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Euphotic representations and rigid automorphic data

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Abstract

We propose a new method to construct rigid G-automorphic representations and rigid $\widehat{G}$-local systems for reductive groups G. The construction involves the notion of *euphotic representations*, and the proof for rigidity involves the geometry of certain Hessenberg varieties.

Keywords Langlands correspondence · Rigid local systems · Automorphic representations · Hessenberg varieties

Mathematics Subject Classification Primary 11F70 , 14D24 , 22E57; Secondary 14F06

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1 Introduction

1.1 Rigid local systems and automorphic representations

Rigid local systems on a punctured curve are those that don't admit deformations that preserve the local monodromy at the punctures. Many well-known local systems in arithmetic are rigid, e.g., Kloosterman sheaves and hypergeometric sheaves. Katz [13, 14] studied rigid local systems systematically, and he gave an algorithm for producing all tame rigid local systems of arbitrary rank. This algorithm has been extended by Arinkin [1] (and Deligne) to cover all rigid local systems.

For a reductive group H over $\overline{\mathbb{Q}}_\ell$ an H-local system on a curve U is a homomorphism from $\pi_1(U, u)$ to $H(\overline{\mathbb{Q}}_\ell)$. There is a notion of rigidity for H-local systems generalizing the rigidity for rank n local systems. Much less is known about rigid H-local systems for general H.

Rigid local systems have seen application in inverse Galois theory and in the construction of motives with exceptional Galois groups, see [4, 5, 26]. In particular it is of interest to construct many examples of rigid H-local systems, especially for exceptional groups H.

While [5] use the Katz algorithm to construct and classify rigid G_2-local systems, this algorithm is unavailable for a general reductive group H. Even for rank n local systems it is often computationally and technically involved. In a series of works [10, 26, 27] a new method of constructing rigid H-local systems is developed, and many examples are given. The key new ingredient in that method is to use the Langlands correspondence for function fields to transport the problem of constructing rigid H-local systems into constructing rigid G-automorphic representations. Here H is identified with the Langlands dual group $\widehat{G}$ of G.

This method has several advantages. When trying construct a rigid H-local system using the Katz algorithm one has to construct a rank n local system and impose conditions on it to force its global monodromy to lie in H. Not all rigid H-local systems can be obtained in this way.

In the geometric Langlands approach one may directly construct H-local systems. In addition it turns out that rigid automorphic representations are sometimes easier to obtain, and techniques from the geometric Langlands program are crucial in passing from rigid automorphic representations to local systems.

This paper aims to expand the zoo of rigid G-automorphic representations and rigid $\widehat{G}$-local systems by generalizing the construction of [27]. We consider $F = k(t)$, the function field of $\mathbb{P}^1$ over $k = \mathbb{F}_q$. The G-automorphic representations we construct have depth zero at 0 and positive depth $1/m$ at ∞. The corresponding $\widehat{G}$-local systems over $\mathbb{P}^1 \backslash \{0, \infty\}$ are tamely ramified at 0 and wildly ramified at ∞, and are expected to be rigid.

1.2 Euphotic representations

In this introduction let G be a split almost simple group over the function field $F = k(t)$ of $\mathbb{P}^1_k$. Let F_0, F_∞ be the local fields of F at $0, \infty \in \mathbb{P}^1$.

The starting point of [27] is a class of supercuspidal representations of $G(F_\infty)$ introduced by Reeder and Yu [21] called *epipelagic representations*. They generalize an earlier construction of *simple supercuspidal representations* by Gross and Reeder [9] that motivated the construction of Kloosterman sheaves in [10].

