isomorphic to the Grothendieck alteration $\pi_{\mathrm{O}_d} : [\widetilde{\mathfrak{o}}_d/\mathrm{O}_d] \to [\mathfrak{o}_d/\mathrm{O}_d]$.*

(2) *In general, $\widetilde{\mathrm{Herm}}_d(X'/X)$ is smooth of dimension 0 and π_d^{Herm} is a small map. In particular, the complex*

$$\mathrm{Spr}_d^{\mathrm{Herm}} := R\pi_{d,*}^{\mathrm{Herm}} \overline{\mathbf{Q}}_\ell$$

is the middle extension perverse sheaf of its restriction to $\mathrm{Herm}_d(X'/X)^{\circ}$.

Proof (1) The proof is similar to that of Corollary 3.4. For a σ-equivariant étale chart (U, f) for $X'_{\overline{k}}$ we have a diagram with Cartesian squares and étale horizontal maps by Lemma 4.3

$$\begin{array}{ccccc}
\widetilde{\mathrm{Herm}}_d(X'/X)_{\overline{k}} & \longleftarrow & \widetilde{\mathrm{Herm}}_d(U'/U)^f & \longrightarrow & \widetilde{\mathrm{Herm}}_d(\mathbf{A}^1_{\sqrt{t}}/\mathbf{A}^1_t)_{\overline{k}} \\
\downarrow{\scriptstyle \pi^{\mathrm{Herm}}_{d,X'/X}} & & \downarrow{\scriptstyle \pi^{\mathrm{Herm}}_{d,U'/U}} & & \downarrow{\scriptstyle \pi^{\mathrm{Herm}}_{d,\mathbf{A}^1_{\sqrt{t}}/\mathbf{A}^1_t}} \\
\mathrm{Herm}_d(X'/X)_{\overline{k}} & \longleftarrow & \mathrm{Herm}_d(U'/U)^f & \xrightarrow{f_d^{\mathrm{Herm}}} & \mathrm{Herm}_d(\mathbf{A}^1_{\sqrt{t}}/\mathbf{A}^1_t)_{\overline{k}}
\end{array}$$

Using the isomorphism in Lemma 4.2, we identify $\pi^{\mathrm{Herm}}_{d,\mathbf{A}^1_{\sqrt{t}}/\mathbf{A}^1_t}$ with the Grothendieck alteration π_{O_d}. Since $\mathrm{Herm}_d(U'/U)^f$ cover $\mathrm{Herm}_d(X'/X)$ by Lemma 4.3(2), $\pi^{\mathrm{Herm}}_{d,X'/X}$ is étale locally isomorphic to $\pi^{\mathrm{Herm}}_{d,\mathbf{A}^1_{\sqrt{t}}/\mathbf{A}^1_t} = \pi_{\mathrm{O}_d}$.

(2) We use the notation from the proof of Lemma 4.3(3). We may assume $|R| \geq 2$. For each function $\delta : R \to \mathbf{Z}_{\geq 0}$ satisfying $\sum_{x\in R}\delta(x) = d$, the base change of π^{Herm}_d along $\mathfrak{Y}^{\heartsuit}_\delta \to (X'_d)^\sigma$ is a disjoint union of the restriction of

$$\prod_x \pi^{\mathrm{Herm}}_{\delta(x)} : \prod_{x\in R} \widetilde{\mathrm{Herm}}_{\delta(x)}(Y'_x/Y_x) \to \prod_{x\in R} \mathrm{Herm}_{\delta(x)}(Y'_x/Y_x)$$

to $\mathfrak{Y}^{\heartsuit}_\delta$. The disjoint union comes from different ways to distribute $\mathrm{supp}(\mathcal{Q}_i/\mathcal{Q}_{i-1})$ among various factors in the product $\prod_x (Y'_{x,\delta(x)})^\sigma$. By (1), $\pi^{\mathrm{Herm}}_{\delta(x)} : \widetilde{\mathrm{Herm}}_{\delta(x)}(Y'_x/Y_x) \to \mathrm{Herm}_{\delta(x)}(Y'_x/Y_x)$ has smooth 0-dimensional source and is small for each $x \in R$, the same holds true for the base change of π^{Herm}_d to $\mathfrak{Y}^{\heartsuit}_\delta$. Since $\{\mathfrak{Y}^{\heartsuit}_\delta\}_\delta$ form an étale covering of $(X'_d)^\sigma$, the same is true for π^{Herm}_d. □

4.3 The action of W_d

From now on we assume that $\nu : X' \to X$ is an étale double cover. In this case, $(X'_{2d})^\sigma$ can be identified with X_d via $\nu^{-1}(D) \leftrightarrow D$. Let

$$W_d := (\mathbf{Z}/2\mathbf{Z})^d \rtimes S_d$$

be the Weyl group for O_{2d}. Then π^{Herm}_{2d} is a W_d-torsor over $\mathrm{Herm}_{2d}(X'/X)^\circ$.

Corollary 4.6 (of Proposition 4.5(1)) *If $\nu : X' \to X$ is an étale double cover, then there is a canonical action of W_d on $\mathrm{Spr}^{\mathrm{Herm}}_{2d}$ extending the geometric action on its restriction to $\mathrm{Herm}_{2d}(X'/X)^\circ$.*

Definition 4.7 (1) For any representation ρ of W_d, we define $\mathrm{Spr}^{\mathrm{Herm}}_{2d}[\rho]$ to be the perverse sheaf on $\mathrm{Herm}_{2d}(X'/X)$:

$$\mathrm{Spr}^{\mathrm{Herm}}_{2d}[\rho] = (\rho^\vee \otimes \mathrm{Spr}^{\mathrm{Herm}}_{2d})^{W_d} \in D^b(\mathrm{Herm}_{2d}(X'/X), \overline{\mathbf{Q}}_\ell).$$

(2) We define the Hermitian analog of the Springer sheaf Spr as

$$\mathrm{HSpr}_d := (\mathrm{Spr}^{\mathrm{Herm}}_{2d})^{(\mathbf{Z}/2\mathbf{Z})^d} \in D^b(\mathrm{Herm}_{2d}(X'/X), \overline{\mathbf{Q}}_\ell).$$

Note that the notation shifts from the subscript $2d$ to d. By Corollary 4.6, HSpr_d carries a canonical S_d-action.

Remark 4.8 In the case ν is ramified with ramification locus $R \subset X(\overline{k})$, the stack $\mathrm{Herm}_{2d}(X'/X)_{\overline{k}}$ decomposes into the disjoint union of open and closed substacks $\mathrm{Herm}^\varepsilon_{2d}(X'/X)_{\overline{k}}$ indexed by $\varepsilon : R \to \{0,1\}$ where the length of $\mathcal{Q}_x$ has parity $\varepsilon(x)$ for all $x \in R$. Then $\mathrm{Spr}^{\mathrm{Herm}}_{2d}|_{\mathrm{Herm}^\varepsilon_{2d}(X'/X)_{\overline{k}}}$ carries a canonical action of $W_{d'}$ where $d' = (d - \sum_{x\in R}\varepsilon(x))/2$.

4.4 The Springer fibers over Herm_{2d}

Let $(\mathcal{Q}, h) \in \mathrm{Herm}_{2d}(X'/X)(\overline{k})$ and consider its Hermitian Springer fiber

$$\mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}} := \pi_{2d}^{\mathrm{Herm},-1}(\mathcal{Q}, h).$$

This is a proper scheme over $\overline{k}$. In this subsection we prove the Hermitian analogs of results in §3.3.

Let $Z' = \mathrm{supp}\, \mathcal{Q} \subset X'(\overline{k})$. Let $D' = s_{2d}^{\mathrm{Herm}}(\mathcal{Q}) \in (X'_{2d})^{\sigma}(\overline{k})$, which is of the form $D' = \nu^{-1}(D)$ for some $D \in X_d(\overline{k})$. Let $Z = \nu(Z')$, the support of D. Write $D' = \sum_{z \in Z'} d_z z$.

Let $\Sigma(Z')$ be the set of maps $y' : \{1, 2, \ldots, 2d\} \to Z'$ satisfying $y'(2d + 1 - i) = \sigma(y'(i))$ for all i and $\sum_{i=1}^{2d} y'(i) = D'$. Identifying W_d with permutations of $\{1, 2, \ldots, 2d\}$ commuting with the involution $i \mapsto 2d + 1 - i$, we get an action of W_d on $\Sigma(Z')$ by $w : y' \mapsto y' \circ w^{-1}$.

Similarly let $\Sigma(Z)$ be the set of maps $y : \{1, \ldots, d\} \to Z$ such that $\sum_{i=1}^{d} y(i) = D$. Then the natural map $\Sigma(Z') \to \Sigma(Z)$ (sending y' to y defined by $y(i) = \nu(y'(i))$) is a $(\mathbf{Z}/2\mathbf{Z})^d$-torsor.

For $y' \in \Sigma(Z')$, let $\mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}}(y')$ be the subscheme of $\mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}}$ consisting of isotropic flags $\mathcal{Q}_\bullet$ such that $\mathrm{supp}(\mathcal{Q}_i/\mathcal{Q}_{i-1}) = y'(i)$ for all $1 \le i \le 2d$. Then we have a decomposition into open and closed subschemes

$$\mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}} = \coprod_{y' \in \Sigma(Z')} \mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}}(y').$$

Accordingly we get a decomposition of cohomology

$$\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}}) = \bigoplus_{y' \in \Sigma(Z')} \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}}(y')).$$

Lemma 4.9 *The action of $w \in W_d$ on $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}})$ sends the direct summand $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}}(y'))$ to the direct summand $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}}(y' \circ w^{-1}))$.*

Proof It suffices to check the statement for each simple reflection s_i, $i = 1, \ldots, d$. Here, for $1 \le i \le d - 1$, $s_i = (i, i + 1)(2d - i, 2d + 1 - i)$; for $i = d$, $s_d = (d, d + 1)$. For $1 \le i \le d$, let $\widetilde{\mathrm{Herm}}_{2d}^{i}$ be the moduli stack classifying isotropic flags that only misses the terms $\mathcal{Q}_i$ and $\mathcal{Q}_{2d-i}$ (for $i = d$ only misses $\mathcal{Q}_d$). Then we have a factorization

$$\pi_{2d}^{\mathrm{Herm}} : \widetilde{\mathrm{Herm}}_{2d} \xrightarrow{\rho_i} \widetilde{\mathrm{Herm}}_{2d}^{i} \xrightarrow{\pi_i} \mathrm{Herm}_{2d}\,.$$

The map ρ_i is an étale double cover over the open dense locus $\widetilde{\mathrm{Herm}}_{2d}^{i,\heartsuit}$ where $\mathcal{Q}_{i+1}/\mathcal{Q}_{i-1}$ (which has length 2) is supported at two distinct points. The map ρ_i is small, and $R\rho_{i*}\overline{\mathbf{Q}}_\ell$ carries an involution $\widetilde{s}_i$, which induces an involution $\widetilde{s}_i$ on $R\pi_{i*}R\rho_{i*}\overline{\mathbf{Q}}_\ell \cong R\pi_{2d*}^{\mathrm{Herm}}\overline{\mathbf{Q}}_\ell$. This action coincides with the action of s_i over $\mathrm{Herm}_{2d}^{\circ}$, hence coincides with s_i everywhere.

Let $(\mathcal{Q}, h) \in \mathrm{Herm}_{2d}(\overline{k})$, and $\mathcal{B}^i_{\mathcal{Q}} = \pi_i^{-1}(\mathcal{Q}, h)$. We have a decomposition of $\mathcal{B}^i_{\mathcal{Q}}$ by the orbit set $\Sigma(Z')/\langle s_i \rangle$. When $y' \in \Sigma(Z')$ satisfies $y' \neq y' \circ s_i$, the s_i-orbit of $\eta' = \{y', y' \circ s_i\}$ gives an open closed substack $\mathcal{B}^i_{\mathcal{Q}}(\eta') \subset \mathcal{B}^i_{\mathcal{Q}}$, such that $\rho_i^{-1}(\mathcal{B}^i_{\mathcal{Q}}(\eta')) = \mathcal{B}_{\mathcal{Q}}(y') \coprod \mathcal{B}_{\mathcal{Q}}(y' \circ s_i)$, and $\mathcal{B}^i_{\mathcal{Q}}(\eta') \subset \widetilde{\mathrm{Herm}}^{i,\diamondsuit}_{2d}$. Therefore in this case the action of $\widetilde{s}_i$ on $\mathrm{H}^*(\rho_i^{-1}(\mathcal{B}^i_{\mathcal{Q}}(\eta')))$ comes from the involution on $\mathcal{B}_{\mathcal{Q}}(y') \coprod \mathcal{B}_{\mathcal{Q}}(y' \circ s_i)$ that interchanges the two components. Since $\widetilde{s}_i = s_i$, this proves the statement for s_i and y' such that $y' \neq y' \circ s_i$. For $y' = y' \circ s_i$ the statement is vacuous. This finishes the proof. □

Choose $Z^\sharp \subset Z'$ such that $Z^\sharp \coprod \sigma(Z^\sharp) = Z'$. Then for each $x \in Z$ there is a unique $x^\sharp \in Z^\sharp$ above x. For $y' \in \Sigma(Z')$ with image $y \in \Sigma(Z)$, we have an isomorphism

$$\gamma_{Z^\sharp, y'} : \mathcal{B}^{\mathrm{Herm}}_{\mathcal{Q}}(y') \xrightarrow{\sim} \mathcal{B}_{\nu_*(\mathcal{Q}|_{Z^\sharp})}(y) \cong \prod_{x \in Z} \mathcal{B}_{\mathcal{Q}_{x^\sharp}} \tag{4.1}$$

mapping $(\mathcal{Q}_i)_{1 \leq i \leq 2d}$ to the (non-strictly increasing) flag $(\mathcal{Q}_{i,x^\sharp})$ of $\mathcal{Q}_{x^\sharp}$.

If $y', y'' \in \Sigma(Z')$, the composition

$$\gamma_{y',y''} := \gamma^{-1}_{Z^\sharp, y''} \circ \gamma_{Z^\sharp, y'} : \mathcal{B}^{\mathrm{Herm}}_{\mathcal{Q}}(y') \xrightarrow{\sim} \mathcal{B}^{\mathrm{Herm}}_{\mathcal{Q}}(y'')$$

is independent of the choice of $Z^\sharp$.

Lemma 4.10 *Let $y', y'' \in \Sigma(Z')$ and let $w \in W_d$ be a minimal length element such that $y'' = y' \circ w^{-1}$. Then the Springer action $w : \mathrm{H}^*(\mathcal{B}^{\mathrm{Herm}}_{\mathcal{Q}}(y')) \to \mathrm{H}^*(\mathcal{B}^{\mathrm{Herm}}_{\mathcal{Q}}(y''))$ is induced by the isomorphism $\gamma_{y',y''}$.*

Proof Similar to the proof of Lemma 3.6. □

4.5 Comparing stalks of HSpr_d and Spr_d

In this subsection we abbreviate $\mathrm{Herm}_{2d}(X'/X)$ by Herm_{2d}. Consider the stack Lagr_{2d} classifying pairs $(\mathcal{L} \subset \mathcal{Q})$ where $\mathcal{Q} \in \mathrm{Herm}_{2d}$ and $\mathcal{L} \subset \mathcal{Q}$ is a Lagrangian subsheaf, i.e., $\mathcal{L}$ has length d and the composition $\mathcal{L} \hookrightarrow \mathcal{Q} \xrightarrow{h} \sigma^*\mathcal{Q}^\vee \to \sigma^*\mathcal{L}^\vee$ is zero. We have natural maps

$$\mathrm{Herm}_{2d} \xleftarrow{\upsilon_{2d}} \mathrm{Lagr}_{2d} \xrightarrow{\varepsilon'_d} \mathrm{Coh}_d(X') \xrightarrow{\nu_*} \mathrm{Coh}_d(X)$$

where $\upsilon_{2d}(\mathcal{L} \subset \mathcal{Q}) = \mathcal{Q}$ and $\varepsilon'_d(\mathcal{L} \subset \mathcal{Q}) = \mathcal{L}$. Let $\varepsilon_d = \nu_* \circ \varepsilon'_d : \mathrm{Lagr}_{2d} \to \mathrm{Coh}_d(X)$.

Let $(X'_d)^\diamond \subset X'_d$ be the open subscheme parametrizing $D \in X'_d$ such that $D \cap \sigma(D) = \varnothing$. Let $\mathrm{Lagr}^\diamond_{2d} \subset \mathrm{Lagr}_{2d}$ be the preimage of $(X'_d)^\diamond$ under the map $\mathrm{Lagr}_{2d} \xrightarrow{\varepsilon'_d} \mathrm{Coh}_d(X') \xrightarrow{s^{\mathrm{Coh}}_{d,X'}} X'_d$. It is easy to see that ε'_d restricts to an isomorphism

$$\mathrm{Lagr}^\diamond_{2d} \cong \mathrm{Coh}_d(X')^\diamond$$

whose inverse is given by $\mathcal{L} \mapsto (\mathcal{L} \subset \mathcal{Q} = \mathcal{L} \oplus \sigma^*\mathcal{L}^\vee)$.

Let $\upsilon_{2d}^{\diamondsuit}$ and $\varepsilon_d^{\diamondsuit}$ be the restrictions of υ_{2d} and ε_d to $\mathrm{Lagr}_{2d}^{\diamondsuit}$. Thus we view $\mathrm{Lagr}_{2d}^{\diamondsuit} \cong \mathrm{Coh}_d(X')^{\diamondsuit}$ as a correspondence between Herm_{2d} and $\mathrm{Coh}_d(X)$

$$\mathrm{Herm}_{2d} \xleftarrow{\upsilon_{2d}^{\diamondsuit}} \mathrm{Lagr}_{2d}^{\diamondsuit} \xrightarrow{\varepsilon_d^{\diamondsuit}} \mathrm{Coh}_d(X)$$

Note that both $\upsilon_{2d}^{\diamondsuit}$ and $\varepsilon_d^{\diamondsuit}$ are surjective. Now both Herm_{2d} and $\mathrm{Coh}_d(X)$ carry Springer sheaves HSpr_d and Spr_d with S_d-actions. The following proposition says that they become isomorphic after pullback to $\mathrm{Lagr}_{2d}^{\diamondsuit}$.

Proposition 4.11 *There is a canonical S_d-equivariant isomorphism of perverse sheaves on* $\mathrm{Lagr}_{2d}^{\diamondsuit}$

$$\upsilon_{2d}^{\diamondsuit,*}\,\mathrm{HSpr}_d \cong \varepsilon_d^{\diamondsuit,*}\,\mathrm{Spr}_d\,.$$

Proof The map π_{2d}^{Herm} factors as

$$\pi_{2d}^{\mathrm{Herm}} : \widetilde{\mathrm{Herm}}_{2d} \xrightarrow{\lambda_{2d}} \mathrm{Lagr}_{2d} \xrightarrow{\upsilon_{2d}} \mathrm{Herm}_{2d}\,.$$

Let $\lambda_{2d}^{\diamondsuit} : \widetilde{\mathrm{Herm}}_{2d}^{\diamondsuit} \to \mathrm{Lagr}_{2d}^{\diamondsuit}$ be the restriction of λ_{2d} to $\mathrm{Lagr}_{2d}^{\diamondsuit}$. We have a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Herm}_{2d} \times_{X_d} X^d & \longleftarrow & \widetilde{\mathrm{Herm}}_{2d}^{\diamondsuit} & \xrightarrow{\widetilde{\varepsilon}_d'^{\diamondsuit}} & \widetilde{\mathrm{Coh}}_d(X')^{\diamondsuit} & \xrightarrow{\nu_*} & \widetilde{\mathrm{Coh}}_d(X) \\
\downarrow & \swarrow{\scriptstyle \pi_{2d}^{\mathrm{Herm},\diamondsuit}} & \downarrow{\scriptstyle \lambda_{2d}^{\diamondsuit}} & & \downarrow{\scriptstyle \pi_{X',d}^{\mathrm{Coh},\diamondsuit}} & & \downarrow{\scriptstyle \pi_d^{\mathrm{Coh}}} \\
\mathrm{Herm}_{2d} & \xleftarrow{\upsilon_{2d}^{\diamondsuit}} & \mathrm{Lagr}_{2d}^{\diamondsuit} & \xrightarrow{\varepsilon_d'^{\diamondsuit}} & \mathrm{Coh}_d(X')^{\diamondsuit} & \xrightarrow{\nu_*} & \mathrm{Coh}_d(X) \\
 & & & \searrow{\scriptstyle \varepsilon_d^{\diamondsuit}} & & & \\
\end{array} \qquad (4.2)$$

Here $\mathrm{Coh}_d(X')^{\diamondsuit}$ and $\widetilde{\mathrm{Coh}}_d(X')^{\diamondsuit}$ are the preimages of $(X'_d)^{\diamondsuit}$ under the support map. We have:

- The middle square is Cartesian. This is true even before restricting to the $\diamondsuit$ locus.
- Since $\varepsilon_d'^{\diamondsuit}$ is an isomorphism, so is $\widetilde{\varepsilon}_d'^{\diamondsuit}$.
- The rightmost square is Cartesian.

From these properties we get maps

$$\begin{aligned}\alpha : \upsilon_{2d}^{\diamondsuit *}\,\mathrm{HSpr}_d &\to \upsilon_{2d}^{\diamondsuit *}\,\mathrm{Spr}_{2d}^{\mathrm{Herm}} = \upsilon_{2d}^{\diamondsuit *} R\upsilon_{2d*} R\lambda_{2d*}\overline{\mathbf{Q}}_\ell \to \upsilon_{2d}^{\diamondsuit *} R\upsilon_{2d*}^{\diamondsuit} R\lambda_{2d*}^{\diamondsuit}\overline{\mathbf{Q}}_\ell \\ &\to R\lambda_{2d*}^{\diamondsuit}\overline{\mathbf{Q}}_\ell \cong \varepsilon_d^{\diamondsuit *}\pi_{d*}^{\mathrm{Coh}}\overline{\mathbf{Q}}_\ell = \varepsilon_d^{\diamondsuit *}\,\mathrm{Spr}_d\,.\end{aligned}$$

To check α is an isomorphism, it suffices to check on geometric stalks. Let $\mathcal{L} \in \mathrm{Coh}_d(X')^{\diamondsuit}(\overline{k})$ with support $Z^\sharp \subset X'(\overline{k})$. Let $\mathcal{Q} = \mathcal{L} \oplus \sigma^*\mathcal{L}^\vee \in \mathrm{Herm}_{2d}(\overline{k})$. The support of $\mathcal{Q}$ is $Z' = Z^\sharp \coprod \sigma(Z^\sharp)$, with image $Z \subset X(\overline{k})$. We have $(\mathcal{L} \subset \mathcal{Q}) \in \mathrm{Lagr}_{2d}^{\diamondsuit}(\overline{k})$,

with image $v_*\mathcal{L} \in \mathrm{Coh}_d(X)(\overline{k})$. The stalk of α at $(\mathcal{L} \subset \mathcal{Q})$ is

$$\alpha_{(\mathcal{L}\subset\mathcal{Q})} : \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}})^{(\mathbf{Z}/2\mathbf{Z})^d} \to \mathrm{H}^*(\lambda_{2d}^{-1}(\mathcal{L} \subset \mathcal{Q})) \xrightarrow{\sim} \mathrm{H}^*(\mathcal{B}_{v_*\mathcal{L}}) \tag{4.3}$$

Recall $\mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}} = \coprod_{y' \in \Sigma(Z')} \mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}}(y')$. Let $\Sigma(Z^\sharp) \subset \Sigma(Z')$ be the set of y' such that $y'(i) \in Z^\sharp$ for $1 \le i \le d$. Then we have a natural bijection $\Sigma(Z) \leftrightarrow \Sigma(Z^\sharp)$, $y \mapsto y^\sharp$. The fiber $\lambda_{2d}^{-1}(\mathcal{L} \subset \mathcal{Q})$ is the disjoint union $\coprod_{y\in\Sigma(Z)} \mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}}(y^\sharp)$. Recall the isomorphism $\gamma_{Z^\sharp, y^\sharp} : \mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}}(y^\sharp) \xrightarrow{\sim} \mathcal{B}_{v_*\mathcal{L}}(y)$ from (4.1). Using these descriptions we may rewrite (4.3) as

$$\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}})^{(\mathbf{Z}/2\mathbf{Z})^d} \twoheadrightarrow \oplus_{y' \in \Sigma(Z^\sharp)} \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}}(y')) \cong \oplus_{y\in\Sigma(Z)} \mathrm{H}^*(\mathcal{B}_{v_*\mathcal{L}}(y)) = \mathrm{H}^*(\mathcal{B}_{v_*\mathcal{L}}).$$

It remains to show that the first map above is an isomorphism. But this follows from the fact that $(\mathbf{Z}/2\mathbf{Z})^d$ acts freely on $\Sigma(Z')$ with orbit representatives $\Sigma(Z^\sharp)$, and Lemma 4.9. This shows that α is an isomorphism.

Finally we show that α is S_d-equivariant. By Proposition 4.5, π_{2d}^{Herm} and hence λ_{2d} is small, $R\lambda_{2d*}\overline{\mathbf{Q}}_\ell$ is the middle extension from a dense open substack of Lagr_{2d}. Therefore the same is true for $R\lambda_{2d*}^{\diamondsuit}\overline{\mathbf{Q}}_\ell$. Since α is an isomorphism, both $\upsilon_{2d}^{\diamondsuit,*}\mathrm{HSpr}_d$ and $\varepsilon_d^{\diamondsuit *}\mathrm{Spr}_d$ are middle extension perverse sheaves from a dense open substack of $\mathrm{Lagr}_{2d}^{\diamondsuit}$. To check that α is S_d-equivariant it suffices to check it over the dense open substack which is the preimage of the multiplicity-free locus X_d°. Over X_d°, all squares in (4.2) are Cartesian, and all vertical maps are S_d-torsors. The S_d-actions on $\mathrm{HSpr}_d|_{\mathrm{Herm}_{2d}^\circ}$ and $\mathrm{Spr}_d|_{\mathrm{Coh}_d(X)^\circ}$ come from the vertical S_d-torsors in the diagram, so α is S_d-equivariant when restricted over X_d°. This finishes the proof. □

4.6 Comparing Frobenius traces of HSpr_d and Spr_d

In this subsection we will prove a relationship between Frobenius trace functions for HSpr_d and for Spr_d. Since these sheaves live on different stacks, to make sense of the comparison we first need to identify the isomorphism classes of the k-points of these stacks.

For a groupoid $\mathcal{G}$, let $|\mathcal{G}|$ denote its set of isomorphism classes.

Lemma 4.12 *There is a canonical bijection of sets*

$$|\mathrm{Herm}_{2d}(X'/X)(k)| \cong |\mathrm{Coh}_d(X)(k)|$$

respecting the support maps to $X_d(k)$.

Proof Let $\mathcal{P}(d)$ be the set of partitions of $d \in \mathbf{Z}_{\ge 0}$, and $\mathcal{P} = \coprod_{d\ge 0}\mathcal{P}(d)$. Let $\mathcal{P}_{|X|}$ be the set of functions $\lambda : |X| \to \mathcal{P}$ such that $\lambda(v)$ is the zero partition for almost all $v \in |X|$. For $\lambda \in \mathcal{P}_{|X|}$, let $|\lambda| = \sum_v |\lambda(v)| \deg(v)$. Let $\mathcal{P}_{|X|}(d)$ be the subset of those $\lambda \in \mathcal{P}_{|X|}$ with $|\lambda| = d$. Let $s_d : \mathcal{P}_{|X|}(d) \to X_d(k)$ be the map sending λ to the divisor $\sum_v |\lambda(v)| v$.

By taking the Jordan type of a torsion sheaf at each closed point, we get a canonical bijection $\Lambda_d^{\mathrm{Coh}} : |\mathrm{Coh}_d(X)(k)| \xrightarrow{\sim} \mathcal{P}_{|X|}(d)$. The map $s_d^{\mathrm{Coh}} : |\mathrm{Coh}_d(X)(k)| \to X_d(k)$ corresponds to s_d under this bijection.

We define a map $\Lambda_d^{\mathrm{Herm}} : |\mathrm{Herm}_{2d}(X'/X)(k)| \to \mathcal{P}_{|X|}(d)$ as follows. For $(\mathcal{Q}, h) \in \mathrm{Herm}_{2d}(X'/X)(k)$ and $v \in |X|$, let $\lambda(v)$ be the Jordan type of $\mathcal{Q}_{v'}$ (the summand $\mathcal{Q}_{v'}$ supported at v') for any $v' \in |X'|$ above v. When v is split in X', the two choices of v' give the same Jordan type. The support map $s_{2d}^{\mathrm{Herm}} : |\mathrm{Herm}_{2d}(X'/X)(k)| \to X_d(k)$ is the composition $s_d \circ \Lambda_d^{\mathrm{Herm}}$.

We claim that $\Lambda_d^{\mathrm{Herm}}$ is a bijection. Then $\Lambda_d^{\mathrm{Coh},-1} \circ \Lambda_d^{\mathrm{Herm}} : |\mathrm{Herm}_{2d}(X'/X)(k)| \xrightarrow{\sim} |\mathrm{Coh}_d(X)(k)|$ is the desired bijection.

To prove the claim, since any torsion coherent sheaf splits as a direct sum over the finitely many points in its support, it suffices to fix a closed point $v \in |X|$ and show that the set of isomorphism classes $|\mathrm{Herm}_v|$ of Hermitian torsion sheaves supported above v maps bijectively (by the restriction of $\Lambda_d^{\mathrm{Herm}}$) to $\mathcal{P}$ which are supported at v.

If v splits into v' and v'' in $|X'|$, then any $(\mathcal{Q}, h) \in |\mathrm{Herm}_v|$ has the form $\mathcal{Q}_{v'} \oplus \sigma^* \mathcal{Q}_{v'}$ equipped with the canonical Hermitian structure. In this case we see that the isomorphism class of $(\mathcal{Q}, h)$ is determined by the Jordan type of $\mathcal{Q}_{v'}$, and conversely each Jordan type arises from some $(\mathcal{Q}, h)$.

If v is inert with preimage $v' \in |X'|$, let $(\mathcal{Q}, h) \in |\mathrm{Herm}_v|$ be of length d over $k_{v'}$. Then $V = \Gamma(X', \mathcal{Q})$ is a d-dimensional Hermitian $k_{v'}$-vector space with a self-adjoint nilpotent endomorphism e given by the action of a uniformizer $\varpi \in \mathcal{O}_v$. Fix a d-dimensional Hermitian space (V_d, h) over $k_{v'}$ (unique up to isomorphism), then the isomorphism classes of $(\mathcal{Q}, h) \in |\mathrm{Herm}_v|$ with length d over $k_{v'}$ is in bijection with the adjoint orbits of the unitary group $\mathrm{U}(V_d, h)(k_v)$ acting on the nilpotent cone $\mathcal{N}(V_d, h)(k_v)$ of self-adjoint nilpotent endomorphisms of V_d. Being a Galois twisted version of the usual nilpotent orbits under GL_d, the orbits $\mathcal{N}(V_d, h)(k_v)/\mathrm{U}(V_d, h)(k_v)$ are again classified by partitions of d according the Jordan types of $e \in \mathcal{N}(V_d, h)(k_v)$ (here we use that the centralizer $C_{\mathrm{GL}_d}(e)$ is connected, and Lang's theorem implies $\mathrm{H}^1(k_v, C_{\mathrm{GL}_d}(e)) = \{1\}$). Therefore the isomorphism class of $(\mathcal{Q}, h) \in |\mathrm{Herm}_v|$ is determined by the Jordan type of $\mathcal{Q}$, and conversely each Jordan type arises from a $(\mathcal{Q}, h)$. This shows that $\Lambda_d^{\mathrm{Herm}}$ is a bijection. □

4.6.1 Further notations

Now let $(\mathcal{Q}, h) \in \mathrm{Herm}_{2d}(k)$. We write $\mathcal{Q}_{\overline{k}}$ for the base change of $\mathcal{Q}$ over $X'_{\overline{k}}$, and adapt the notations $Z' \subset X'(\overline{k})$, $Z \subset X(\overline{k})$, $\Sigma(Z')$, $\Sigma(Z)$ from §4.4. Let $|Z'|$ and $|Z|$ be the set of closed points contained in Z' and Z. We have a decomposition

$$|Z| = |Z|_s \coprod |Z|_i$$

into split and inert places. For each closed point $v \in |Z|$ we choose a geometric point $x'_v \in Z'$ above v and denote its image in Z by x_v.

Let $\mathrm{Fr} : X' \to X'$ be the Frobenius morphism. Let $Z^\sharp$ be the following subset of Z'

$$Z^\sharp = \left\{ F^i(x'_v) : v \in |Z|, 0 \le i < \deg(v) \right\}.$$

When v splits into v', v'' in $|X'|$, with $x'_v | v'$, then $Z^\sharp$ contains all geometric points above v' and not any above v''. When v is inert with preimage $v' \in |X'|$, $Z^\sharp$ contains

half of the geometric points above v' which form a chain under the Frobenius, starting with x'_v. Therefore $Z' = Z^\sharp \coprod \sigma(Z^\sharp)$. For $x \in Z$ let $x^\sharp \in Z^\sharp$ be the unique element above x. This induces a section $\Sigma(Z) \xrightarrow{\sim} \Sigma(Z^\sharp) \subset \Sigma(Z')$ which we denote $y \mapsto y^\sharp$.

Let $\mathcal{Q}^\flat \in \mathrm{Coh}_d(X)(k)$ be the point corresponding to the isomorphism class of $(\mathcal{Q}, h)$ under the bijection in Lemma 4.12. Then $\mathcal{Q}^\flat_{\overline{k}} \cong \nu_*(\mathcal{Q}|_{Z^\sharp})$. Recall the isomorphism (3.2) for each $y \in \Sigma(Z)$

$$\beta_y : \mathcal{B}_{\mathcal{Q}^\flat}(y) \xrightarrow{\sim} \prod_{x \in Z} \mathcal{B}_{\mathcal{Q}^\flat_x}.$$

From this and the Künneth formula we get an identification

$$\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat}(y)) = \bigotimes_{x \in Z} \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat_x}).$$

By [9, Corollary 2.3(1)], $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat_x})$ is concentrated in even degrees. Let $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat}(y))^+$ be the direct sum of all $\otimes_{x \in Z} \mathrm{H}^{2i_x}(\mathcal{B}_{\mathcal{Q}^\flat_x})$ (for varying $(i_x)_{x \in Z}$) such that

$$\sum_{v \in |Z|_i} i_{x_v} \text{ is even.} \tag{4.4}$$

Similarly, let $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat}(y))^-$ be the direct sum of all $\otimes_{x \in Z} \mathrm{H}^{2i_x}(\mathcal{B}_{\mathcal{Q}^\flat_x})$ such that the quantity in (4.4) is odd. We have

$$\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat}(y)) = \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat}(y))^+ \oplus \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat}(y))^-. \tag{4.5}$$

Taking direct sum over all $y \in \Sigma(Z)$ we get a decomposition

$$\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat}) = \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat})^+ \oplus \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat})^-. \tag{4.6}$$

Note that this decomposition depends on the choice of a geometric point x_v over each inert v. By Corollary 3.7, the action of $S_y \subset S_d$ on $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat}(y))$ preserves the decomposition (4.5) since it is the same as the tensor product of the Springer actions on each factor $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat_x})$. Therefore the decomposition (4.6) is stable under the S_d-action.

Now $(\mathcal{Q}|_{Z^\sharp} \subset \mathcal{Q})$ gives a geometric point of $\mathrm{Lagr}^\diamondsuit_{2d}$, which is not defined over k if $|Z|_i \neq \varnothing$. Using this geometric point in $\mathrm{Lagr}^\diamondsuit_{2d}$, Proposition 4.11 gives an isomorphism $\alpha^\sharp := \alpha_{(\mathcal{Q}|_{Z^\sharp} \subset \mathcal{Q})}$ on the level of stalks (see (4.3)):

$$\alpha^\sharp : \mathrm{H}^*(\mathcal{B}^{\mathrm{Herm}}_{\mathcal{Q}})^{(\mathbf{Z}/2\mathbf{Z})^d} \cong \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat}).$$

This isomorphism is S_d-equivariant. Both sides now carry geometric Frobenius actions which we denote by $\mathrm{Frob}_{\mathcal{Q}}$ and $\mathrm{Frob}_{\mathcal{Q}^\flat}$, which are not necessarily intertwined under $\alpha^\sharp$ because the point $(\mathcal{Q}|_{Z^\sharp} \subset \mathcal{Q})$ is not necessarily defined over k. The next result gives the relation between the two Frobenius actions.

Proposition 4.13 *Let θ be the involution on $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat})$ which is 1 on $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat})^+$ and -1 on $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat})^-$. Then under the isomorphism $\alpha^\sharp$, $\mathrm{Frob}_{\mathcal{Q}}$ corresponds to $\mathrm{Frob}_{\mathcal{Q}^\flat} \circ \theta$.*

Proof Recall from the proof of Proposition 4.11 that $\alpha^\sharp$ is the composition

$$\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}})^{(\mathbf{Z}/2\mathbf{Z})^d} \cong \bigoplus_{y\in\Sigma(Z)} \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}}(y^\sharp)) \xrightarrow{\oplus\gamma_{Z^\sharp,y^\sharp}} \bigoplus_{y\in\Sigma(Z)} \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat}(y)).$$

Here $\gamma_{Z^\sharp,y^\sharp}$ is defined in (4.1).

Then Fr^* on $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}})$ maps $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}}(\mathrm{Fr}(y^\sharp)))$ to $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}}(y^\sharp))$. Note that $\mathrm{Fr}(y^\sharp)$ and $(\mathrm{Fr}\, y)^\sharp$ are in general different: if $y(i) = \mathrm{Fr}^{-1}(x_v)$ for some inert v, then $\mathrm{Fr}(y^\sharp)(i) = \sigma(x_v')$ while $(\mathrm{Fr}\, y)^\sharp(i) = x_v'$. In other words, the only difference between $\mathrm{Fr}(y^\sharp)$ and $(\mathrm{Fr}\, y)^\sharp$ is the switch of all x_v' and $\sigma(x_v')$ for all inert v. Therefore there is a unique element $\tau_y \in (\mathbf{Z}/2\mathbf{Z})^d$ such that $(\mathrm{Fr}\, y)^\sharp = \mathrm{Fr}(y^\sharp) \circ \tau_y$.

Identifying $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}})^{(\mathbf{Z}/2\mathbf{Z})^d}$ with $\oplus_{y\in\Sigma(Z)}\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}}(y^\sharp))$, the geometric Frobenius endomorphism $\mathrm{Frob}_{\mathcal{Q}}$ on $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}})^{(\mathbf{Z}/2\mathbf{Z})^d}$ is the direct sum of the following compositions

$$\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}}((\mathrm{Fr}\, y)^\sharp)) \xrightarrow{\tau_y} \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}}(\mathrm{Fr}(y^\sharp))) \xrightarrow{\mathrm{Fr}^*} \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}}(y^\sharp))$$

where the first map is the Springer action of $\tau_y \in W_d$ on $(\mathrm{Spr}_{2d}^{\mathrm{Herm}})_{\mathcal{Q}}$.

On the other hand, let $w_y \in W_d$ be the *minimal length* element such that $(\mathrm{Fr}\, y)^\sharp = \mathrm{Fr}(y^\sharp) \circ w_y$. Write

$$\tau_y = w_y u_y, \text{ for a unique } u_y \in \mathrm{Stab}_{W_d}((\mathrm{Fr}\, y)^\sharp) = \mathrm{Stab}_{S_d}(\mathrm{Fr}\, y) \subset S_d.$$

Note that $\mathrm{Stab}_{S_d}(Fy) = \prod_{x\in Z} S_{I_x}$ where $I_x \subset \{1, 2, \ldots, d\}$ is the preimage of x under y. An easy calculation shows that $u_y = (u_{y,x})_{x\in Z}$ where $u_{y,x} \in S_{I_x}$ is

$$u_{y,x} = \begin{cases} w_{I_x}, & \text{if } x = x_v, v \in |Z|_i, \\ 1, & \text{otherwise.} \end{cases}$$

Here $w_{I_x} \in S_{I_x}$ is the involution that reverses the order of I_x.