In this paper, we define a more general class of representations of the p-adic group $G(F_\infty)$ than epipelagic ones which we call *euphotic* representations.[1]

The data needed to construct a euphotic representation is a triple $(\mathbf{P}_\infty, \psi, \chi)$. Here

- $\mathbf{P}_\infty$ is a parahoric subgroup of $G(F_\infty)$;
- ψ is a linear function on the vector space $V_\mathbf{P} = \mathbf{P}_\infty^+/\mathbf{P}_\infty^{++}$ (the first nontrivial associated graded of the pro-unipotent radical $\mathbf{P}_\infty^+$ under the Moy-Prasad filtration). We require ψ to be *semisimple* in the sense that its orbit under $L_\mathbf{P} = \mathbf{P}_\infty/\mathbf{P}_\infty^+$ is closed.
- Let L_ψ be the stabilizer of ψ under $L_\mathbf{P}$, and B_ψ be a Borel subgroup of L_ψ with quotient Cartan T_ψ. Then χ is a character $\chi : T_\psi(k) \to \overline{\mathbb{Q}}_\ell^\times$.

A euphotic representation π of type $(\mathbf{P}_\infty, \psi, \chi)$ is an irreducible representation of $G(F_\infty)$ that contains an eigenvector under $B_\psi(k)\mathbf{P}_\infty^+$ on which $B_\psi(k)$ acts via χ (inflated from $T_\psi(k)$) and $\mathbf{P}_\infty^+$ acts via $\psi_k \circ \psi$ (inflated from $V_\mathbf{P}$, and $\psi_k : k \to \overline{\mathbb{Q}}_\ell^\times$ is a fixed nontrivial additive character).

Compared to the notion of epipelagic representations in [21], we have relaxed the condition on ψ: it is only required to have a closed orbit under $L_\mathbf{P}$ and *not* required to have finite stabilizer under $L_\mathbf{P}$. Functionals on $V_\mathbf{P}$ with closed orbit that are not stable are also encountered in work of Kamgarpour and Yi on the geometric Langlands correspondence for hypergeometric sheaves [12].

1.3 Euphotic automorphic data

To construct rigid G-automorphic representations, we start with a triple $(\mathbf{P}_\infty, \psi, \chi)$ as above, and choose a parahoric subgroup $\mathbf{Q}_0$ of $G(F_0)$. We impose several conditions on these data (see Definition 3.4.1) which are of geometric nature (i.e., they only have to do with the situation over $\overline{k}$). Among these conditions is the requirement that certain Hessenberg varieties coming from cyclic gradings on $\mathfrak{g}$ have a stabilizer property under a group action, which we call "spectrally meager" (see Definition 3.3.1), a notion that we believe is of independent interest.

We prove that a euphotic automorphic datum $(\mathbf{P}_\infty, \psi, \chi, \mathbf{Q}_0)$ is weakly rigid in the following sense: there is a small (but nonzero) number of irreducible cuspidal automorphic representations π of $G(\mathbb{A}_F)$ such that π_∞ is euphotic of type $(\mathbf{P}_\infty, \psi, \chi)$, π_0 contains a nonzero $\mathbf{Q}_0$-fixed vector, and π is unramified otherwise; and the number of such cuspidal automorphic representations is uniformly bounded when k is replaced with any finite extension. This is proved by analyzing the space of automorphic functions cut out by the eigen-conditions at 0 and ∞, and Hessenberg varieties naturally show up in this analysis.

[1] In oceanography euphotic is synonymous to epipelagic, stressing the role of light. Depending on the transparency of the ocean water the euphotic zone may vary in depth - in analogy there are more possibilities for the depth of a euphotic representation than for the depth of an epipelagic representation.

1.4 Hecke eigensheaves and local systems

To construct the $\widehat{G}$-local systems out of these automorphic representations, we consider automorphic sheaves instead of functions. The automorphic datum $(\mathbf{P}_\infty, \psi, \chi, \mathbf{Q}_0)$ gives rise to an abelian category $\mathcal{P} = \mathcal{P}(\psi, \chi)$ of perverse sheaves on a certain moduli stack of G-bundles on $\mathbb{P}^1_{\overline{k}}$ with level structures given by $\mathbf{P}_\infty^{++}$ and $\mathbf{Q}_0$. This category has only finitely many simple objects, which is an indication of rigidity.