We use abbreviated notation

$$H(y') := \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}}(y')), \text{ for } y' \in \Sigma(Z'),$$
$$C(y) := \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat}(y)), \text{ for } y \in \Sigma(Z).$$

For each $y \in \Sigma(Z)$, consider the following diagram

$$\begin{array}{ccccccc}
H((\mathrm{Fr}\, y)^\sharp) & \xrightarrow{u_y} & H((\mathrm{Fr}\, y)^\sharp) & \xrightarrow{w_y} & H(\mathrm{Fr}(y^\sharp)) & \xrightarrow{\mathrm{Fr}^*} & H(y^\sharp) \\
\downarrow{\scriptstyle \gamma_{Z^\sharp,(\mathrm{Fr}\, y)^\sharp}} & & \downarrow{\scriptstyle \gamma_{Z^\sharp,(\mathrm{Fr}\, y)^\sharp}} & \swarrow{\scriptstyle \delta^*\gamma_{Z^\sharp,\mathrm{Fr}(y^\sharp)}} & \downarrow{\scriptstyle \gamma_{\mathrm{Fr}(Z^\sharp),\mathrm{Fr}(y^\sharp)}} & & \downarrow{\scriptstyle \gamma_{Z^\sharp,y^\sharp}} \\
C(Fy) & \xrightarrow{u_y} & C(Fy) & & C(Fy) & \xrightarrow{\mathrm{Fr}^*} & C(y)
\end{array} \tag{4.7}$$

The left square is commutative by the S_d-equivariance of $\alpha^\sharp$ proved in Proposition 4.11 (here $u_y \in \mathrm{Stab}_{W_d}((\mathrm{Fr}\, y)^\sharp) \subset S_d$). The middle upper triangle is commutative by Lemma 4.10. The map δ^* is defined to make the lower middle triangle commutative. The right square is clearly commutative. The composite of the upper row is the restriction of $\mathrm{Fr}_{\mathcal{Q}}$ to $H((\mathrm{Fr}\, y)^\sharp)$. Let us compute the composite of the lower row.

The map δ^* is the pullback along the automorphism δ of $\mathcal{B}_{\mathcal{Q}^\flat}(\mathrm{Fr}\, y)$ that makes the following diagram commutative

$$\begin{array}{ccc} & & \mathcal{B}^{\mathrm{Herm}}_{\mathcal{Q}}(\mathrm{Fr}(y^\sharp)) \\ & {\scriptstyle \gamma_{Z^\sharp,\mathrm{Fr}(y^\sharp)}} \swarrow & \downarrow {\scriptstyle \gamma_{\mathrm{Fr}(Z^\sharp),\mathrm{Fr}(y^\sharp)}} \\ \mathcal{B}_{\mathcal{Q}^\flat}(\mathrm{Fr}\, y) & \xleftarrow{\ \delta\ } & \mathcal{B}_{\mathcal{Q}^\flat}(\mathrm{Fr}\, y) \end{array}$$

Under the isomorphism $\beta_{\mathrm{Fr}\, y} : \mathcal{B}_{\mathcal{Q}^\flat}(\mathrm{Fr}\, y) \xrightarrow{\sim} \prod_{x\in Z} \mathcal{B}_{\mathcal{Q}^\flat_x}$, δ is the product of automorphisms δ_x for each $\mathcal{B}_{\mathcal{Q}^\flat_x}$. If x is not of the form $x = x_v$ for $v \in |Z|_i$, δ_x is the identity. If $x = x_v$ for some $v \in |Z|_i$, then $\mathcal{Q}^\flat_x = \mathcal{Q}_{x'_v}$ and the Hermitian structure on $\mathcal{Q}$ gives an isomorphism $\iota : \mathcal{Q}_{\sigma(x'_v)} \cong \mathcal{Q}^\vee_{x'_v}$. On the other hand, $\mathrm{Fr}^{\deg(v)}$ gives an isomorphism $\phi : \mathcal{Q}_{x'_v} \cong \mathcal{Q}_{\sigma(x'_v)}$ since $\sigma(x'_v) = \mathrm{Fr}^{\deg(v)}(x'_v)$. Combining ι and ϕ we get a perfect symmetric pairing $(\cdot,\cdot)_{x'_v}$ on $\mathcal{Q}_{x'_v}$ itself. Then δ_x sends a full flag $\mathcal{R}_\bullet$ of $\mathcal{Q}^\flat_{x_v} = \mathcal{Q}_{x'_v}$ to $\mathcal{R}^\perp_\bullet$ under the pairing $(\cdot,\cdot)_{x'_v}$.

By the description of u_y and δ above, under the isomorphism β_{Fy}, the composition $\delta^* \circ u_y$ takes the form

$$\otimes(\delta_x^* \circ u_{y,x}) : \bigotimes_{x\in Z} \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat_x}) \to \bigotimes_{x\in Z} \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat_x}).$$

The automorphisms $\delta_x^* \circ u_{y,x}$ are the identity maps except when $x = x_v$ for some $v \in |Z|_i$, in which case $u_{y,x} = w_{I_x}$. Let us compute $\delta_x^* \circ w_{I_x}$ on $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat_x})$ for $x = x_v$ and $v \in |Z|_i$. For this we switch to the following notation. Let $V = \mathcal{Q}_{x'_v} = \mathcal{Q}^\flat_{x_v}$, a vector space of dimension m over $\overline{k}$. We have argued that V carries a symmetric self-duality $(\cdot,\cdot)$; the action of a uniformizer at x_v gives a nilpotent element $e \in \mathrm{End}_{\overline{k}}(V)$, which is self-adjoint under $(\cdot,\cdot)$. Let $\mathcal{B}$ be the flag variety of $\mathrm{GL}(V)$ and $\mathcal{B}_e$ be the Springer fiber of e. Then S_m acts on $\mathrm{H}^*(\mathcal{B}_e)$. Let w_0 be the longest element in S_m. Let $\delta : \mathcal{B}_e \xrightarrow{\sim} \mathcal{B}_e$ be the map sending a flag $V_\bullet$ to $V_\bullet^\perp$. We claim that $\delta^* \circ w_0$ acts on $\mathrm{H}^{2i}(\mathcal{B}_e)$ by $(-1)^i$. Indeed, by [9, Corollary 2.3(2)], the restriction map $\mathrm{H}^*(\mathcal{B}) \to \mathrm{H}^*(\mathcal{B}_e)$ is surjective and is clearly equivariant under $\delta^* \circ w_0$, so it suffices to show that $\delta^* \circ w_0$ acts by $(-1)^i$ on $\mathrm{H}^{2i}(\mathcal{B})$. Since $\delta^* \circ w_0$ preserves the cup product on $\mathrm{H}^*(\mathcal{B})$, it suffices to show it acts by -1 on $\mathrm{H}^2(\mathcal{B})$ (which generates all of $\mathrm{H}^{2*}(\mathcal{B})$ under the cup product). For $1 \le j \le m$, let ξ_j be the Chern class of the tautological line bundle on $\mathcal{B}$ whose fiber at $V_\bullet$ is V_j/V_{j-1}. Then $\mathrm{H}^2(\mathcal{B})$ is spanned by ξ_j for $1 \le j \le m$. Now we have $w_0(\xi_j) = \xi_{m+1-j}$ since $\mathrm{H}^2(\mathcal{B})$ is the reflection representation of S_m, and $\delta^*\xi_j = -\xi_{m+1-j}$ by the definition of δ. Therefore $\delta^* \circ w_0(\xi_j) = -\xi_j$ for all

$1 \le j \le m$, which proves that $\delta^* \circ w_0$ acts by -1 on $\mathrm{H}^2(\mathcal{B})$, and hence acts by $(-1)^i$ on $\mathrm{H}^{2i}(\mathcal{B})$ and on $\mathrm{H}^{2i}(\mathcal{B}_e)$.

The above argument shows that $\delta^* \circ u_y$ acts by 1 on $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat}(\mathrm{Fr}\, y))^+$ and acts by -1 on $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat}(\mathrm{Fr}\, y))^-$. Therefore the bottom row of (4.7) is $\mathrm{Fr}_{\mathcal{Q}^\flat} \circ \theta$. By the commutativity of (4.7), $\mathrm{Fr}_{\mathcal{Q}^\flat} \circ \theta$ corresponds under $\alpha^\sharp$ to the composite of the top row, which is $\mathrm{Fr}_{\mathcal{Q}}$. This finishes the proof. □

5 Geometrization of local densities

The goal of this section is to give a sheaf-theoretic interpretation of the formula of Cho-Yamauchi on the representation density of Hermitian lattices, see Theorem 5.3. This will complete the geometrization of the analytic side of our proposed Siegel-Weil formula, at least for non-singular Fourier coefficients. The technical part of the proof of the theorem is a Frobenius trace calculation that uses properties of the Hermitian Springer action proved in §4.6.

5.1 Density function for torsion sheaves

Following Remark 2.4, for any Hermitian torsion sheaf $(\mathcal{Q}, h) \in \mathrm{Herm}_{2d}(X'/X)(k)$, we may define the density polynomial $\mathrm{Den}(T, \mathcal{Q})$ using the Cho-Yamauchi formula as follows. Let $\mathcal{Q}_v$ be the summand of $\mathcal{Q}$ supported over $v \in |X|$, we define

$$\mathrm{Den}(T, \mathcal{Q}) := \prod_{v \in |X|} \mathrm{Den}_v(T^{\deg(v)}, \mathcal{Q}_v)$$

where

$$\mathrm{Den}_v(T, \mathcal{Q}_v) = \sum_{0 \subset \mathcal{I} \subset \mathcal{I}^\perp \subset \mathcal{Q}_v} T^{2\ell'_v(\mathcal{I})} \mathfrak{m}_v(t'_v(\mathcal{I}^\perp/\mathcal{I}); T).$$

Here $I^\perp$ is the orthogonal of I under the Hermitian form on $\mathcal{Q}_v$, so in other words the sum is over all subsheaves of $\mathcal{Q}_v$ that are isotropic under $h_v = h|_{\mathcal{Q}_v}$, and we write $\mathfrak{m}_v(-)$ to emphasize the dependence of $\mathfrak{m}(a; T)$ on F'_v/F_v (see Definition 2.2). The functions $\ell'_v(-)$ and $t'_v(-)$ are the functions $\ell'(-)$ and $t'(-)$ defined in §2.4 for F'_v/F_v.

Expanding the product into a summation, we see

$$\mathrm{Den}(T, \mathcal{Q}) = \sum_{0 \subset \mathcal{I} \subset \mathcal{I}^\perp \subset \mathcal{Q}} T^{\dim_k \mathcal{I}} \prod_{v \in |X|} \mathfrak{m}_v(t'_v(\mathcal{I}^\perp/\mathcal{I}); T^{\deg(v)}). \tag{5.1}$$

Given an injective Hermitian map $a\colon \mathcal{E} \hookrightarrow \sigma^*\mathcal{E}^\vee$, we have a Hermitian structure on the torsion sheaf $\mathrm{coker}(a)$ as follows. Applying $\sigma^* R\underline{\mathrm{Hom}}(-, \omega_{X'})$ to the short exact sequence

$$0 \to \mathcal{E} \xrightarrow{a} \sigma^*\mathcal{E}^\vee \to \mathcal{Q} \to 0 \tag{5.2}$$

yields a short exact sequence

$$0 \to \mathcal{E} \xrightarrow{\sigma^* a^\vee} \sigma^* \mathcal{E}^\vee \to \sigma^* \underline{\mathrm{Ext}}^1(\mathcal{Q}, \omega_{X'}) \to 0. \tag{5.3}$$

Since $\sigma^* a^\vee = a$, we may identify (5.2) and (5.3) and get an isomorphism $h_{\mathcal{Q}} : \mathcal{Q} \xrightarrow{\sim} \sigma^* \mathcal{Q}^\vee$. Using this, we may restate Theorem 2.8 as follows.

Theorem 5.1 *Let $\mathcal{E}$ be a rank n vector bundle over X' and $a : \mathcal{E} \hookrightarrow \sigma^* \mathcal{E}^\vee$ be an injective Hermitian map. Then*

$$E_a(m(\mathcal{E}), s, \Phi) = \chi(\det(\mathcal{E})) q^{-\deg(\mathcal{E})(s-n/2) - \frac{1}{2} n^2 \deg(\omega_X)} \times \mathscr{L}_n(s)^{-1} \mathrm{Den}(q^{-2s}, \mathrm{coker}(a)).$$

5.2 Density sheaves

We will define a graded perverse sheaf on $\mathrm{Herm}_{2d}(X'/X)$ whose Frobenius trace at $\mathcal{Q}$ recovers $\mathrm{Den}(T, \mathcal{Q})$. We will suppress X'/X from the notations.

For $0 \le i \le d$, let $\mathrm{Herm}_{i,2d}$ be the stack classifying

$$\{(\mathcal{I}, (\mathcal{Q}, h)) \in \mathrm{Coh}_i(X') \times \mathrm{Herm}_{2d} : \mathcal{I} \subset \mathcal{Q} \text{ and is isotropic under } h\}.$$

We have the following maps

$$\begin{array}{ccc} & \mathrm{Herm}_{i,2d} & \\ \overleftarrow{f}_i \swarrow & & \searrow \overrightarrow{f}_i = (\overrightarrow{f}'_i, \overrightarrow{f}''_i) \\ \mathrm{Herm}_{2d} & & \mathrm{Coh}_i(X') \times \mathrm{Herm}_{2(d-i)} \end{array}$$

Here $\overleftarrow{f}_i$ takes $(\mathcal{I}, (\mathcal{Q}, h))$ to $(\mathcal{Q}, h)$, $\overrightarrow{f}'_i$ takes it to $\mathcal{I}$ and $\overrightarrow{f}''_i$ takes it to $\mathcal{I}^\perp/\mathcal{I}$ with the Hermitian structure induced from h.

Recall the perverse sheaf HSpr_d on Herm_{2d} from Definition 4.7(2). It is obtained from the Springer sheaf on Herm_{2d} by taking $(\mathbf{Z}/2\mathbf{Z})^d$-invariants, and HSpr_{2d} carries an action of S_d.

Definition 5.2 We define the following graded virtual perverse sheaves on Herm_{2d}. (In the notation below, the degree of the formal variable T encodes the grading.)

(1) $\mathfrak{P}_d(T) = \bigoplus_{j=0}^d (-1)^j (\mathrm{HSpr}_d)^{(S_j \times S_{d-j}, \mathrm{sgn}_j \boxtimes \mathbf{1})} T^j$.

(2) $\mathcal{K}_d^{\mathrm{Eis}}(T) = \bigoplus_{i=0}^d R\overleftarrow{f}_{i,!} R\overrightarrow{f}_i^* (\overline{\mathbf{Q}}_\ell \cdot T^i \boxtimes \mathfrak{P}_{d-i}(T))$.

Theorem 5.3 *For any $\mathcal{Q} \in \mathrm{Herm}_{2d}(k)$, we have*

$$\mathrm{Den}(T, \mathcal{Q}) = \mathrm{Tr}(\mathrm{Fr}, \mathcal{K}_d^{\mathrm{Eis}}(T)_{\mathcal{Q}}).$$

Proof By the Grothendieck-Lefschetz trace formula, we have

$$\mathrm{Tr}(\mathrm{Fr}, \mathcal{K}_d^{\mathrm{Eis}}(T)_{\mathcal{Q}}) = \sum_{i=0}^{d} \sum_{(\mathcal{I},\mathcal{Q})\in \mathrm{Herm}_{i,2d}(k)} T^{\dim_k \mathcal{I}} \mathrm{Tr}(\mathrm{Fr}, \mathfrak{P}_{d-i}(T)_{\mathcal{I}^\perp/\mathcal{I}}).$$

Comparing with the expansion (5.1) for $\mathrm{Den}(T, \mathcal{Q})$, it suffices to prove, changing notation $d-i$ to d and $\mathcal{I}^\perp/\mathcal{I}$ to $\mathcal{Q}$, that

$$\mathrm{Tr}(\mathrm{Fr}, \mathfrak{P}_d(T)_{\mathcal{Q}}) = \prod_{v\in|X|} \mathfrak{m}_v(t'_v(\mathcal{Q}); T^{\deg(v)}). \tag{5.4}$$

This will be proved in Proposition 5.7. □

The rest of the section is devoted to the proof of (5.4). The idea is to relate the Frobenius trace to a similar Frobenius trace of a graded perverse sheaf on $\mathrm{Coh}_d(X)$ using results from §4.6, and then calculate the latter explicitly.

5.3 Comparison of two graded Frobenius modules

For $(\mathcal{Q}, h) \in \mathrm{Herm}_{2d}(k)$, write

$$\mathfrak{P}_{\mathcal{Q}}(T) = \mathfrak{P}_d(T)_{\mathcal{Q}} = \bigoplus_{j=0}^{d} (-1)^j (\mathrm{HSpr}_d)_{\mathcal{Q}}^{(S_j\times S_{d-j}, \mathrm{sgn}_j\boxtimes 1)} T^j$$

$$= \bigoplus_{j=0}^{d} (-1)^j \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}}^{\mathrm{Herm}})^{(W_j\times W_{d-j}, \overline{\mathrm{sgn}}_j\boxtimes 1)} T^j.$$

Here $\overline{\mathrm{sgn}}_j$ is the inflation of the sign representation of S_j under $W_j \twoheadrightarrow S_j$. We view $\mathfrak{P}_{\mathcal{Q}}(T)$ as a $\mathbf{Z}$-graded virtual Frobenius module, with the $\mathbf{Z}$-grading indicated by the power of T. Let $\mathcal{Q}^\flat \in \mathrm{Coh}_d(X)(k)$ be in the isomorphism class that corresponds to $(\mathcal{Q}, h)$ under the bijection in Lemma 4.12. Define

$$\mathfrak{P}_{\mathcal{Q}^\flat}(T) = \bigoplus_{j=0}^{d} (-1)^j (\mathrm{Spr}_d)_{\mathcal{Q}^\flat}^{(S_j\times S_{d-j}, \mathrm{sgn}_j\boxtimes 1)} T^j$$

$$= \bigoplus_{j=0}^{d} (-1)^j \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat})^{(S_j\times S_{d-j}, \mathrm{sgn}_j\boxtimes 1)} T^j.$$

Define the Frobenius traces

$$P_{\mathcal{Q}}(T) := \mathrm{Tr}(\mathrm{Fr}, \mathfrak{P}_{\mathcal{Q}}(T)), \quad P_{\mathcal{Q}^\flat}(T) := \mathrm{Tr}(\mathrm{Fr}, \mathfrak{P}_{\mathcal{Q}^\flat}(T)) \in \overline{\mathbf{Q}}_\ell[T].$$

The goal is to get a relationship between $P_{\mathcal{Q}}(T)$ and $P_{\mathcal{Q}^\flat}(T)$. Note that $\mathfrak{P}_{\mathcal{Q}^\flat}(T)$ is a special case of $\mathfrak{P}_{\mathcal{Q}}(T)$ when the double cover $X' = X \sqcup X$ (and $\mathcal{Q}$ is the direct sum of $\mathcal{Q}^\flat$ on one copy of X and $\mathcal{Q}^{\flat,\vee}$ on the other). We shall apply Proposition 4.13 to

express $P_{\mathcal{Q}}(T)$ in terms of $P_{\mathcal{Q}^\flat}(T)$. For this we need to calculate the decomposition of $\mathfrak{P}_{\mathcal{Q}^\flat}(T)$ given in (4.6).

Recall notations $Z, Z', \Sigma(Z)$ and $\Sigma(Z')$ from §4.4. First we show that $\mathfrak{P}_{\mathcal{Q}^\flat}(T)$ factorizes according to the support of $\mathcal{Q}$ in X. Let $|Z|$ be the set of closed points in Z. For $v \in |Z|$, let $\mathcal{Q}^\flat_v$ denote the direct summand of $\mathcal{Q}^\flat$ whose support is over v.

Lemma 5.4 *We have a natural isomorphism of graded* Fr*-modules*

$$\mathfrak{P}_{\mathcal{Q}^\flat}(T) \cong \bigotimes_{v\in|Z|} \mathfrak{P}_{\mathcal{Q}^\flat_v}(T).$$

In particular,

$$P_{\mathcal{Q}^\flat}(T) = \prod_{v\in|Z|} P_{\mathcal{Q}^\flat_v}(T).$$

Proof Choose any $y \in \Sigma(Z)$ and let $|y|$ be the resulting map $\{1, 2, \ldots, d\} \to |Z|$. Let $S_{|y|} = \mathrm{Stab}_{S_d}(|y|)$, then $S_{|y|} \cong \prod_{v\in|Z|} S_{I_v}$, where $I_v = |y|^{-1}(v)$. Applying Corollary 3.7 to $\mathcal{Q}^\flat$ and to each $\mathcal{Q}^\flat_v$ gives an isomorphism of (S_d, Fr)-modules

$$\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat}) \cong \mathrm{Ind}_{S_{|y|}}^{S_d}\left(\bigotimes_{v\in|Z|} \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat_v})\right). \tag{5.5}$$

Here the factor S_{I_v} of $S_{|y|}$ acts on $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat_v})$ by the Springer action.

Write $H_v = \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat_v})$ as a (S_{I_v}, Fr)-module. Let $d_v = \#I_v$. By Mackey theory, the $(S_j \times S_{d-j}, \mathrm{sgn}_j \boxtimes 1)$-isotypical part of the right side of (5.5) is a direct sum over double cosets $(S_j \times S_{d-j})\backslash S_d / S_{|y|}$, which can be identified with the set of functions $i : |Z| \to \mathbf{Z}_{\geq 0}$, $v \mapsto i_v \leq d_v$, such that $\sum_v i_v = j$. The stabilizer of the $S_j \times S_{d-j}$-action on the orbit indexed by i is isomorphic to $\prod_{v\in|Z|} S_{i_v} \times S_{d_v - i_v}$ (where the S_{i_v} factor lies in S_j, $S_{d_v-i_v}$ lies in S_{d-j}, and $S_{i_v} \times S_{d_v-i_v}$ is naturally a subgroup of $S_{I_v} \cong S_{d_v}$). The contribution of the summand indexed by i is

$$\bigotimes_{v\in|Z|} H_v^{(S_{i_v}\times S_{d_v-i_v}, \mathrm{sgn}_{i_v}\boxtimes 1)}.$$

This implies that

$$\begin{aligned}\mathfrak{P}_{\mathcal{Q}^\flat}(T) &\cong \sum_{i:|Z|\to\mathbf{Z}_{\geq 0}, i_v\leq d_v} (-1)^{\sum i_v}\left(\bigotimes_{v\in|Z|} H_v^{(S_{i_v}\times S_{d_v-i_v}, \mathrm{sgn}_{i_v}\boxtimes 1)}\right) T^{\sum i_v} \\ &\cong \bigotimes_{v\in|Z|}\left(\sum_{i_v=0}^{d_v} (-1)^{i_v} H_v^{(S_{i_v}\times S_{d_v-i_v}, \mathrm{sgn}_{i_v}\boxtimes 1)} T^{i_v}\right) = \bigotimes_{v\in|Z|} \mathfrak{P}_{\mathcal{Q}^\flat_v}(T).\end{aligned}$$

□

Let $\mathcal{Q}_v$ be the direct summand of $\mathcal{Q}$ supported over v. Then $\mathcal{Q}_v$ and $\mathcal{Q}_v^\flat$ correspond under the bijection in Lemma 4.12.

For any Frobenius module M with integral weights, let $\mathrm{Gr}_i^W M$ be the pure of weight i part of M. This notation applies also to graded Frobenius modules by taking Gr_i^W on each graded piece.

Proposition 5.5 *Let $\mathcal{Q} \in \mathrm{Herm}_{2d}(k)$.*

(1) *We have*

$$\mathfrak{P}_{\mathcal{Q}}(T) \cong \bigotimes_{v\in|Z|} \mathfrak{P}_{\mathcal{Q}_v}(T), \quad P_{\mathcal{Q}}(T) = \prod_{v\in|Z|} P_{\mathcal{Q}_v}(T). \tag{5.6}$$

(2) *If $v \in |Z|$ is split in X', then*

$$P_{\mathcal{Q}_v}(T) = P_{\mathcal{Q}_v^\flat}(T). \tag{5.7}$$

(3) *If $v \in |Z|$ is inert in X', then*

$$P_{\mathcal{Q}_v}(T) = \sum_i (-1)^i \operatorname{Tr}(\mathrm{Fr}, \mathrm{Gr}^W_{2i\deg(v)} \mathfrak{P}_{\mathcal{Q}_v^\flat}(T)). \tag{5.8}$$

Moreover, $\operatorname{Tr}(\mathrm{Fr}, \mathrm{Gr}_i^W \mathfrak{P}_{\mathcal{Q}_v^\flat}(T)) = 0$ *if i is not a multiple of* $2\deg(v)$.

Proof We will use the notations from §4.6.1. For each $y \in \Sigma(Z)$, the summand $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat}(y)) \cong \otimes_{x\in Z} \mathrm{H}^*(\mathcal{B}_{\mathcal{Q}_x^\flat})$ is graded by multidegree $\mathbf{i} : Z \to \mathbf{Z}_{\geq 0}$

$$\mathrm{H}^{2\mathbf{i}}(\mathcal{B}_{\mathcal{Q}^\flat}(y)) = \bigotimes_{x\in Z} \mathrm{H}^{2\mathbf{i}(x)}(\mathcal{B}_{\mathcal{Q}_x^\flat}).$$

We define $\mathrm{H}^{2\mathbf{i}}(\mathcal{B}_{\mathcal{Q}^\flat})$ as the direct sum of $\mathrm{H}^{2\mathbf{i}}(\mathcal{B}_{\mathcal{Q}^\flat}(y))$ over all $y \in \Sigma(Z)$. Then each $\mathrm{H}^{2\mathbf{i}}(\mathcal{B}_{\mathcal{Q}^\flat})$ is stable under S_d. Accordingly, $\mathfrak{P}_{\mathcal{Q}^\flat}(T)$ decomposes into the direct sum of $\mathfrak{P}^{2\mathbf{i}}_{\mathcal{Q}^\flat}(T)$, which is by definition $\oplus_{j=0}^d (-1)^j \mathrm{H}^{2\mathbf{i}}(\mathcal{B}_{\mathcal{Q}^\flat})^{(S_j\times S_{d-j}, \mathrm{sgn}_j \boxtimes 1)} T^j$. Let $\mathbf{i}_v$ be the restriction of $\mathbf{i}$ to those $x|v$, then under the factorization isomorphism in Lemma 5.4 we have

$$\mathfrak{P}^{2\mathbf{i}}_{\mathcal{Q}^\flat}(T) \cong \bigotimes_{v\in|Z|} \mathfrak{P}^{2\mathbf{i}_v}_{\mathcal{Q}_v^\flat}(T). \tag{5.9}$$

(1) Recall the involution θ on $\mathrm{H}^*(\mathcal{B}_{\mathcal{Q}^\flat})$ in Proposition 4.13. Using the above notation, we see that θ acts on $\mathrm{H}^{2\mathbf{i}}(\mathcal{B}_{\mathcal{Q}^\flat})$ by $\prod_{v\in|Z|_i} (-1)^{\mathbf{i}(x_v)}$ (where $x_v \in Z$ is a chosen geometric point over v, as in §4.6.1). Because of (5.9), the action of θ on $\mathfrak{P}_{\mathcal{Q}^\flat}(T)$ factorizes as the tensor product of the similarly-defined θ_v on each $\mathfrak{P}_{\mathcal{Q}_v^\flat}(T)$. By Proposition 4.13, $\mathfrak{P}_{\mathcal{Q}}(T)$ is the Frobenius module obtained by modifying the Frobenius action on $\mathfrak{P}_{\mathcal{Q}^\flat}(T)$ by composing with θ. By Lemma 5.4, this modified Frobenius structure on $\mathfrak{P}_{\mathcal{Q}^\flat}(T)$ is the tensor product of the similarly modified Frobenius modules $\mathfrak{P}_{\mathcal{Q}_v^\flat}(T)$, which in turn are isomorphic to $\mathfrak{P}_{\mathcal{Q}_v}(T)$ by Proposition 4.13. This implies (5.6).

(2) From the definition we see that if v is split, then θ_v is the identity on $\mathfrak{P}_{\mathcal{Q}_v^\flat}(T)$. Hence $\mathfrak{P}_{\mathcal{Q}_v}(T) = \mathfrak{P}_{\mathcal{Q}_v^\flat}(T)$ and $P_{\mathcal{Q}_v}(T) = P_{\mathcal{Q}_v^\flat}(T)$.

(3) Since the cohomology of $\mathcal{B}_{\mathcal{Q}_v^\flat}$ is pure by [9, Corollary 2.3(3)], we see that $\mathrm{Gr}_i^W \mathfrak{P}_{\mathcal{Q}_v^\flat}(T)$ is the sum of $\mathfrak{P}^{2\mathbf{i}_v}_{\mathcal{Q}_v^\flat}(T)$ where $\mathbf{i}_v : \{x \in Z : x|v\} \to \mathbf{Z}_{\geq 0}$ satisfies $\sum_{x|v} 2\mathbf{i}_v(x) = i$. The action of Frobenius sends $\mathfrak{P}^{2\mathbf{i}_v}_{\mathcal{Q}_v^\flat}(T)$ to $\mathfrak{P}^{2F^*\mathbf{i}_v}_{\mathcal{Q}_v^\flat}(T)$, where $F^*\mathbf{i}_v$ means precomposing $\mathbf{i}_v$ with Frobenius. Therefore only constant multi-degrees (i.e., constant functions $\mathbf{i}_v$) contribute to the Frobenius trace of $\mathfrak{P}_{\mathcal{Q}_v^\flat}(T)$. This implies $\mathrm{Tr}(\mathrm{Fr}, \mathrm{Gr}_i^W \mathfrak{P}_{\mathcal{Q}_v^\flat}(T)) = 0$ unless i is a multiple of $2\deg(v)$.

By the discussion above,

$$P_{\mathcal{Q}_v}(T) = \mathrm{Tr}(\mathrm{Fr} \circ \theta_v, \mathfrak{P}_{\mathcal{Q}_v^\flat}(T)) = \sum_{i \geq 0} \mathrm{Tr}(\mathrm{Fr} \circ \theta_v, \mathfrak{P}^{(2i,2i,\ldots,2i)}_{\mathcal{Q}_v^\flat}(T)).$$

Since v is inert, θ_v acts by $(-1)^i$ on $\mathfrak{P}^{(2i,2i,\ldots,2i)}_{\mathcal{Q}_v^\flat}(T)$, hence

$$\mathrm{Tr}(\mathrm{Fr} \circ \theta_v, \mathfrak{P}^{(2i,2i,\ldots,2i)}_{\mathcal{Q}_v^\flat}(T)) = (-1)^i \,\mathrm{Tr}(\mathrm{Fr}, \mathfrak{P}^{(2i,2i,\ldots,2i)}_{\mathcal{Q}_v^\flat}(T)).$$

On the other hand, $\mathrm{Tr}(\mathrm{Fr}, \mathrm{Gr}^W_{2i\deg(v)} \mathfrak{P}_{\mathcal{Q}_v^\flat}(T))$ is the sum of $\mathrm{Tr}(\mathrm{Fr}, \mathfrak{P}^{2\mathbf{i}_v}_{\mathcal{Q}_v^\flat}(T))$ with total degree $\sum_{x|v} 2\mathbf{i}_v(x) = 2i\deg(v)$. Since only constant multi-degrees contribute to the trace, we again conclude

$$\mathrm{Tr}(\mathrm{Fr}, \mathrm{Gr}^W_{2i\deg(v)} \mathfrak{P}_{\mathcal{Q}_v^\flat}(T)) = \mathrm{Tr}(\mathrm{Fr}, \mathfrak{P}^{(2i,2i,\ldots,2i)}_{\mathcal{Q}_v^\flat}(T)).$$

Combining the above identities we get (5.8). □

5.4 Calculation of $P_{\mathcal{Q}^\flat}(T)$ and $P_{\mathcal{Q}}(T)$

Let $\mathcal{Q}^\flat \in \mathrm{Coh}_d(X)(k)$ with support $Z \subset X(\overline{k})$. For each $v \in |Z|$ recall $t_v(\mathcal{Q}^\flat)$ from (2.5) for the local field F_v.

Proposition 5.6 *For $\mathcal{Q}^\flat \in \mathrm{Coh}_d(X)(k)$, we have*

$$P_{\mathcal{Q}^\flat}(T) = \prod_{v \in |Z|} (1 - T^{\deg(v)})(1 - q_v T^{\deg(v)}) \cdots (1 - q_v^{t_v(\mathcal{Q}^\flat)-1} T^{\deg(v)}).$$

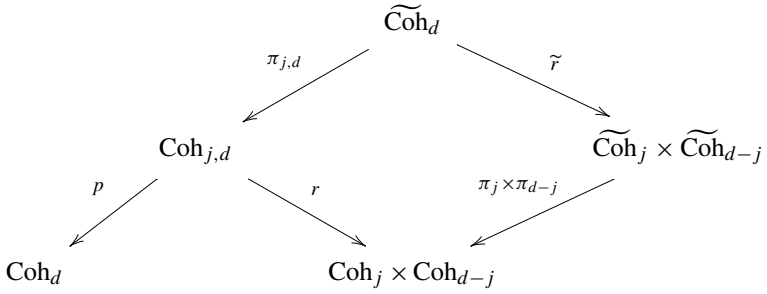
Proof We write $\mathrm{Coh}_d(X)$ simply as Coh_d in the proof. For $0 \leq j \leq d$, consider the correspondence

$$\begin{array}{ccc} & \mathrm{Coh}_{j,d} & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle r} \\ \mathrm{Coh}_d & & \mathrm{Coh}_j \times \mathrm{Coh}_{d-j}\,. \end{array}$$

Here $\mathrm{Coh}_{j,d}$ classifies pairs $(\mathcal{Q}_j \subset \mathcal{Q})$ of torsion sheaves of length j and d respectively, and the map r sends $(\mathcal{Q}_j \subset \mathcal{Q})$ to $(\mathcal{Q}_j, \mathcal{Q}/\mathcal{Q}_j)$. We claim that

$$\mathrm{Spr}_d^{(S_j \times S_{d-j}, \mathrm{sgn}_j \boxtimes 1)} \cong Rp_* Rr^*(\mathrm{St}_j \boxtimes \overline{\mathbf{Q}}_\ell). \tag{5.10}$$

Indeed, consider the following diagram with Cartesian parallelogram



Here $\pi_j = \pi_j^{\mathrm{Coh}}$, etc. The composition $p \circ \pi_{j,d} = \pi_d = \pi_d^{\mathrm{Coh}}$. By the proper base change theorem, we get

$$\mathrm{Spr}_d = Rp_* R\pi_{j,d*} \overline{\mathbf{Q}}_\ell \cong Rp_* Rr^* R(\pi_j \times \pi_{d-j})_* \overline{\mathbf{Q}}_\ell = Rp_* Rr^*(\mathrm{Spr}_j \boxtimes \mathrm{Spr}_{d-j}).$$

This isomorphism is $S_j \times S_{d-j}$-equivariant by checking easily over the open substack Coh_d°. Taking $(S_j \times S_{d-j}, \mathrm{sgn}_j \boxtimes 1)$-isotypical parts of both sides we get (5.10).

By Lemma 5.4 it remains to compute $P_{\mathcal{Q}_v^\flat}(T)$ for each v. Therefore we may assume $\mathcal{Q}^\flat = \mathcal{Q}_v^\flat$. By (5.10) and the Grothendieck-Lefschetz trace formula we have

$$\begin{aligned} P_{\mathcal{Q}_v^\flat}(T) &= \sum_j (-1)^j \operatorname{Tr}(\mathrm{Fr}, (\mathrm{Spr}_d)_{\mathcal{Q}_v^\flat}^{(S_j \times S_{d-j}, \mathrm{sgn}_j \boxtimes 1)}) T^j \\ &= \sum_j (-1)^j \operatorname{Tr}(\mathrm{Fr}, Rp_* Rr^*(\mathrm{St}_j \boxtimes \overline{\mathbf{Q}}_\ell)_{\mathcal{Q}_v^\flat}) T^j \\ &= \sum_{\substack{\mathcal{R} \subset \mathcal{Q}_v^\flat, \\ \dim_{k_v}(\mathcal{R}) = j}} (-1)^{\deg(v) j} \operatorname{Tr}(\mathrm{Fr}, (\mathrm{St}_{\deg(v) j})_{\mathcal{R}}) T^{\deg(v) j}. \end{aligned}$$

By Proposition 3.8, $(\mathrm{St}_{\deg(v) j})_{\mathcal{R}}$ is zero unless $\mathcal{R} \cong k_v^{\oplus j}$, in which case the Frobenius trace is $(-1)^{j(\deg(v)-1)} q_v^{j(j-1)/2}$. Let $\mathcal{Q}_v[\varpi]$ be the kernel of the action of a uniformizer ϖ at v. Note $V := \mathcal{Q}_v^\flat[\varpi]$ has dimension $t = t_v(\mathcal{Q}^\flat)$ over k_v. Then we only need to sum over k_v-subspaces $\mathcal{R}$ of V. The above sum becomes

$$\sum_{j=0}^{t} (-1)^j q_v^{j(j-1)/2} T^{\deg(v) j} \#\mathrm{Gr}(j, V)(k_v).$$

Recall that the "q-binomial theorem" says that $q_v^{j(j-1)/2}\#\mathrm{Gr}(j,V)(k_v)$ is the coefficient of x^j in $(1+x)(1+q_v x)\dots(1+q_v^{t-1}x)$. Making the change of variables $x=-T^{\deg(v)}$, we get

$$\sum_{j=0}^{t}(-1)^j q_v^{j(j-1)/2}T^{\deg(v)j}\#\mathrm{Gr}(j,V)(k_v)$$
$$=(1-T^{\deg(v)})(1-q_vT^{\deg(v)})\dots(1-q_v^{t-1}T^{\deg(v)})$$

as desired. □

Now we are ready to prove (5.4).

Proposition 5.7 *For $\mathcal{Q}\in\mathrm{Herm}_{2d}(X'/X)(k)$ with support Z, we have*

$$P_{\mathcal{Q}}(T)=\prod_{v\in|X|}\mathfrak{m}_v(t_v'(\mathcal{Q});T^{\deg(v)})=\prod_{v\in|Z|}\prod_{j=0}^{t_v'(\mathcal{Q})-1}(1-(\eta(\varpi_v)q_v)^jT^{\deg(v)}).$$

Proof By (5.6) it suffices to treat the case $\mathcal{Q}$ is supported over a single place v. Let $\mathcal{Q}^\flat\in\mathrm{Coh}_d(X)(k)$ be the corresponding point. If v is split, we have $P_{\mathcal{Q}}(T)=P_{\mathcal{Q}^\flat}(T)$ by (5.7), and the formula follows from Proposition 5.6.

If v is inert, let $t=t_v'(\mathcal{Q})=t_v(\mathcal{Q}^\flat)$. From the form of $P_{\mathcal{Q}^\flat}(T)$ computed in Proposition 5.6, which is valid for any extension of k, the trace of the pure weight pieces of $\mathfrak{P}_{\mathcal{Q}^\flat}(T)$ are separated by different powers of q_v, i.e., $q_v^{-i}\,\mathrm{Tr}(\mathrm{Fr},\mathrm{Gr}^W_{2i\deg(v)}\mathfrak{P}_{\mathcal{Q}^\flat}(T))$ is the coefficient of q_v^i in $\prod_{j=0}^{t-1}(1-q_v^jT^{\deg(v)})$. By (5.8),

$$P_{\mathcal{Q}}(T)=\sum_i(-1)^i\,\mathrm{Tr}(\mathrm{Fr},\mathrm{Gr}^W_{2i\deg(v)}\mathfrak{P}_{\mathcal{Q}^\flat}(T))=\prod_{j=0}^{t-1}(1-(-q_v)^jT^{\deg(v)})$$

which is what we want because $\eta(\varpi_v)=-1$ in this case. □

Part 2. The geometric side

6 Moduli of Hermitian shtukas

In this section we introduce some of the fundamental geometric objects in our story, in particular the moduli stacks of unitary (also called Hermitian) shtukas, which play an analogous role to that of unitary Shimura varieties in the work of Kudla-Rapoport.

6.1 Hermitian bundles

We adopt the notation of §1.3, and in particular for the remainder of the paper enforce the assumption that X is *proper*, and $\nu\colon X'\to X$ is a finite étale double cover (possibly trivial).

Definition 6.1 A rank n *Hermitian* (also called *unitary*) bundle on $X \times S$ with respect to $\nu : X' \to X$ is a vector bundle $\mathcal{F}$ of rank n on $X' \times S$, equipped with an isomorphism $h \colon \mathcal{F} \xrightarrow{\sim} \sigma^* \mathcal{F}^\vee$ such that $\sigma^* h^\vee = h$. We refer to h as the *Hermitian structure* on $\mathcal{F}$.

We denote by $\mathrm{Bun}_{U(n)}$ the moduli stack of rank n unitary bundles on X, which sends a test scheme S to the groupoid of rank n unitary bundles on $X \times S$. The notation is justified by the following remark.

Remark 6.2 There is an equivalence of categories between the groupoid of Hermitian bundles on $X \times S$, and the groupoid of G-torsors for the group scheme $G = U(n)$ over X defined as

$$\{g \in \mathrm{Res}_{X'/X} \mathrm{GL}_n \colon \sigma({}^t g^{-1}) = g\}.$$

Indeed, we choose a square root $\omega_X^{1/2}$ of ω_X (which exists over $k = \mathbf{F}_q$ by [27, p.291, Theorem 13]). Then $\mathcal{F}_1 := \nu^* \omega_{X'}^{1/2}$ is equipped with the canonical Hermitian structure $h_1 : \mathcal{F}_1 \cong \sigma^* \mathcal{F}_1 \cong \sigma^* \mathcal{F}_1^\vee$, and $(\mathcal{F}_n, h_n) := (\mathcal{F}_1, h_1)^{\oplus n}$ is a rank n Hermitian bundle on X whose automorphism group scheme is $U(n)$. To a Hermitian bundle $(\mathcal{F}, h)$ on $X \times S$, $\underline{\mathrm{Isom}}_{X \times S}((\mathcal{F}_n \boxtimes \mathcal{O}_S, h_n \boxtimes \mathrm{Id}), (\mathcal{F}, h))$ (the scheme of unitary isometries) is a right torsor for $U(n)$ over $X \times S$. Conversely, for a right $U(n)$-torsor $\mathcal{G}$ over $X \times S$, the contracted product $\mathcal{G} \overset{U(n)}{\times} \mathcal{F}_n$ is a Hermitian bundle on $X \times S$.

6.2 Hecke stacks

We now define some particular Hecke correspondences for $\mathrm{Bun}_{U(n)}$.

Definition 6.3 Let $r \geq 0$ be an integer. The *Hecke stack* $\mathrm{Hk}^r_{U(n)}$ has as S-points the groupoid of the following data:

(1) $x_i' \in X'(S)$ for $i = 1, \ldots, r$, with graphs denoted by $\Gamma_{x_i'} \subset X' \times S$.
(2) A sequence of vector bundles $\mathcal{F}_0, \ldots, \mathcal{F}_r$ of rank n on $X' \times S$, each equipped with Hermitian structure $h_i \colon \mathcal{F}_i \xrightarrow{\sim} \sigma^* \mathcal{F}_i^\vee$.
(3) Isomorphisms $f_i \colon \mathcal{F}_{i-1}|_{X' \times S - \Gamma_{x_i'} - \Gamma_{\sigma(x_i')}} \xrightarrow{\sim} \mathcal{F}_i|_{X' \times S - \Gamma_{x_i'} - \Gamma_{\sigma(x_i')}}$, for $1 \leq i \leq r$, compatible with the Hermitian structures, with the following property: there exists a rank n vector bundle $\mathcal{F}^\flat_{i-1/2}$ and a diagram of vector bundles

$$\begin{array}{ccc} & \mathcal{F}^\flat_{i-1/2} & \\ {\scriptstyle f_i^{\leftarrow}} \swarrow & & \searrow {\scriptstyle f_i^{\rightarrow}} \\ \mathcal{F}_{i-1} & & \mathcal{F}_i \end{array}$$

such that $\mathrm{coker}(f_i^{\leftarrow})$ is locally free of rank 1 over $\Gamma_{x_i'}$, and $\mathrm{coker}(f_i^{\rightarrow})$ is locally free of rank 1 over $\Gamma_{\sigma(x_i')}$. In particular, $f_i^{\leftarrow}$ and $f_i^{\rightarrow}$ are invertible upon

restriction to $X' \times S - \Gamma_{x_i'} - \Gamma_{\sigma(x_i')}$, and the composition

$$\mathcal{F}_{i-1}|_{X'\times S-\Gamma_{x_i'}-\Gamma_{\sigma(x_i')}} \xrightarrow{(f_i^{\leftarrow})^{-1}} \mathcal{F}^{\flat}_{i-1/2}|_{X'\times S-\Gamma_{x_i'}-\Gamma_{\sigma(x_i')}} \xrightarrow{f_i^{\rightarrow}} \mathcal{F}_i|_{X'\times S-\Gamma_{x_i'}-\Gamma_{\sigma(x_i')}}$$

agrees with f_i.

Remark 6.4 Condition (3) above is equivalent to asking for the existence of a diagram

$$\begin{array}{ccccc} \mathcal{F}_{i-1} & & & & \mathcal{F}_i \\ & \searrow^{h_i^{\leftarrow}} & & \swarrow_{h_i^{\rightarrow}} & \\ & & \mathcal{F}^{\sharp}_{i-1/2} & & \end{array}$$

such that $\operatorname{coker}(h_i^{\leftarrow})$ is flat of length 1 over $\Gamma_{\sigma(x_i')}$, and $\operatorname{coker}(h_i^{\rightarrow})$ is flat of length 1 over $\Gamma_{x_i'}$. In particular, $h_i^{\leftarrow}$ and $h_i^{\rightarrow}$ are invertible upon restriction to $X' \times S - \Gamma_{x_i'} - \Gamma_{\sigma(x_i')}$, and the composition

$$\mathcal{F}_{i-1}|_{X'\times S-\Gamma_{x_i'}-\Gamma_{\sigma(x_i')}} \xrightarrow{h_i^{\leftarrow}} \mathcal{F}^{\sharp}_{i-1/2}|_{X'\times S-\Gamma_{x_i'}-\Gamma_{\sigma(x_i')}} \xrightarrow{(h_i^{\rightarrow})^{-1}} \mathcal{F}_i|_{X'\times S-\Gamma_{x_i'}-\Gamma_{\sigma(x_i')}}$$

agrees with f_i.

Definition 6.5 (Terminology for modifications of vector bundles) Given two vector bundles $\mathcal{F}$ and $\mathcal{F}'$ on $X' \times S$, we will refer to an isomorphism between $\mathcal{F}$ and $\mathcal{F}'$ on the complement of a relative Cartier divisor $D \subset X' \times S$ as a "modification" between $\mathcal{F}$ and $\mathcal{F}'$, and denote such a modification by $\mathcal{F} \dashrightarrow \mathcal{F}'$. Given $x, y \in X'(S)$, we say that the modification is *"lower" of length* 1 *at x and "upper" of length* 1 *at y* if it is as in Definition 6.3 (3), i.e. if there exists a diagram

$$\begin{array}{ccccc} & & \mathcal{F}^{\flat} & & \\ & \swarrow^{f^{\leftarrow}} & & \searrow^{f^{\rightarrow}} & \\ \mathcal{F} & & & & \mathcal{F}' \end{array}$$

such that $\operatorname{coker}(f^{\leftarrow})$ is flat of length 1 over Γ_x, and $\operatorname{coker}(f^{\rightarrow})$ is flat of length 1 over Γ_y, and $\mathcal{F} \dashrightarrow \mathcal{F}'$ agrees with the composition

$$\mathcal{F}|_{X'\times S-\Gamma_x-\Gamma_y} \xrightarrow{(f^{\leftarrow})^{-1}} \mathcal{F}^{\flat}|_{X'\times S-\Gamma_x-\Gamma_y} \xrightarrow{f^{\rightarrow}} \mathcal{F}|_{X'\times S-\Gamma_x-\Gamma_y}.$$

The condition admits a reformulation as in Remark 6.4.

6.3 Hermitian shtukas

For a vector bundle $\mathcal{F}$ on $X' \times S$, we denote by ${}^{\tau}\mathcal{F} := (\mathrm{Id}_{X'} \times \mathrm{Fr}_S)^*\mathcal{F}$. If $\mathcal{F}$ has a Hermitian structure $h\colon \mathcal{F} \xrightarrow{\sim} \sigma^*\mathcal{F}^{\vee}$, then ${}^{\tau}\mathcal{F}$ is equipped with the Hermitian structure ${}^{\tau}h$; we may suppress this notation when we speak of the "Hermitian bundle" ${}^{\tau}\mathcal{F}$.

Definition 6.6 Let $r \geq 0$ be an integer. We define $\mathrm{Sht}^r_{U(n)}$ by the Cartesian diagram

$$\begin{array}{ccc} \mathrm{Sht}^r_{U(n)} & \longrightarrow & \mathrm{Hk}^r_{U(n)} \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{U(n)} & \xrightarrow{(\mathrm{Id},\mathrm{Fr})} & \mathrm{Bun}_{U(n)} \times \mathrm{Bun}_{U(n)} \end{array}$$

A point of $\mathrm{Sht}^r_{U(n)}$ will be called a "$U(n)$-shtuka".

Concretely, the S-points of $\mathrm{Sht}^r_{U(n)}$ are given by the groupoid of the following data:

(1) $x'_i \in X'(S)$ for $i = 1, \ldots, r$, with graphs denoted $\Gamma_{x'_i} \subset X \times S$. These are called the *legs* of the shtuka.
(2) A sequence of vector bundles $\mathcal{F}_0, \ldots, \mathcal{F}_n$ of rank n on $X' \times S$, each equipped with a Hermitian structure $h_i : \mathcal{F}_i \xrightarrow{\sim} \sigma^* \mathcal{F}_i^\vee$.
(3) Isomorphisms $f_i : \mathcal{F}_{i-1}|_{X' \times S - \Gamma_{x'_i} - \Gamma_{\sigma(x'_i)}} \xrightarrow{\sim} \mathcal{F}_i|_{X' \times S - \Gamma_{x'_i} - \Gamma_{\sigma(x'_i)}}$ compatible with the Hermitian structure, which as modifications of the underlying vector bundles on $X' \times S$ are lower of length 1 at x'_i and upper of length 1 at $\sigma(x'_i)$.
(4) An isomorphism $\varphi : \mathcal{F}_r \cong {}^\tau \mathcal{F}_0$ compatible with the Hermitian structure.

Lemma 6.7 *The stack $\mathrm{Sht}^r_{U(n)}$ is empty if and only if r is odd.*

Proof We first treat the case $n = 1$. Let $\mathrm{Nm}_{X'/X} : \mathrm{Pic}_{X'} \to \mathrm{Pic}_X$ be the norm map. Then $\mathrm{Bun}_{U(1)} \cong \mathrm{Nm}^{-1}_{X'/X}(\omega_X)$, hence it is a torsor under $\mathrm{Prym}(X'/X) = \ker(\mathrm{Nm}_{X'/X})$. Moreover, $\mathrm{Sht}^r_{U(1)}$ fits into a Cartesian square

$$\begin{array}{ccc} \mathrm{Sht}^r_{U(1)} & \longrightarrow & \mathrm{Bun}_{U(1)} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Lang}} \\ (X')^r & \longrightarrow & \mathrm{Prym}(X'/X) \end{array} \qquad \begin{array}{c} \mathcal{F} \\ \downarrow \\ \mathcal{F}^{-1} \otimes {}^\tau \mathcal{F} \end{array}$$

with the bottom horizontal map sending $D \mapsto \mathcal{O}(D - \sigma D)$. If X' is geometrically connected, then the stack $\mathrm{Prym}(X'/X)$ has two connected components, and by a result of Wirtinger, explained in [21, §2], the bottom horizontal map lands in the identity component if and only if r is even. If X' is geometrically disconnected (i.e. it is either $X \coprod X$ or $X_{k'}$), then we have $\pi_0(\mathrm{Prym}(X'/X)_{\overline{k}}) \cong \mathbf{Z}$, the Lang map lands in (therefore surjects onto, by Lang's Theorem) the identity component, and the bottom horizontal map hits the identity component if and only if r is even. This shows that, in all cases, $\mathrm{Sht}^r_{U(1)}$ is empty if and only if r is odd.

For general n, taking determinant of a hermitian shtuka gives a map $\mathrm{Sht}^r_{U(n)} \to \mathrm{Sht}^r_{U(1)}$. From this we see that if r is odd, then $\mathrm{Sht}^r_{U(n)}$ is empty for any n since $\mathrm{Sht}^r_{U(1)}$ is empty.

On the other hand if r is even, then $\mathrm{Sht}^r_{U(1)}$ is non-empty. If $n > 1$, from an S point of $\mathrm{Sht}^r_{U(1)}$, we can produce an S-point of $\mathrm{Sht}^r_{U(n)}$ by formation of direct sum with (the base change to $X \times S$ of) a unitary bundle of rank $n - 1$ on X (e.g. we can take $(\mathcal{F}_{n-1}, h_{n-1})$ from Remark 6.2). □

6.4 Geometric properties

Lemma 6.8 *The stack* $\mathrm{Bun}_{U(n)}$ *is smooth and equidimensional.*

Proof The standard tangent complex argument, cf. [7, Prop. 1]. □

Lemma 6.9 (1) *The projection map* $(\mathrm{pr}_X, \mathrm{pr}_r) : \mathrm{Hk}^r_{U(n)} \to (X')^r \times \mathrm{Bun}_{U(n)}$ *recording* $\{x_i\}$ *and* $(\mathcal{F}_r, h_r)$ *is smooth of relative dimension* $r(n-1)$.

(2) $\mathrm{Sht}^r_{U(n)}$ *is a Deligne-Mumford stack locally of finite type. The map* $\mathrm{Sht}^r_{U(n)} \to (X')^r$ *is smooth, separated, equidimensional of relative dimension* $r(n-1)$.

Proof The statements about $\mathrm{Sht}^r_{U(n)}$ being locally finite type and separated are well-known properties of moduli of G-shtukas for general G [24, Proposition 2.16 and Theorem 2.20].[3]

Part (2) follows from (1) by [14, Lemma 2.13].

So it suffices to check (1). As a self-correspondence of $\mathrm{Bun}_{U(n)}$, $\mathrm{Hk}^r_{U(n)}$ is the r-fold composition of $\mathrm{Hk}^1_{U(n)}$. This allows us to reduce to the case $r = 1$. In this case, the map $(\mathrm{pr}_X, \mathrm{pr}_1) : \mathrm{Hk}^1_{U(n)} \to X' \times \mathrm{Bun}_{U(n)}$ exhibits $\mathrm{Hk}^1_{U(n)}$ as a $\mathbf{P}^{n-1}$-bundle whose fiber over $(x', \mathcal{F}_1, h_1)$ classifies hyperplanes in $\mathcal{F}_{1,\sigma(x')}$. Indeed, a hyperplane in $\mathcal{F}_{1,\sigma(x')}$ determines a lower modification at $\sigma(x')$, and the upper modification at x' is then determined from the lower modification by the Hermitian structure. This shows that $(\mathrm{pr}_X, \mathrm{pr}_1)$ is smooth, separated and equidimensional of relative dimension $(n-1)$ in the case $r = 1$, and the general case follows. □

7 Special cycles: basic properties

In this section we define special cycles over the moduli stacks of hermitian shtukas, and construct corresponding cycle classes. The latter task is rather subtle, as the cycles are in most cases of a highly "derived" nature, with their "virtual dimension" differing significantly from their actual dimension.

7.1 Special cycles

Definition 7.1 Let $\mathcal{E}$ be a rank m vector bundle on X'.

We define the stack $\mathcal{Z}^r_{\mathcal{E}}$ whose S-points are the groupoid of the following data:

- A $U(n)$-shtuka with $(\{x'_1, \ldots, x'_r\}, \{\mathcal{F}_0, \ldots, \mathcal{F}_r\}, \{f_1, \ldots, f_r\}, \varphi) \in \mathrm{Sht}^r_{U(n)}(S)$.
- Maps of coherent sheaves $t_i : \mathcal{E} \boxtimes \mathcal{O}_S \to \mathcal{F}_i$ on $X' \times S$ such that the isomorphism $\varphi : \mathcal{F}_r \cong {}^{\tau}\mathcal{F}_0$ intertwines t_r with ${}^{\tau}t_0$, and the maps t_{i-1}, t_i are intertwined by the

[3] See also [28, paragraph after Theorem 5.4] for a sketch of the separatedness in a similar situation, which readily adapts here.

modification $f_i : \mathcal{F}_{i-1} \dashrightarrow \mathcal{F}_i$ for each $i = 1, \ldots, r$, i.e. the diagram below commutes.

$$\begin{array}{ccccccccc}
\mathcal{E}\boxtimes\mathcal{O}_S & = & \mathcal{E}\boxtimes\mathcal{O}_S & = & \cdots & = & \mathcal{E}\boxtimes\mathcal{O}_S & \xrightarrow{\sim} & {}^{\tau}(\mathcal{E}\boxtimes\mathcal{O}_S) \\
\downarrow{\scriptstyle t_0} & & \downarrow{\scriptstyle t_1} & & \downarrow & & \downarrow{\scriptstyle t_r} & & \downarrow{\scriptstyle {}^{\tau}t_0} \\
\mathcal{F}_0 & \overset{f_0}{\dashrightarrow} & \mathcal{F}_1 & \overset{f_1}{\dashrightarrow} & \cdots & \overset{f_r}{\dashrightarrow} & \mathcal{F}_r & \xrightarrow{\sim} & {}^{\tau}\mathcal{F}_0
\end{array}$$

In the sequel, when writing such diagrams we will usually just omit the "$\boxtimes\mathcal{O}_S$" factor from the notation.

We will call the $\mathcal{Z}^r_{\mathcal{E}}$ (or their connected components) *special cycles of corank m* (with r legs).

There is an evident map $\mathcal{Z}^r_{\mathcal{E}} \to \mathrm{Sht}^r_{U(n)}$ projecting to the data in the first bullet point. When $\mathrm{rank}\,\mathcal{E} = 1$, the $\mathcal{Z}^r_{\mathcal{E}}$ are function field analogues (with multiple legs) of the *Kudla-Rapoport divisors* introduced in [12, 13].

7.2 Indexing via Hermitian maps

Definition 7.2 Let $\mathcal{A}^{\mathrm{all}}_{\mathcal{E}}(k)$ be the k-vector space of Hermitian maps[4] $a : \mathcal{E} \to \sigma^*\mathcal{E}^\vee$ such that $\sigma(a)^\vee = a$. Let $\mathcal{A}_{\mathcal{E}}(k) \subset \mathcal{A}^{\mathrm{all}}_{\mathcal{E}}(k)$ be the subset where the map $a : \mathcal{E} \to \sigma^*\mathcal{E}^\vee$ is injective (as a map of coherent sheaves).

Let $(\{x'_i\}, \{\mathcal{F}_i\}, \{f_i\}, \varphi, \{t_i\}) \in \mathcal{Z}^r_{\mathcal{E}}(S)$. By the compatibilities between the t_i in the definition of $\mathcal{Z}^r_{\mathcal{E}}$, the compositions

$$\mathcal{E}\boxtimes\mathcal{O}_S \xrightarrow{t_i} \mathcal{F}_i \xrightarrow{h_i} \sigma^*\mathcal{F}_i^\vee \xrightarrow{\sigma^* t_i^\vee} \sigma^*\mathcal{E}^\vee \boxtimes \mathcal{O}_S \tag{7.1}$$

agree for each i, and (7.1) for $i = r$ also agrees with the Frobenius twist of (7.1) for $i = 0$. Hence (7.1) for every i gives the same map $\mathcal{E}\boxtimes\mathcal{O}_S \to \mathcal{E}^\vee \boxtimes \mathcal{O}_S$, which moreover must come by pullback from $\mathcal{A}^{\mathrm{all}}_{\mathcal{E}}(k)$. This defines a map $\mathcal{Z}^r_{\mathcal{E}} \to \mathcal{A}^{\mathrm{all}}_{\mathcal{E}}(k)$. For $a \in \mathcal{A}^{\mathrm{all}}_{\mathcal{E}}(k)$, we denote by $\mathcal{Z}^r_{\mathcal{E}}(a)$ the fiber of $\mathcal{Z}^r_{\mathcal{E}}$ over a. We have

$$\mathcal{Z}^r_{\mathcal{E}} = \coprod_{a \in \mathcal{A}^{\mathrm{all}}_{\mathcal{E}}(k)} \mathcal{Z}^r_{\mathcal{E}}(a).$$

Definition 7.3 For $a \in \mathcal{A}_{\mathcal{E}}(k)$, let D_a be the effective divisor on X such that $\nu^{-1}(D_a)$ is the divisor of the Hermitian map $\det(a) : \det(\mathcal{E}) \to \sigma^* \det(\mathcal{E})^\vee$.

Definition 7.4 For any $a \in \mathcal{A}^{\mathrm{all}}_{\mathcal{E}}(k)$, we define:

- $\mathcal{Z}^r_{\mathcal{E}}(a)^\circ \subset \mathcal{Z}^r_{\mathcal{E}}(a)$ to be the open substack classifying $(\{x'_i\}, \{\mathcal{E} \xrightarrow{t_i} \mathcal{F}_i\})$ with the additional condition that all t_i are *injective* when restricted to $X'_{\overline{s}}$ for any geometric point $\overline{s}$ of the test scheme S.

[4] We will later in §8.2 introduce a space $\mathcal{A}^{\mathrm{all}}$ over $\mathrm{Bun}_{\mathrm{GL}'_m}$ for which $\mathcal{A}^{\mathrm{all}}_{\mathcal{E}}(k)$ is the k-rational points of the fiber over $\mathcal{E} \in \mathrm{Bun}_{\mathrm{GL}'_m}(k)$, justifying the notation.

- $\mathcal{Z}^r_{\mathcal{E}}(a)^* \subset \mathcal{Z}^r_{\mathcal{E}}(a)$ to be the open substack classifying $(\{x'_i\}, \{\mathcal{E} \xrightarrow{t_i} \mathcal{F}_i\})$ with the additional condition that all t_i are *non-zero* when restricted to $X'_{\overline{s}}$ for any geometric point $\overline{s}$ of the test scheme S.

Note that if $\mathrm{rank}(\mathcal{E}) = 1$, then the inclusion $\mathcal{Z}^r_{\mathcal{E}}(a)^\circ \hookrightarrow \mathcal{Z}^r_{\mathcal{E}}(a)^*$ is an isomorphism, and both include isomorphically into $\mathcal{Z}^r_{\mathcal{E}}(a)$ unless $a = 0$.

7.3 Finiteness properties

We next establish that the projection map $\mathcal{Z}^r_{\mathcal{E}}(a) \to \mathrm{Sht}^r_{U(n)}$ is finite, which will eventually allow us to construct cycle classes on $\mathrm{Sht}^r_{U(n)}$ associated to $\mathcal{Z}^r_{\mathcal{E}}(a)$.

Proposition 7.5 *Let $\mathcal{E}$ be any vector bundle of rank m on X' and let $a \in \mathcal{A}^{\mathrm{all}}_{\mathcal{E}}(k)$. Then the projection maps $\mathcal{Z}^r_{\mathcal{E}}(a) \to \mathrm{Sht}^r_{U(n)}$ and $\mathcal{Z}^r_{\mathcal{E}}(a)^* \to \mathrm{Sht}^r_{U(n)}$ are both finite.*

Proof Note that $\mathcal{Z}^r_{\mathcal{E}}(a)$ has a closed substack where the map $t\colon \mathcal{E} \to \mathcal{F}$ is 0, which projects isomorphically to $\mathrm{Sht}^r_{U(n)}$. The complement of this closed substack is $\mathcal{Z}^r_{\mathcal{E}}(a)^*$, so it suffices to show the finiteness of $\mathcal{Z}^r_{\mathcal{E}}(a)^* \to \mathrm{Sht}^r_{U(n)}$. We will show that it is proper and quasi-finite. First we establish the properness. It suffices to show this locally on the target, so we pick a Harder-Narasimhan polygon P for $\mathrm{Bun}_{U(n)}$ and consider the truncation $\mathrm{Bun}^{\leq P}_{U(n)}$. Define $\mathrm{Sht}^{r,\leq P}_{U(n)}$ to be the open substack of $\mathrm{Sht}^r_{U(n)}$ obtained as the pullback of $\mathrm{Bun}^{\leq P}_{U(n)} \hookrightarrow \mathrm{Bun}_{U(n)}$ via the tautological projection $\mathrm{pr}_0 : \mathrm{Sht}^r_{U(n)} \to \mathrm{Bun}_{U(n)}$ recording $\mathcal{F}_0$, and $\mathcal{Z}^{r,\leq P}_{\mathcal{E}}(a) \hookrightarrow \mathcal{Z}^r_{\mathcal{E}}(a)$ the analogous pullback.

We can then pick a sufficiently anti-ample vector bundle $\mathcal{E}'$ of rank m on X' and an injection $\iota\colon \mathcal{E}' \hookrightarrow \mathcal{E}$ so that the stack $\underline{\mathrm{Hom}}(\mathcal{E}', -)^{\leq P}$ parametrizing $\{(\mathcal{F} \in \mathrm{Bun}^{\leq P}_{U(n)}, t \in \mathrm{Hom}(\mathcal{E}', \mathcal{F}))\}$ forms a vector bundle over $\mathrm{Bun}^{\leq P}_{U(n)}$, with respect to the obvious projection map. Let $a' := (\sigma^* \iota^\vee) \circ a \circ \iota\colon \mathcal{E}' \to \mathcal{E}'^\vee$. Then we have a closed embedding $\mathcal{Z}^{r,\leq P}_{\mathcal{E}}(a) \hookrightarrow \mathcal{Z}^{r,\leq P}_{\mathcal{E}'}(a')$ cut out by the condition that the map $t\colon \mathcal{E}' \to \mathcal{F}$ factors through ι, which fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{Z}^{r,\leq P}_{\mathcal{E}}(a) & \hookrightarrow & \mathcal{Z}^{r,\leq P}_{\mathcal{E}'}(a') \\ & \searrow & \downarrow \\ & & \mathrm{Sht}^{r,\leq P}_{U(n)}\,. \end{array}$$

It suffices to show the open substack $\mathcal{Z}^{r,\leq P}_{\mathcal{E}'}(a')^*$ defined by the condition that $t_0 \neq 0$ fiberwise over the test scheme is proper over $\mathrm{Sht}^{r,\leq P}_{U(n)}$. We can factorize this map as the composition of two maps in the diagram below:

$$\begin{array}{ccc} \mathcal{Z}^{r,\leq P}_{\mathcal{E}'}(a')^* & \xrightarrow{j} & \mathbf{P}(\underline{\mathrm{Hom}}(\mathcal{E}', -)^{\leq P}) \times_{\mathrm{Bun}^{\leq P}_{U(n)}} \mathrm{Sht}^{r,\leq P}_{U(n)} \\ & \searrow & \downarrow \mathrm{pr}_2 \\ & & \mathrm{Sht}^{r,\leq P}_{U(n)} \end{array}$$

where j is determined by the map $\mathcal{Z}^{r,\leq P}_{\mathcal{E}'}(a')^* \to \mathbf{P}(\underline{\mathrm{Hom}}(\mathcal{E}', -)^{\leq P})$ sending

$$(\mathcal{F}_0 \dashrightarrow \ldots \dashrightarrow \mathcal{F}_r \cong {}^{\tau}\mathcal{F}_0, (t_i)_{i=0}^r) \mapsto (\mathcal{F}_0, t_0 \colon \mathcal{E}' \to \mathcal{F}_0).$$

The map pr_2 is a projective bundle by design, so in order to establish that $\mathrm{pr}_2 \circ j$ is proper it suffices to show that j is finite. Indeed, since data of all the t_i is determined by t_0, the analogous map $\mathcal{Z}^{r,\leq P}_{\mathcal{E}'}(a')^* \to \underline{\mathrm{Hom}}(\mathcal{E}', -)^{\leq P} \times_{\mathrm{Bun}^{\leq P}_{U(n)}} \mathrm{Sht}^{r,\leq P}_{U(n)}$ is a closed embedding. The requirement $t_0 = {}^{\tau}t_0$ in the definition of $\mathcal{Z}^r_{\mathcal{E}'}$ therefore implies that the map j is a $k^{\times}$-torsor onto its image, which is a closed substack of $\mathbf{P}(\underline{\mathrm{Hom}}(\mathcal{E}', -)^{\leq P})$. This completes the proof of properness.

It remains to show that $\mathcal{Z}^r_{\mathcal{E}}(a) \to \mathrm{Sht}^r_{U(n)}$ is quasi-finite. Since the map is already established to be proper, it suffices by [23, Tag 01TC] to check that the fibers over field-valued points are finite. Let

$$(\{x_i'\}_{1\leq i\leq r}, (\mathcal{F}_0, h_0) \dashrightarrow (\mathcal{F}_1, h_1) \dashrightarrow \ldots (\mathcal{F}_r, h_r) \xrightarrow{\sim} ({}^{\tau}\mathcal{F}_0, {}^{\tau}h_0)) \in \mathrm{Sht}^r_{U(n)}(\kappa)$$

be such a point valued in a field κ. Its fiber in $\mathcal{Z}^r_{\mathcal{E}}(a)(\kappa)$ consists of $\{t_i \colon \mathcal{E} \to \mathcal{F}_i\}_{0\leq i\leq r}$ fitting into commutative diagrams

$$\begin{array}{ccccccccc}
\mathcal{E}_\kappa & = & \mathcal{E}_\kappa & = & \ldots & = & \mathcal{E}_\kappa & \xrightarrow{\sim} & {}^{\tau}\mathcal{E}_\kappa \\
\downarrow{\scriptstyle t_0} & & \downarrow{\scriptstyle t_1} & & & & \downarrow{\scriptstyle t_{r-1}} & & \downarrow{\scriptstyle {}^{\tau}t_0} \\
\mathcal{F}_0 & \dashrightarrow & \mathcal{F}_1 & \longrightarrow & \ldots & \dashrightarrow & \mathcal{F}_{r-1} & \xrightarrow{\sim} & {}^{\tau}\mathcal{F}_0
\end{array}$$

such that $\sigma^* t_i^{\vee} \circ h_i \circ t_i = a \in \mathcal{A}^{\mathrm{all}}_{\mathcal{E}}(\kappa)$ for each $i = 0, \ldots, r$. We want to show that there are finitely many possibilities for such $t_i \in \mathrm{H}^0(X'_\kappa, \mathcal{E}_\kappa^{\vee} \otimes \mathcal{F}_i)$.

The situation can be abstracted to the following semi-linear algebra problem.

Lemma 7.6 *Suppose that κ is any field over k, and we have finite-dimensional κ-vector spaces $V_1, V_2 \subset V$ with an injective* Fr*-semi-linear map $\tau \colon V_1 \hookrightarrow V_2$.*

$$\begin{array}{ccc}
V_1 & \xrightarrow{\tau} & V_2 \\
& \searrow \quad \swarrow & \\
& V &
\end{array}$$

Then the set $\{x \in V_1 \colon \tau(x) = x \in V\}$ is finite.

We assume Lemma 7.6 for the moment and use it to conclude the proof of Proposition 7.5. We apply it to the situation above with $V_1 := \mathrm{Hom}_{X'_\kappa}(\mathcal{E}_\kappa, \mathcal{F}_0)$, $V_2 := \mathrm{Hom}_{X'_\kappa}({}^{\tau}\mathcal{E}_\kappa, {}^{\tau}\mathcal{F}_0)$, which are both viewed as subspaces of

$$V := \mathrm{Hom}_{X'_\kappa}\left(\mathcal{E}_\kappa, \mathcal{F}_0(\sum_{j=1}^{r}(x'_j + \sigma(x'_j)))\right)$$

by the obvious inclusion. The map $V_1 \to V_2$ is the twist by τ. Then Lemma 7.6 shows that there are finitely many possibilities for t_0 since $\tau(t_0) = t_0$. The other t_i

are determined by t_0 (if they exist) because the t_i as well as the modifications $\mathcal{F}_i \dashrightarrow \mathcal{F}_{i+1}$ are all isomorphisms over an open subset of X'_κ. □

Proof of Lemma 7.6 By replacing κ with an algebraic closure, it suffices to consider the case when κ is algebraically closed. Let us call a subspace $V'_1 \subset V_1$ "τ-fixed" if $\tau(V'_1) = V'_1 \subset V$. Since a sum of τ-fixed subspaces is evidently τ-fixed, there is a well-defined largest τ-fixed subspace $V_1^\circ \subset V_1$. It is a sub κ-vector space of V_1, hence necessarily finite-dimensional. Since $\tau : V_1 \to V_2$ is injective, the restriction of τ to V_1° is a Fr-semi-linear bijection. The set $\{x \in V_1 : \tau(x) = x \in V\}$ is evidently contained in $(V_1^\circ)^\tau$, which is an k-form of V_1° (because κ is algebraically closed) and therefore finite-dimensional over k. □

7.4 Variation with $\mathcal{E}$

Let $\mathcal{E}', \mathcal{E}$ be two vector bundles (with possibly distinct ranks) on X' and $s : \mathcal{E}' \to \mathcal{E}$ be a map of coherent sheaves. Given $a : \mathcal{E} \to \sigma^*\mathcal{E}^\vee$ in $\mathcal{A}^{\mathrm{all}}_{\mathcal{E}}(k)$, let $a' = (\sigma^* s^\vee) \circ a \circ s : \mathcal{E}' \to \sigma^*(\mathcal{E}')^\vee$ be the corresponding element in $\mathcal{A}^{\mathrm{all}}_{\mathcal{E}'}(k)$. Therefore, composing with s defines a map

$$z_s : \mathcal{Z}^r_{\mathcal{E}}(a) \to \mathcal{Z}^r_{\mathcal{E}'}(a') \tag{7.2}$$

sending

$$\begin{array}{ccccccccc}
\mathcal{E} & = & \cdots & \dashrightarrow & \mathcal{E} & = & {}^\tau\mathcal{E} & & \mathcal{E}' & = & \cdots & \dashrightarrow & \mathcal{E}' & = & {}^\tau\mathcal{E}' \\
\downarrow & & & & \downarrow & & \downarrow & \mapsto & \downarrow & & & & \downarrow & & \downarrow \\
\mathcal{F}_0 & \dashrightarrow & \cdots & \dashrightarrow & \mathcal{F}_r & \xrightarrow{\sim} & {}^\tau\mathcal{F}_0 & & \mathcal{F}_0 & \dashrightarrow & \cdots & \dashrightarrow & \mathcal{F}_r & \xrightarrow{\sim} & {}^\tau\mathcal{F}_0
\end{array}$$

The following lemma follows directly from definitions.

Lemma 7.7 *If $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$, and $a_i \in \mathcal{A}^{\mathrm{all}}_{\mathcal{E}_i}(k)$ for $i = 1, 2$, then there is a canonical isomorphism*

$$\mathcal{Z}^r_{\mathcal{E}_1}(a_1) \times_{\mathrm{Sht}^r_{U(n)}} \mathcal{Z}^r_{\mathcal{E}_2}(a_2) \cong \coprod_{a = \begin{pmatrix} a_1 & * \\ * & a_2 \end{pmatrix}} \mathcal{Z}^r_{\mathcal{E}}(a)$$

where the union runs over all Hermitian maps $a : \mathcal{E} \to \sigma^*\mathcal{E}^\vee$ whose restriction to $\mathcal{E}_i$ is a_i (for $i = 1, 2$). The map from the right side to the left is given by $(z_{\iota_1}, z_{\iota_2})$, where $\iota_i : \mathcal{E}_i \hookrightarrow \mathcal{E}$ is the inclusion.

Lemma 7.8 *Under the notations of the beginning of this subsection,*

(1) *If $s : \mathcal{E}' \to \mathcal{E}$ is generically surjective, then $z_s : \mathcal{Z}^r_{\mathcal{E}}(a) \to \mathcal{Z}^r_{\mathcal{E}'}(a')$ is a closed embedding.*

(2) *Suppose that s is generically an isomorphism (in particular $\mathcal{E}$ and $\mathcal{E}'$ have the same rank). Let $D_s \subset X'$ be the divisor of* $\det(s)$. *Then the restriction of z_s over $(X\backslash\nu(D_s))^r$*

$$\mathcal{Z}^r_{\mathcal{E}}(a)|_{(X\backslash\nu(D_s))^r} \subset \mathcal{Z}^r_{\mathcal{E}'}(a')|_{(X\backslash\nu(D_s))^r}$$

is open and closed. Here we write $\mathcal{Z}^r_{\mathcal{E}}(a)|_{(X\backslash\nu(D_s))^r}$ for the preimage of $(X\backslash\nu(D_s))^r$ under the leg map $\mathcal{Z}^r_{\mathcal{E}}(a) \to \mathrm{Sht}^r_{U(n)} \to X'^r \xrightarrow{\nu^r} X^r$.

Proof (1) Let $\overline{\mathcal{E}}' = \mathcal{E}'/\ker(s) \xrightarrow{\bar{s}} \mathcal{E}$, equipped with the induced Hermitian map $\overline{a}' : \overline{\mathcal{E}}' \to \sigma^*(\overline{\mathcal{E}}')^\vee$. Then s^* factors as $\mathcal{Z}^r_{\mathcal{E}}(a) \xrightarrow{z_{\bar{s}}} \mathcal{Z}^r_{\overline{\mathcal{E}}'}(\overline{a}') \subset \mathcal{Z}^r_{\mathcal{E}'}(a')$, the latter being evidently a closed embedding. Therefore it suffices to show $z_{\bar{s}}$ is a closed embedding. We thus reduce to the case s is generically an isomorphism.

Let D be an effective divisor on X' such that $\mathcal{E}(-D) \hookrightarrow \mathcal{E}' \xrightarrow{s} \mathcal{E}$. Let $\mathcal{F}^{\mathrm{univ}}$ be the universal Hermitian bundle over $X' \times \mathrm{Bun}_{U(n)}$, and $\mathcal{F}^{\mathrm{univ}}_D$ its restriction to $D \times \mathrm{Bun}_{U(n)}$. Let $\mathcal{V}_D = \mathrm{pr}_{2*}\,\underline{\mathrm{Hom}}(\mathrm{pr}_1^*\,\mathcal{E}(-D)|_D, \mathcal{F}^{\mathrm{univ}}_D)$, where $\mathrm{pr}_1, \mathrm{pr}_2$ are the projections of $D \times \mathrm{Bun}_{U(n)}$ to the two factors. Then $\mathcal{V}_D$ is a vector bundle of rank equal to $n\,\mathrm{rank}(\mathcal{E})\deg(D)$ over $\mathrm{Bun}_{U(n)}$. Let $\mathcal{V}_{i,D}$ be the pullback of $\mathcal{V}_D$ over $\mathcal{Z}^r_{\mathcal{E}'}(a')$ via the map $\mathrm{pr}_i : \mathcal{Z}^r_{\mathcal{E}'}(a') \to \mathrm{Bun}_{U(n)}$. Then $\mathcal{V}_{i,D}$ has a section v_i whose value at $(\{x'_j\}, \{\mathcal{F}_j\}, \{t_j : \mathcal{E}' \to \mathcal{F}_j\}) \in \mathcal{Z}^r_{\mathcal{E}'}(a')$ is the restriction of t_i to $\mathcal{E}(-D)|_D \to \mathcal{F}_i|_D$. Then t_i extends to $\mathcal{E}$ if and only if v_i vanishes at the point $(\{x'_j\}, \{\mathcal{F}_j\}, \{t_j\})$. This identifies $\mathcal{Z}^r_{\mathcal{E}}(a)$ as the common zero locus of the sections $(v_i)_{0\le i\le r-1}$ of the vector bundles $\mathcal{V}_{i,D}$ over $\mathcal{Z}^r_{\mathcal{E}'}(a')$, hence closed in $\mathcal{Z}^r_{\mathcal{E}'}(a')$.