Here comes a crucial difference with all previous work on rigid automorphic representations. Previously the analogous categories $\mathcal{P}$ always decomposed into Hecke-stable pieces with one simple object (a Hecke eigensheaf) in each piece, and the framework of [26] allowed us to extract a $\widehat{G}$-local system from each Hecke eigensheaf. In the euphotic situation, the category $\mathcal{P}$ sometimes has more than one simple object yet there isn't an obvious way to decompose it further. We remark that this is a feature rather than a bug: it is likely that these more complicated categories $\mathcal{P}$ give nontrivial global L-packets. To deal with this situation, we extend the framework of [26] to extract eigen local systems from a Hecke eigencategory rather than a Hecke eigensheaf. The extra work needed is of categorical nature: we need to analyze the structure of semisimple factorizable module categories under a neutral Tannakian category. We give a self-contained treatment of this issue in "Appendix A", proving a classification result in Theorem A.4.1.

The main results of general nature in this paper can be summarized as follows. For simplicity we state the results in the case G is split. For notations, see Sect. 4.

1.4.1 Theorem (See Theorem 4.2.2 and Prop. 4.3.2) *Assume G is split. Let $(\mathbf{P}_\infty, \psi, \chi, \mathbf{Q}_0)$ be a euphotic automorphic datum. Consider the category $\mathcal{P}(\psi, \chi)$ of perverse sheaves on $\mathrm{Bun}_G(\mathbf{Q}_0, \mathbf{P}_\infty^{++})_{\overline{k}}$ that are $(B_\psi \ltimes V_{\mathbf{P}}, \mathrm{AS}_\psi \boxtimes \mathcal{K}_\chi)$-equivariant. Then*

(1) *The category $\mathcal{P}(\psi, \chi)$ has finitely many simple objects up to isomorphisms, and all of them are clean extensions from an explicit locally closed substack of $\mathrm{Bun}_G(\mathbf{Q}_0, \mathbf{P}_\infty^{++})$.*
(2) *There are finitely many semisimple $\widehat{G}$-local systems $\{E_\sigma\}_{\sigma\in\Sigma}$ over $\mathbb{G}_{m,\overline{k}}$ (for some index set Σ), and a decomposition of $\mathcal{P}^{ss}(\psi, \chi)$ (semisimple objects in $\mathcal{P}(\psi, \chi)$)*

$$\mathcal{P}^{ss}(\psi, \chi) = \bigoplus_{\sigma\in\Sigma} \mathcal{P}_\sigma$$

into Hecke-stable subcategories, such that each $\mathcal{P}_\sigma$ is an E_2-module category under $\mathrm{Rep}(\widehat{G}_\sigma)$ for $\widehat{G}_\sigma = \mathrm{Aut}_{\widehat{G}}(E_\sigma)$ (whose action on $\mathcal{P}_\sigma$ is denoted by $\bullet$), and the action of the geometric Hecke operator $\mathbf{T}_V$ (where $V \in \mathrm{Rep}(\widehat{G})$) on $\mathcal{A} \in \mathcal{P}_\sigma$ is given by

$$\mathbf{T}_V(\mathcal{A}) \cong \bigoplus_E E \boxtimes ([E_\sigma(V) : E] \bullet \mathcal{A})$$

in $\mathrm{Perv}(\mathbb{G}_{m,\overline{k}} \times \mathrm{Bun}_G(\mathbf{Q}_0, \mathbf{P}_\infty^{++})_{\overline{k}})$. Here the direct sum is over all irreducible local systems E over $\mathbb{G}_{m,\overline{k}}$, $E_\sigma(V) \in \mathrm{Loc}(\mathbb{G}_{m,\overline{k}})$ is the (semisimple) local system

on $\mathbb{G}_{m,\bar{k}}$ *associated to* E_σ *and* V, $[E_\sigma(V) : E]$ *is the multiplicity space of* E *in* $E_\sigma(V)$, *viewed as an object in* $\mathrm{Rep}(\widehat{G}_\sigma)$.

(3) *The geometric monodromy of each* $\widehat{G}$*-local system* E_σ *is tame and unipotent at* 0. *Under Lusztig's bijection between unipotent classes in* $\widehat{G}$ *and two-sided cells of the affine Weyl group* W_{aff}, *the unipotent monodromy of* E_σ *at* 0 *corresponds to the two-sided cell* $c_{\mathbf{Q}}$ *containing the longest element of* $W_{\mathbf{Q}_0}$.