(2) By (1), it remains to show the openness of z_s when restricted to $(X\backslash\nu(D_s))^r$. Let $U' = X'\backslash\mathrm{supp}(D_s + \sigma D_s)$. Let $\mathrm{Sht}^r_{U(n),D_s}$ be the moduli stack of Hermitian shtukas $(\{x'_i\}, \{\mathcal{F}_i\})$ of rank n with legs in U'^r, and trivializations of $\mathcal{F}_i|_{D_s}$ (as a vector bundle over D_s of rank n) compatible with the shtuka structures. Then $\lambda : \mathrm{Sht}^r_{U(n),D_s} \to \mathrm{Sht}^r_{U(n)}|_{U'^r}$ is a $\mathrm{GL}_n(\mathcal{O}_{D_s})$-torsor. Let $\mathcal{Z}^r_{\mathcal{E}}(a)_{D_s}$ and $\mathcal{Z}^r_{\mathcal{E}'}(a')_{D_s}$ be the base changes of $\mathcal{Z}^r_{\mathcal{E}}(a)$ and $\mathcal{Z}^r_{\mathcal{E}'}(a')$ along λ. Since $\mathcal{Z}^r_{\mathcal{E}'}(a')_{D_s} \to \mathcal{Z}^r_{\mathcal{E}'}(a')|_{U'^r}$ is finite étale surjective, it suffices to show that the inclusion $\mathcal{Z}^r_{\mathcal{E}}(a)_{D_s} \hookrightarrow \mathcal{Z}^r_{\mathcal{E}'}(a')_{D_s}$ is open. Using the trivializations of $\mathcal{F}_i|_{D_s}$, we get an evaluation map

$$\mathrm{ev}_{D_s} : \mathcal{Z}^r_{\mathcal{E}'}(a')_{D_s} \to \mathrm{Hom}_{D_s}(\mathcal{E}'|_{D_s}, \mathcal{O}_{D_s}^{\oplus n})$$

where the target is a discrete set. Then $\mathcal{Z}^r_{\mathcal{E}}(a)_{D_s}$ is the preimage of the image of

$$\mathrm{Hom}_{D_s}(\mathcal{E}|_{D_s}, \mathcal{O}_{D_s}^{\oplus n}) \xrightarrow{(-)\circ s} \mathrm{Hom}_{D_s}(\mathcal{E}'|_{D_s}, \mathcal{O}_{D_s}^{\oplus n})$$

under ev_{D_s}. Indeed, a map $\mathcal{E}' \to \mathcal{F}_i$ extends to $\mathcal{E} \to \mathcal{F}_i$ if and only if $\mathcal{E}'|_{D_s} \to \mathcal{F}_i|_{D_s}$ vanishes on $\ker(\mathcal{E}'|_{D_s} \to \mathcal{E}|_{D_s})$ (this can be checked locally using elementary divisors). Since the target of ev_{D_s} is discrete, $\mathcal{Z}^r_{\mathcal{E}}(a)_{D_s} \subset \mathcal{Z}^r_{\mathcal{E}'}(a')_{D_s}$ is open and closed. □

7.5 Corank 1 special cycles

A special role is played by the case $m = 1$, i.e. where $\mathcal{E}$ is a line bundle on X', because it is only in this case that we can appropriately control the dimension of the cycles $\mathcal{Z}^r_{\mathcal{E}}$. We will write $\mathcal{L} := \mathcal{E}$ to emphasize that it is a line bundle.

Note that in this case $a \in \mathcal{A}_{\mathcal{L}}(k)$ if and only if $a \neq 0$.

Proposition 7.9 *We have* $\dim \mathcal{Z}^r_{\mathcal{L}}(a)^\circ \leq r(n-1)$.

This is established later, in Proposition 9.1 (for $a \neq 0$) and Proposition 9.5 (for $a = 0$), as a consequence of a more refined study of the geometry of $\mathcal{Z}^r_{\mathcal{L}}(a)^\circ$.

Remark 7.10 One can show that when $a \neq 0$, in fact $\mathcal{Z}^r_{\mathcal{L}}(a)^\circ$ is LCI of pure dimension $r(n-1)$. This will appear in a future paper; it relies on some ideas from [4]. This fact will not be used in the present paper, but it may be psychologically helpful.

Definition 7.11 For $a \in \mathcal{A}^{\mathrm{all}}_{\mathcal{L}}(k)$, we define $[\mathcal{Z}^r_{\mathcal{L}}(a)^\circ] \in \mathrm{Ch}_{r(n-1)}(\mathcal{Z}^r_{\mathcal{L}}(a)^\circ)$ to be the cycle class of the union of the irreducible components of $\mathcal{Z}^r_{\mathcal{L}}(a)^\circ$ with dimension $r(n-1)$, throwing away the irreducible components of dimension $< r(n-1)$. (According to Remark 7.10, there are no such components to be thrown away at least when $a \neq 0$, but we neither prove nor use this in the present paper.)

7.6 Corank n special cycles

In this paper we are mainly concerned with the case where the rank of $\mathcal{E}$ is $m = n$. The following proposition contains basic geometric information about $\mathcal{Z}^r_{\mathcal{E}}(a)$.

Lemma 7.12 *Let $\mathcal{E}$ be a vector bundle on X' of rank n, and $a \in \mathcal{A}_{\mathcal{E}}(k)$. Then the map $\mathcal{Z}^r_{\mathcal{E}}(a) \to X'^r$ recording the legs has image in* $(\mathrm{supp}\, \nu^{-1}(D_a))^r$.

Proof Let $(\{x'_i\}, \{\mathcal{F}_i\}, \{t_i\})$ be a geometric point of $\mathcal{Z}^r_{\mathcal{E}}(a)$. For each $1 \leq i \leq r$, the Hermitian map a factorizes as $\mathcal{E} \hookrightarrow \mathcal{F}^\flat_{i-1/2} \hookrightarrow \mathcal{F}_{i-1} \hookrightarrow \sigma^*\mathcal{E}^\vee$, we see that x'_i (the support of $\mathcal{F}_{i-1}/\mathcal{F}^\flat_{i-1/2}$) is in the support of $\sigma^*\mathcal{E}^\vee/a(\mathcal{E})$, i.e., $x'_i \in \mathrm{supp}\, \nu^{-1}(D_a)$. □

Proposition 7.13 *Let $\mathcal{E}$ be a vector bundle on X' of rank n, and $a \in \mathcal{A}_{\mathcal{E}}(k)$. Then $\mathcal{Z}^r_{\mathcal{E}}(a)$ is a proper scheme over k that depends only on the torsion sheaf $\mathcal{Q} = \mathrm{coker}(a) = \sigma^*\mathcal{E}^\vee/\mathcal{E}$ together with the Hermitian structure $\overline{a}$ on $\mathcal{Q}$ induced from a (see §4.1 for the notion of Hermitian structure on a torsion sheaf).*

The proof involves a few ideas not yet introduced, and will be given later in §8.4.1.

The next goal is to equip the proper scheme $\mathcal{Z}^r_{\mathcal{E}}(a)$ with a 0-cycle class in its Chow group. The "virtual dimension" of $\mathcal{Z}^r_{\mathcal{E}}(a)$ is at most zero, for if $\mathcal{E}$ is a direct sum of line bundles $\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$, then $\mathcal{Z}^r_{\mathcal{E}}(a)$ is contained in the intersection of $Z^r_{\mathcal{L}_i}(a_{ii})$ for $1 \leq i \leq r$, each having codimension at least r in $\mathrm{Sht}^r_{U(n)}$ by Proposition 7.9 (which can be shown to be an equality, cf. Remark 7.10). However the actual dimension of $\mathcal{Z}^r_{\mathcal{E}}(a)$ can be strictly positive. Our task is to find the correct virtual fundamental class of $\mathcal{Z}^r_{\mathcal{E}}(a)$.

7.7 Intersection theory on stacks

Recall the discussion of intersection theory on Deligne-Mumford stacks from [28, Appendix A]. Let Y be a smooth, separated, locally finite type Deligne-Mumford stack over k of pure dimension d. Let $Y_1, \dots, Y_n$ be Deligne-Mumford stacks with maps $f_i : Y_i \to Y$. Then there is an intersection product

$$\begin{aligned}(-) \cdot_Y (-) \cdot_Y \cdots \cdot_Y (-) \colon\ & \mathrm{Ch}_{i_1}(Y_1) \times \mathrm{Ch}_{i_2}(Y_2) \times \cdots \times \mathrm{Ch}_{i_n}(Y_n) \\ & \to \mathrm{Ch}_{i_1+\cdots+i_n-d(n-1)}(Y_1 \times_Y \cdots \times_Y Y_n).\end{aligned}$$

For $\zeta_i \in \mathrm{Ch}_*(Y_i)$, the intersection product $\zeta_1 \cdot_Y \cdots \cdot_Y \zeta_n$ is defined as the Gysin pull-back of the external product $\zeta_1 \times \cdots \times \zeta_n \in \mathrm{Ch}_*(Y_1 \times \cdots \times Y_n)$ along the diagonal map $\Delta : Y \to Y^n$, which is a regular embedding of codimension $d(n-1)$.

7.8 Intersection problem: the case of a direct sum of line bundles

We now formulate the cycle classes which enter into our intersection problem. We first consider the case $\mathcal{E}$ a direct sum of m line bundles on X',

$$\mathcal{E} \cong \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_m.$$

Let $a \in \mathcal{A}_{\mathcal{E}}(k)$. We write a as an $m \times m$-matrix with entries $a_{ij} \in \mathrm{Hom}(\mathcal{L}_j, \sigma^* \mathcal{L}_i^\vee)$.

Let

$$\mathcal{Z}^r_{\mathcal{L}_1,\dots,\mathcal{L}_m}(a_{11},\dots,a_{mm})^\circ := \mathcal{Z}^r_{\mathcal{L}_1}(a_{11})^\circ \times_{\mathrm{Sht}^r_{U(n)}} \cdots \times_{\mathrm{Sht}^r_{U(n)}} \mathcal{Z}^r_{\mathcal{L}_m}(a_{mm})^\circ.$$

In Definition 7.11 we defined a fundamental class $[\mathcal{Z}^r_{\mathcal{L}}(a)^\circ] \in \mathrm{Ch}_{r(n-1)}(\mathcal{Z}^r_{\mathcal{L}}(a)^\circ)$. Applying the intersection product construction in §7.7 for $Y = \mathrm{Sht}^r_{U(n)}$ (the hypotheses apply by Lemma 6.9 (2)), we obtain a class

$$\begin{aligned}&[\mathcal{Z}^r_{\mathcal{L}_1}(a_{11})^\circ] \cdot_{\mathrm{Sht}^r_{U(n)}} \cdots \cdot_{\mathrm{Sht}^r_{U(n)}} [\mathcal{Z}^r_{\mathcal{L}_m}(a_{mm})^\circ] \\ &\quad \in \mathrm{Ch}_{r(n-m)}(\mathcal{Z}^r_{\mathcal{L}_1,\dots,\mathcal{L}_m}(a_{11},\dots,a_{mm})^\circ). \end{aligned} \tag{7.3}$$

Let $\mathcal{A}^{\mathrm{all}}_{\mathcal{E}}(a_{11},\dots,a_{mm})(k)$ be the finite set of Hermitian maps $a : \mathcal{E} \to \sigma^*\mathcal{E}^\vee$ (not assumed to be injective) such that its restriction to $\mathcal{L}_i$ is a_{ii} for $i = 1, \dots, m$. By Lemma 7.7, there is a map

$$\mathcal{Z}^r_{\mathcal{L}_1}(a_{11}) \times_{\mathrm{Sht}^r_{U(n)}} \cdots \times_{\mathrm{Sht}^r_{U(n)}} \mathcal{Z}^r_{\mathcal{L}_m}(a_{mm}) \to \mathcal{A}^{\mathrm{all}}_{\mathcal{E}}(a_{11},\dots,a_{mm})(k)$$

such that the fiber over $a \in \mathcal{A}_{\mathcal{E}}(k)$ identifies with $\mathcal{Z}^r_{\mathcal{E}}(a)$. Since a is injective, the image of $\mathcal{Z}^r_{\mathcal{E}}(a) \to \mathcal{Z}^r_{\mathcal{L}_i}(a_{ii})$ lies in $\mathcal{Z}^r_{\mathcal{L}_i}(a_{ii})^\circ$. In particular,

$$\mathcal{Z}^r_{\mathcal{E}}(a) \subset \mathcal{Z}^r_{\mathcal{L}_1,\dots,\mathcal{L}_m}(a_{11},\dots,a_{mm})^\circ$$

is open and closed. Restricting (7.3) to $\mathcal{Z}^r_{\mathcal{E}}(a)$ gives a cycle class

$$\begin{aligned}\zeta^r_{\mathcal{L}_1,\dots,\mathcal{L}_m}(a) := &\left([\mathcal{Z}^r_{\mathcal{L}_1}(a_{11})^\circ] \cdot_{\mathrm{Sht}^r_{U(n)}} \cdots \cdot_{\mathrm{Sht}^r_{U(n)}} [\mathcal{Z}^r_{\mathcal{L}_m}(a_{mm})^\circ]\right) |_{\mathcal{Z}^r_{\mathcal{E}}(a)} \\ &\in \mathrm{Ch}_{r(n-m)}(\mathcal{Z}^r_{\mathcal{E}}(a)).\end{aligned}$$

Remark 7.14 Our notation suggests that $\zeta^r_{\mathcal{L}_1,\dots,\mathcal{L}_m}(a)$ (as a cycle class on $\mathcal{Z}^r_{\mathcal{E}}(a)$) depends, at least a priori, on the decomposition of $\mathcal{E}$ into a direct sum of line bundles $\mathcal{L}_1,\dots,\mathcal{L}_m$. However, we will show later in Theorem 10.1 that, at least when $m=n$, it only depends on $\mathcal{E}$, and is equal to the cycle class $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ that we will define for general rank n bundle $\mathcal{E}$.

7.9 Intersection problem: $m=n$ and $\mathcal{E}$ arbitrary

To define a 0-cycle $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ for general rank n vector bundle $\mathcal{E}$ on X', we need to make some auxiliary choice first; eventually we will show that the definition is independent of the choice.

Definition 7.15 Let $\mathcal{E}$ be a rank n vector bundle over X' and $a\in\mathcal{A}_{\mathcal{E}}(k)$. A *good framing* of $(\mathcal{E},a)$ is an n-tuple $(s_i:\mathcal{L}_i\to\mathcal{E})_{1\le i\le n}$ of $\mathcal{O}_{X'}$-linear maps from line bundles $\mathcal{L}_i\in\mathrm{Pic}(X')$ satisfying:

(1) The map $s=\oplus s_i:\mathcal{E}':=\oplus_{i=1}^n\mathcal{L}_i\to\mathcal{E}$ is injective.
(2) Let D_s be the divisor of the nonzero map $\det(s):\otimes_{i=1}^n\mathcal{L}_i\to\det\mathcal{E}$. Then $\nu(D_s)$ (image in X) is disjoint from D_a (see Definition 7.3).

Lemma 7.16 *For any rank n bundle $\mathcal{E}$ on X' and $a\in\mathcal{A}_{\mathcal{E}}(k)$, there exists a good framing for $(\mathcal{E},a)$ in the sense of Definition 7.15.*

Proof For notational convenience we give the argument for X' connected; the case $X'=X\coprod X$ can be proved with obvious changes.

We strengthen condition (2) on $s:\oplus_{i=1}^n\mathcal{L}_i\to\mathcal{E}$ slightly by asking $\nu(D_s)$ to avoid a prescribed divisor D_0 on X, instead of D_a. We prove the existence of s satisfying this stronger condition by induction on n.

The base case $n=1$ is trivial: take $\mathcal{L}_1=\mathcal{E}$.

For the inductive step, start by picking any saturated line bundle $\mathcal{L}_1\hookrightarrow\mathcal{E}$. Then $\mathcal{E}_{n-1}:=\mathcal{E}/\mathcal{L}_1$ is a vector bundle of rank $n-1$. By induction hypothesis we may pick $\overline{s}:\oplus_{i=2}^n\mathcal{L}'_i\hookrightarrow\mathcal{E}_{n-1}$ satisfying the conditions of Definition 7.15 and such that $\nu(D_{\overline{s}})$ avoids the given divisor D_0. Let $D_2,\dots,D_n$ be effective divisors on X' such that

(1) $\nu(D_2),\dots,\nu(D_n)$ are disjoint from $\nu(D_0)$, and
(2) $\deg\mathcal{L}'_i-D_i+2g'-2<\deg\mathcal{L}_1$ for $i=2,\dots,n$.

Let $\mathcal{L}_i=\mathcal{L}'_i(-D_i)$. By the inequality above we see that $\mathrm{Ext}^1(\mathcal{L}_i,\mathcal{L}_1)=0$, so the map $\overline{s}_i:\mathcal{L}_i\hookrightarrow\mathcal{E}_{n-1}=\mathcal{E}/\mathcal{L}_1$ lifts to a map $s_i:\mathcal{L}_i\hookrightarrow\mathcal{E}$, $i=2,\dots,n$.

Now we have an injection $s:\oplus_{i=1}^n\mathcal{L}_i\hookrightarrow\mathcal{E}$ whose divisor D_s satisfies $D_s=D_{\overline{s}}+D_2+\cdots+D_n$. Since $\nu(D_2),\dots,\nu(D_n)$, $\nu(D_{\overline{s}})$ are disjoint from D_0 by construction, the same is true for $\nu(D_s)$. □

Corollary 7.17 *If $s:\mathcal{E}'=\oplus_{i=1}^n\mathcal{L}_i\hookrightarrow\mathcal{E}$ is a good framing, then the map (7.2) realizes $\mathcal{Z}_{\mathcal{E}}(a)$ as an open and closed subscheme of $\mathcal{Z}_{\mathcal{E}'}(a')$.*

Proof Closedness is proved in Lemma 7.8(1). Only the openness requires an argument. By the definition of a good framing, $\nu(D_s)$ is disjoint from D_a, and therefore

disjoint from all legs of all points of $\mathcal{Z}^r_{\mathcal{E}}(a)$ by Lemma 7.12. Let $U' = X' \setminus \mathrm{supp}(D_s + \sigma D_s)$, then $\mathcal{Z}^r_{\mathcal{E}}(a) = \mathcal{Z}^r_{\mathcal{E}}(a)|_{U'^r}$. By Lemma 7.8(2), the inclusion

$$\mathcal{Z}^r_{\mathcal{E}}(a) = \mathcal{Z}^r_{\mathcal{E}}(a)|_{U'^r} \hookrightarrow \mathcal{Z}_{\mathcal{E}'}(a')|_{U'^r}$$

is open, hence the inclusion $\mathcal{Z}^r_{\mathcal{E}}(a) \hookrightarrow \mathcal{Z}^r_{\mathcal{E}'}(a')$ is open. □

Definition-Proposition 7.18 *Let $\mathcal{E}$ be a vector bundle of rank n over X' and $a \in \mathcal{A}_{\mathcal{E}}(k)$. Let $s : \oplus_{i=1}^n \mathcal{L}_i \hookrightarrow \mathcal{E}$ be a good framing of $(\mathcal{E}, a)$. Let*

$$[\mathcal{Z}^r_{\mathcal{E}}(a)] := \zeta^r_{\mathcal{L}_1,\dots,\mathcal{L}_n}(a')|_{\mathcal{Z}^r_{\mathcal{E}}(a)} \in \mathrm{Ch}_0(\mathcal{Z}^r_{\mathcal{E}}(a)).$$

Here we are using Corollary 7.17 to make sense of the restriction, as it implies that $\mathcal{Z}^r_{\mathcal{E}}(a)$ is a union of connected components of $\mathcal{Z}^r_{\mathcal{E}'}(a')$. Then the cycle class $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ thus defined is independent of the good framing $s : \mathcal{E}' = \oplus_{i=1}^n \mathcal{L}_i \hookrightarrow \mathcal{E}$.

The independence of good framing will be proved in Theorem 10.1 after some preparation in §8. The idea is to construct another 0-cycle class on $\mathcal{Z}^r_{\mathcal{E}}(a)$ without making auxiliary choices (which is done by introducing Hitchin shtukas), and show that the two constructions agree.

By Proposition 7.13, $\mathcal{Z}^r_{\mathcal{E}}(a)$ is proper over k, therefore the *degree* of the 0-cycle of $[\mathcal{Z}^r_{\mathcal{E}}(a)] \in \mathrm{Ch}_0(\mathcal{Z}^r_{\mathcal{E}}(a))$ is a well-defined number in $\mathbf{Q}$. The main problem we are concerned with in this Part is to determine $\deg[\mathcal{Z}^r_{\mathcal{E}}(a)] \in \mathbf{Q}$.

8 Hitchin-type moduli spaces

In this section we introduce certain "Hitchin-type moduli stacks" which will help to analyze the special cycles. In particular, we will be able to use these to give an alternative construction of the cycle classes associated to special divisors, that is manifestly independent of auxiliary choices.

8.1 Hitchin stacks

Until §8.4, we fix an arbitrary positive integer m.

Definition 8.1 The *Hitchin stack* $\mathcal{M}^{\mathrm{all}}(m, n)$ (sometimes denoted $\mathcal{M}^{\mathrm{all}}$ when m, n are understood) has S-points the groupoid consisting of the following data.

- $\mathcal{E}$ a rank m vector bundle on $X' \times S$.
- $\mathcal{F}$ a rank n vector bundle on $X' \times S$, equipped with a Hermitian map $h : \mathcal{F} \xrightarrow{\sim} \sigma^*\mathcal{F}^\vee$.
- A map of underlying coherent sheaves $t : \mathcal{E} \to \mathcal{F}$ over $X' \times S$.

We define $\mathcal{M}(m, n) \subset \mathcal{M}^{\mathrm{all}}$ (sometimes denoted $\mathcal{M}$ when m, n are understood) to be the open substack where the map t base changes to an injective map on $X'_{\overline{s}}$ for each geometric point $\overline{s} \to S$.

Let us emphasize that *both* $\mathcal{E}$ and $(\mathcal{F}, h)$ are varying in this definition. We will usually suppress the dependence on m, n from the notation.

8.2 Hitchin base

Definition 8.2 We define the following two versions of the *Hitchin base*.

(1) $\mathcal{A}^{\mathrm{all}}(m)$ (sometimes denoted $\mathcal{A}^{\mathrm{all}}$ when m is understood) to be the stack whose S-points is the groupoid of the following data:

- $\mathcal{E}$ a rank m vector bundle on $X' \times S$;
- $a\colon \mathcal{E} \to \sigma^*\mathcal{E}^\vee$ is a map of coherent sheaves on $X' \times S$ such that $\sigma(a^\vee) = a$.

(2) We define $\mathcal{A} \subset \mathcal{A}^{\mathrm{all}}$ to be the open substack where $a\colon \mathcal{E} \to \sigma^*\mathcal{E}^\vee$ is injective after base change to $X'_{\overline{s}}$ for every geometric point $\overline{s} \to S$.

Definition 8.3 For integers $1 \le m \le n$, we define the *Hitchin fibration* for $\mathcal{M}^{\mathrm{all}} = \mathcal{M}^{\mathrm{all}}(m,n)$ to be the map $f\colon \mathcal{M}^{\mathrm{all}} \to \mathcal{A}^{\mathrm{all}}$ sending $(\mathcal{E}, (\mathcal{F}, h), t)$ to the composition

$$a\colon \mathcal{E} \xrightarrow{t} \mathcal{F} \xrightarrow{h} \sigma^*\mathcal{F}^\vee \xrightarrow{\sigma^* t^\vee} \sigma^*\mathcal{E}^\vee.$$

Remark 8.4 In general the Hitchin fibration does not send $\mathcal{M}(m,n)$ to $\mathcal{A}(m)$ even when $m \le n$. However, in the special case $m = n$, the Hitchin map does send $\mathcal{M}(n,n)$ to $\mathcal{A}(n)$ because when $t : \mathcal{E} \to \mathcal{F}$ is generically injective, it is generically an isomorphism for rank reasons, hence the induced Hermitian map a on $\mathcal{E}$ is generically non-degenerate.

8.3 Hitchin shtukas

We now discuss a notion of shtukas for Hitchin stacks. Throughout, $\mathcal{M} = \mathcal{M}(m,n)$.

Definition 8.5 (Hecke stacks for Hitchin spaces) For $r \ge 0$, we define $\mathrm{Hk}^r_{\mathcal{M}^{\mathrm{all}}}$ to be the stack whose S-points are given by the groupoid of the following data:

(1) $(\{x'_i\}_{1\le i\le r}, \{(\mathcal{F}_i, h_i)\}_{0\le i\le r}) \in \mathrm{Hk}^r_{U(n)}(S)$.
(2) A vector bundle $\mathcal{E}$ of rank m on $X' \times S$.
(3) Maps $t_i : \mathcal{E} \to \mathcal{F}_i$ fitting into the commutative diagram

$$\begin{array}{ccccccc} \mathcal{E} & = & \mathcal{E} & = & \cdots & = & \mathcal{E} \\ \downarrow{\scriptstyle t_0} & & \downarrow{\scriptstyle t_1} & & & & \downarrow{\scriptstyle t_r} \\ \mathcal{F}_0 & \dashrightarrow & \mathcal{F}_1 & \dashrightarrow & \cdots & \dashrightarrow & \mathcal{F}_r \end{array}$$

We define the open substack $\mathrm{Hk}^r_{\mathcal{M}} \subset \mathrm{Hk}^r_{\mathcal{M}^{\mathrm{all}}}$ by the condition that t_0 base changes to an injective map along every geometric point $\overline{s} \to S$ (equivalently, every t_i has this property). Let $\mathrm{pr}_i : \mathrm{Hk}^r_{\mathcal{M}} \to \mathcal{M}$ (resp. $\mathrm{pr}_i^{\mathrm{all}} : \mathrm{Hk}^r_{\mathcal{M}^{\mathrm{all}}} \to \mathcal{M}^{\mathrm{all}}$) be the map recording $(\mathcal{E}, \mathcal{F}_i, h_i, t_i)$, for $0 \le i \le r$.

Definition 8.6 (Shtukas for Hitchin stacks) For $r \geq 0$, we define $\mathrm{Sht}^r_{\mathcal{M}^{\mathrm{all}}}$ as the fibered product

$$\begin{array}{ccc} \mathrm{Sht}^r_{\mathcal{M}^{\mathrm{all}}} & \longrightarrow & \mathrm{Hk}^r_{\mathcal{M}^{\mathrm{all}}} \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{pr}_0^{\mathrm{all}}, \mathrm{pr}_r^{\mathrm{all}})} \\ \mathcal{M}^{\mathrm{all}} & \xrightarrow{(\mathrm{Id},\mathrm{Fr})} & \mathcal{M}^{\mathrm{all}} \times \mathcal{M}^{\mathrm{all}} \end{array}$$

and the open substack $\mathrm{Sht}^r_{\mathcal{M}} \subset \mathrm{Sht}^r_{\mathcal{M}^{\mathrm{all}}}$ as the fibered product

$$\begin{array}{ccc} \mathrm{Sht}^r_{\mathcal{M}} & \longrightarrow & \mathrm{Hk}^r_{\mathcal{M}} \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{pr}_0, \mathrm{pr}_r)} \\ \mathcal{M} & \xrightarrow{(\mathrm{Id},\mathrm{Fr})} & \mathcal{M} \times \mathcal{M} \end{array} \tag{8.1}$$

Explicitly, the stack $\mathrm{Sht}^r_{\mathcal{M}^{\mathrm{all}}}$ parametrizes diagrams of the form below, with notation as in Definition 8.5,

$$\begin{array}{ccccccccc} \mathcal{E} & = & \mathcal{E} & = & \cdots & = & \mathcal{E} & \xrightarrow{\sim} & {}^{\tau}\mathcal{E} \\ \downarrow{\scriptstyle t_0} & & \downarrow{\scriptstyle t_1} & & & & \downarrow{\scriptstyle t_r} & & \downarrow{\scriptstyle {}^{\tau}t_0} \\ \mathcal{F}_0 & \dashrightarrow & \mathcal{F}_1 & \dashrightarrow & \cdots & \dashrightarrow & \mathcal{F}_r & \xrightarrow{\sim} & {}^{\tau}\mathcal{F}_0 \end{array} \tag{8.2}$$

and $\mathrm{Sht}^r_{\mathcal{M}}$ is the open substack where t_0 base changes to an injective map along every geometric point $\overline{s} \to S$ (equivalently, the same property holds for every t_i).

In particular, $\mathcal{E} \xrightarrow{\sim} {}^{\tau}\mathcal{E}$ is a shtuka with no legs, exhibiting $\mathcal{E}$ as arising from a rank m vector bundle on X', i.e. coming from $\mathrm{Bun}_{\mathrm{GL}'_m}(k)$. Note that in (8.2), $\mathcal{E}$ is not fixed, so the automorphisms of $\mathcal{E}$ are present in the functor of points of $\mathrm{Sht}^r_{\mathcal{M}^{\mathrm{all}}}$. Therefore, if we define

$$\overline{\mathcal{Z}}^r_{\mathcal{E}} := [\mathcal{Z}^r_{\mathcal{E}} / (\mathrm{Aut}(\mathcal{E})(k)],$$

then $\mathrm{Sht}^r_{\mathcal{M}^{\mathrm{all}}}$ decomposes as a disjoint union of special cycles

$$\mathrm{Sht}^r_{\mathcal{M}^{\mathrm{all}}} = \coprod_{\mathcal{E} \in \mathrm{Bun}_{\mathrm{GL}'_m}(k)} \overline{\mathcal{Z}}^r_{\mathcal{E}}.$$

This decomposition can be refined. The compositions $f \circ \mathrm{pr}_i \colon \mathrm{Hk}^r_{\mathcal{M}^{\mathrm{all}}} \to \mathcal{A}^{\mathrm{all}}$ all coincide, and they induce a map

$$\mathrm{Sht}^r_{\mathcal{M}^{\mathrm{all}}} \to \mathcal{A}^{\mathrm{all}}(k).$$

Here $\mathcal{A}^{\mathrm{all}}(k)$ is the groupoid of pairs $\{\mathcal{E}, \mathcal{E} \xrightarrow{a} \sigma^*\mathcal{E}^\vee\}$. Let us write $\mathrm{Aut}(a) := \mathrm{Aut}(\mathcal{E} \xrightarrow{a} \sigma^*\mathcal{E}^\vee) \subset \mathrm{Aut}(\mathcal{E})$. Then each $\mathcal{E} \xrightarrow{a} \sigma^*\mathcal{E}^\vee$ defines a map $a : \mathrm{Spec}\, k \to$

$B\,\mathrm{Aut}(a)(k) \hookrightarrow \mathcal{A}^{\mathrm{all}}(k)$, the latter map being an open-closed inclusion, and we have a Cartesian square

$$\begin{array}{ccc} \mathcal{Z}^r_{\mathcal{E}}(a) & \longrightarrow & \mathrm{Sht}^r_{\mathcal{M}^{\mathrm{all}}} \\ \downarrow & & \downarrow f \\ \mathrm{Spec}\ k & \xrightarrow{a} & \mathcal{A}^{\mathrm{all}}(k) \end{array}$$

In particular, $\mathcal{Z}^r_{\mathcal{E}}(a)$ is a finite étale cover of an open-closed substack of $\mathrm{Sht}^r_{\mathcal{M}^{\mathrm{all}}}$ isomorphic to $[\mathcal{Z}^r_{\mathcal{E}}(a)/\mathrm{Aut}(a)(k)]$. Similarly, restricting to the injective locus (see Definition 7.4) we have a Cartesian square

$$\begin{array}{ccc} \mathcal{Z}^r_{\mathcal{E}}(a)^\circ & \longrightarrow & \mathrm{Sht}^r_{\mathcal{M}} \\ \downarrow & & \downarrow f \\ \mathrm{Spec}\ k & \xrightarrow{a} & \mathcal{A}^{\mathrm{all}}(k) \end{array} \tag{8.3}$$

If $m = n$ and $a \in \mathcal{A}(k)$, then we have $\mathcal{Z}^r_{\mathcal{E}}(a)^\circ = \mathcal{Z}^r_{\mathcal{E}}(a)$ and we can replace $\mathcal{A}^{\mathrm{all}}(k)$ by $\mathcal{A}(k)$ in the above diagram (see Remark 8.4).

8.4 From vector bundles to torsion sheaves

For the rest of the section, we concentrate on the case $m = n$. In this case, we will relate $\mathcal{M} = \mathcal{M}(n, n)$ to the moduli stack of Hermitian torsion sheaves introduced in §4.1. We introduce the following abbreviated notations.

Definition 8.7 Let $d \in \mathbf{Z}_{\geq 0}$.

(1) Let $\mathcal{M}_d = \mathcal{M}(n, n)_d$ be the open-closed substack of $\mathcal{M} = \mathcal{M}(n, n)$ consisting of $(\mathcal{E} \xrightarrow{t} \mathcal{F})$ where $d = \frac{\deg \mathcal{E}^\vee - \deg(\mathcal{E})}{2} = -\chi(X', \mathcal{E})$.
(2) Let $\mathcal{A}_d = \mathcal{A}(n)_d$ be the open-closed substack of $\mathcal{A} = \mathcal{A}(n)$ consisting of $(\mathcal{E}, a)$ where $d = \frac{\deg \mathcal{E}^\vee - \deg(\mathcal{E})}{2} = -\chi(X', \mathcal{E})$.

By Remark 8.4, the Hitchin map for $\mathcal{M}^{\mathrm{all}} = \mathcal{M}^{\mathrm{all}}(n, n)$ restricts to a map

$$f_d : \mathcal{M}_d \to \mathcal{A}_d.$$

When d is understood, we abbreviate f for f_d.

Recall that $\mathrm{Herm}_{2d} = \mathrm{Herm}_{2d}(X'/X)$ is the moduli stack of length $2d$ torsion coherent sheaves $\mathcal{Q}$ on X' equipped with a Hermitian structure $h_{\mathcal{Q}} \colon \mathcal{Q} \xrightarrow{\sim} \sigma^*\mathcal{Q}^\vee$, where $\mathcal{Q}^\vee := \underline{\mathrm{Ext}}^1(\mathcal{Q}, \omega_{X'})$ such that $\sigma^* h^\vee_{\mathcal{Q}} = h_{\mathcal{Q}}$. Alternatively we may think of $h_{\mathcal{Q}}$ as the datum of a perfect pairing

$$h'_{\mathcal{Q}} : \mathcal{Q} \otimes_{\mathcal{O}_{X'}} \sigma^*\mathcal{Q} \to \omega_{X',F'}/\omega_{X'}$$

where $\omega_{X',F'}$ is the constant Zariski sheaf of rational differential form on X'. The Hermitian condition is equivalent to $h'_{\mathcal{Q}}(u, v) = \sigma^* h'_{\mathcal{Q}}(v, u)$ for local sections u, v of $\mathcal{Q}$.

In §4.5 we have also introduced the moduli stack $\mathrm{Lagr}_{2d} = \mathrm{Lagr}_{2d}(X'/X)$ classifying $(\mathcal{Q}, h_{\mathcal{Q}}, \mathcal{L})$ where $(\mathcal{Q}, h_{\mathcal{Q}}) \in \mathrm{Herm}_{2d}$ and $\mathcal{L} \subset \mathcal{Q}$ is a Lagrangian subsheaf.

There is a canonical map $g : \mathcal{A}_d \to \mathrm{Herm}_{2d}$ sending $(\mathcal{E}, a)$ to the torsion sheaf $\mathcal{Q} = \sigma^*\mathcal{E}^\vee/\mathcal{E}$ together with the Hermitian structure $h_{\mathcal{Q}}$ defined in §5.1.

We have a map $g_{\mathcal{M}} : \mathcal{M}_d \to \mathrm{Lagr}_{2d}$ sending $(\mathcal{E} \xrightarrow{t} \mathcal{F}) \in \mathcal{M}_d$ to the torsion Hermitian sheaf $(\mathcal{Q} = \sigma^*\mathcal{E}^\vee/\mathcal{E}, h_{\mathcal{Q}})$ constructed above together with the Lagrangian $\mathcal{L} = \mathcal{F}/\mathcal{E}$.

Lemma 8.8 *The maps defined above fit into a Cartesian diagram*

$$\begin{array}{ccc} \mathcal{M}_d & \xrightarrow{g_{\mathcal{M}}} & \mathrm{Lagr}_{2d} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \upsilon_{2d}} \\ \mathcal{A}_d & \xrightarrow{g} & \mathrm{Herm}_{2d} \end{array} \tag{8.4}$$

Proof Given $a : \mathcal{E} \to \sigma^*\mathcal{E}^\vee$ that is injective, the datum of a subsheaf $\mathcal{L} \subset \sigma^*\mathcal{E}^\vee/\mathcal{E}$ is the same as a coherent sheaf $\mathcal{F}$ such that $\mathcal{E} \subset \mathcal{F} \subset \sigma^*\mathcal{E}^\vee$. It is easy to see that $\mathcal{L}$ is Lagrangian if and only if $\mathcal{F}$ is self-dual under the Hermitian map a. □

Corollary 8.9 *The Hitchin fibration $f_d : \mathcal{M}_d \to \mathcal{A}_d$ is proper.*

Proof Apply Lemma 8.8 and the fact that υ_{2d} is proper. □

Lemma 8.10 *The map $\upsilon_{2d} : \mathrm{Lagr}_{2d} \to \mathrm{Herm}_{2d}$ is small.*

Proof The map π^{Herm}_{2d} from §4.2 factors as

$$\pi^{\mathrm{Herm}}_{2d} : \widetilde{\mathrm{Herm}}_{2d} \xrightarrow{\lambda_{2d}} \mathrm{Lagr}_{2d} \xrightarrow{\upsilon_{2d}} \mathrm{Herm}_{2d}\,.$$

Since λ_{2d} is surjective and π^{Herm}_{2d} is small by Proposition 4.5, we get the desired statement. □

8.4.1 Proof of Proposition 7.13

Let $\mathrm{Lagr}(\mathcal{Q})$ be the moduli space of Lagrangian subsheaves of $\mathcal{Q}$. Let $\mathrm{Hk}^r_{\mathrm{Lagr}(\mathcal{Q})}$ be its Hecke version, classifying points $\{x'_i\}_{1\le i\le r}$ of X' and chains of Lagrangian subsheaves of $\mathcal{Q}$

$$\mathcal{L}_0 \overset{f_1}{\dashrightarrow} \mathcal{L}_1 \overset{f_2}{\dashrightarrow} \cdots \overset{f_r}{\dashrightarrow} \mathcal{L}_r$$

where the dashed arrow f_i are modifications at $x'_i \cup \sigma(x'_i)$, similar to those in Definition 6.3. There is a natural map $\mathcal{Z}^r_{\mathcal{E}}(a) \to \mathrm{Hk}^r_{\mathrm{Lagr}(\mathcal{Q})}$ sending a point $(\{x'_i\}, \{t_i : \mathcal{E} \to \mathcal{F}_i\})$ of $\mathcal{Z}^r_{\mathcal{E}}(a)$ to the collection of (necessarily Lagrangian) subsheaves $\overline{\mathcal{F}}_i =$

$\mathrm{coker}(t_i) \subset \mathcal{Q} = \sigma^*\mathcal{E}^\vee/\mathcal{E}$. This map fits into a Cartesian diagram

$$\begin{array}{ccc} \mathcal{Z}^r_{\mathcal{E}}(a) & \longrightarrow & \mathrm{Hk}^r_{\mathrm{Lagr}(\mathcal{Q})} \\ \downarrow & & \downarrow \\ \mathrm{Lagr}(\mathcal{Q}) & \xrightarrow{(\mathrm{Id},\mathrm{Fr})} & \mathrm{Lagr}(\mathcal{Q}) \times \mathrm{Lagr}(\mathcal{Q}) \end{array}$$

Now both $\mathrm{Lagr}(\mathcal{Q})$ and $\mathrm{Hk}^r_{\mathrm{Lagr}(\mathcal{Q})}$ are proper schemes over k, hence the same is true for $\mathcal{Z}^r_{\mathcal{E}}(a)$. The diagram also makes it clear that $\mathcal{Z}^r_{\mathcal{E}}(a)$ only depends on $(\mathcal{Q}, \overline{a})$.

8.5 Smoothness

Lemma 8.11 *The map*

$$\beta_d : \mathcal{M}_d \to \mathrm{Coh}_d(X') \times \mathrm{Bun}_{U(n)}$$

sending $(\mathcal{E} \xrightarrow{t} \mathcal{F}, h)$ *to* $(\mathrm{coker}(t), (\mathcal{F}, h))$ *is smooth of relative dimension* dn. *In particular,* $\mathcal{M}_d$ *is smooth of pure dimension* $dn + n^2(g-1)$.

Proof Consider the stack $\mathcal{M}'_d$ classifying $(\mathcal{T}, \mathcal{F}, h, s)$ where $\mathcal{T} \in \mathrm{Coh}_d(X')$, $(\mathcal{F}, h) \in \mathrm{Bun}_{U(n)}$ and an $\mathcal{O}_{X'}$-linear map $s : \mathcal{F} \to \mathcal{T}$. Let $\beta'_d : \mathcal{M}'_d \to \mathrm{Coh}_d(X') \times \mathrm{Bun}_{U(n)}$ be the natural map. Due to the vanishing of $\mathrm{Ext}^1(\mathcal{F}, \mathcal{T})$, β'_d exhibits $\mathcal{M}'_d$ as a vector bundle over $\mathrm{Coh}_d(X') \times \mathrm{Bun}_{U(n)}$ of rank $dn = \dim \mathrm{Hom}_{X'}(\mathcal{F}, \mathcal{T})$. Now $\mathcal{M}_d$ is the open substack of $\mathcal{M}'_d$ where s is surjective. Therefore β_d is also smooth of relative dimension dn. □

Proposition 8.12 *The map* $g : \mathcal{A}_d \to \mathrm{Herm}_{2d}$ *is smooth.*

Proof We have a map $s^{\mathrm{Lagr}}_{2d} : \mathrm{Lagr}_{2d} \to X'_d$ sending $(\mathcal{Q}, h_{\mathcal{Q}}, \mathcal{L})$ to the divisor of $\mathcal{L}$. Recall also the map $s^{\mathrm{Herm}}_{2d} : \mathrm{Herm}_{2d} \to X_d$ sending $(\mathcal{Q}, h_{\mathcal{Q}})$ to the descent of the divisor of $\mathcal{Q}$ to X.

Recall in §4.5 we introduced the open subset

$$(X'_d)^\diamond = \{D' \subset X' : D' \cap \sigma(D') = \varnothing\} \subset X'_d.$$

Let $\mathrm{Lagr}^\diamond_{2d} \subset \mathrm{Lagr}_{2d}$ and $\mathcal{M}^\diamond_d \subset \mathcal{M}_d$ be the preimages of $(X'_d)^\diamond$ under the maps s^{Lagr}_{2d} and $s^{\mathrm{Lagr}}_{2d} \circ g_{\mathcal{M}}$.

We claim that both squares in the diagram

$$\begin{array}{ccccc} \mathcal{M}^\diamond_d & \longrightarrow & \mathrm{Lagr}^\diamond_{2d} & \longrightarrow & (X'_d)^\diamond \\ \downarrow{\scriptstyle f_d} & & \downarrow & & \downarrow \\ \mathcal{A}_d & \xrightarrow{g} & \mathrm{Herm}_{2d} & \longrightarrow & X_d \end{array}$$

are Cartesian. The left square is Cartesian by definition. Now we show that the right square is Cartesian. Let $(\mathcal{Q}, h_{\mathcal{Q}}) \in \mathrm{Herm}_{2d}$, $D' \in (X'_d)^\diamond$ lying over $D =$

$s_{2d}^{\mathrm{Herm}}(\mathcal{Q}, h_{\mathcal{Q}})$. Since $D' \cap \sigma(D') = \varnothing$, there is a unique Lagrangian subsheaf $\mathcal{L} \subset \mathcal{Q}$ supported on the support of D', namely $\mathcal{L} = \mathcal{Q}|_{\mathrm{supp}\, D'}$. This gives the unique point $(\mathcal{Q}, h_{\mathcal{Q}}, \mathcal{L}) \in \mathrm{Lagr}_{2d}^{\diamondsuit}$ mapping to $(\mathcal{Q}, h_{\mathcal{Q}}) \in \mathrm{Herm}_{2d}$ and $D' \in (X'_d)^{\diamondsuit}$.

Note that the map $(X'_d)^{\diamondsuit} \to X_d$ is faithfully flat: it is clearly surjective, and the map $\nu_d : X'_d \to X_d$ is a finite morphism between smooth schemes, hence flat. We will show that $\mathcal{M}_d^{\diamondsuit} \to \mathrm{Lagr}_d^{\diamondsuit}$ is smooth. By fppf descent it then follows that $\mathcal{A}_d \to \mathrm{Herm}_d$ is also smooth.

Recall from §4.5 that $\varepsilon'_d : \mathrm{Lagr}_{2d} \to \mathrm{Coh}_d(X')$ (recording only $\mathcal{L}$) restricts to an isomorphism $\mathrm{Lagr}_{2d}^{\diamondsuit} \xrightarrow{\sim} \mathrm{Coh}_d(X')^{\diamondsuit} := \mathrm{Coh}_d(X')|_{(X'_d)^{\diamondsuit}}$. Therefore it suffices to show that the composition $\mathcal{M}_d^{\diamondsuit} \xrightarrow{g_{\mathcal{M}}} \mathrm{Lagr}_{2d}^{\diamondsuit} \xrightarrow{\varepsilon'_d} \mathrm{Coh}_d(X')^{\diamondsuit}$ is smooth. This follows from the smoothness of $\mathcal{M}_d \to \mathrm{Coh}_d(X')$ proved in Lemma 8.11. □

Corollary 8.13 *The Hitchin fibration $f : \mathcal{M}_d \to \mathcal{A}_d$ is small. The complex $Rf_*\overline{\mathbf{Q}}_\ell$ is a shifted perverse sheaf that is the middle extension from any dense open substack of $\mathcal{A}_d$.*

Proof By the smoothness of g in Proposition 8.12 and the Cartesian diagram in Lemma 8.8, the smallness of f follows from that of $\upsilon_{2d} : \mathrm{Lagr}_{2d} \to \mathrm{Herm}_{2d}$, which is proved in Lemma 8.10. □

8.6 Cycle class from Hitchin shtukas

In this subsection we take $m = n$, so $\mathcal{M} = \mathcal{M}(n, n)$ and $\mathcal{A} = \mathcal{A}(n)$. Now consider the Hitchin shtukas for $\mathcal{M}_d \subset \mathcal{M}$. Let $N = \dim \mathcal{M}_d$. By Corollary 8.13, $\dim \mathcal{A}_d = N$. The Cartesian diagram (8.1) restricts to a Cartesian diagram

$$\begin{array}{ccc} \mathrm{Sht}^r_{\mathcal{M}_d} & \longrightarrow & \mathrm{Hk}^r_{\mathcal{M}_d} \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{pr}_0, \mathrm{pr}_r)} \\ \mathcal{M}_d & \xrightarrow{(\mathrm{Id}, \mathrm{Fr})} & \mathcal{M}_d \times \mathcal{M}_d \end{array} \tag{8.5}$$

We would like to define a 0-cycle class on $\mathrm{Sht}^r_{\mathcal{M}_d}$ as the Gysin pullback of a cycle on $\mathrm{Hk}^r_{\mathcal{M}_d}$ along the Frobenius graph of $\mathcal{M}_d$. Although the virtual dimension of $\mathrm{Hk}^r_{\mathcal{M}_d}$ is the same as $\dim \mathcal{M}_d$, its actual dimension may be larger. For this reason we have to define in a roundabout way a virtual fundamental cycle on $\mathrm{Hk}^r_{\mathcal{M}_d}$ of the virtual dimension by relating it to $\mathrm{Hk}^1_{\mathcal{M}_d}$, which we show is smooth below.

Lemma 8.14 *The stack $\mathrm{Hk}^1_{\mathcal{M}_d}$ is smooth and equidimensional of the same dimension as $\mathcal{M}_d$.*

Proof Let $(x', \mathcal{F}_0 \dashrightarrow \mathcal{F}_1) \in \mathrm{Hk}^1_{U(n)}$. Let $\mathcal{F}^\flat := \mathcal{F}^\flat_{1/2} = \mathcal{F}_0 \cap \mathcal{F}_1$ as in Definition 6.3. The generically compatible Hermitian structures on $\mathcal{F}_0$ and $\mathcal{F}_1$ equip this intersection with a Hermitian structure $h^\flat : \mathcal{F}^\flat \hookrightarrow \sigma^*(\mathcal{F}^\flat)^\vee$ whose cokernel has length 1 at x' and

at $\sigma(x')$. We call such a bundle *almost Hermitian* with defect at the ordered pair of conjugate points $(x', \sigma(x'))$. Conversely, given $(\mathcal{F}^\flat, h^\flat)$ almost Hermitian with defect at $(x', \sigma(x'))$, one can uniquely recover $\mathcal{F}_0$ (resp. $\mathcal{F}_1$) as the upper modification of $\mathcal{F}^\flat$ at x' (resp. $\sigma(x')$) inside $\sigma^*(\mathcal{F}^\flat)^\vee$.

Let $\mathrm{Bun}^\flat_{U(n)}$ be the moduli stack parametrizing $(x' \in X', \mathcal{F}^\flat, h^\flat)$ where $(\mathcal{F}^\flat, h^\flat)$ is an almost Hermitian bundle with defect at $(x', \sigma(x'))$. The discussion in the previous paragraph shows that there is an isomorphism $\mathrm{Hk}^1_{U(n)} \xrightarrow{\sim} \mathrm{Bun}^\flat_{U(n)}$ over X'. Let $\mathcal{M}^\flat_d$ be the moduli stack of $(x', \mathcal{E} \xrightarrow{t} \mathcal{F}^\flat, h^\flat)$ where $(\mathcal{F}^\flat, h^\flat)$ is almost Hermitian with defect at $(x', \sigma(x'))$, $\mathcal{E}$ is a vector bundle on X' of rank n and $\chi(X', \mathcal{E}) = -d$, and t is injective. Then we have an isomorphism $\mathrm{Hk}^1_{\mathcal{M}_d} \cong \mathcal{M}^\flat_d$.

We have a natural map

$$\beta^\flat_d : \mathcal{M}^\flat_d \to \mathrm{Coh}_{d-1}(X') \times \mathrm{Bun}^\flat_{U(n)}$$

sending $(x', \mathcal{E} \xrightarrow{t} \mathcal{F}^\flat, h^\flat)$ to $(\mathrm{coker}(t), (x', \mathcal{F}^\flat, h^\flat))$. The same argument as Lemma 8.11 shows that $\beta^\flat_d$ exhibits $\mathcal{M}^\flat_d$ as an open substack in a vector bundle of rank $n(d-1)$ over $\mathrm{Coh}_{d-1}(X') \times \mathrm{Bun}^\flat_{U(n)}$. Now $\dim \mathrm{Coh}_{d-1}(X') = 0$ and $\dim \mathrm{Bun}^\flat_{U(n)} = \dim \mathrm{Hk}^1_{U(n)} = \dim \mathrm{Bun}_{U(n)} + n$ by Lemma 6.9(1). Therefore $\mathrm{Hk}^1_{\mathcal{M}_d} \cong \mathcal{M}^\flat_d$ is of pure dimension $\dim \mathrm{Bun}_{U(n)} + dn$, which is the same as $\dim \mathcal{M}_d$ by Lemma 8.11. □

Definition 8.15 For any stack S over k we define a morphism

$$\Phi^r_S : S^{r+1} \to S^{2r+2}$$

by the formula $\Phi^r_S(\xi_0, \ldots, \xi_r) = (\xi_0, \xi_1, \xi_1, \xi_2, \xi_2, \ldots, \xi_{r-1}, \xi_r, \xi_r, \mathrm{Fr}(\xi_0))$. When r is fixed in the context, we simply write Φ_S.

We rewrite $\mathrm{Sht}^r_{\mathcal{M}_d}$ as the fiber product

$$\begin{array}{ccc} \mathrm{Sht}^r_{\mathcal{M}_d} & \longrightarrow & (\mathrm{Hk}^1_{\mathcal{M}_d})^r \times \mathcal{M}_d \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{pr}_0, \mathrm{pr}_1)^r \times \Delta} \\ (\mathcal{M}_d)^{r+1} & \xrightarrow{\Phi^r_{\mathcal{M}_d}} & (\mathcal{M}_d)^{2r+2} = (\mathcal{M}_d)^{2r} \times (\mathcal{M}_d)^2 \end{array} \tag{8.6}$$

Here the vertical map $(\mathrm{pr}_0, \mathrm{pr}_1)^r$ sends $(h_1, \ldots, h_r) \in (\mathrm{Hk}^1_{\mathcal{M}_d})^r$ to $(\mathrm{pr}_0(h_1), \mathrm{pr}_1(h_1), \ldots, \mathrm{pr}_0(h_r), \mathrm{pr}_1(h_r)) \in (\mathcal{M}_d)^{2r}$, while Δ is the diagonal map.

Definition 8.16 We define a 0-cycle classes $[\mathrm{Sht}^r_{\mathcal{M}_d}] \in \mathrm{Ch}_0(\mathrm{Sht}^r_{\mathcal{M}_d})$ as the image of the fundamental class of $(\mathrm{Hk}^1_{\mathcal{M}_d})^r \times \mathcal{M}_d$ (which is smooth of the same dimension as $(\mathcal{M}_d)^{r+1}$ by Lemma 8.14) under the refined Gysin map along $\Phi_{\mathcal{M}_d} : (\mathcal{M}_d)^{r+1} \to (\mathcal{M}_d)^{2r+2}$ (which is defined since $\mathcal{M}_d$ is smooth and equidimensional by Lemma 8.11; see [10, Theorem 2.1.12(xi)])

$$[\mathrm{Sht}^r_{\mathcal{M}_d}] := (\Phi^r_{\mathcal{M}_d})^![(\mathrm{Hk}^1_{\mathcal{M}_d})^r \times \mathcal{M}_d] \in \mathrm{Ch}_0(\mathrm{Sht}^r_{\mathcal{M}_d}).$$

9 Special cycles of corank one

In this section we prove geometric properties of the special cycles $\mathcal{Z}^r_{\mathcal{E}}(a)$ when $m = \operatorname{rank}\mathcal{E} = 1$ (where the number field analogues are called "Kudla-Rapoport divisors"). In particular, we show that for $a \neq 0$, $\mathcal{Z}^r_{\mathcal{E}}(a)$ are local complete intersections of dimension r less than $\mathrm{Sht}^r_{U(n)}$. When $a = 0$, we show that $\mathcal{Z}^r_{\mathcal{E}}(a)^*$ has dimension at most $\dim \mathrm{Sht}^r_{U(n)} - r$. These geometric properties are proved by studying stratifications introduced and analyzed in §9.1 and §9.2.

9.1 Stratification of the special cycle when $m = 1$

In this section we fix a line bundle $\mathcal{L}$ on X'_k and $a \in \mathcal{A}_{\mathcal{L}}(k)$, i.e., $a : \mathcal{L} \to \sigma^*\mathcal{L}^\vee$ is nonzero Hermitian. We now define a stratification of $\mathcal{Z} := \mathcal{Z}^r_{\mathcal{L}}(a)_{\overline{k}}$ and estimate the dimension of each stratum.

When $n = 1$, we have $\dim \mathcal{Z} = 0$. Indeed, Lemma 7.12 implies that there is a 0-dimensional closed subscheme $D \subset (X')^r$ such that the morphism $\mathcal{Z} \to \mathrm{Sht}^r_{U(1)}$ takes values in the preimage of D under the leg map $\mathrm{Sht}^r_{U(1)} \to (X')^r$. This preimage is 0-dimensional by Lemma 6.9 (2). As $Z \to \mathrm{Sht}^r_{U(1)}$ is finite (by Proposition 7.5) with 0-dimensional image, we conclude that $\dim \mathcal{Z} = 0$.

Therefore in the rest of this subsection we will assume $n \geq 2$.

For each $(\mathcal{L} \xrightarrow{t_i} \mathcal{F}_i)_{0\leq i\leq r} \in \mathcal{Z}(\overline{k})$ with legs $(x'_i)_{1\leq i\leq r} \in X'(\overline{k})^r$, let D_i $(0 \leq i \leq r)$ be the divisor on $X'_{\overline{k}}$ such that $t_i : \mathcal{L}(D_i) \hookrightarrow \mathcal{F}_i$ is saturated. For each $1 \leq i \leq r$, we have one of the four cases:

(0) $D_i = D_{i-1}$;
(+) $D_i = D_{i-1} + \sigma(x'_i)$;
(−) $D_i = D_{i-1} - x'_i$;
(±) $D_i = D_{i-1} - x'_i + \sigma(x'_i)$.

Since the composition $\mathcal{L} \xrightarrow{t_i} \mathcal{F}_i \xrightarrow{h} \sigma^*\mathcal{F}_i^\vee \xrightarrow{\sigma^* t_i^\vee} \sigma^*\mathcal{L}^\vee$ is equal to a, we see that $D_i + \sigma(D_i)$ is a subdivisor of the divisor of a. Therefore $\nu(D_i) \leq D_a$ as divisors on $X(\overline{k})$.

9.1.1 Indexing set for strata

Consider the set $\mathfrak{D}$ of sequences of effective divisors $(D_i)_{0\leq i\leq r}$ on $X'_{\overline{k}}$ satisfying

- $\nu(D_i) \leq D_a$ for all $0 \leq i \leq r$.
- For each $1 \leq i \leq r$, the pair (D_{i-1}, D_i) falls into one of the cases (0), (+), (−), (±) above for some $x'_i \in X'(\overline{k})$.
- $D_r = {}^\tau D_0$.

It is clear that $\mathfrak{D}$ is a finite set. This will be the index set for our stratification of $\mathcal{Z}$.

9.1.2 Definition of strata

Fix $D_\bullet = (D_i)_{0\leq i\leq r} \in \mathfrak{D}$. Let $I_0 := \{1 \leq i \leq r | D_i = D_{i-1}\}$. Similarly we define I_+, I_- and $I_\pm$ as the set of those i such that (D_{i-1}, D_i) falls into case (+), (−)

and $(\pm)$ respectively. Let $\mathcal{Z}[D_\bullet]$ be the substack of $\mathcal{Z}_{\overline{k}}$ classifying

$$(\{x_i'\} \in X'^r; \{\mathcal{F}_i\} \in \mathrm{Hk}^r_{U(n)}; \{\mathcal{L}(D_i) \xrightarrow{t_i'} \mathcal{F}_i\}_{0\le i\le r})$$

such that every t_i' is saturated. Let

$$\pi[D_\bullet] : \mathcal{Z}[D_\bullet] \to (X'_{\overline{k}})^{I_0}$$

be the map recording those x_i' for $i \in I_0$. Note that for $i \in I_+ \cup I_- \cup I_\pm$, x_i' is determined by $D_\bullet$.

Proposition 9.1 *Let $n \ge 2$.*

(1) *The substacks $\mathcal{Z}[D_\bullet]$ for $D_\bullet \in \mathfrak{D}$ give a partition of $\mathcal{Z}$.*
(2) *Each geometric fiber of $\pi[D_\bullet]$ has dimension $\le (n-1)|I_+| + (n-2)|I_0|$.*
(3) *We have $\dim \mathcal{Z}[D_\bullet] \le r(n-1)$. The equality can only be achieved when $I_0 = \{1, 2, \dots, r\}$, i.e., all D_i are equal to the same divisor of X', which is then necessarily defined over k.* [5]

Proof (1) Each geometric point of $z \in \mathcal{Z}$ defines a (unique) point $D_\bullet \in \mathfrak{D}$ by taking the zero divisor of t_i, and then $z \in \mathcal{Z}[D_\bullet]$ by definition.

(2) Let $\mathcal{H}[D_\bullet]$ be the substack of the fiber of $(\mathrm{Hk}^r_{\mathcal{M}})_{\overline{k}}$ over $(\mathcal{L}, a) \in \mathcal{A}_d(k)$ classifying data $(\{x_i'\}, \mathcal{L} \xrightarrow{t_i} \mathcal{F}_i)$ such that t_i extends to a map $t_i' : \mathcal{L}(D_i) \to \mathcal{F}_i$ which is saturated. Note that for $i \in I_+ \cup I_- \cup I_\pm$, x_i' is determined by $D_\bullet$. Let $\mathcal{M}[D_i]$ be the substack of the fiber of $\mathcal{M}(1,n)_{\overline{k}}$ over $(\mathcal{L}, a) \in \mathcal{A}(1,n)(k)$ classifying maps $t : \mathcal{L} \to \mathcal{F}$ that extend to a saturated map $t' : \mathcal{L}(D_i) \to \mathcal{F}$. Then we have a Cartesian diagram of stacks over $\overline{k}$

$$\begin{array}{ccc} \mathcal{Z}[D_\bullet] & \longrightarrow & \mathcal{H}[D_\bullet] \\ \downarrow & & \downarrow{\scriptstyle (p_0, p_r)} \\ \mathcal{M}[D_0] & \xrightarrow{(\mathrm{Id},\mathrm{Fr})} & \mathcal{M}[D_0] \times \mathcal{M}[D_r]. \end{array} \tag{9.1}$$

Note since $D_r = {}^\tau D_0$, the Frobenius morphism sends $\mathcal{M}[D_0]$ to $\mathcal{M}[D_r]$.

Let

$$\Pi[D_\bullet] : \mathcal{H}[D_\bullet] \to \mathcal{M}[D_r] \times X'^{I_0}_{\overline{k}}$$

be the projection p_r and the map recording x_i' for $i \in I^0$.

Claim 9.2 *The map $\Pi[D_\bullet]$ is smooth and representable of relative dimension $(n-1)|I_+| + (n-2)|I_0|$.*

[5] In this case, $\mathcal{Z}[D_\bullet]$ can be identified with the open substack $\mathring{\mathcal{Z}}^r_{\mathcal{L}(D_0)}(a') \subset \mathcal{Z}^r_{\mathcal{L}(D_0)}(a')$ (where a' is the map $\mathcal{L}(D_0) \to \sigma^*\mathcal{L}(D_0)^\vee$ induced from a) defined by requiring all the maps $t_i' : \mathcal{L}(D_0) \to \mathcal{F}_i$ be saturated.

Assuming the claim, we finish the proof of (2). Indeed, we may apply Lemma 9.3 below to the Cartesian diagram (9.1) to conclude, except that $\mathcal{M}[D_0]$ is generally not a scheme. To remedy, we may restrict to a finite type open substack $\mathcal{M}[D_0]^{\leq P} \subset \mathcal{M}[D_0]$ by bounding Harder-Narasimhan polygon of $(\mathcal{F}, h)$, and impose level structures on the Hermitian bundle $(\mathcal{F}, h)$ at a closed point $x \in |X|$ to arrive at a scheme $\widetilde{\mathcal{M}}[D_0]^{\leq P}$ which is a torsor over $\mathcal{M}[D_0]^{\leq P}$ by an algebraic group H. Truncating and imposing the same level structures to $\mathcal{H}[D_\bullet]$ gives a scheme $\widetilde{\mathcal{H}}[D_\bullet]^{\leq P}$ (with legs away from x) such that $\widetilde{\mathcal{H}}[D_\bullet]^{\leq P}/H = \mathcal{H}[D_\bullet]^{\leq P}$. Let $\widetilde{\mathcal{Z}}[D_\bullet]^{\leq P}$ be defined by a Cartesian diagram similar to (9.1), with $\mathcal{H}[D_\bullet]$ replaced by $\widetilde{\mathcal{H}}[D_\bullet]^{\leq P}$ and $\mathcal{M}[D_i]$ replaced by $\widetilde{\mathcal{M}}[D_i]^{\leq P}$ for $i = 0, r$. We apply Lemma 9.3 to conclude that the fibers of $\widetilde{\mathcal{Z}}[D_\bullet]^{\leq P} \to X'^{I_0}_{\overline{k}}$ have dimension $\leq (n-1)|I_+| + (n-2)|I_0|$. Now $\widetilde{\mathcal{Z}}[D_\bullet]^{\leq P}/H(k) \xrightarrow{\sim} \mathcal{Z}[D_\bullet]^{\leq P}$ and for varying P and x, $\mathcal{Z}[D_\bullet]^{\leq P}$ cover $\mathcal{Z}[D_\bullet]$, hence the same dimension estimate holds for $\mathcal{Z}[D_\bullet]$.

It remains to prove the claim.

For $r \geq j \geq 0$, let $\mathcal{H}_{\geq j}$ be the moduli stack defined similarly to $\mathcal{H}[D_\bullet]$ but classifying saturated maps $\{t_i : \mathcal{L}(D_i) \to \mathcal{F}_i\}_{j \leq i \leq r}$ (lying over a) only for i in the indicated range. We can factorize $\Pi[D_\bullet]$ as

$$\Pi[D_\bullet] : \mathcal{H}[D_\bullet] = \mathcal{H}_{\geq 0} \xrightarrow{\Pi_1} \mathcal{H}_{\geq 1} \xrightarrow{\Pi_2} \cdots \xrightarrow{\Pi_r} \mathcal{H}_{\geq r} = \mathcal{M}[D_r] \times X'^{I_0}_{\overline{k}}.$$

The desired smoothness and relative dimension claims would follow from the following four statements:

($H0$) If $i \in I_0$, then Π_i exhibits $\mathcal{H}_{\geq i-1}$ as an open substack in a $\mathbf{P}^{n-2}$-bundle over $\mathcal{H}_{\geq i}$.
($H+$) If $i \in I_+$, then Π_i exhibits $\mathcal{H}_{\geq i-1}$ as an open substack in a $\mathbf{P}^{n-1}$-bundle over $\mathcal{H}_{\geq i}$.
($H-$) If $i \in I_-$, then Π_i is an isomorphism.
($H\pm$) If $i \in I_\pm$, then Π_i is an open immersion.

We next establish each of these statements.

Proof of ($H0$). When $i \in I_0$, $D_{i-1} = D_i$. We write the modification $\mathcal{F}_{i-1} \dashrightarrow \mathcal{F}_i$ as

$$\mathcal{F}_{i-1} \xhookleftarrow{x'_i} \mathcal{F}^\flat_{i-1/2} \xhookrightarrow{\sigma(x'_i)} \mathcal{F}_i \tag{9.2}$$

Here both arrows have cokernel of length one supported at the labelled points. Such modifications of $\mathcal{F}_i$ are parametrized by a hyperplane H in the fiber $\mathcal{F}_i|_{\sigma(x'_i)}$. The requirement that $t_i : \mathcal{L}(D_i) \to \mathcal{F}_i$ should land in $\mathcal{F}^\flat_{i-1/2}$ is equivalent to the (closed) condition that H should contain the line given by the image of $\mathcal{L}(D_i)|_{\sigma(x'_i)}$. This cuts out a $\mathbf{P}^{n-2}$ in the space of hyperplanes $H \subset \mathcal{F}_i|_{\sigma(x'_i)}$. The further requirement that $t_{i-1} : \mathcal{L}(D_i) \to \mathcal{F}^\flat_{i-1/2} \to \mathcal{F}_{i-1}$ be saturated is an open condition.

This argument globalizes in the evident way, exhibiting that Π_i as an open substack in a $\mathbf{P}^{n-2}$-bundle. This applies similarly for the analogous arguments below for the other cases.

Proof of $(H+)$. When $i \in I_+$, we have $D_{i-1} = D_i - \sigma(x_i')$. We write the modification of $\mathcal{F}_i$ as in (9.2). This time the choice of the $\mathcal{F}_i \leftarrow \mathcal{F}_{i-1/2}^\flat$ is the open subset of those hyperplanes $H \in \mathbf{P}(\mathcal{F}_i|_{\sigma(x_i')})$ that do not contain the image of $\mathcal{L}(D_i)|_{\sigma(x_i')}$. The requirement that $\mathcal{L}(D_{i-1}) = \mathcal{L}(D_i - \sigma(x_i')) \to \mathcal{F}_{i-1/2}^\flat \to \mathcal{F}_{i-1}$ be saturated at x_i' imposes a further open condition.

Proof of $(H-)$. When $i \in I_-$, we have $D_{i-1} = D_i + x_i'$. We write the modification as

$$\mathcal{F}_{i-1} \xrightarrow{\sigma(x_i')} \mathcal{F}_{i-1/2}^\sharp \xleftarrow{x_i'} \mathcal{F}_i \tag{9.3}$$

where both arrows have cokernel of length one supported at the labelled points. Now $t_i : \mathcal{L}(D_i) \to \mathcal{F}_i$ is required to extend to $\mathcal{L}(D_i + x_i') \to \mathcal{F}_{i-1/2}^\sharp$. This determines the upper modification $\mathcal{F}_i \to \mathcal{F}_{i-1/2}^\sharp$ uniquely, which in turn determines the lower modification $\mathcal{F}_{i-1/2}^\sharp \leftarrow \mathcal{F}_{i-1}$ as well. We get a map $t_{i-1}' : \mathcal{L}(D_{i-1}) = \mathcal{L}(D_i + x_i') \to \mathcal{F}_{i-1/2}^\sharp$. We claim that t_{i-1}' automatically lands in $\mathcal{F}_{i-1}$. Indeed, the claim is equivalent to saying that under the pairing between $\mathcal{F}_i|_{x_i'}$ and $\mathcal{F}_i|_{\sigma(x_i')}$, the images of $t_i(x_i')$ and $t_i(\sigma(x_i'))$ pair to zero. The latter statement is equivalent to saying that the induced Hermitian map $a_i' : \mathcal{L}(D_i) \to \sigma^*(\mathcal{L}(D_i))^\vee$ vanishes at x_i'. But we know that the divisor of a_i' is $\nu^* D_a - D_i - \sigma(D_i)$. Since $D_{i-1} = D_i + x_i'$ satisfies $\nu(D_{i-1}) \le D_a$ by assumption, we see that $\nu^* D_a - D_i - \sigma(D_i) \ge x_i' + \sigma(x_i')$, hence a_i' is guaranteed to vanish at x_i' and $\sigma(x_i')$. This shows that there is a unique lifting of any point of $\mathcal{H}_{\ge i}$ to $\mathcal{H}_{\ge i-1}$, hence Π_i is an isomorphism.

Proof of $(H\pm)$. When $i \in I_\pm$, we have $D_i + x_i' = D_{i-1} + \sigma(x_i')$. We write the modification as in (9.3). As in the case $(H-)$, the requirement that $t_i : \mathcal{L}(D_i) \to \mathcal{F}_i$ should extend to $\mathcal{L}(D_i + x_i') \to \mathcal{F}_{i-1/2}^\sharp$ determines the modification. Then we automatically get a map $t_{i-1} : \mathcal{L}(D_{i-1}) = \mathcal{L}(D_i + x_i' - \sigma(x_i')) \to \mathcal{F}_{i-1}$; the requirement that t_{i-1} be saturated is an open condition. Therefore Π_i is an open immersion in this case.

(3) By (2) we have

$$\dim \mathcal{Z}[D_\bullet] \le (n-1)|I_+| + (n-2)|I_0| + |I_0| = (n-1)(|I_+ \cup I_0|) \le r(n-1).$$

Equality holds only if I_- and $I_\pm$ are empty. However, by degree reasons we have $|I_+| = |I_-|$, so in the equality case we must have $I_+ = \varnothing$ as well. We conclude that equality can only be achieved if $I_0 = \{1, 2, \ldots, r\}$; in other words, all D_i must be the same. In particular, since $D_r = {}^\tau D_0$, this forces D_0 to be defined over k. □

The Lemma below, a slight variant of [14, Lemma 2.13], was used above.

Lemma 9.3 (Variant of [14, Lemma 2.13]) *Let W, Z, T be schemes of finite type over $\overline{k}$. Let $Z^{(1)}$ be the Frobenius twist of Z (i.e., the pullback of Z under the q-Frobenius* Spec $\overline{k} \to$ Spec $\overline{k}$*). Let $\widetilde{h} = (h_1, h_T) \colon W \to Z^{(1)} \times T$ be smooth of relative dimension*

d, and $h_0\colon W \to Z$ be an arbitrary map. Define V as the fibered product

$$\begin{array}{ccc} V & \longrightarrow & W \\ \downarrow & & \downarrow{\scriptstyle (h_0,h_1)} \\ Z & \xrightarrow{(\mathrm{Id},\mathrm{Fr})} & Z \times Z^{(1)} \end{array}$$

Then each fiber of the composition map $V \to W \xrightarrow{h_T} T$ has dimension $\leq d$.

Proof Restricting W over a point $t \in T(\overline{k})$, we reduce to the case T itself is the point $\operatorname{Spec} \overline{k}$. We may assume $Z = \operatorname{Spec} R$ where $R = \overline{k}[x_1, \ldots, x_l]/I$. Let $R^{(1)} = \overline{k} \otimes_{\overline{k}} R \cong \overline{k}[\xi_1, \ldots, \xi_l]/I^{(1)}$ be the base change of R under Fr_q, where $\xi_i = 1 \otimes x_i$. Since $h_1 : W \to Z^{(1)}$ is smooth of relative dimension d, by Zariski localizing we may assume $W = \overline{k}[\xi_1, \ldots, \xi_l, y_1, \ldots, y_{m+d}]/(I^{(1)}, r_1, \ldots, r_m)$, with $(\frac{\partial r_i}{\partial y_j})_{j=1}^m$ having rank m ($r_i \in \overline{k}[\xi_1, \ldots, \xi_l, y_1, \ldots, y_{m+d}]$). Under $h_0 : W \to Z$, the coordinates x_i of Z pullback to functions f_i on W, $1 \leq i \leq l$. We lift f_i to polynomials $f_i \in \overline{k}[\xi_1, \ldots, \xi_l, y_1, \ldots, y_{m+d}]$.

By definition, V has the form

$$V \cong \operatorname{Spec} \frac{\overline{k}[\xi_1, \ldots, \xi_l, y_1, \ldots, y_m, y_{m+1} \ldots, y_{m+d}]}{(I^{(1)}(\xi), g_1, \ldots, g_l, r_1, \ldots, r_m)}$$

where $g_i = \xi_i - f_i^q$. In particular, V is a closed subscheme of

$$U := \operatorname{Spec} \left(\overline{k}[\xi_1, \ldots, \xi_l, y_1, \ldots, y_m, y_{m+1} \ldots, y_{m+d}]/(g_1, \ldots, g_l, r_1, \ldots, r_m) \right).$$

The Jacobian matrix for the defining equations of U has the form

$$\begin{pmatrix} \left[\frac{\partial g_i}{\partial \xi_j}\right]_{j=1}^{l} & \left[\frac{\partial g_i}{\partial y_j}\right]_{j=1}^{m} & \left[\frac{\partial g_i}{\partial y_j}\right]_{j=m+1}^{m+d} \\ \left[\frac{\partial r_i}{\partial \xi_j}\right]_{j=1}^{l} & \left[\frac{\partial r_i}{\partial y_j}\right]_{j=1}^{m} & \left[\frac{\partial r_i}{\partial y_j}\right]_{j=m+1}^{m+d} \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_l & 0 & 0 \\ * & \text{invertible}_m & * \end{pmatrix},$$

which evidently has rank $l + m$. Hence U is smooth of dimension d. Since $V \hookrightarrow U$, $\dim V \leq d$. □

9.2 The case $m = 1$ and $a = 0$

We keep the notations from §9.1. In this subsection we extend the discussion in §9.1 to the case $a = 0$. Fix a line bundle $\mathcal{L} \in \mathrm{Pic}(X')$. Recall from Definition 7.4 that $\mathcal{Z}^r_{\mathcal{L}}(0)^\circ$ is the moduli stack classifying hermitian shtukas $(\{x'_i\}, \{\mathcal{F}_i\})$ together with compatible maps $\{\mathcal{L} \xrightarrow{t_i} \mathcal{F}_i\}$) with t_i injective (fiberwise over the test scheme S) and the image of t_i being isotropic. In this subsection, let

$$\mathcal{Z} := \mathcal{Z}^r_{\mathcal{L}}(0)^\circ_{\overline{k}}.$$

If $n = 1$ then $\mathcal{Z}^r_{\mathcal{L}}(0)^\circ = \varnothing$. We always assume $n \geq 2$ below.

9.2.1 Indexing set for strata

Let $I_0 \sqcup I_+ \sqcup I_-$ be a partition of $\{1, 2, \ldots, r\}$ such that $|I_+| = |I_-|$. We denote this partition simply by $I_\bullet$. For any $N \in \mathbf{Z}_{\geq 0}$, define $\mathfrak{D}(N; I_\bullet)$ to be the moduli space of sequences of effective divisors $(D_i)_{0 \leq i \leq r}$ on $X'_{\overline{k}}$ such that

(1) $\deg(D_0) \leq N$.
(2) For $? = 0, +$ or $-$, and $i \in I_?$, the pair (D_{i-1}, D_i) belongs to the corresponding Case (?) listed in the beginning of §9.1 (for some $x'_i \in X'_{\overline{k}}$ in the case $? = +$ or $-$).
(3) $D_r = {}^\tau D_0$.

We have a map recording the points x'_i for $i \in I_+ \cup I_-$:

$$(\pi_+, \pi_-) : \mathfrak{D}(N; I_\bullet) \to (X'_{\overline{k}})^{I_+} \times (X'_{\overline{k}})^{I_-}.$$

Lemma 9.4 *The map $\pi_+ : \mathfrak{D}(N; I_\bullet) \to (X'_{\overline{k}})^{I_+}$ is quasi-finite.*

Proof For a fixed geometric point $(x'_i)_{i \in I_+} \in (X'_{\overline{k}})^{I_+}(\overline{k})$, its fiber in $\mathfrak{D}(N; I_\bullet)$ consists of $(D_0, \{x'_i\}_{i \in I_-})$ such that $\deg D_0 \leq N$ and

$$D_0 + \sum_{i \in I_+} \sigma(x'_i) = {}^\tau D_0 + \sum_{i \in I_-} x'_i. \tag{9.4}$$

Let $v \in |X'|$ be a closed point that intersects $\operatorname{supp} D_0$. If $\deg(v) > N$, then D_0 cannot contain all geometric points over v and hence there exists a geometric point $y|v$ such that $y \in \operatorname{supp} {}^\tau D_0$ but $y \notin \operatorname{supp} D_0$. By (9.4), $y = \sigma(x'_i)$ for some $i \in I_+$. Therefore points in D_0 are either over closed points of degree $\leq N$, or in the Galois orbit of $\sigma(x'_i)$ for some $i \in I_+$. This leaves finitely many possibilities for D_0, hence for $\{x'_i\}_{i \in I_-}$ as well. □

9.2.2 Definition of strata

For a partition $I = (I_0, I_+, I_-)$ of $\{1, 2, \ldots, r\}$, define $\mathcal{Z}[N; I_\bullet]$ to be the stack classifying

$$(\{D_i\}_{0 \leq i \leq r}, (\{x'_i\}_{1 \leq i \leq r}, \{\mathcal{F}_i\}_{0 \leq i \leq r}) \in \mathrm{Hk}^r_{U(n)}, \{\mathcal{L} \xrightarrow{t_i} \mathcal{F}_i\}_{0 \leq i \leq r})$$

such that $\{D_i\} \in \mathfrak{D}(N; I_\bullet)$ with image $\{x'_i\}_{i \in I_?}$ under $\pi_?$ $(? = +, -)$, and t_i extends to a saturated embedding $\mathcal{L}(D_i) \hookrightarrow \mathcal{F}_i$. We have a map

$$\pi[N; I_\bullet] : \mathcal{Z}[N; I_\bullet] \to (X'_{\overline{k}})^{I_0} \times \mathfrak{D}(N; I_\bullet).$$

The following is the analog of Proposition 9.1 when $a = 0$.

Proposition 9.5 *Let $n \geq 2$.*

(1) *For varying $N \in \mathbf{Z}_{\geq 0}$ and partitions $I_\bullet$ of $\{1, 2, \ldots, r\}$ such that $|I_+| = |I_-|$, the substacks $\mathcal{Z}[N; I_\bullet]$ give a partition of $\mathcal{Z}$.*

(2) *The fibers of the map* $\pi[N; I_\bullet]$ *have dimension* $\leq (n-1)|I_+| + (n-2)|I_0|$.
(3) *We have* $\dim \mathcal{Z}[N; I_\bullet] \leq r(n-1)$. *Moreover, when* $n \geq 3$, *the equality can only be achieved when* $I_0 = \{1, 2, \ldots, r\}$, *i.e., all* D_i *are equal to the same divisor of* X' *defined over* k.

Proof (1) is similar to Proposition 9.1(1), except we have to argue that the strata are only non-empty for $|I_+| = |I_-|$, and that Case $(\pm)$ cannot appear for points in $\mathcal{Z} = \mathcal{Z}^r_{\mathcal{L}}(0)^\circ$. The first statement follows from the assumption that $D_r = {}^\tau D_0$ has the same degree as D_0. For the second statement, suppose Case $(\pm)$ happens for the modification

$$\mathcal{F}_{i-1} \leftarrow \mathcal{F}^\flat_{i-1/2} \rightarrow \mathcal{F}_i,$$

and let $H \subset \mathcal{F}_{i-1,x'_i}$ be the hyperplane that is the image of $\mathcal{F}^\flat_{i-1/2}$. Then $H^\perp \subset \mathcal{F}_{i-1,\sigma(x'_i)}$ is the line along which the upper modification $\mathcal{F}^\flat_{i-1/2} \hookrightarrow \mathcal{F}_i$ is performed. Let $\ell_{x'_i}$ (resp. $\ell_{\sigma(x'_i)}$) be the image of $\mathcal{L}(D_{i-1}) \rightarrow \mathcal{F}_{i-1}$ at x'_i (resp. at $\sigma(x'_i)$). Since the image of $\mathcal{L}(D_i - 1)$ is isotropic (because $a = 0$), $(\ell_{x'_i}, \ell_{\sigma(x'_i)}) = 0$ under the pairing between $\mathcal{F}_{i-1,x'_i}$ and $\mathcal{F}_{i-1,\sigma(x'_i)}$. The condition $D_i + x'_i = D_{i-1} + \sigma(x'_i)$ happens only if $\ell_{x'_i} \not\subset H$ and $\ell_{\sigma(x'_i)} = H^\perp$. This contradicts the fact that $(\ell_{x'_i}, \ell_{\sigma(x'_i)}) = 0$.

(2) is proved in the same way as Proposition 9.1(2).

(3) Applying (2) and Lemma 9.4 we get

$$\begin{aligned}\dim \mathcal{Z}[N; I_\bullet] &= (n-1)|I_+| + (n-2)|I_0| + |I_0| + \dim \mathfrak{D}(N; I_\bullet) \\ &\leq (n-1)|I_+| + (n-1)|I_0| + |I_+| = (n-1)|I_0| + n|I_+|.\end{aligned}$$

Since $n \geq 2$, we have $n \leq 2(n-1)$, therefore the above is $\leq (n-1)(|I_0| + 2|I_+|) = r(n-1)$. When $n \geq 3$, we have strict inequalities $n < 2(n-1)$, so equality can only be achieved when I_+, hence I_-, are all empty. □

10 Comparison of two cycle classes

The goal in this section is to show the following theorem.

Theorem 10.1 *Let* $\mathcal{E} \in \mathrm{Bun}_{\mathrm{GL}'_n}(k)$ *and* $a \in \mathcal{A}_{\mathcal{E}}(k)$. *Let* $s : \mathcal{E}' = \oplus_{i=1}^n \mathcal{L}_i \hookrightarrow \mathcal{E}$ *be a good framing of* $(\mathcal{E}, a)$ *in the sense of Definition 7.15. Let* $a' : \mathcal{E}' \rightarrow \sigma^*(\mathcal{E}')^\vee$ *be the Hermitian map induced from* a. *Then we have an equality in the Chow group*

$$\zeta^r_{\mathcal{L}_1,\ldots,\mathcal{L}_n}(a')|_{\mathcal{Z}^r_{\mathcal{E}}(a)} = [\mathrm{Sht}^r_{\mathcal{M}(n,n)}]|_{\mathcal{Z}^r_{\mathcal{E}}(a)} \in \mathrm{Ch}_0(\mathcal{Z}^r_{\mathcal{E}}(a)). \tag{10.1}$$

Here the restriction on the RHS is via (8.3), *noting that* $\mathcal{Z}^r_{\mathcal{E}}(a)^\circ = \mathcal{Z}^r_{\mathcal{E}}(a)$, *and the class* $[\mathrm{Sht}^r_{\mathcal{M}(n,n)}]$ *is as in Definition 8.16.*

In particular, the cycle class $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ *as in Definition-Proposition 7.18 is well-defined* (*i.e., independent of the choice of a good framing*).

Below we consider the case where X' is geometrically connected. At the end of this section (§10.5) we comment on how to modify the argument in the case $X' = X \coprod X$ or $X' = X_{k'}$, where k'/k is the quadratic extension.

10.1 First reductions

For a vector bundle $\mathcal{E}$ on X' let $\mu_{\min}(\mathcal{E}) \in \mathbf{Q}$ be the smallest slope that appears in the Harder-Narasimhan filtration of $\mathcal{E}$. For $\mathcal{E}$ of rank n and $a \in \mathcal{A}_{\mathcal{E}}(k)$, a good framing $s : \oplus_{i=1}^n \mathcal{L}_i \hookrightarrow \mathcal{E}$ for $(\mathcal{E}, a)$ is called *very good* if it satisfies the additional condition

(3) $\mu_{\min}(\mathcal{E}) > \max\{\deg \mathcal{L}_i + 2g' - 1\}_{1 \le i \le n}$.

Most of the work in this section will be devoted to proving the slightly weaker statement below.

Theorem 10.2 *Suppose X' is connected. Then the identity (10.1) holds if $s : \mathcal{E}' = \oplus_{i=1}^n \mathcal{L}_i \hookrightarrow \mathcal{E}$ is a very good framing of $(\mathcal{E}, a)$.*

Lemma 10.3 *Theorem 10.2 implies Theorem 10.1.*

Proof Choose effective divisors D_i on X' $(1 \le i \le n)$ such that $\nu(D_1 + \cdots + D_n)$ is multiplicity-free and disjoint from $\nu^{-1}(D_a)$. Let $\mathcal{L}_i' = \mathcal{L}_i(-D_i)$. When the D_i's have sufficiently large degree, the resulting map

$$s' : \oplus_{i=1}^n \mathcal{L}_i' \hookrightarrow \oplus_{i=1}^n \mathcal{L}_i \hookrightarrow \mathcal{E}$$

is a very good framing. Let a'' be the induced Hermitian map $\oplus \mathcal{L}_i' \to \sigma^*(\oplus \mathcal{L}_i')^\vee$. By Theorem 10.2 we have

$$\zeta^r_{\mathcal{L}_1', \ldots, \mathcal{L}_n'}(a'')|_{\mathcal{Z}^r_{\mathcal{E}}(a)} = [\mathrm{Sht}^r_{\mathcal{M}(n,n)}]|_{\mathcal{Z}^r_{\mathcal{E}}(a)}.$$

Therefore, to prove (10.1) it suffices to show

$$\zeta^r_{\mathcal{L}_1, \ldots, \mathcal{L}_n}(a')|_{\mathcal{Z}^r_{\mathcal{E}}(a)} = \zeta^r_{\mathcal{L}_1', \ldots, \mathcal{L}_n'}(a'')|_{\mathcal{Z}^r_{\mathcal{E}}(a)} \in \mathrm{Ch}_0(\mathcal{Z}^r_{\mathcal{E}}(a)).$$

Let U' be the complement of $\cup_{i=1}^n \mathrm{supp}(D_i + \sigma D_i)$ in X'. By construction, U' contains $\nu^{-1}(D_a)$, therefore $\mathcal{Z}^r_{\mathcal{E}}(a)|_{U'^r} = \mathcal{Z}^r_{\mathcal{E}}(a)$ by Lemma 7.12. Let $\zeta_i = [\mathcal{Z}^r_{\mathcal{L}_i}(a'_{ii})^\circ]|_{U'^r} \in \mathrm{Ch}_{r(n-1)}(\mathcal{Z}^r_{\mathcal{L}_i}(a'_{ii})^\circ|_{U'^r})$. Similarly define ζ_i' using $(\mathcal{L}_i', a''_{ii})$. Then it suffices to show the equality

$$(\zeta_1 \cdot \zeta_2 \cdot \cdots \cdot \zeta_n)|_{\mathcal{Z}^r_{\mathcal{E}}(a)} = (\zeta_1' \cdot \zeta_2' \cdot \cdots \cdot \zeta_n')|_{\mathcal{Z}^r_{\mathcal{E}}(a)} \in \mathrm{Ch}_0(\mathcal{Z}^r_{\mathcal{E}}(a)), \tag{10.2}$$

where the intersection products are taken over $\mathrm{Sht}^r_{U(n)}|_{U'^r}$. Applying Lemma 7.8 to each injection $\mathcal{L}_i' \hookrightarrow \mathcal{L}_i$, we see that $\mathcal{Z}^r_{\mathcal{L}_i}(a'_{ii})|_{U'^r} \hookrightarrow \mathcal{Z}^r_{\mathcal{L}_i'}(a''_{ii})|_{U'^r}$ is open and closed. Therefore $\mathcal{Z}^r_{\mathcal{L}_i}(a'_{ii})^\circ|_{U'^r} \hookrightarrow \mathcal{Z}^r_{\mathcal{L}_i'}(a''_{ii})^\circ|_{U'^r}$ is open. This shows that the fundamental class ζ_i is the open restriction of ζ_i' to $\mathcal{Z}^r_{\mathcal{L}_i}(a'_{ii})^\circ|_{U'^r}$. The equality (10.2) then follows. □

10.2 Auxiliary moduli spaces

Let $\underline{d} = (d_i)_{1 \le i \le n} \in \mathbf{Z}^n_{\ge 0}$ and $e \in \mathbf{Z}_{\ge 0}$. Write $d = \sum d_i$.

Recall that $\mathcal{M}_e \subset \mathcal{M}(n,n)$ is the open-closed substack where $\chi(X', \mathcal{E}) = -e$. Let $\mathcal{M}_{\underline{d}}$ be the moduli stack classifying $(\{\mathcal{L}_i\}_{1 \le i \le n}, (\mathcal{F}, h), \{t_i' : \mathcal{L}_i \to \mathcal{F}\}_{1 \le i \le n})$ where

- $\mathcal{L}_i$ is a line bunde on X' with $\chi(X', \mathcal{L}_i) = -d_i$ for $1 \le i \le n$;
- $(\mathcal{F}, h) \in \mathrm{Bun}_{U(n)}$ satisfying

$$\mu_{\min}(\mathcal{F}) > \max\{-d_i + 3g' - 2\}_{1\le i\le n}. \tag{10.3}$$

- For each $1 \le i \le n$, $t_i' : \mathcal{L}_i \to \mathcal{F}$ is an injective map (fiberwise over the test scheme).

We define $\mathrm{Hk}^r_{\mathcal{M}_{\underline{d}}}$ to be the moduli stack classifying

$$(\{\mathcal{L}_i\}_{1\le i\le n}, (\{x_i'\}_{1\le i\le r}, \{(\mathcal{F}_j, h_j)\}_{0\le j\le r}) \in \mathrm{Hk}^r_{U(n)}, \{t_{ij}' : \mathcal{L}_i \to \mathcal{F}_j\})$$

where $\mathcal{L}_i$ are the same as in $\mathcal{M}_{\underline{d}}$, each $\mathcal{F}_j$ satisfies the analogue of (10.3) with $\mathcal{F}$ replaced by $\mathcal{F}_j$, and fiberwise injective maps $t_{ij}' : \mathcal{L}_i \to \mathcal{F}_j$ are compatible with the isomorphisms between $\mathcal{F}_{j-1}$ and $\mathcal{F}_j$ away from x_i'.

Lemma 10.4 *The stacks $\mathcal{M}_{\underline{d}}$ and $\mathrm{Hk}^1_{\mathcal{M}_{\underline{d}}}$ are smooth stacks of pure dimension $dn - (n^2 - 2n)(g - 1)$.*

Proof We first prove the statement for $\mathcal{M}_{\underline{d}}$. Consider the map $\gamma_{\underline{d}} : \mathcal{M}_{\underline{d}} \to \prod_{i=1}^n \mathrm{Pic}_{X'}^{-d_i+g'-1} \times \mathrm{Bun}_{U(n)}$ sending $(\{\mathcal{L}_i\}, (\mathcal{F}, h), \{t_i\})$ to $(\{\mathcal{L}_i\}, (\mathcal{F}, h))$. For $(\mathcal{F}, h) \in \mathrm{Bun}_{U(n)}$ and $\mathcal{L}_i \in \mathrm{Pic}_{X'}^{-d_i+g'-1}$, the condition $\mu_{\min}(\mathcal{F}) \ge \max\{-d_i + 3g' - 2\}_{1\le i\le n} = \max\{\deg \mathcal{L}_i + 2g' - 1\}_{1\le i\le n}$ guarantees that $\mathrm{Ext}^1(\mathcal{L}_i, \mathcal{F}) = \mathrm{Hom}(\mathcal{F}, \mathcal{L}_i \otimes \omega_{X'})^\vee = 0$. Noting that $\deg \mathcal{F} = n(g' - 1)$, the Riemann-Roch formula implies that $\gamma_{\underline{d}}$ exhibits $\mathcal{M}_{\underline{d}}$ as an open substack of a vector bundle of rank $\dim \mathrm{Hom}(\oplus \mathcal{L}_i, \mathcal{F}) = -n \sum_i \deg \mathcal{L}_i = dn - n^2(g' - 1)$ over the base. In particular, $\mathcal{M}_{\underline{d}}$ is smooth and equidimensional. Since $\dim \mathrm{Pic}_{X'}^{-d_i+g'-1} = g' - 1$ and $\dim \mathrm{Bun}_{U(n)} = n^2(g - 1)$, we conclude that

$$\dim \mathcal{M}_{\underline{d}} = dn - n^2(g' - 1) + n(g' - 1) + n^2(g - 1) = dn - (n^2 - 2n)(g - 1).$$

The argument for $\mathrm{Hk}^1_{\mathcal{M}_{\underline{d}}}$ is similar. The natural map

$$\mathrm{Hk}^1_{\mathcal{M}_{\underline{d}}} \to \prod_{i=1}^n \mathrm{Pic}_{X'}^{-d_i+g'-1} \times \mathrm{Hk}^1_{U(n)}$$

exhibits $\mathrm{Hk}^1_{\mathcal{M}_{\underline{d}}}$ as an open substack of a vector bundle of rank $\dim \mathrm{Hom}(\oplus \mathcal{L}_i, \mathcal{F}_0 \cap \mathcal{F}_1) = dn - n$ (using that $\deg(\mathcal{F}_0 \cap \mathcal{F}_1) = n(g' - 1) - 1$) over the base. Here we need the stronger inequality $\mu_{\min}(\mathcal{F}_0) > \max\{-d_i + 3g' - 2\}$ to guarantee $\mu_{\min}(\mathcal{F}_0 \cap \mathcal{F}_1) \ge \max\{-d_i + 3g' - 2\}$. In particular, $\mathrm{Hk}^1_{\mathcal{M}_{\underline{d}}}$ is smooth and equidimensional, and

$$\begin{aligned}\dim \mathrm{Hk}^1_{\mathcal{M}_{\underline{d}}} &= dn - n - n^2(g' - 1) + n(g' - 1) + \dim \mathrm{Hk}^1_{U(n)} \\ &= dn - n - 2n^2(g - 1) + 2n(g - 1) + n + n^2(g - 1) = \dim \mathcal{M}_{\underline{d}},\end{aligned}$$

as desired. □

Let $\mathcal{M}_{\underline{d},e}$ be the moduli stack of $(\{\mathcal{L}_i\}_{1\le i\le n}, \mathcal{E}, \mathcal{F}, h, s, t)$ where

- $\mathcal{L}_i \in \mathrm{Pic}_{X'}$ satisfies $\chi(X', \mathcal{L}_i) = -d_i$ for $i = 1, \dots, n$;
- $\mathcal{E} \in \mathrm{Bun}_{\mathrm{GL}'_n}$ satisfies $\chi(X', \mathcal{E}) = -e$;
- $(\mathcal{F}, h) \in \mathrm{Bun}_{U(n)}$;
- $t : \mathcal{E} \to \mathcal{F}$ is an injective map;
- $s : \oplus_{i=1}^n \mathcal{L}_i \to \mathcal{E}$ is a very good framing for $(\mathcal{E}, a)$, where $a = \sigma^* t^\vee \circ h \circ t$ is the induced Hermitian map on $\mathcal{E}$.

Note that being a very good framing requires $-d_i < \mu_{\min}(\mathcal{E}) - (3g' - 2)$ for all i, which imposes an open condition on $\mathcal{E}$. We view $\mathcal{M}_{\underline{d},e}$ as a correspondence

$$\begin{array}{ccccc} & & \mathcal{M}_{\underline{d},e} & & \\ & \overset{w_1}{\swarrow} & & \overset{w_2}{\searrow} & \\ \mathcal{M}_{\underline{d}} & & & & \mathcal{M}_e\,. \end{array} \tag{10.4}$$

Here w_1 records $\oplus_{i=1}^n \mathcal{L}_i \xrightarrow{t\circ s} \mathcal{F}$ and w_2 records $\mathcal{E} \xrightarrow{t} \mathcal{F}$.

We denote the Hitchin bases for $\mathcal{M}_{\underline{d}}, \mathcal{M}_e$ and $\mathcal{M}_{\underline{d},e}$ by $\mathcal{A}_{\underline{d}}$, $\mathcal{A}_e$ and $\mathcal{A}_{\underline{d},e}$ respectively. Here $\mathcal{A}_{\underline{d}}$ parametrizes $(\{\mathcal{L}_i\}_{1\le i\le n}, a' = (a'_{ij}))$ (where $a'_{ij} : \mathcal{L}_j \to \sigma^* \mathcal{L}_i^\vee$) such that $a' : \oplus \mathcal{L}_i \to \sigma^*(\oplus \mathcal{L}_i)^\vee$ is an injective Hermitian map. The base $\mathcal{A}_e$ classifies $(\mathcal{E}, a)$ with a an injective Hermitian map. The base $\mathcal{A}_{\underline{d},e}$ is the moduli stack of $(\{\mathcal{L}_i\}_{1\le i\le n}, \mathcal{E}, s, a)$ where $(\mathcal{E}, a) \in \mathcal{A}_e$, $s : \oplus_{i=1}^n \mathcal{L}_i \to \mathcal{E}$ is a very good framing of $(\mathcal{E}, a)$ and $\mathcal{L}_i \in \mathrm{Pic}_{X'}$ with $\chi(X', \mathcal{L}_i) = -d_i$. We view $\mathcal{A}_{\underline{d},e}$ as a correspondence

$$\begin{array}{ccccc} & & \mathcal{A}_{\underline{d},e} & & \\ & \overset{u_1}{\swarrow} & & \overset{u_2}{\searrow} & \\ \mathcal{A}_{\underline{d}} & & & & \mathcal{A}_e\,. \end{array} \tag{10.5}$$

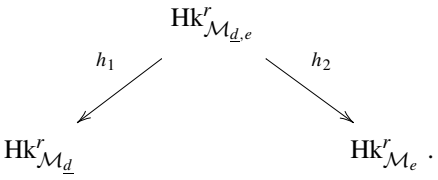
We have Hitchin maps

$$\begin{aligned} f_{\underline{d}} &: \ \mathcal{M}_{\underline{d}} \to \mathcal{A}_{\underline{d}}, \\ f_e &: \ \mathcal{M}_e \to \mathcal{A}_e, \\ f_{\underline{d},e} &: \mathcal{M}_{\underline{d},e} \to \mathcal{A}_{\underline{d},e}. \end{aligned}$$

These maps together give a map of correspondences (10.4) to (10.5). Note $f_{\underline{d}}$ is not necessarily proper because we have imposed an open condition on the minimal slope of $\mathcal{F}$.

Similarly we define the Hecke version $\mathrm{Hk}^r_{\mathcal{M}_{\underline{d},e}}$ of $\mathcal{M}_{\underline{d},e}$ as the moduli stack of $(\{x'_i\}_{1\le i\le r}, \{\mathcal{E} \to \mathcal{F}_i\}_{0\le i\le r}) \in \mathrm{Hk}^r_{\mathcal{M}_e}$ together with a very good framing $s : \oplus \mathcal{L}_i \hookrightarrow$

$\mathcal{E}$ for $(\mathcal{E}, a)$ with $\chi(X', \mathcal{L}_i) = -d_i$. Again we view $\mathrm{Hk}^r_{\mathcal{M}_{\underline{d},e}}$ as a correspondence



Lemma 10.5 *The maps w_1, u_1 and h_1 are étale.*

Proof We first prove that u_1 is étale. Let $(X_{d-e} \times X_e)^\heartsuit$ be the open subscheme of divisors $(D_1, D_2) \in X_{d-e} \times X_e$ such that D_1 is multiplicity-free and disjoint from D_2; let $(X'_{d-e} \times X_e)^\heartsuit$ be the preimage of $(X_{d-e} \times X_e)^\heartsuit$ in $X'_{d-e} \times X_e$. We have a map $\alpha : X'_{d-e} \times X_e \to X_d$ sending (D'_1, D_2) to $\nu(D_1) + D_2$. Let $\alpha^\heartsuit$ be the restriction of α to $(X'_{d-e} \times X_e)^\heartsuit$. By factorizing $\alpha^\heartsuit$ as the composition

$$(X'_{d-e} \times X_e)^\heartsuit \xrightarrow{\nu_{d-e} \times \mathrm{Id}} (X_{d-e} \times X_e)^\heartsuit \xrightarrow{\mathrm{add}} X_d$$

we see that $\alpha^\heartsuit$ is étale. From the definition we have a map

$$j = (u_1, j'_{d-e}, j_e) : \mathcal{A}_{\underline{d},e} \to \mathcal{A}_{\underline{d}} \times_{X_d} (X'_{d-e} \times X_e)^\heartsuit.$$

where $j'_{d-e} : \mathcal{A}_{\underline{d},e} \to X'_{d-e}$ sends $(\{\mathcal{L}_i\}, \oplus \mathcal{L}_i \xrightarrow{s} \mathcal{E}, a)$ to $\mathrm{Div}(s)$ (the divisor of $\det(s)$) and $j_e : \mathcal{A}_{\underline{d},e} \to X_e$ sends it to D_a (see Definition 7.3). The map $\mathcal{A}_{\underline{d}} \to X_d$ used in the fiber product records the divisor $D_{a'}$ of the Hermitian map a' on $\oplus \mathcal{L}_i$. We claim that j is an open immersion. Indeed, given $(\{\mathcal{L}_i\}, a') \in \mathcal{A}_{\underline{d}}$ and $(D'_1, D_2) \in (X'_{d-e} \times X_e)^\heartsuit$ such that $\nu(D_1) + D_2 = D_{a'}$, by the disjointness of $D'_1, \sigma(D'_1)$ and $\nu^{-1}(D_2)$, there is one and only one coherent sheaf $\mathcal{E}$ such that $\oplus \mathcal{L}_i \subset \mathcal{E} \subset \sigma^*(\oplus \mathcal{L}_i)^\vee$, $\mathcal{E}/\oplus \mathcal{L}_i$ is supported on D'_1, and $\sigma^*(\oplus \mathcal{L}_i)^\vee/\mathcal{E}$ is supported away from D'_1. This would give a very good framing of $\mathcal{E}$ if the open condition $\mu_{\min}(\mathcal{E}) > \max\{-d_i + 3g' - 2\}_{1 \le i \le n}$ is satisfied. This shows that j is an open immersion. Since $\alpha^\heartsuit$ is étale, we conclude that $\mathcal{A}_{\underline{d},e}$ is étale over $\mathcal{A}_{\underline{d}}$.

To show w_1 is étale, we observe that $\mathcal{M}_{\underline{d},e} \cong \mathcal{M}_{\underline{d}} \times_{\mathcal{A}_{\underline{d}}} \mathcal{A}_{\underline{d},e}$. Since u_1 is étale, so is w_1.

Finally, $\mathrm{Hk}^r_{\mathcal{M}_{\underline{d},e}}$ is the open substack of $\mathrm{Hk}^r_{\mathcal{M}_{\underline{d}}} \times_{\mathcal{A}_{\underline{d}}} \mathcal{A}_{\underline{d},e}$ where the legs avoid $\mathrm{Div}(s)$. Since u_1 is étale, so is h_1. □

10.3 Auxiliary Hitchin shtukas

We define $\mathrm{Sht}^r_{\mathcal{M}_{\underline{d}}}$ and $\mathrm{Sht}^r_{\mathcal{M}_{\underline{d},e}}$ as the fiber product

$$\begin{array}{ccc} \mathrm{Sht}^r_{\mathcal{M}_{\underline{d}}} & \longrightarrow & \mathrm{Hk}^r_{\mathcal{M}_{\underline{d}}} \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{pr}_0,\mathrm{pr}_r)} \\ \mathcal{M}_{\underline{d}} & \xrightarrow{(\mathrm{Id},\mathrm{Fr})} & \mathcal{M}_{\underline{d}} \times \mathcal{M}_{\underline{d}} \end{array} \qquad \begin{array}{ccc} \mathrm{Sht}^r_{\mathcal{M}_{\underline{d},e}} & \longrightarrow & \mathrm{Hk}^r_{\mathcal{M}_{\underline{d},e}} \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{pr}_0,\mathrm{pr}_r)} \\ \mathcal{M}_{\underline{d},e} & \xrightarrow{(\mathrm{Id},\mathrm{Fr})} & \mathcal{M}_{\underline{d},e} \times \mathcal{M}_{\underline{d},e} \end{array} \tag{10.6}$$

The maps w_i and h_i induce maps

$$\begin{array}{ccccc} & & \mathrm{Sht}^r_{\mathcal{M}_{\underline{d},e}} & & \\ & \swarrow^{u_1} & & \searrow^{u_2} & \\ \mathrm{Sht}^r_{\mathcal{M}_{\underline{d}}} & & & & \mathrm{Sht}^r_{\mathcal{M}_e} \end{array} \tag{10.7}$$

The stack $\mathrm{Sht}^r_{\mathcal{M}_{\underline{d},e}}$ decomposes into a disjoint union of open-closed substacks indexed by $(\{\mathcal{L}_i\}, \oplus\mathcal{L}_i \xrightarrow{s} \mathcal{E}, a) \in \mathcal{A}_{\underline{d},e}(k)$

$$\mathrm{Sht}^r_{\mathcal{M}_{\underline{d},e}} = \coprod_{(\oplus\mathcal{L}_i \to \mathcal{E}, a) \in \mathcal{A}_{\underline{d},e}(k)} \mathcal{Z}^r_{\mathcal{E}}(a).$$

Correspondingly, the diagram (10.7) decomposes into the disjoint union indexed by $\mathcal{A}_{\underline{d},e}(k)$ of diagrams of the form

$$\begin{array}{ccccc} & & \mathcal{Z}^r_{\mathcal{E}}(a) & & \\ & \swarrow^{u_1} & & \searrow^{u_2} & \\ \mathcal{Z}^r_{\oplus\mathcal{L}_i}(a')^{\heartsuit} & & & & \mathcal{Z}^r_{\mathcal{E}}(a) \end{array}$$

Here $\mathcal{Z}^r_{\oplus\mathcal{L}_i}(a')^{\heartsuit} \subset \mathcal{Z}^r_{\oplus\mathcal{L}_i}(a')$ (where a' is the Hermitian map on $\oplus\mathcal{L}_i$ induced from a) is cut by the open condition $\mu_{\min}(\mathcal{F}_j) > \max\{-d_i + 3g' - 2\}$ for all $0 \le j \le r$. From this description and Corollary 7.17, we see that:

Lemma 10.6 *The map u_1 (resp. u_2), when restricted to each connected component of $\mathrm{Sht}^r_{\mathcal{M}_{\underline{d},e}}$, is an isomorphism onto a connected component of $\mathrm{Sht}^r_{\mathcal{M}_{\underline{d}}}$ (resp. $\mathrm{Sht}^r_{\mathcal{M}_e}$).*

10.4 Zero cycles on auxiliary Hitchin shtukas

Similar to the definition of $[\mathrm{Sht}^r_{\mathcal{M}_e}]$ given in §8.6, we define 0-cycles supported on $\mathrm{Sht}^r_{\mathcal{M}_{\underline{d},e}}$ and $\mathrm{Sht}^r_{\mathcal{M}_{\underline{d}}}$ as follows.

We rewrite $\mathrm{Sht}^r_{\mathcal{M}_{\underline{d}}}$ as the fiber product

$$\begin{array}{ccc} \mathrm{Sht}^r_{\mathcal{M}_{\underline{d}}} & \longrightarrow & (\mathrm{Hk}^1_{\mathcal{M}_{\underline{d}}})^r \times \mathcal{M}_{\underline{d}} \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{pr}_0,\mathrm{pr}_1)^r\times\Delta} \\ (\mathcal{M}_{\underline{d}})^{r+1} & \xrightarrow{\Phi_{\mathcal{M}_{\underline{d}}}} & (\mathcal{M}_{\underline{d}})^{2r+2} \end{array}$$

Here $\Phi_{\mathcal{M}_{\underline{d}}} = \Phi^r_{\mathcal{M}_{\underline{d}}}$ and the vertical maps are defined as in Definition 8.15. By the smoothness of $\mathrm{Hk}^1_{\mathcal{M}_{\underline{d}}}$ and $\mathcal{M}_{\underline{d}}$ proved in Lemma 10.4 and the dimension calculation there, we define

$$[\mathrm{Sht}^r_{\mathcal{M}_{\underline{d}}}] := \Phi^!_{\mathcal{M}_{\underline{d}}}[(\mathrm{Hk}^1_{\mathcal{M}_{\underline{d}}})^r \times \mathcal{M}_{\underline{d}}] \in \mathrm{Ch}_0(\mathrm{Sht}^r_{\mathcal{M}_{\underline{d}}}).$$

Similarly, using the Cartesian diagram

$$\begin{array}{ccc} \mathrm{Sht}^r_{\mathcal{M}_{\underline{d},e}} & \longrightarrow & (\mathrm{Hk}^1_{\mathcal{M}_{\underline{d},e}})^r \times \mathcal{M}_{\underline{d},e} \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{pr}_0,\mathrm{pr}_1)^r\times\Delta} \\ (\mathcal{M}_{\underline{d},e})^{r+1} & \xrightarrow{\Phi_{\mathcal{M}_{\underline{d},e}}} & (\mathcal{M}_{\underline{d},e})^{2r+2} \end{array} \tag{10.8}$$

and the smoothness and dimension calculations of $\mathrm{Hk}^1_{\mathcal{M}_{\underline{d},e}}$ and $\mathcal{M}_{\underline{d},e}$ (which follow from Lemma 10.5 and Lemma 10.4), we define

$$[\mathrm{Sht}^r_{\mathcal{M}_{\underline{d},e}}] := \Phi^!_{\mathcal{M}_{\underline{d},e}}[(\mathrm{Hk}^1_{\mathcal{M}_{\underline{d},e}})^r \times \mathcal{M}_{\underline{d},e}] \in \mathrm{Ch}_0(\mathrm{Sht}^r_{\mathcal{M}_{\underline{d},e}}).$$

Lemma 10.7 *We have* $u_1^*[\mathrm{Sht}^r_{\mathcal{M}_{\underline{d}}}] = [\mathrm{Sht}^r_{\mathcal{M}_{\underline{d},e}}] \in \mathrm{Ch}_0(\mathrm{Sht}^r_{\mathcal{M}_{\underline{d},e}})$.

Proof This is because the maps w_1 and h_1 are both étale by Lemma 10.5. □

Lemma 10.8 *We have* $u_2^*[\mathrm{Sht}^r_{\mathcal{M}_e}] = [\mathrm{Sht}^r_{\mathcal{M}_{\underline{d},e}}] \in \mathrm{Ch}_0(\mathrm{Sht}^r_{\mathcal{M}_{\underline{d},e}})$.

Proof The diagram (10.8) is obtained from (8.6) (for $\mathcal{M}_d$ replaced by $\mathcal{M}_e$) by base changing termwise along the map of the following two Cartesian diagrams induced by $u_2 : \mathcal{A}_{\underline{d},e} \to \mathcal{A}_e$:

$$\begin{array}{ccc} \mathcal{A}_{\underline{d},e}(k) & \longrightarrow & (\mathcal{A}_{\underline{d},e})^{r+1} \\ \downarrow & & \downarrow{\scriptstyle \Delta^{r+1}} \\ (\mathcal{A}_{\underline{d},e})^{r+1} & \xrightarrow{\Phi_{\mathcal{A}_{\underline{d},e}}} & (\mathcal{A}_{\underline{d},e})^{2r+2} \end{array} \qquad \begin{array}{ccc} \mathcal{A}_e(k) & \longrightarrow & (\mathcal{A}_e)^{r+1} \\ \downarrow & & \downarrow{\scriptstyle \Delta^{r+1}} \\ (\mathcal{A}_e)^{r+1} & \xrightarrow{\Phi_{\mathcal{A}_e}} & (\mathcal{A}_e)^{2r+2} \end{array}$$

Note that $u_2 : \mathcal{A}_{\underline{d},e} \to \mathcal{A}_e$ is smooth since it exhibits $\mathcal{A}_{\underline{d},e}$ as an open substack of a vector bundle over $\mathcal{A}_e$ (using the condition $\mu_{\min}(\mathcal{E}) > \max\{-d_i + 3g' - 2\}$). We conclude by applying Proposition 10.9 below. □

10.4.1 Compatibility of cycle classes under Hitchin base change

To state the next result, we need some notations. Suppose we are given:

- stacks S, M and H that are locally of finite type over k and can be stratified into locally closed substacks that are global quotient stacks;
- the stack M is smooth of pure dimension N with a map $f : M \to S$;
- a map $\widetilde{h} : H \to S^r \times_{\Delta^r, S^{2r}} M^{2r}$ (the fiber product uses the r-fold product of the diagonal $\Delta^r : S^r \to S^{2r}$).

Define $h\colon H \to M^{2r}$ as $\widetilde{h}$ followed by the second projection. Form the Cartesian square

$$\begin{array}{ccc} \mathrm{Sht}_H & \longrightarrow & H \times M \\ \downarrow & & \downarrow{\scriptstyle h\times\Delta} \\ M^{r+1} & \xrightarrow{\Phi_M} & M^{2r+2} \end{array}$$

Let $u : S' \to S$ be a smooth representable morphism of pure relative dimension D. Let $M' = M \times_S S'$, $H' = H \times_{S^r} S'^r$ with natural maps $\widetilde{h'} : H' \to S'^r \times_{\Delta^r, S'^{2r}} M'^{2r}$. Let $h' : H' \to M'^{2r}$ be the resulting map. Let $u_M : M' \to M$ and $u_H : H' \to H$ be the natural maps. Form the Cartesian square

$$\begin{array}{ccc} \mathrm{Sht}'_H & \longrightarrow & H' \times M' \\ \downarrow & & \downarrow{\scriptstyle h'\times\Delta} \\ M'^{r+1} & \xrightarrow{\Phi_{M'}} & M'^{2r+2} \end{array} \tag{10.9}$$

Since $S^{r+1} \times_{\Phi_S, S^{2r+2}, \Delta^{r+1}} S^{r+1} = S(k)$, Sht_H decomposes as

$$\mathrm{Sht}_H = \coprod_{s \in S(k)} \mathrm{Sht}_H(s).$$

Similarly Sht'_H decomposes into the disjoint union of $\mathrm{Sht}'_H(s')$ indexed by $s' \in S'(k)$. Then the natural map $u_{\mathrm{Sht}} : \mathrm{Sht}'_H \to \mathrm{Sht}_H$ lifts to an isomorphism

$$\mathrm{Sht}'_H = \mathrm{Sht}_H \times_{S(k)} S'(k) = \coprod_{s' \in S'(k)} \mathrm{Sht}_H(u(s')).$$

Proposition 10.9 *Let $\zeta \in \mathrm{Ch}_*(H)$ and $[M]$ be the fundamental class of M. Then we have*

$$u^*_{\mathrm{Sht}} \Phi^!_M(\zeta \times [M]) = \Phi^!_{M'} u^*_H(\zeta \times [M]) \in \mathrm{Ch}_{*-rN}(\mathrm{Sht}'_H).$$

Proof Consider first the diagram where all squares are Cartesian

$$\begin{array}{ccccc}
\mathrm{Sht}_H \times_{S^r} S'^r & \longrightarrow & H' \times M' & \longrightarrow & S'^{r+1} \\
\downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u_H \times u_M} & & \downarrow{\scriptstyle u^{r+1}} \\
\mathrm{Sht}_H & \longrightarrow & H \times M & \longrightarrow & S^{r+1} \\
\downarrow & & \downarrow{\scriptstyle h \times \Delta} & & \\
M^{r+1} & \xrightarrow{\Phi_M} & M^{2r+2} & &
\end{array}$$

Here the top vertical arrows are smooth and representable. By the compatibility of Gysin map with flat pullback [10, Theorem 2.1.12(ix)], we have

$$\begin{aligned} v^* \Phi_M^!(\zeta \times [M]) &= \Phi_M^! u_H^*(\zeta \times [M]) \\ &\in \mathrm{Ch}_{*-rN+(r+1)D}(\mathrm{Sht}_H \times_{S^{r+1}} S'^{r+1}). \end{aligned} \tag{10.10}$$

Here we recall that D is the relative dimension of u. We have

$$\mathrm{Sht}_H \times_{S^{r+1}} S'^{r+1} = \coprod_{s \in S(k)} \mathrm{Sht}_H(s) \times (S'_s)^{r+1} \tag{10.11}$$

where $S'_s = u^{-1}(s)$, which is a smooth scheme over k. Factorize u_{Sht} as the composition

$$\mathrm{Sht}'_H \xrightarrow{i} \mathrm{Sht}_H \times_{S^r} (S'^r) \xrightarrow{v} \mathrm{Sht}_H$$

From (10.11) we see that i is a regular embedding of codimension $(r+1)D$. Now v and u_{Sht} are both smooth. Applying [10, Theorem 2.1.12(ix)] we have $u_{\mathrm{Sht}}^*(-) = i^! v^*$ as maps $\mathrm{Ch}_*(\mathrm{Sht}_H) \to \mathrm{Ch}_*(\mathrm{Sht}'_H)$. Therefore

$$u_{\mathrm{Sht}}^* \Phi_M^!(\zeta \times [M]) = i^! v^* \Phi_M^!(\zeta \times [M]) \in \mathrm{Ch}_{*-rN}(\mathrm{Sht}'_H). \tag{10.12}$$

On the other hand, consider the following diagram where all squares are Cartesian

$$\begin{array}{ccccccc}
\mathrm{Sht}'_H & \xrightarrow{i} & \mathrm{Sht}_H \times_{S^{r+1}} S'^{r+1} & \longrightarrow & H' \times M' & & \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle h' \times \Delta} & & \\
M'^{r+1} & \xrightarrow{\Phi_1} & M^{r+1} \times_{\Phi_S \circ f^{r+1}, S^{2r+2}} S'^{2r+2} & \xrightarrow{\Phi_2} & M'^{2r+2} & & \\
 & & \downarrow & & \downarrow{\scriptstyle u_M^{2r+2}} & & \\
 & & M^{r+1} & \xrightarrow{\Phi_M} & M^{2r+2} & &
\end{array}$$

Here Φ_1 is the base change of $\Phi_{S'}$ and Φ_2 is the base change of Φ_M. The outer square of the top rows give (10.9). By the transitivity of Gysin maps, we have

$$\Phi'^{!}_{M'} u_H^*(\zeta \times [M]) = \Phi_1^! \Phi_2^! u_H^*(\zeta \times [M]).$$

Since u_M^{2r+2} is smooth representable, we have $\Phi_2^! u_H^*(\zeta \times [M]) = \Phi_M^! u_H^*(\zeta \times [M])$. Hence

$$\Phi'^{!}_{M'} u_H^*(\zeta \times [M]) = \Phi_1^! \Phi_M^! u_H^*(\zeta \times [M]).$$

Since both Φ_1 and i are regular embeddings of the same codimension, we have $\Phi_1^!(-) = i^!(-)$ as maps $\mathrm{Ch}_*(\mathrm{Sht}_H \times_{S^{r+1}} S'^{r+1}) \to \mathrm{Ch}_{*-(r+1)D}(\mathrm{Sht}'_H)$, by [10, Theorem 2.1.12(xi)] and [5, Theorem 6.2(c)]. Therefore

$$\Phi'^{!}_{M'} u_H^*(\zeta \times [M]) = i^! \Phi_M^! u_H^*(\zeta \times [M]) \in \mathrm{Ch}_{*-rN}(\mathrm{Sht}'_H). \tag{10.13}$$

Combining (10.10), (10.12) and (10.13) we conclude

$$\begin{aligned} u_{\mathrm{Sht}}^* \Phi_M^!(\zeta \times [M]) &= i^! v^* \Phi_M^!(\zeta \times [M]) = i^! \Phi_M^! u_H^*(\zeta \times [M]) \\ &= \Phi'^{!}_{M'} u_H^*(\zeta \times [M]) \in \mathrm{Ch}_{*-rN}(\mathrm{Sht}'_H). \end{aligned}$$

□

Lemma 10.10 *Let* $(\{\mathcal{L}_i\}_{1\le i\le n}, a') \in \mathcal{A}_{\underline{d}}(k)$ *and* $\mathcal{E}' := \oplus_{i=1}^n \mathcal{L}_i$. *Then we have an equality*

$$[\mathrm{Sht}^r_{\mathcal{M}_{\underline{d}}}]|_{\mathcal{Z}^r_{\mathcal{E}'}(a')} = \zeta^r_{\mathcal{L}_1,\dots,\mathcal{L}_n}(a') \in \mathrm{Ch}_0(\mathcal{Z}^r_{\mathcal{E}'}(a')).$$

For the definition of $\zeta^r_{\mathcal{L}_1,\dots,\mathcal{L}_n}(a')$ *see §7.8.*

Proof We will apply the Octahedron Lemma [28, Theorem A.10] to a diagram of moduli stacks in our setting. Since the Octahedron Lemma requires certain stacks in question to be Deligne-Mumford, we need to rigidify our moduli stacks to satisfy these requirements. This is a minor technical issue which we encourage the reader to ignore: it is simply because the Octahedron Lemma in [28] is not stated and proved in the most general form.

Let $v \in |X'|$. Let P_v be the moduli space (a scheme!) of line bundles on X' together with a trivialization of their fibers over v. Let $\mathbf{G}_v = \mathrm{Res}^{k_v}_k \mathbf{G}_m$. Then $P_v \to \mathrm{Pic}_{X'}$ is a $\mathbf{G}_v$-torsor.

Now for each moduli stacks $\mathcal{M}_{\underline{d}}$, $\mathcal{A}_{\underline{d}}$ and $\mathrm{Hk}^r_{\mathcal{M}_{\underline{d}}}$ that involve an n-tuple of line bundles $\{\mathcal{L}_i\}$, we write $\dot{\mathcal{M}}_{\underline{d}}$, $\dot{\mathcal{A}}_{\underline{d}}$ and $\dot{\mathrm{Hk}}^r_{\dot{\mathcal{M}}_{\underline{d}}}$ to mean their rigidified versions where $\mathcal{L}_i \in \mathrm{Pic}_{X'}$ is replaced by $\dot{\mathcal{L}}_i \in P_v$. Note that we do not impose any compatibility condition between the rigidifcation on $\mathcal{L}_i$ and the rest of the structures classified by these moduli stacks. Define $\mathrm{Sht}^r_{\dot{\mathcal{M}}_{\underline{d}}}$ using the dotted version of the left one of the Cartesian diagrams in (10.6).

Note that $\dot{\mathcal{M}}_{\underline{d}}$, $\dot{\mathcal{A}}_{\underline{d}}$ and $\dot{\mathrm{Hk}}^r_{\dot{\mathcal{M}}_{\underline{d}}}$ are now schemes, and they are $\mathbf{G}_v^n$-torsors over their undotted counterparts. The dotted version of Lemma 10.4 remains valid if we add

$n\deg(v) = \dim \mathbf{G}_v^n$ to the dimensions. Also, $\mathrm{Sht}^r_{\dot{\mathcal{M}}_{\underline{d}}} \simeq \mathrm{Sht}^r_{\mathcal{M}_{\underline{d}}} \times_{\mathcal{A}_{\underline{d}}(k)} \dot{\mathcal{A}}_{\underline{d}}(k)$. Since $\dot{\mathcal{A}}_{\underline{d}}(k) \to \mathcal{A}_{\underline{d}}(k)$ is surjective, to prove the Lemma, it suffices to prove its dotted version: for any $(\{\dot{\mathcal{L}}_i\}, a') \in \dot{\mathcal{A}}_{\underline{d}}(k)$, writing $\mathcal{E}' := \oplus_{i=1}^n \mathcal{L}_i$, then there is an open and closed embedding $\mathcal{Z}^r_{\mathcal{E}'}(a') \hookrightarrow \mathrm{Sht}^r_{\dot{\mathcal{M}}_{\underline{d}}}$ (using the rigifications $\dot{\mathcal{L}}_i$ of $\mathcal{L}_i$); then we shall prove

$$[\mathrm{Sht}^r_{\dot{\mathcal{M}}_{\underline{d}}}]|_{\mathcal{Z}^r_{\mathcal{E}'}(a')} = \zeta^r_{\mathcal{L}_1,\ldots,\mathcal{L}_n}(a') \in \mathrm{Ch}_0(\mathcal{Z}^r_{\mathcal{E}'}(a')). \tag{10.14}$$

For $i = 1, \ldots, n$, let $\mathcal{N}_{d_i}$ be the open substack of $\mathcal{M}(1,n)$ consisting of points $(\mathcal{L} \hookrightarrow \mathcal{F}, h)$ where $\chi(X', \mathcal{L}) = -d_i$ and $\mu_{\min}(\mathcal{F}) > -d_i + 3g' - 2$. Similarly define $\mathrm{Hk}^1_{\mathcal{N}_{d_i}}$ and $\mathrm{Sht}^r_{\mathcal{N}_{d_i}}$; these are open substacks of $\mathrm{Hk}^1_{\mathcal{M}(1,n)}$ and $\mathrm{Sht}^r_{\mathcal{M}(1,n)}$ respectively.

Let $\dot{\mathcal{N}}_{d_i}, \mathrm{Hk}^1_{\dot{\mathcal{N}}_{d_i}}$ be the rigidified versions of $\mathcal{N}_{d_i}$ and $\mathrm{Hk}^1_{\mathcal{N}_{d_i}}$ where $\mathcal{L} \in \mathrm{Pic}_{X'}$ is replaced by $\dot{\mathcal{L}} \in P_v$. Let $\omega_i : \dot{\mathcal{N}}_{d_i} \to \mathrm{Bun}_{U(n)}$ and $\widetilde{\omega}_i : \mathrm{Hk}^1_{\dot{\mathcal{N}}_{d_i}} \to \mathrm{Hk}^1_{U(n)}$ be the forgetful maps.

We shall apply the Octahedron Lemma [28, Theorem A.10] to the following diagram:

$$\begin{array}{ccccc}
(\mathrm{Hk}^1_{U(n)})^r \times \mathrm{Bun}_{U(n)} & \xrightarrow{\Delta} & \prod_{i=1}^n ((\mathrm{Hk}^1_{U(n)})^r \times \mathrm{Bun}_{U(n)}) & \xleftarrow{\prod(\widetilde{\omega}_i^r \times \omega_i)} & \prod_{i=1}^n ((\mathrm{Hk}^1_{\dot{\mathcal{N}}_{d_i}})^r \times \dot{\mathcal{N}}_{d_i}) \\
\downarrow{\scriptstyle (\mathrm{pr}_0,\mathrm{pr}_1)^r \times \Delta} & & \downarrow{\scriptstyle \prod((\mathrm{pr}_0,\mathrm{pr}_1)^r \times \Delta)} & & \downarrow{\scriptstyle \prod((\mathrm{pr}_0,\mathrm{pr}_1)^r \times \Delta)} \\
\mathrm{Bun}^{2r+2}_{U(n)} & \xrightarrow{\Delta} & \prod_{i=1}^n \mathrm{Bun}^{2r+2}_{U(n)} & \xleftarrow{\prod \omega_i^{2r+2}} & \prod_{i=1}^n \dot{\mathcal{N}}^{2r+2}_{d_i} \\
\uparrow{\scriptstyle \Phi_{\mathrm{Bun}_{U(n)}}} & & \uparrow{\scriptstyle \prod \Phi_{\mathrm{Bun}_{U(n)}}} & & \uparrow{\scriptstyle \prod \Phi_{\dot{\mathcal{N}}_{d_i}}} \\
\mathrm{Bun}^{r+1}_{U(n)} & \xrightarrow{\Delta} & \prod_{i=1}^n \mathrm{Bun}^{r+1}_{U(n)} & \xleftarrow{\prod \omega_i^{r+1}} & \prod_{i=1}^n \dot{\mathcal{N}}^{r+1}_{d_i}
\end{array} \tag{10.15}$$

The fiber products of the three columns are

$$\mathrm{Sht}^r_{U(n)} \xrightarrow{\Delta} \prod_{i=1}^n \mathrm{Sht}^r_{U(n)} \xleftarrow{} \prod_{i=1}^n \mathrm{Sht}^r_{\dot{\mathcal{N}}_{d_i}} \tag{10.16}$$

where

$$\mathrm{Sht}^r_{\dot{\mathcal{N}}_{d_i}} = \coprod_{\substack{\dot{\mathcal{L}}_i \in P_v(k), \\ \chi(X', \mathcal{L}_i) = -d_i}} \left(\mathcal{Z}^r_{\mathcal{L}_i}(0)^* \coprod \left(\coprod_{a'_{ii} \in \mathcal{A}_{\mathcal{L}_i}(k)} \mathcal{Z}^r_{\mathcal{L}_i}(a'_{ii}) \right) \right). \tag{10.17}$$

Let $\overline{\mathcal{M}}_{\underline{d}}$ be the moduli stack of $(\{\dot{\mathcal{L}}_i\}, \oplus_{i=1}^n \mathcal{L}_i \xrightarrow{t'} \mathcal{F}, h)$ defined similarly as $\dot{\mathcal{M}}_{\underline{d}}$ but without the condition that t' be injective, only that $t'_i = t|_{\mathcal{L}_i}$ be injective. Then $\dot{\mathcal{M}}_{\underline{d}} \hookrightarrow \overline{\mathcal{M}}_{\underline{d}}$ is open. Similarly define $\mathrm{Hk}^1_{\overline{\mathcal{M}}_{\underline{d}}}$ and $\mathrm{Sht}^r_{\overline{\mathcal{M}}_{\underline{d}}}$. Note that $\overline{\mathcal{M}}_{\underline{d}}$ is exactly

the fiber product of

$$\mathrm{Bun}_{U(n)} \xrightarrow{\Delta} \prod_{i=1}^n \mathrm{Bun}_{U(n)} \xleftarrow{\prod \omega_i} \prod_{i=1}^n \dot{\mathcal{N}}_{d_i}. \tag{10.18}$$

Similar remarks apply to $\mathrm{Hk}^1_{\overline{\mathcal{M}}_{\underline{d}}}$. Therefore the fiber products of the three rows of (10.15) are

$$\begin{array}{c} (\mathrm{Hk}^1_{\overline{\mathcal{M}}_{\underline{d}}})^r \times \overline{\mathcal{M}}_{\underline{d}} \\ \downarrow{\scriptstyle (\mathrm{pr}_0,\mathrm{pr}_1)^r\times\Delta} \\ \overline{\mathcal{M}}_{\underline{d}}^{2r+2} \\ \uparrow{\scriptstyle \Phi_{\overline{\mathcal{M}}_{\underline{d}}}} \\ \overline{\mathcal{M}}_{\underline{d}}^{r+1} \end{array} \tag{10.19}$$

The common fiber product of (10.16) and (10.19) is $\mathrm{Sht}^r_{\overline{\mathcal{M}}_{\underline{d}}}$, which decomposes as a disjoint union over the groupoid $\mathcal{B}(k)$ of $(\dot{\mathcal{L}}_i, a'_{ii})_{1\le i\le n}$ where $\dot{\mathcal{L}}_i \in P_v(k)$ with $\chi(X', \mathcal{L}_i) = -d_i$ and $a'_{ii} : \mathcal{L}_i \to \sigma^*\mathcal{L}_i^\vee$ injective Hermitian. For a point $(\dot{\mathcal{L}}_i, a'_{ii})_{1\le i\le n}$ of $\mathcal{B}(k)$, we have

$$\mathrm{Sht}^r_{\overline{\mathcal{M}}_{\underline{d}}}|_{(\dot{\mathcal{L}}_i, a'_{ii})_{1\le i\le n}} = \mathcal{Z}^r_{\mathcal{L}_1,\dots,\mathcal{L}_n}(a'_{11},\dots,a'_{nn})^\circ,$$

where $\mathrm{Sht}^r_{\overline{\mathcal{M}}_{\underline{d}}}|_{(\dot{\mathcal{L}}_i, a'_{ii})_{1\le i\le n}}$ means the pullback of $\mathrm{Sht}^r_{\overline{\mathcal{M}}_{\underline{d}}}$ to Spec k along the corresponding Spec $k \to \mathcal{B}(k)$.

We check that the assumptions for applying the Octahedron Lemma are satisfied (the numbering below refers to that in [28, Theorem A.10]).

(1) All members in the diagram (10.15) are smooth and equidimensional. This is clear for $\mathrm{Bun}_{U(n)}$ and $\mathrm{Hk}^1_{U(n)}$. The same argument as in Lemma 10.4 proves that $\dot{\mathcal{N}}_{d_i}$ and $\mathrm{Hk}^1_{\dot{\mathcal{N}}_{d_i}}$ are smooth of pure dimension $d_i n + (n^2 - 2n + 2)(g-1) + \deg(v)$.

(2) We check that, in forming the fiber products of the middle and bottom rows and the left and middle columns, the intersections are proper intersections with smooth equidimensional outcomes with the expected dimension. Here we use Lemma 6.8 and 6.9 to argue for the left and middle columns. For the rows, the same argument as in Lemma 10.4 proves that $\overline{\mathcal{M}}_{\underline{d}}$ and $\mathrm{Hk}^1_{\overline{\mathcal{M}}_{\underline{d}}}$ are smooth of the same dimension as $\dot{\mathcal{M}}_{\underline{d}}$, which is $dn - (n^2 - 2n)(g-1) + n\deg(v)$. This is the virtual dimension for $\overline{\mathcal{M}}_{\underline{d}}$ as the fiber product of (10.18), since

$$\sum_{i=1}^n \dim \mathcal{N}_{d_i} - (n-1)\dim \mathrm{Bun}_{U(n)}$$

$$= \sum_{i=1}^{n} \Big(d_i n + (n^2 - 2n + 2)(g-1) + \deg(v) \Big) - n^2(n-1)(g-1)$$
$$= dn - (n^2 - 2n)(g-1) + n \deg(v).$$

(3) We check the fiber products of the top row and right column of (10.15) satisfy the conditions for [28, A.2.10]. The fiber product of the top row is also a proper intersection: this follows from the same calculation as for the middle and bottom rows. The fiber product of the right column is also a proper intersection: this uses the decomposition (10.17) and the calculation of the dimension of $\mathcal{Z}^r_{\mathcal{L}_i}(0)^*$ in Proposition 9.5 and the dimension of $\mathcal{Z}^r_{\mathcal{L}_i}(a'_{ii})^*$ in Proposition 9.1.

The only issue is that $(\mathrm{Hk}^1_{\overline{\mathcal{M}}_{\underline{d}}})^{r+1}$ may not be a Deligne-Mumford stack, which was part of the requirement of [28, A.2.10]. However, we argue that this is not really an issue. The proof of the Octahedron Lemma allows the following flexibility: since eventually we only care about the 0-cycles restricted to $\mathrm{Sht}^r_{\dot{\mathcal{M}}_{\underline{d}}}$, in the middle steps of forming the fiber products, we may restrict to open substacks as long as the final fiber product contains $\mathrm{Sht}^r_{\dot{\mathcal{M}}_{\underline{d}}}$ and only need check the relevant requirements there. Now in (10.19) we may restrict to the open substack $(\mathrm{Hk}^1_{\dot{\mathcal{M}}_{\underline{d}}})^{r+1} \subset (\mathrm{Hk}^1_{\overline{\mathcal{M}}_{\underline{d}}})^{r+1}$, which is a scheme.

(4) The same remark as above shows that it suffices to check that the fiber squares obtained from (10.16) and (10.19), after replacing $\overline{\mathcal{M}}_{\underline{d}}$ by $\dot{\mathcal{M}}_{\underline{d}}$, each satisfy the condition [28, A.2.8]. Therefore it suffices to check

- $\mathrm{Sht}^r_{\dot{\mathcal{M}}_{\underline{d}}}$ admits a finite flat presentation in the sense of [28, Definition A.1]. This is true because $\mathrm{Sht}^r_{\dot{\mathcal{M}}_{\underline{d}}}$ is a scheme.
- The diagonal map $\Delta : \mathrm{Sht}^r_{U(n)} \hookrightarrow \prod_{i=1}^n \mathrm{Sht}^r_{U(n)}$ is a regular local immersion. This is true because $\mathrm{Sht}^r_{U(n)}$ is a smooth Deligne-Mumford stack.
- The map $\Phi_{\dot{\mathcal{M}}_{\underline{d}}} : \dot{\mathcal{M}}^{r+1}_{\underline{d}} \to \dot{\mathcal{M}}^{2r+2}_{\underline{d}}$ is a regular local immersion. This is true because $\dot{\mathcal{M}}_{\underline{d}}$ is a smooth equidimensional scheme by the dotted version of Lemma 10.4.

The conclusion of the (variant of) Octahedron Lemma says that the following two elements in $\mathrm{Ch}_0(\mathrm{Sht}^r_{\overline{\mathcal{M}}_{\underline{d}}})$

$$\Phi^!_{\overline{\mathcal{M}}_{\underline{d}}} \Delta^!_{(\mathrm{Hk}^1_{U(n)})^r} [\prod_{i=1}^n (\mathrm{Hk}^1_{\dot{\mathcal{N}}_{d_i}})^r \times \dot{\mathcal{N}}_{d_i}] \quad \text{and} \quad \Delta^!_{\mathrm{Sht}^r_{U(n)}} (\prod \Phi_{\dot{\mathcal{N}}_{d_i}})^! [\prod_{i=1}^n (\mathrm{Hk}^1_{\dot{\mathcal{N}}_{d_i}})^r \times \dot{\mathcal{N}}_{d_i}]$$

become the same when restricted to $\mathrm{Sht}^r_{\dot{\mathcal{M}}_{\underline{d}}}$. Further restricting to $\mathcal{Z}^r_{\mathcal{E}'}(a')$ we get the desired identity (10.14). □

Proof of Theorem 10.2 Restricting the equality in Lemma 10.10 to $\mathcal{Z}^r_{\mathcal{E}}(a)$, which is open and closed in $\mathcal{Z}^r_{\mathcal{E}'}(a')$ by Corollary 7.17, we get

$$\zeta^r_{\mathcal{L}_1,\dots,\mathcal{L}_n}(a')|_{\mathcal{Z}^r_{\mathcal{E}}(a)} = [\mathrm{Sht}^r_{\mathcal{M}_{\underline{d}}}]|_{\mathcal{Z}^r_{\mathcal{E}}(a)}. \tag{10.20}$$

For fixed $(\oplus\mathcal{L}_i \hookrightarrow \mathcal{E}, a) \in \mathcal{A}_{\underline{d},e}(k)$, $\mathcal{Z}^r_{\mathcal{E}}(a)$ can be viewed as a finite étale cover of an open-closed substack in $\mathrm{Sht}^r_{\mathcal{M}_{\underline{d}}}$, $\mathrm{Sht}^r_{\mathcal{M}_{\underline{d},e}}$ and $\mathrm{Sht}^r_{\mathcal{M}_e}$ by Lemma 10.6. By Lemma 10.7 and Lemma 10.8 we have

$$[\mathrm{Sht}^r_{\mathcal{M}_{\underline{d}}}]|_{\mathcal{Z}^r_{\mathcal{E}}(a)} = [\mathrm{Sht}^r_{\mathcal{M}_{\underline{d},e}}]|_{\mathcal{Z}^r_{\mathcal{E}}(a)} = [\mathrm{Sht}^r_{\mathcal{M}_e}]|_{\mathcal{Z}^r_{\mathcal{E}}(a)}.$$

Combining this with (10.20) proves the theorem. □

10.5 Proof of Theorem 10.1 for $X' = X \coprod X$ or $X_{k'}$

Here k'/k is the quadratic extension.

In the case $X' = X \coprod X$, we have $\mathrm{Bun}_{U(n)} \cong \mathrm{Bun}_{\mathrm{GL}_n}$. We shall identify a Hermitian bundle $\mathcal{F}$ on X' with a pair of vector bundles $(\mathcal{F}_1, \mathcal{F}_2)$ equipped with an isomorphism $\mathcal{F}_2 \cong \mathcal{F}_1^\vee$, each living on one copy of X. A vector bundle $\mathcal{E}$ on X' of rank n corresponds to two rank n vector bundles $(\mathcal{E}_1, \mathcal{E}_2)$, each living on one copy of X. Now $\mathcal{A}_{\mathcal{E}}(k)$ is the set of injective maps $a : \mathcal{E}_1 \to \mathcal{E}_2^\vee$. A good framing $s : \oplus_{i=1}^n \mathcal{L}_i \hookrightarrow \mathcal{E}$ for $(\mathcal{E}, a)$ now consists of line bundles $\mathcal{L}_i = (\mathcal{L}_{i,1}, \mathcal{L}_{i,2})$ $(1 \le i \le n)$ satisfying the same conditions in Definition 7.15; it is called *very good* if it satisfies the additional conditions

(3_1) $\mu_{\min}(\mathcal{E}_1) > \max\{\deg \mathcal{L}_{i,1} + 2g - 1\}_{1\le i\le n}$, and
(3_2) $\mu_{\min}(\mathcal{E}_2) > \max\{\deg \mathcal{L}_{i,2} + 2g - 1\}_{1\le i\le n}$.

The same argument of Lemma 10.3 shows that it suffices to prove the analogue of Theorem 10.2, i.e., prove Theorem 10.1 for very good framings.

In both the $X' = X \coprod X$ and $X' = X_{k'}$ case, we need to modify the definitions of $\mathcal{M}_{\underline{d}}$ and $\mathcal{M}_{\underline{d},e}$ as follows. In the definition of $\mathcal{M}_{\underline{d},e}$, we use the notion of very good framing just defined over geometric fibers of $X'_S \to S$ (which are of the form $X_{\overline{s}} \coprod X_{\overline{s}}$). In the definition of $\mathcal{M}_{\underline{d}}$, we change the inequality (10.3) to two inequalities over the geometric fibers of $X'_S \to S$

$$\mu_{\min}(\mathcal{F}_1) > \max\{\deg \mathcal{L}_{i,1} + 2g - 1\}_{1\le i\le n},$$
$$\mu_{\min}(\mathcal{F}_2) > \max\{\deg \mathcal{L}_{i,2} + 2g - 1\}_{1\le i\le n}.$$

The same inequalities should be imposed in the definition of $\mathcal{N}_{d_i}$ that appear in the proof of Lemma 10.10. With these changes, the argument for proving Theorem 10.2 goes through.

11 Local intersection number and trace formula

11.1 Local nature of the intersection problem

Recall from Proposition 7.13 that $\mathcal{Z}^r_{\mathcal{E}}(a)$ only depends on the Hermitian torsion sheaf $\mathcal{Q} = \mathrm{coker}(a)$. In this subsection we show that the 0-cycle class $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ also only depends on $\mathcal{Q}$.

Recall the stacks $\mathrm{Herm}_{2d} = \mathrm{Herm}_{2d}(X'/X)$ and Lagr_{2d} from §4. We have a self-correspondence $\mathrm{Hk}^r_{\mathrm{Lagr}_{2d}}$ of Lagr_{2d} over Herm_{2d}: it classifies $(\mathcal{Q}, h, \{\mathcal{L}_i\}_{0\le i\le r})$ where $(\mathcal{Q}, h) \in \mathrm{Herm}_{2d}$, $\mathcal{L}_i \subset \mathcal{Q}$ are Lagrangian subsheaves such that $\mathcal{L}_i/(\mathcal{L}_i \cap \mathcal{L}_{i-1})$ has length one for $1 \le i \le r$. Define the local version $\mathrm{Sht}^r_{\mathrm{Lagr}_{2d}}$ of $\mathrm{Sht}^r_{\mathcal{M}_d}$ by the Cartesian diagram

$$\begin{array}{ccc} \mathrm{Sht}^r_{\mathrm{Lagr}_{2d}} & \longrightarrow & \mathrm{Hk}^r_{\mathrm{Lagr}_{2d}} \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{pr}_0, \mathrm{pr}_r)} \\ \mathrm{Lagr}_{2d} & \xrightarrow{(\mathrm{Id},\mathrm{Fr})} & \mathrm{Lagr}_{2d} \times \mathrm{Lagr}_{2d} \end{array} \tag{11.1}$$

We have a decomposition into open-closed substacks

$$\mathrm{Sht}^r_{\mathrm{Lagr}_{2d}} = \coprod_{(\mathcal{Q},h)\in \mathrm{Herm}_{2d}(k)} \mathcal{Z}^r_{\mathcal{Q}}.$$

Lemma 11.1 *The stack* $\mathrm{Hk}^1_{\mathrm{Lagr}_{2d}}$ *is smooth of dimension zero.*

Proof We may identify $\mathrm{Hk}^1_{\mathrm{Lagr}_{2d}}$ with the moduli stack of $(0 \subset \mathcal{L}' \subset \mathcal{L} \subset \mathcal{Q}, h)$ where $(\mathcal{L} \subset \mathcal{Q}, h) \in \mathrm{Lagr}_{2d}$ and $\mathcal{L}/\mathcal{L}'$ has length one. Under the local chart for Herm_{2d} described in Lemma 4.3, $\mathrm{Hk}^1_{\mathrm{Lagr}_{2d}}$ becomes $[\mathfrak{p}/P]$, where $P \subset \mathrm{O}_{2d} = \mathrm{O}(V)$ is the parabolic subalgebra stabilizing a pair of subspaces $L' \subset L$ with L Lagrangian and $\dim L' = d - 1$. This local description implies that $\mathrm{Hk}^1_{\mathrm{Lagr}_{2d}}$ is smooth of dimension zero. □

Rewriting $\mathrm{Sht}^r_{\mathrm{Lagr}_{2d}}$ as the fibered product (cf. Definition 8.15)

$$\begin{array}{ccc} \mathrm{Sht}^r_{\mathrm{Lagr}_{2d}} & \longrightarrow & (\mathrm{Hk}^1_{\mathrm{Lagr}_{2d}})^r \times \mathrm{Lagr}_{2d} \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{pr}_0, \mathrm{pr}_1)^r \times \Delta} \\ (\mathrm{Lagr}_{2d})^{r+1} & \xrightarrow{\Phi_{\mathrm{Lagr}_{2d}}} & (\mathrm{Lagr}_{2d})^{2r+2} \end{array} \tag{11.2}$$

we define a 0-cycle class

$$[\mathrm{Sht}^r_{\mathrm{Lagr}_{2d}}] := \Phi^!_{\mathrm{Lagr}_{2d}}[(\mathrm{Hk}^1_{\mathrm{Lagr}_{2d}})^r \times \mathrm{Lagr}_{2d}] \in \mathrm{Ch}_0(\mathrm{Sht}^r_{\mathrm{Lagr}_{2d}}).$$

Restricting to $\mathcal{Z}^r_{\mathcal{Q}}$ we get

$$[\mathcal{Z}^r_{\mathcal{Q}}] := [\mathrm{Sht}^r_{\mathrm{Lagr}_{2d}}]|_{\mathcal{Z}^r_{\mathcal{Q}}} \in \mathrm{Ch}_0(\mathcal{Z}^r_{\mathcal{Q}}).$$

Recall the maps $g_{\mathcal{M}} : \mathcal{M}_d \to \mathrm{Lagr}_{2d}$ and $g : \mathcal{A}_d \to \mathrm{Herm}_{2d}$ defined in §8.4. We also have a map $g_{\mathrm{Hk}} : \mathrm{Hk}^r_{\mathcal{M}_d} \to \mathrm{Hk}^r_{\mathrm{Lagr}_{2d}}$ sending $(\{x'_i\}, \{\mathcal{E} \xrightarrow{t_i} \mathcal{F}_i\})$ to $(\mathcal{Q} = \mathrm{coker}(a), h_{\mathcal{Q}}, \{\mathrm{coker}(t_i)\})$ (a is the induced Hermitian map on $\mathcal{E}$). The maps g_{Hk}, $g_{\mathcal{M}}$ and g exhibit the diagram (8.5) as the pullback of the diagram (11.1) via the base change $g : \mathcal{A}_d \to \mathrm{Herm}_{2d}$. In particular we have a natural map

$$g_{\mathrm{Sht}} : \mathrm{Sht}^r_{\mathcal{M}_d} \to \mathrm{Sht}^r_{\mathrm{Lagr}_{2d}}.$$

For fixed $(\mathcal{E}, a) \in \mathcal{A}_d(k)$ with image $\mathcal{Q} = \mathrm{coker}(a) \in \mathrm{Herm}_{2d}(k)$, g_{Sht} restricts to an isomorphism to the open-closed subschemes

$$g_{\mathrm{Sht}}|_{\mathcal{Z}^r_{\mathcal{E}}(a)} : \mathcal{Z}^r_{\mathcal{E}}(a) \xrightarrow{\sim} \mathcal{Z}^r_{\mathcal{Q}}. \tag{11.3}$$

Proposition 11.2 *We have an equality*

$$[\mathrm{Sht}^r_{\mathcal{M}_d}] = g^*_{\mathrm{Sht}}[\mathrm{Sht}^r_{\mathrm{Lagr}_{2d}}] \in \mathrm{Ch}_0(\mathrm{Sht}^r_{\mathcal{M}_d}). \tag{11.4}$$

Proof Apply Proposition 10.9 to the diagram (11.2), the fundamental class $\zeta = [(\mathrm{Hk}^1_{\mathrm{Lagr}_{2d}})^r \times \mathrm{Lagr}_{2d}]$ and the base change map $u = g : \mathcal{A}_d \to \mathrm{Herm}_{2d}$. By Proposition 8.12, g is smooth. We then have

$$\Phi^!_{\mathcal{M}_d} g^*_{\mathrm{Hk}}[(\mathrm{Hk}^1_{\mathrm{Lagr}_{2d}})^r \times \mathrm{Lagr}_{2d}] = g^*_{\mathrm{Sht}} \Phi^!_{\mathrm{Lagr}_{2d}}[(\mathrm{Hk}^1_{\mathrm{Lagr}_{2d}})^r \times \mathrm{Lagr}_{2d}] \in \mathrm{Ch}_0(\mathrm{Sht}^r_{\mathcal{M}_d}).$$

Since g_{Hk} is smooth, $g^*_{\mathrm{Hk}}[(\mathrm{Hk}^1_{\mathrm{Lagr}_{2d}})^r \times \mathrm{Lagr}_{2d}] = [(\mathrm{Hk}^1_{\mathcal{M}_d})^r \times \mathrm{Hk}^1_{\mathcal{M}_d}]$. The above equality then becomes (11.4). □

Combined with Theorem 10.1, we get a local description of the cycle class $[\mathcal{Z}^r_{\mathcal{E}}(a)]$:

Corollary 11.3 *For any $(\mathcal{E}, a) \in \mathcal{A}_d(k)$ with image $\mathcal{Q} = \mathrm{coker}(a) \in \mathrm{Herm}_{2d}(k)$, $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ is the same as $[\mathcal{Z}^r_{\mathcal{Q}}]$ under the isomorphism (11.3). In particular,*

$$\deg[\mathcal{Z}^r_{\mathcal{E}}(a)] = \deg[\mathcal{Z}^r_{\mathcal{Q}}].$$

11.2 Sheaves on Herm_{2d}

To describe the direct image complex $Rf_*\overline{\mathbf{Q}}_\ell$ on $\mathcal{A}_d$, by the Cartesian diagram (8.4), we first need to understand $R(\upsilon_{2d})_*\overline{\mathbf{Q}}_\ell$ on Herm_{2d}.

Lemma 11.4 *The perverse sheaf $R(\upsilon_{2d})_*\overline{\mathbf{Q}}_\ell$ on Herm_{2d} is canonically isomorphic to $(\mathrm{Spr}^{\mathrm{Herm}}_{2d})^{S_d}$ (see Proposition 4.5(2)). Here the S_d-action on $\mathrm{Spr}^{\mathrm{Herm}}_{2d}$ is the restriction of the Springer W_d-action.*

Proof We have a Cartesian diagram

$$\begin{array}{ccc} \widetilde{\mathrm{Herm}}_{2d} & \xrightarrow{\widetilde{\varepsilon}_d} & \widetilde{\mathrm{Coh}}_d(X') \\ \downarrow{\scriptstyle \lambda_{2d}} & & \downarrow{\scriptstyle \pi^{\mathrm{Coh}}_{X',d}} \\ \mathrm{Lagr}_{2d} & \xrightarrow{\varepsilon'_d} & \mathrm{Coh}_d(X') \end{array}$$

where ε'_d sends $(\mathcal{Q}, h_{\mathcal{Q}}, \mathcal{L})$ to $\mathcal{L}$ and $\widetilde{\varepsilon}_d$ sends $(\mathcal{Q}_1 \subset \cdots \subset \mathcal{Q}_d \subset \cdots \subset \mathcal{Q}, h)$ to $(\mathcal{Q}_1 \subset \cdots \subset \mathcal{Q}_d)$. By proper base change

$$R\lambda_{2d*}\overline{\mathbf{Q}}_\ell \cong \varepsilon'^*_d \,\mathrm{Spr}_{d,X'}\,.$$

In particular, $R\lambda_{2d*}\overline{\mathbf{Q}}_\ell$ carries an action of S_d. Moreover, the induced S_d-action on $\mathrm{Spr}^{\mathrm{Herm}}_{2d} \cong R(\upsilon_{2d})_* R\lambda_{2d*}\overline{\mathbf{Q}}_\ell$ is the restriction of the Springer W_d-action to S_d: this can be easily checked over Herm°_{2d}, and then the statement holds over Herm_{2d} since $\mathrm{Spr}^{\mathrm{Herm}}_{2d}$ is the middle extension from its restriction to Herm°_{2d} by Proposition 4.5(2). Since $(R\lambda_{2d*}\overline{\mathbf{Q}}_\ell)^{S_d} \cong \varepsilon'^*_d(\mathrm{Spr}_{d,X'})^{S_d} \cong \overline{\mathbf{Q}}_\ell$, we conclude that $(\mathrm{Spr}^{\mathrm{Herm}}_{2d})^{S_d} \cong R(\upsilon_{2d})_*(R\lambda_{2d*}\overline{\mathbf{Q}}_\ell)^{S_d} \cong R(\upsilon_{2d})_*\overline{\mathbf{Q}}_\ell$, as desired. □

It is an elementary exercise to see that $\mathrm{Ind}^{W_d}_{S_d}\mathbf{1}$ decomposes into irreducible representations

$$\mathrm{Ind}^{W_d}_{S_d}\mathbf{1} = \bigoplus_{i=0}^{d} \rho_i \tag{11.5}$$

where $\rho_i = \mathrm{Ind}^{W_d}_{(\mathbf{Z}/2\mathbf{Z})^d \rtimes (S_i \times S_{d-i})}(\chi_i \boxtimes \mathbf{1})$, and $\chi_i : (\mathbf{Z}/2\mathbf{Z})^d \rtimes (S_i \times S_{d-i}) \to \{\pm 1\}$ is the character that is nontrivial on the first i factors of $(\mathbf{Z}/2\mathbf{Z})^i$, trivial on the rest and trivial on $S_i \times S_{d-i}$. The decomposition also shows up in [28, §8.1.1].

Recall the notation $\mathrm{Spr}^{\mathrm{Herm}}_{2d}[\rho]$ from Definition 4.7.

Corollary 11.5 *There is a canonical decomposition*

$$R(\upsilon_{2d})_*\overline{\mathbf{Q}}_\ell \cong \bigoplus_{i=0}^{d} \mathrm{Spr}^{\mathrm{Herm}}_{2d}[\rho_i]. \tag{11.6}$$

Proof By Lemma 11.4 and Frobenius reciprocity, we have

$$R(\upsilon_{2d})_*\overline{\mathbf{Q}}_\ell \cong \mathrm{Hom}_{W_d}(\mathrm{Ind}^{W_d}_{S_d}\mathbf{1}, \mathrm{Spr}^{\mathrm{Herm}}_{2d}) = \mathrm{Spr}^{\mathrm{Herm}}_{2d}[\mathrm{Ind}^{W_d}_{S_d}\mathbf{1}].$$

The desired decomposition then follows from (11.5). □

Definition 11.6 Define the graded perverse sheaf on $\mathrm{Herm}_{2d}(X'/X)$

$$\mathcal{K}^{\mathrm{Int}}_d(T) := \bigoplus_{i=0}^{d} \mathrm{Spr}^{\mathrm{Herm}}_{2d}[\rho_i]T^i.$$

The fundamental class of the self-correspondence $\mathrm{Hk}^1_{\mathrm{Lagr}_{2d}}$ of Lagr_{2d} is viewed as a cohomological correspondence of the constant sheaf on Lagr_{2d} with itself. It induces an endomorphism (see notation from [28, A.4.1])

$$(\upsilon_{2d})_![\mathrm{Hk}^1_{\mathrm{Lagr}_{2d}}] : R(\upsilon_{2d})_*\overline{\mathbf{Q}}_\ell \to R(\upsilon_{2d})_*\overline{\mathbf{Q}}_\ell.$$

Proposition 11.7 *The action of* $(\upsilon_{2d})_![\mathrm{Hk}^1_{\mathrm{Lagr}_{2d}}]$ *on* $R(\upsilon_{2d})_*\overline{\mathbf{Q}}_\ell$ *preserves the decomposition* (11.6), *and it acts on* $\mathrm{Spr}^{\mathrm{Herm}}_{2d}[\rho_i]$ *by multiplication by* $(d-2i)$.

Proof By Proposition 4.5(2), $\mathrm{Spr}^{\mathrm{Herm}}_{2d}[\rho_i]$ is the middle extension from its restriction to Herm°_{2d}, it suffices to prove the same statement on Herm°_{2d}. Let $I'_d \subset X'_d \times X'$ be the universal divisor $\{(D \in X'_d, y \in X') : y \in D\}$. The maps $\mathrm{pr}(D, y) = D$ and $q(D, y) = D - y + \sigma(y)$ define the incidence correspondence as in [28, proof of Proposition 8.3],

$$\begin{array}{ccc} \mathrm{Sht}^r_{\mathrm{Lagr}_{2d}} & \longrightarrow & (\mathrm{Hk}^1_{\mathrm{Lagr}_{2d}})^r \times \mathrm{Lagr}_{2d} \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{pr}_0,\mathrm{pr}_1)^r\times\Delta} \\ (\mathrm{Lagr}_{2d})^{r+1} & \xrightarrow{\Phi_{\mathrm{Lagr}_{2d}}} & (\mathrm{Lagr}_{2d})^{2r+2} \end{array} \tag{11.7}$$

Now over Herm°_{2d}, the map $\mathrm{Herm}^\circ_{2d} \to X^\circ_d$ is smooth, and $\mathrm{Hk}^1_{\mathrm{Lagr}_{2d}}$ is the pullback of (11.7). We therefore reduce to checking the statement for the action of $[I'_d]$ on the direct image sheaf of $\nu_d : X'_d \to X_d$, which is done in the proof of [28, Proposition 8.3]. □

11.3 Lefschetz trace formula

We shall give a slight generalization of the Lefschetz trace formula [28, Proposition A.12] expressing the intersection number of a cycle with the graph of Frobenius as a trace. Instead of the graph of Frobenius, we need to intersect along $\Phi_M : M^{r+1} \to M^{2r+2}$. Consider the following situation:

- Let S be an algebraic stack locally of finite type over $k = \mathbf{F}_q$. Assume S can be stratified by locally closed substacks that are global quotients.
- Let M be a smooth equidimensional stack over $k = \mathbf{F}_q$ of dimension N with a proper representable map $f : M \to S$.
- For $1 \le i \le r$, let $(\mathrm{pr}^i_0, \mathrm{pr}^i_1) : C_i \to M \times_S M$ be a self-correspondence of M over S. Assume pr^i_0 is proper and representable.

Form the Cartesian diagram

$$\begin{array}{ccc} \mathrm{Sht}_C & \longrightarrow & (\prod_{i=1}^r C_i) \times M \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{pr}^i_0,\mathrm{pr}^i_1)_{1\le i\le r}} \\ M^{r+1} & \xrightarrow{\Phi_M} & M^{2r+2}. \end{array} \tag{11.8}$$

Then Sht_C decomposes as

$$\mathrm{Sht}_C = \coprod_{s \in S(k)} \mathrm{Sht}_C(s)$$

where $\mathrm{Sht}_C(s)$ is the fibered product of $\mathrm{Sht}_C \to S(k)$ against $s \in S(k)$. For $s \in S(k)$, we write $\mathrm{Sht}_C|_s$ for the fibered product

$$\begin{array}{ccc} \mathrm{Sht}_C|_s & \longrightarrow & \mathrm{Sht}_C \\ \downarrow & & \downarrow \\ \mathrm{Spec}\ k & \xrightarrow{s} & S(k) \end{array}$$

Then $\mathrm{Sht}_C|_s \to \mathrm{Sht}_C(s)$ is a torsor for $\mathrm{Aut}(s)$, and in particular a finite étale cover.

Suppose we are given cycle classes

$$\zeta_i \in \mathrm{Ch}_N(C_i), \quad 1 \le i \le r.$$

The cycle class $\mathrm{cl}(\zeta_i) \in \mathrm{H}^{\mathrm{BM}}_{2N}(C_i, \overline{\mathbf{Q}}_\ell(-N))$ is viewed as a cohomological correspondence between the constant sheaf on M and itself. Therefore it induces an endomorphism of $Rf_!\overline{\mathbf{Q}}_\ell$ which we denote by $f_!\,\mathrm{cl}(\zeta_i)$.

Proposition 11.8 *For each $s \in S(k)$ we have*

$$\deg((\Phi_M^!(\zeta_1 \times \cdots \times \zeta_r \times [M]))|_{\mathrm{Sht}_C|_s}) = \mathrm{Tr}(f_!\,\mathrm{cl}(\zeta_1) \circ \cdots \circ f_!\,\mathrm{cl}(\zeta_r) \circ \mathrm{Fr}_s, (Rf_!\overline{\mathbf{Q}}_\ell)_s).$$

Proof We first prove the formula when S is a scheme of finite type. In this case M, C_i are also schemes of finite type over k. Let $C = C_1 \times_M \times \cdots \times_M C_r$ be the composition correspondence, with maps $\mathrm{pr}_i : C \to M$ for $0 \le i \le r$. Consider the diagram where all squares are Cartesian

$$\begin{array}{ccccc} \mathrm{Sht}_C & \longrightarrow & C & \longrightarrow & (\prod_{i=1}^r C_i) \times M \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{pr}_0,\ldots,\mathrm{pr}_r,\mathrm{pr}_r)} & & \downarrow{\scriptstyle \prod(\mathrm{pr}_0^i,\mathrm{pr}_1^i)\times\Delta} \\ M^{r+1} & \xrightarrow{\Phi_1} & M^{r+2} & \xrightarrow{\Phi_2} & M^{2r+2} \\ {\scriptstyle \mathrm{pr}_0}\downarrow & & \downarrow{\scriptstyle (\mathrm{pr}_0,\mathrm{pr}_{r+1})} & & \\ M & \xrightarrow[(\mathrm{Id},\mathrm{Fr}_M)]{} & M \times M & & \end{array}$$

Here

$$\Phi_1(\xi_0, \ldots, \xi_{r-1}, \xi_r) = (\xi_0, \ldots, \xi_r, \mathrm{Fr}_M(\xi_0)),$$
$$\Phi_2(\xi_0, \ldots, \xi_r, \xi_{r+1}) = (\xi_0, \xi_1, \xi_1, \ldots, \xi_r, \xi_r, \xi_{r+1}).$$

We have $\Phi_M = \Phi_2 \circ \Phi_1$. Let $\zeta = \Phi_2^!(\zeta_1 \times \zeta_2 \times \cdots \times \zeta_r \times [M]) \in \mathrm{Ch}_N(C)$.

On the one hand, by the transitivity of the Gysin maps,

$$\Phi^!_M(\zeta_1 \times \cdots \times \zeta_r \times [M]) = \Phi^!_1(\zeta) = (\mathrm{Id}, \mathrm{Fr}_M)^!(\zeta). \tag{11.9}$$

Applying the Lefschetz trace formula [28, Proposition A.12], we get

$$\deg((\mathrm{Id}, \mathrm{Fr}_M)^!(\zeta)|_{\mathrm{Sht}_C|_s}) = \mathrm{Tr}(f_! \,\mathrm{cl}(\zeta) \circ \mathrm{Fr}, (Rf_!\overline{\mathbf{Q}}_\ell)_s). \tag{11.10}$$

One the other hand, by a diagram chase, we see that $\mathrm{cl}(\zeta)$ is the composition of the cohomological correspondences $\mathrm{cl}(\zeta_i)$ $(1 \le i \le r)$, hence $f_! \,\mathrm{cl}(\zeta) \in \mathrm{End}(Rf_!\overline{\mathbf{Q}}_\ell)$ is the composition of $f_! \,\mathrm{cl}(\zeta_1) \circ f_! \,\mathrm{cl}(\zeta_2) \circ \cdots \circ f_! \,\mathrm{cl}(\zeta_r)$. Combining this fact with (11.9) and (11.10) we get the desired formula.

Now consider the general case where S is a stack locally of finite type over k and we aim to prove the formula for $s \in S(k)$. We claim that there exists a scheme S' of finite type over k and a smooth map $u : S' \to S$ such that $u(S'(k))$ contains s. Indeed, pick any smooth map $u_1 : S_1 \to S$ with S_1 a scheme of finite type over k, such that s is contained in the image of u. Let $s_1 \in S_1(\mathbf{F}_{q^m})$ be a point that maps to s. Let $(S_1/S)^m$ be the m-fold fibered product of S_1 over S, based changed to $\overline{k}$. We equip $(S_1/S)^m$ with the Fr-descent datum given by $(x_1, \ldots, x_m) \mapsto (\mathrm{Fr}(x_m), \mathrm{Fr}(x_1), \ldots, \mathrm{Fr}(x_{m-1}))$. This gives a descent of $(S_1/S)^m$ to a scheme S' over k equipped with a map $u : S' \to S$ which is still smooth since u_1 is. Now s_1 gives rise to a k-point $s' = (s_1, \mathrm{Fr}(s_1), \ldots, \mathrm{Fr}^{m-1}(s_1)) \in S'(k)$ such that $u(s') = s$.

Let $M' = M \times_S S'$, $C'_i = C_i \times_S S'$ and let $u_{C_i} : C'_i \to C_i$ be the projection. Define Sht'_C using the analog of the diagram (11.8) with M and C_i replaced by M' and C'_i. Then $\mathrm{Sht}'_C \cong \mathrm{Sht}_C \times_{S(k)} S'(k)$. For $s' \in S'(k)$ such that $u(s') = s$, we get an isomorphism $\mathrm{Sht}'_C|_{s'} \xrightarrow{\sim} \mathrm{Sht}_C|_s$. Let $\zeta'_i = u^*_{C_i}\zeta_i$. Now we apply Proposition 10.9 to the diagram (11.8) along the base change map $u : S' \to S$ to get

$$u^*_{\mathrm{Sht}}\Phi^!_M(\zeta_1 \times \cdots \times \zeta_r \times [M]) \cong \Phi^!_{M'}(\zeta'_1 \times \cdots \times \zeta'_r \times [M]) \in \mathrm{Ch}_0(\mathrm{Sht}'_C).$$

Restricting to $\mathrm{Sht}'_C|_{s'} \cong \mathrm{Sht}_C|_s$ and taking degrees we get

$$\begin{aligned} &\deg(\Phi^!_M(\zeta_1 \times \cdots \times \zeta_r \times [M])|_{\mathrm{Sht}_C|_s}) \\ &\quad = \deg(\Phi^!_{M'}(\zeta'_1 \times \cdots \times \zeta'_r \times [M])|_{\mathrm{Sht}'_C|_{s'}}). \end{aligned} \tag{11.11}$$

On the other hand, letting $f' : M' \to S'$, by smooth base change we have

$$\begin{aligned} &\mathrm{Tr}(f_! \,\mathrm{cl}(\zeta_1) \circ \cdots \circ f_! \,\mathrm{cl}(\zeta_r) \circ \mathrm{Fr}_s, (Rf_!\overline{\mathbf{Q}}_\ell)_s) \\ &\quad = \mathrm{Tr}(f'_! \,\mathrm{cl}(\zeta'_1) \circ \cdots \circ f'_! \,\mathrm{cl}(\zeta'_r) \circ \mathrm{Fr}_{s'}, (Rf'_!\overline{\mathbf{Q}}_\ell)_{s'}). \end{aligned} \tag{11.12}$$

Since the right sides of (11.11) and (11.12) are equal by the scheme case that is already proven, the left sides of (11.11) and (11.12) are also equal, proving the proposition in general. □

Recall the graded perverse sheaf $\mathcal{K}^{\mathrm{Int}}_d(T)$ on Herm_{2d} from Definition 11.6.

Corollary 11.9 *Let* $(\mathcal{Q}, h_{\mathcal{Q}}) \in \mathrm{Herm}_{2d}(k)$. *We have*

$$\begin{aligned}\deg[\mathcal{Z}^r_{\mathcal{Q}}] &= \mathrm{Tr}([\mathrm{Hk}^1_{\mathrm{Lagr}_{2d}}]^r \circ \mathrm{Fr}, (R(\upsilon_{2d})_*\overline{\mathbf{Q}}_\ell)_{\mathcal{Q}}) \\ &= \sum_{i=0}^{d} (d-2i)^r \,\mathrm{Tr}(\mathrm{Fr}, \mathrm{Spr}^{\mathrm{Herm}}_{2d}[\rho_i]_{\mathcal{Q}}) \\ &= \frac{1}{(\log q)^r}\left(\frac{d}{ds}\right)^r\Big|_{s=0}\left(q^{ds}\,\mathrm{Tr}(\mathrm{Fr}, \mathcal{K}^{\mathrm{Int}}_d(q^{-2s})_{\mathcal{Q}})\right).\end{aligned}$$

Proof The first equality is an application of Proposition 11.8 to the case $S = \mathrm{Herm}_{2d}$, $M = \mathrm{Lagr}_{2d}$, $C_i = \mathrm{Hk}^1_{\mathrm{Lagr}_{2d}}$ and $\zeta_i = [\mathrm{Hk}^1_{\mathrm{Lagr}_{2d}}]$. The second equality follows from Proposition 11.7. The third one is a direct calculation. □

Combining Corollary 11.9 with Corollary 11.3 we get:

Corollary 11.10 *Let* $(\mathcal{E}, a) \in \mathcal{A}_d(k)$ *with image* $(\mathcal{Q}, h_{\mathcal{Q}}) \in \mathrm{Herm}_{2d}(k)$. *Then we have*

$$\deg[\mathcal{Z}^r_{\mathcal{E}}(a)] = \frac{1}{(\log q)^r}\left(\frac{d}{ds}\right)^r\Big|_{s=0}\left(q^{ds}\,\mathrm{Tr}(\mathrm{Fr}, \mathcal{K}^{\mathrm{Int}}_d(q^{-2s})_{\mathcal{Q}})\right).$$

11.4 Symmetry

This subsection is not used in the proof of the main theorem. The graded perverse sheaf $\mathcal{K}^{\mathrm{Int}}_d(T)$ has a palindromic symmetry that we spell out. First, the étale double covering $\nu : X' \to X$ gives a local system $\eta_{X'/X}$ on X with monodromy in ± 1. It induces a local system η_d on X_d with monodromy in ± 1: its stalk at a divisor $x_1 + \cdots + x_d \in X_d(\overline{k})$ is $\otimes_{i=1}^d (\eta_{X'/X})_{x_i}$. Let

$$\eta^{\mathrm{Herm}}_{2d} := s^{\mathrm{Herm}*}_{2d}\eta_d,$$

where $s^{\mathrm{Herm}}_{2d} : \mathrm{Herm}_{2d} \to X_d$ is the support map. This is a rank-one local system on Herm_{2d} with monodromy in ± 1.

Lemma 11.11 *We have a canonical isomorphism of perverse sheaves on* Herm_{2d}:

$$\mathrm{Spr}^{\mathrm{Herm}}_{2d}[\rho_d] \cong \eta^{\mathrm{Herm}}_{2d}.$$

Proof By Proposition 4.5(2), $\mathrm{Spr}^{\mathrm{Herm}}_{2d}[\rho_d]$ is the middle extension of its restriction to the open dense substack Herm°_{2d} (preimage of X°_d). The same is true for η_d because it is a local system and Herm_{2d} is smooth. Therefore it suffices to check their equality over Herm°_{2d}, over which both are obtained by pushing out the W_d-torsor $(X'^d)^\circ \to X^\circ_d$ along the character $\chi_d : W_d \to \{\pm 1\}$. □

Lemma 11.12 *There is an isomorphism of graded perverse sheaves on* Herm_{2d}

$$T^d \mathcal{K}^{\mathrm{Int}}_d(T^{-1}) \cong \eta^{\mathrm{Herm}}_{2d} \otimes \mathcal{K}^{\mathrm{Int}}_d(T).$$

Proof The equality amounts to

$$\mathrm{Spr}_{2d}^{\mathrm{Herm}}[\rho_{d-i}] \cong \eta_{2d}^{\mathrm{Herm}} \otimes \mathrm{Spr}_{2d}^{\mathrm{Herm}}[\rho_i].$$

Both sides are middle extensions from $\mathrm{Herm}_{2d}^{\circ}$ by Proposition 4.5(2), over which they correspond to representations ρ_{d-i} and $\chi_d \otimes \rho_i$ of W_d. By definition,

$$\chi_d \otimes \rho_i \cong \chi_d \otimes \mathrm{Ind}_{W_i \times W_{d-i}}^{W_d}(\chi_i \boxtimes \mathbf{1}).$$

Inserting $\chi_d|_{W_i \times W_{d-i}} \cong \chi_i \boxtimes \chi_{d-i}$ to the right side above gives

$$\mathrm{Ind}_{W_i \times W_{d-i}}^{W_d}(\chi_d|_{W_i \times W_{d-i}} \otimes (\chi_i \boxtimes \mathbf{1})) \cong \mathrm{Ind}_{W_i \times W_{d-i}}^{W_d}(\mathbf{1} \boxtimes \chi_{d-i}) \cong \rho_{d-i}.$$

Lemma 11.13 *If $(\mathcal{Q}, h_{\mathcal{Q}}) \in \mathrm{Herm}_{2d}(k)$ is the image of some $(\mathcal{E}, a) \in \mathcal{A}_d(k)$, then* $\mathrm{Tr}(\mathrm{Fr}, \eta_{2d}^{\mathrm{Herm}}|_{\mathcal{Q}}) = 1$.

Proof If X'/X is split, then $\eta_{2d}^{\mathrm{Herm}}$ is trivial, and there is nothing to prove. Below we assume X'/X is nonsplit. The local system η_d on X_d is pulled back from a local system η_{Pic} on Pic_X via the Abel-Jacobi map $\mathrm{AJ}_d : X_d \to \mathrm{Pic}_X^d \subset \mathrm{Pic}_X$. The Frobenius trace function of η_{Pic} is the idèle class character

$$\eta_{F'/F} : F^\times \backslash \mathbb{A}_F^\times / \widehat{\mathcal{O}}^\times = \mathrm{Pic}_X(k) \to \{\pm 1\}$$

trivial on the image of $\mathrm{Nm}_{X'/X} : \mathrm{Pic}_{X'}(k) \to \mathrm{Pic}_X(k)$. Denote by $\det_X(\mathcal{Q})$ the image of $\mathcal{Q}$ under $\mathrm{Herm}_{2d} \to X_d \xrightarrow{\mathrm{AJ}_d} \mathrm{Pic}_X^d$. We have $\eta_{2d}^{\mathrm{Herm}}|_{\mathcal{Q}} \cong \eta_{\mathrm{Pic}}|_{\det_X(\mathcal{Q})}$ as Fr-modules. Now $(\mathcal{Q}, h_{\mathcal{Q}})$ comes from $(\mathcal{E}, a)$, which implies

$$\det_X(\mathcal{Q}) \cong \mathrm{Nm}_{X'/X}(\det \mathcal{E})^{-1} \otimes \omega_X^{\otimes n}.$$

By [27, p.291, Theorem 13], ω_X is a square in $\mathrm{Pic}_X(k)$, hence $\eta_{F'/F}(\omega_X^{\otimes n}) = 1$. Since $\eta_{F'/F}(\mathrm{Nm}_{X'/X}(\det \mathcal{E})) = 1$, we see that $\eta_{F'/F}(\det_X(\mathcal{Q})) = 1$, hence $\mathrm{Tr}(\mathrm{Fr}, \eta_{2d}^{\mathrm{Herm}}|_{\mathcal{Q}}) = 1$. □

Corollary 11.14 *Let $(\mathcal{E}, a) \in \mathcal{A}_d(k)$ with image $(\mathcal{Q}, h_{\mathcal{Q}}) \in \mathrm{Herm}_{2d}(k)$. Then $s \mapsto q^{ds}\,\mathrm{Tr}(\mathrm{Fr}, \mathcal{K}_d^{\mathrm{Int}}(q^{-2s})_{\mathcal{Q}})$ is an even function in s. In particular, its odd order derivatives at $s = 0$ vanish.*

By Corollary 11.10, this implies $\deg[\mathcal{Z}_{\mathcal{E}}^r(a)] = 0$ for r odd. However, we know from Lemma 6.7 that $\mathrm{Sht}_{U(n)}^r = \varnothing$ when r is odd, which implies $\mathcal{Z}_{\mathcal{E}}^r(a) = \varnothing$.

Part 3. The comparison

12 Matching of sheaves

12.1 Recap

Let

$$\widetilde{E}_a(m(\mathcal{E}), s, \Phi) = E_a(m(\mathcal{E}), s, \Phi) \cdot \chi(\det(\mathcal{E}))^{-1} q^{\deg(\mathcal{E})(s-\frac{n}{2})+\frac{1}{2}n^2 \deg(\omega_X)} \cdot \mathscr{L}_n(s)$$

$$= \mathrm{Den}(q^{-2s}, (\mathcal{E}, a)) \tag{12.1}$$

where the notation is as in Theorem 2.8, being a renormalization of the a^{th} Fourier coefficient of $E_a(m(\mathcal{E}), s, \Phi)$.

We emphasize that, in keeping with §1.3, X is proper and $\nu\colon X' \to X$ is a finite étale double cover (possibly trivial).

Theorem 12.1 *Keep the notations above. Let $(\mathcal{E}, a) \in \mathcal{A}_d(k)$. Then we have*

$$\deg[\mathcal{Z}^r_{\mathcal{E}}(a)] = \frac{1}{(\log q)^r}\left(\frac{d}{ds}\right)^r\Big|_{s=0}\left(q^{ds}\,\widetilde{E}_a(m(\mathcal{E}), s, \Phi)\right). \tag{12.2}$$

In the previous parts, we have found sheaves on $\mathcal{A}_d$ which correspond to the two sides of (12.2), in the sense of the function-sheaf dictionary. Let us summarize the situation.

On the analytic side, we proved a formula expressing the non-singular Fourier coefficient of the Siegel–Eisenstein series in terms of the Frobenius trace of a graded virtual perverse sheaf $\mathcal{K}^{\mathrm{Eis}}_d(T)$ on $\mathrm{Herm}_{2d}(X'/X)$.

Theorem 12.2 (Combination of Theorems 2.8 and 5.3) *Let $(\mathcal{E}, a) \in \mathcal{A}_d(k)$. Then we have*

$$\widetilde{E}_a(m(\mathcal{E}), s, \Phi) = \mathrm{Tr}(\mathrm{Fr}, \mathcal{K}^{\mathrm{Eis}}_d(q^{-2s})_{\mathcal{Q}}). \tag{12.3}$$

On the geometric side, in Corollary 11.10, we found a formula expressing the degree of the special 0-cycle in terms of r^{th} derivative of the Frobenius trace of another graded perverse sheaf $\mathcal{K}^{\mathrm{Int}}_d(T)$ on $\mathrm{Herm}_{2d}(X'/X)$, repeated below:

$$\deg[\mathcal{Z}^r_{\mathcal{E}}(a)] = \frac{1}{(\log q)^r}\left(\frac{d}{ds}\right)^r\Big|_{s=0}\left(q^{ds}\,\mathrm{Tr}(\mathrm{Fr}, \mathcal{K}^{\mathrm{Int}}_d(q^{-2s})_{\mathcal{Q}})\right). \tag{12.4}$$

12.2 Proof of the main theorem

Comparing (12.3) and (12.4), we see that in order to prove Theorem 12.1, it remains to match the graded sheaves $\mathcal{K}^{\mathrm{Int}}_d(T)$ and $\mathcal{K}^{\mathrm{Eis}}_d(T)$ on $\mathrm{Herm}_{2d}(X'/X)$.

Proposition 12.3 *We have $\mathcal{K}^{\mathrm{Int}}_d(T) \cong \mathcal{K}^{\mathrm{Eis}}_d(T)$ as graded perverse sheaves on $\mathrm{Herm}_{2d}(X'/X)$.*

Proof Both sides can be written as $\mathrm{Spr}^{\mathrm{Herm}}_{2d}[\rho]$ for some graded virtual representation ρ of W_d. By definition (Definition 11.6), the sheaf $\mathcal{K}^{\mathrm{Int}}_d(T)$ corresponds to

$$\rho_{\mathcal{K}^{\mathrm{Int}}_d}(T) = \sum_{i=0}^{d} \mathrm{Ind}^{W_d}_{W_i \times W_{d-i}}(\chi_i \boxtimes 1)T^i.$$

We calculate the (a priori virtual) representation of W_d which corresponds under Springer theory to the $\mathcal{K}^{\mathrm{Eis}}_d(T)$ from Definition 5.2. The operation $R\overleftarrow{f}_{i,!}R\overrightarrow{f}_i^*$

corresponds under Springer theory to $\mathrm{Ind}_{W_{d-i}\times S_i}^{W_d}$ and $\mathfrak{P}_{d-i}(T)$ corresponds to $\sum_{j=0}^{d-i}(-1)^j \mathrm{Ind}_{W_j\times W_{d-i-j}}^{W_{d-i}}(\mathrm{sgn}_j \boxtimes \mathbf{1})T^j$ (cf. Definition 4.7 for HSpr_d). Hence $\mathcal{K}_d^{\mathrm{Eis}}(T)$ corresponds to

$$\sum_{i=0}^{d} \mathrm{Ind}_{W_{d-i}\times S_i}^{W_d}\left(\sum_{j=0}^{d-i}(-1)^j \mathrm{Ind}_{W_j\times W_{d-i-j}}^{W_{d-i}}(\mathrm{sgn}_j \boxtimes \mathbf{1}_{d-i-j}) \boxtimes \mathbf{1}_i T^{i+j}\right).$$

After re-indexing, we have

$$\rho_{\mathcal{K}_d^{\mathrm{Eis}}}(T) = \sum_{i=0}^{d}\sum_{j=0}^{i}(-1)^j \mathrm{Ind}_{S_{i-j}\times W_j\times W_{d-i}}^{W_d}(\mathbf{1}\boxtimes \overline{\mathrm{sgn}}_j \boxtimes \mathbf{1})T^i.$$

The desired statement then follows from the Lemma below (whose notation has been re-indexed) by comparing each coefficient. □

Lemma 12.4 *We have the identity of virtual representations of W_d:*

$$\chi_d = \sum_{j=0}^{d}(-1)^j \mathrm{Ind}_{S_{d-j}\times W_j}^{W_d}(1\boxtimes \overline{\mathrm{sgn}}_j).$$

Proof We will prove this by comparing traces of an arbitrary element $g \in W_d$. For $g \in W_d$,

$$\mathrm{Tr}(g, \mathrm{Ind}_{S_{d-j}\times W_j}^{W_d}(1\times \overline{\mathrm{sgn}}_j)) = \sum_{\substack{w\in W_d/(S_{d-j}\times W_j)\\ w^{-1}gw\in S_{d-j}\times W_j}} \overline{\mathrm{sgn}}_j(g''). \tag{12.5}$$

Here, when $w^{-1}gw \in S_{d-j}\times W_j$, we write $w^{-1}gw = (g', g'')$ for $g' \in S_{d-j}$ and $g'' \in W_j$.

Identify W_d with the group of permutations of $\{\pm 1, \ldots, \pm d\}$ that commute with the involution σ exchanging $j \leftrightarrow -j$ for all $1 \le j \le d$. The subgroup $S_{d-j}\times W_j$ is the stabilizer of $\{1, 2, \ldots, d-j\}$. Therefore the coset space $W_d/(S_{d-j}\times W_j)$ is in natural bijection with subsets $J \subset \{\pm 1, \ldots, \pm d\}$ such that $|J| = d-j$ and $J \cap (-J) = \varnothing$. Let $\mathfrak{J}_g$ be the set of $J \subset \{\pm 1, \ldots, \pm d\}$ such that $|J| = d-j$, $J \cap (-J) = \varnothing$ and $gJ = J$. Let g''_J be the permutation of g on $\{\pm 1, \ldots, \pm d\}\backslash(J \cup (-J))$. Combining this with (12.5), we obtain

$$\sum_{j=0}^{d}(-1)^j \mathrm{Tr}(g, \mathrm{Ind}_{S_{d-j}\times W_j}^{W_d}(1\times \overline{\mathrm{sgn}}_j)) = \sum_{J\in\mathfrak{J}_g}(-1)^{d-|J|}\overline{\mathrm{sgn}}(g''_J).$$

For any $g \in W_d$, the cycle decomposition of g can be grouped into a decomposition $g = g_1 \ldots g_r$ (unique up to reordering) where g_i is one of the two forms:

- (positive bicycle) g_i is a product of two disjoint cycles $c_i\sigma(c_i)$ (in particular, no two elements appearing in c_i are negatives of each other).

- (negative cycle) g_i is a single cycle invariant under the involution σ.

Let C_g^+ be the set of cycles of g that are part of a positive bicycle (i.e., C_g^+ contains both c_i and $\sigma(c_i)$ for each positive bicycle g_i). For any $x \in W_d$ we denote by $\underline{x} \subset \{\pm 1, \ldots, \pm d\}$ the set of elements that are not fixed by x. For a cycle c we let $|c|$ be its length. From this description we see that $J \in \mathfrak{I}_g$ if and only if J is a union of $\underline{c}$ for a subset of cycles $c \in C_g^+$. In other words, consider the set $\mathfrak{I}_g$ of subsets $I \subset C_g^+$ such that $\underline{I}$ is disjoint from $\sigma(\underline{I})$. Then we have a bijection $\mathfrak{I}_g \xrightarrow{\sim} \mathfrak{I}_g$ sending $I \in \mathfrak{I}_g$ to $J := \cup_{c \in I} \underline{c}$.

For $I \in \mathfrak{I}_g$, let g_I' be the product of g_i such that g_i contains a cycle in common with I; let g_I'' be the product of the remaining g_i's. The above discussion allows us to rewrite

$$\sum_{j=0}^{d} (-1)^j \operatorname{Tr}(g, \operatorname{Ind}_{S_{d-j} \times W_j}^{W_d} (1 \times \overline{\operatorname{sgn}}_j)) = \sum_{I \in \mathfrak{I}_g} (-1)^{\sum_{c \in C_g^+} |c|} \overline{\operatorname{sgn}}(g_I'').$$

This sum factorizes as a product over the g_i with individual factors as follows:

- For a positive bicycle $g_i = c_i \sigma(c_i)$, the local factor is the sum of three contributions, corresponding to whether $c_i \in I$, $\sigma(c_i) \in I$ or neither c_i nor $\sigma(c_i)$ is in I. The first two cases each contribute 1. The last case leads to a contribution of $(-1)^{|c_i|} \operatorname{sgn}(c_i) = -1$. The total contribution of the factor corresponding to a positive bicycle g_i is therefore $1 + 1 + (-1) = 1$.
- For each negative cycle g_i, since it always appears in g_I'', its contribution is $(-1)^{|g_i|/2} \overline{\operatorname{sgn}}(g_i)$. Let $\overline{g}_i$ be the image of g_i in S_d, which is a cycle of length $|g_i|/2$. Then $(-1)^{|g_i|/2} \overline{\operatorname{sgn}}(g_i) = (-1)^{|\overline{g}_i|} \operatorname{sgn}(\overline{g}_i) = -1$. Therefore the contribution of the factor corresponding to a negative cycle g_i is -1.

Summarizing, we have found

$$\sum_{j=0}^{d} (-1)^j \operatorname{Tr}(g, \operatorname{Ind}_{S_{d-j} \times W_j}^{W_d} (1 \times \overline{\operatorname{sgn}}_j)) = \left(\prod_{g_i \text{ positive}} 1 \right) \cdot \left(\prod_{g_i \text{ negative}} (-1) \right). \quad (12.6)$$

On the other hand, we have

$$\chi_d(g_i) = \begin{cases} 1, & g_i \text{ is positive;} \\ -1, & g_i \text{ is negative.} \end{cases}$$

Indeed, if $g_i = c_i \sigma(c_i)$ is positive, then we have $\chi_d(c_i) = \chi_d(\sigma(c_i)) = 1$ because both c_i and $\sigma(c_i)$ can be conjugated into S_d. If g_i is negative, then up to conjugacy we may assume g_i is the cyclic permutation $(1, 2, \ldots, m, -1, \ldots, -m)$ for some $1 \le m \le d$. Then $g_i = (1, -1)(1, 2, \ldots, m)(-1, -2, \ldots, -m)$, from which we see $\chi_d(g_i) = -1$.

We conclude that the right side of (12.6) is $\prod \chi_d(g_i) = \chi_d(g)$. This completes the proof. □

12.3 The split case $X' = X \coprod X$

We make our result more explicit in the split case $X' = X \coprod X$.

On the analytic side in §2.1, the group $H_n = \mathrm{GL}_{2n,F}$ and P_n is the standard parabolic corresponding to the partition (n, n), with Levi $M_n \simeq \mathrm{GL}_{n,F} \times \mathrm{GL}_{n,F}$. We then have the degenerate principal series

$$I_n(s) = \mathrm{Ind}_{P_n(\mathbb{A})}^{H_n(\mathbb{A})}(|\cdot|_F^{s+n/2} \times |\cdot|_F^{-s-n/2}), \quad s \in \mathbb{C}.$$

Let $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2) \in \mathrm{Bun}_{M_n}(k) \simeq \mathrm{Bun}_{\mathrm{GL}_n}(k) \times \mathrm{Bun}_{\mathrm{GL}_n}(k)$, and let $a : \mathcal{E}_1 \to \mathcal{E}_2^\vee$ be an injective map of $\mathcal{O}_X$-modules. Then by §2.6 the Siegel–Eisenstein series has a well-defined a^{th} Fourier coefficient $E_a(m(\mathcal{E}), s, \Phi)$ at $(\mathcal{E}_1, \mathcal{E}_2)$. By Theorem 2.8 and 5.1 we have

$$E_a(m(\mathcal{E}), s, \Phi) = q^{-(\deg(\mathcal{E}_1)+\deg(\mathcal{E}_2))(s-n/2)-\frac{1}{2}n^2 \deg \omega_X} \mathscr{L}_n(s)^{-1} \mathrm{Den}(q^{-2s}, \mathcal{E}_2^\vee/\mathcal{E}_1),$$

where $\mathscr{L}_n(s) = \prod_{i=1}^n \zeta_F(i+2s)$ and, for a torsion $\mathcal{O}_X$-module $\mathcal{Q}$, the density polynomial is given by

$$\mathrm{Den}(T, \mathcal{Q}) = \sum_{0 \subset \mathcal{I}_1 \subset \mathcal{I}_2 \subset \mathcal{Q}} T^{\dim_k \mathcal{I}_1 + \dim_k \mathcal{Q}/\mathcal{I}_2} \prod_{v \in |X|} \mathfrak{m}_v(t_v(\mathcal{I}_2/\mathcal{I}_1); T^{\deg(v)}).$$

Here see (2.2) for $\mathfrak{m}_v(t_v; T)$. The normalized Fourier coefficient (12.1) is

$$\widetilde{E}_a(m(\mathcal{E}), s, \Phi) = \mathrm{Den}(q^{-2s}, \mathcal{E}_2^\vee/\mathcal{E}_1).$$

Next we come to the geometric side. We have a natural partition

$$(X')^r = \coprod_{\mu \in \{\pm 1\}^r} X^r.$$

The moduli of hermitian shtukas $\mathrm{Sht}^r_{U(n)}$ defined in §6 is then partitioned into

$$\mathrm{Sht}^r_{U(n)} = \coprod_{\mu \in \{\pm 1\}^r} \mathrm{Sht}^\mu_{U(n)},$$

and there is a natural isomorphism

$$\mathrm{Sht}^\mu_{U(n)} \simeq \mathrm{Sht}^\mu_{\mathrm{GL}_n}.$$

Here we recall that $\mathrm{Sht}^\mu_{\mathrm{GL}_n}$ is the moduli of shtukas for GL_n, cf. [28], whose S-points are given by the groupoid of the following data:

(1) $x_i \in X(S)$ for $i = 1, \ldots, r$.
(2) $\mathcal{F}_0, \ldots, \mathcal{F}_n \in \mathrm{Bun}_{\mathrm{GL}_n}(S)$.
(3) An elementary modification $f_i : \mathcal{F}_{i-1} \dashrightarrow \mathcal{F}_i$ at the graph of x_i, which is of upper of length 1 if $\mu_i = +1$ and of lower of length 1 if $\mu_i = -1$.
(4) An isomorphism $\varphi : \mathcal{F}_r \cong {}^\tau\mathcal{F}_0$.

In particular, $\mathrm{Sht}^\mu_{\mathrm{GL}_n}$ is empty unless $\sum_{i=1}^r \mu_i = 0$.

For the special cycle $\mathcal{Z}^r_{\mathcal{E}}$ (cf. Definition 7.1) associated to $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2)$ above, we have a partition

$$\mathcal{Z}^r_{\mathcal{E}} = \coprod_{\mu \in \{\pm 1\}^r} \mathcal{Z}^{\mu}_{\mathcal{E}},$$

where an object in $\mathcal{Z}^{\mu}_{\mathcal{E}}(S)$ is an object as above in $\mathrm{Sht}^{\mu}_{\mathrm{GL}_n}(S)$ together with maps

$$\mathcal{E}_1 \boxtimes \mathcal{O}_S \xrightarrow{t_i^{(1)}} \mathcal{F}_i \xrightarrow{t_i^{(2)}} \mathcal{E}_2^{\vee} \boxtimes \mathcal{O}_S, \quad i = 1, \dots, r, \tag{12.7}$$

such that the diagram commutes

$$\begin{array}{ccccccccc}
\mathcal{E}_1 \boxtimes \mathcal{O}_S & = & \mathcal{E}_1 \boxtimes \mathcal{O}_S & = & \dots & = & \mathcal{E}_1 \boxtimes \mathcal{O}_S & \xrightarrow{\sim} & {}^{\tau}(\mathcal{E}_1 \boxtimes \mathcal{O}_S) \\
\downarrow{\scriptstyle t_0^{(1)}} & & \downarrow{\scriptstyle t_1^{(1)}} & & \downarrow & & \downarrow{\scriptstyle t_r^{(1)}} & & \downarrow{\scriptstyle {}^{\tau}t_0^{(1)}} \\
\mathcal{F}_0 & \xrightarrow{f_0} & \mathcal{F}_1 & \xrightarrow{f_1} & \dots & \xrightarrow{f_r} & \mathcal{F}_r & \xrightarrow{\sim} & {}^{\tau}\mathcal{F}_0 \\
\downarrow{\scriptstyle t_0^{(2)}} & & \downarrow{\scriptstyle t_1^{(2)}} & & \downarrow & & \downarrow{\scriptstyle t_r^{(2)}} & & \downarrow{\scriptstyle {}^{\tau}t_0^{(2)}} \\
\mathcal{E}_2^{\vee} \boxtimes \mathcal{O}_S & = & \mathcal{E}_2^{\vee} \boxtimes \mathcal{O}_S & = & \dots & = & \mathcal{E}_2^{\vee} \boxtimes \mathcal{O}_S & \xrightarrow{\sim} & {}^{\tau}(\mathcal{E}_2^{\vee} \boxtimes \mathcal{O}_S)
\end{array}$$

Let $a : \mathcal{E}_1 \to \mathcal{E}_2^{\vee}$ be a map of $\mathcal{O}_X$-modules. Then $\mathcal{Z}^{\mu}_{\mathcal{E}}(a)$ is the open-closed subscheme of $\mathcal{Z}^{\mu}_{\mathcal{E}}$ such that the common composition (12.7) is equal to $a \boxtimes \mathrm{Id}_{\mathcal{O}_S}$.

For an injective $a : \mathcal{E}_1 \to \mathcal{E}_2^{\vee}$, our §7 shows that $\mathcal{Z}^{\mu}_{\mathcal{E}}(a)$ is proper over Spec k and defines a class $[\mathcal{Z}^{\mu}_{\mathcal{E}}(a)] \in \mathrm{Ch}_0(\mathcal{Z}^{\mu}_{\mathcal{E}}(a))$ for each $\mu \in \{\pm 1\}^r$. Then our main Theorem asserts

$$\sum_{\mu \in \{\pm 1\}^r} \deg[\mathcal{Z}^{\mu}_{\mathcal{E}}(a)] = \frac{1}{(\log q)^r} \left(\frac{d}{ds}\right)^r \Big|_{s=0} \left(q^{ds} \widetilde{E}_a(m(\mathcal{E}), s, \Phi)\right),$$

where $d = -(\chi(X, \mathcal{E}_1) + \chi(X, \mathcal{E}_2))$. We remark that $\deg[\mathcal{Z}^{\mu}_{\mathcal{E}}(a)]$ is not independent of $\mu \in \{\pm 1\}^r$, even if we restrict our attention to those μ with $\sum_{i=1}^r \mu_i = 0$.

Acknowledgements We thank Michael Harris, Chao Li, Alan Peng, Zhiyu Zhang, and the anonymous referee for comments on a draft. We thank the referee for helpful suggestions and corrections.

Funding TF was supported by NSF Postdoctoral Fellowship DMS-1902927, NSF Grant DMS-2302520, and the Friends of the Institute for Advanced Study. ZY was partially supported by the Packard Fellowship, and the Simons Investigator grant. WZ is partially supported by the NSF grant DMS #1901642 and the Simons Investigator grant.

References

1. Borho, W., MacPherson, R.: Partial resolutions of nilpotent varieties. In: Analysis and Topology on Singular Spaces, II, III, Luminy, 1981. Astérisque, vol. 101, pp. 23–74. Soc. Math. France, Paris (1983)
2. Bruinier, J.H., Howard, B., Kudla, S.S., Rapoport, M., Yang, T.: Modularity of generating series of divisors on unitary Shimura varieties. In: Diviseurs Arithmétiques sur les Variétés Orthogonales et Unitaires de Shimura. Astérisque, pp. 7–125 (2020)
3. Cho, S., Yamauchi, T.: A reformulation of the Siegel series and intersection numbers. Math. Ann. **377**(3–4), 1757–1826 (2020)
4. Feng, T., Yun, Z., Zhang, W.: Higher theta series for unitary groups over function fields (2021). Available at https://arxiv.org/abs/2110.07001
5. Fulton, W.: Intersection Theory, 2nd edn. Ergebnisse der Mathematik und Ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2. Springer, Berlin (1998)
6. Garcia, L.E., Sankaran, S.: Green forms and the arithmetic Siegel-Weil formula. Invent. Math. **215**(3), 863–975 (2019)
7. Heinloth, J.: Uniformization of $\mathscr{G}$-bundles. Math. Ann. **347**(3), 499–528 (2010)
8. Hironaka, Y.: Spherical functions on $U(2n)/(U(n) \times U(n))$ and Hermitian Siegel series, geometry and analysis of automorphic forms of several variables. In: Ser. Number Theory Appl., vol. 7, pp. 120–159. World Sci. Publ., Hackensack, NJ (2012)
9. Hotta, R., Springer, T.A.: A specialization theorem for certain Weyl group representations and an application to the Green polynomials of unitary groups. Invent. Math. **41**(2), 113–127 (1977)
10. Kresch, A.: Cycle groups for Artin stacks. Invent. Math. **138**(3), 495–536 (1999)
11. Kudla, S.S.: Central derivatives of Eisenstein series and height pairings. Ann. Math. (2) **146**(3), 545–646 (1997)
12. Kudla, S., Rapoport, M.: Special cycles on unitary Shimura varieties I. Unramified local theory. Invent. Math. **184**(3), 629–682 (2011)
13. Kudla, S., Rapoport, M.: Special cycles on unitary Shimura varieties II: global theory. J. Reine Angew. Math. **697**, 91–157 (2014)
14. Lafforgue, V.: Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale. J. Am. Math. Soc. **31**(3), 719–891 (2018)
15. Laumon, G.: Correspondance de Langlands géométrique pour les corps de fonctions. Duke Math. J. **54**(2), 309–359 (1987)
16. Li, C., Liu, Y.: Chow groups and L-derivatives of automorphic motives for unitary groups. Ann. Math. (2) **194**(3), 817–901 (2021)
17. Li, C., Liu, Y.: Chow groups and L-derivatives of automorphic motives for unitary groups, II. Forum Math. Pi **10**, Article ID e5 (2022)
18. Li, C., Zhang, W.: Kudla-Rapoport cycles and derivatives of local densities. J. Am. Math. Soc. **35**(3), 705–797 (2022)
19. Li, C., Zhang, W.: On the arithmetic Siegel-Weil formula for GSpin Shimura varieties. Invent. Math. **228**(3), 1353–1460 (2022)
20. Liu, Y.: Arithmetic theta lifting and L-derivatives for unitary groups, I. Algebra Number Theory **5**(7), 849–921 (2011)
21. Mumford, D.: Theta characteristics of an algebraic curve. Ann. Sci. Éc. Norm. Supér. (4) **4**, 181–192 (1971)
22. Siegel, C.L.: Indefinite quadratische Formen und Funktionentheorie. I. Math. Ann. **124**, 17–54 (1951)
23. The Stacks Project Authors: *Stacks Project* (2020). https://stacks.math.columbia.edu
24. Varshavsky, Y.: Moduli spaces of principal F-bundles. Sel. Math. New Ser. **10**(1), 131–166 (2004)
25. Wei, F.-T.: On derivatives of Siegel-Eisenstein series over global function fields. J. Lond. Math. Soc. (2) **100**(2), 518–544 (2019)
26. Weil, A.: Sur la formule de Siegel dans la théorie des groupes classiques. Acta Math. **113**, 1–87 (1965)
27. Weil, A.: Basic Number Theory. Classics in Mathematics. Springer, Berlin (1995). Reprint of the second (1973) edition
28. Yun, Z., Zhang, W.: Shtukas and the Taylor expansion of L-functions. Ann. Math. (2) **186**(3), 767–911 (2017)
29. Yun, Z., Zhang, W.: Shtukas and the Taylor expansion of L-functions (II). Ann. Math. (2) **189**(2), 393–526 (2019)