1.5 Examples

More than half of the paper is devoted to various examples of euphotic automorphic data that are not epipelagic.

The starting point of our work is a new rigid G_2-connection on $\mathbb{P}^1\backslash\{0, \infty\}$ found by the first-named author [11]. We looked for the automorphic representation corresponding to the ℓ-adic counterpart of that G_2-connection, and arrived at the notion of euphotic automorphic data in general. This G_2 example is presented in detail in Sect. 5.

In Sect. 6 we give a complete list of euphotic automorphic data when the parahoric subgroup $\mathbf{P}_\infty$ is the hyperspecial parahoric $G(\mathcal{O}_\infty)$. The list in this case turns out to be closely related to the classification of double partial flag varieties $G/P_1 \times G/P_2$ that are spherical as a G-variety. The latter problem has been solved by Stembridge [24], and we use his results. We then study in Sect. 7 the Hessenberg varieties that appear in these examples in more detail, in order to conclude that they are spectrally meager, thereby verifying that the list in Sect. 6 indeed gives euphotic automorphic data.

In Sect. 8 we give some potential examples of euphotic automorphic data, mostly for exceptional groups. For these groups we have only checked one of the conditions in the definition of euphotic automorphic data.

1.6 Questions for further study

(1) The most complicated part in verifying a euphotic automorphic datum is to show that certain Hessenberg varieties are spectrally meager, a condition on the stabilizers of a certain solvable group action. In Sect. 7 we deal with Hessenberg varieties arising from the adjoint representation of G, and we show they are spectrally meager by relating them to Springer fibers. What is still missing is an effective criterion for Hessenberg varieties to be spectrally meager in general.

(2) We make predictions on the Langlands parameters of euphotic representations in Sect. 2.4, especially about their slopes. This prediction is closely related to another open problem of showing that the $\widehat{G}$-local systems we obtain from euphotic automorphic data are indeed cohomologically rigid (see Prop. 4.3.3 for evidence).

(3) Give a complete list of $(\mathbf{P}_\infty, \psi, \mathbf{Q}_0)$ for each almost simple quasi-split G, such that there exists χ making $(\mathbf{P}_\infty, \psi, \chi, \mathbf{Q}_0)$ into a euphotic automorphic datum.

1.7 Notation and preliminaries

1.7.1 The curve X

Fix a finite field k of characteristic p and let F be the field of rational functions on $X = \mathbb{P}^1_k$. Fix an affine coordinate t on $X\backslash\{\infty\}$, and we identify $F = k(t)$. The closed points $|X|$ of X are in bijection with the places of F. For $x \in |X|$ we denote by $\mathcal{O}_x$ the completed local ring of X at x and by F_x its field of fractions. The ring of addèles of F is the restricted product

$$\mathbb{A}_F = \prod_{x\in|X|}{}' F_x.$$

1.7.2 The split group $\mathbb{G}$

Let $\mathbb{G}$ be a split, connected semisimple group over k which is almost simple over $\overline{k}$. Fix a maximal split torus $\mathbb{T} \subset \mathbb{G}$, a Borel subgroup $\mathbb{B}$ containing $\mathbb{T}$, and extend these choices into a pinning $\dagger = (\mathbb{T}, \mathbb{B}, \{x_i\}_{i\in I})$ of $\mathbb{G}$ (where I indexes the set of simple roots). Let $\mathrm{Aut}^\dagger(\mathbb{G})$ be the finite group of pinned automorphisms of $\mathbb{G}$, which is identified with the outer automorphism group of $\mathbb{G}$. Let $\mathfrak{g} = \mathrm{Lie}\,\mathbb{G}$.

We make the following assumption:

There exists a non-degenerate $\mathrm{Ad}(\mathbb{G})$-invariant symmetric bilinear form on$\mathfrak{g}$.

This assumption is satisfied when $\mathrm{char}(k)$ is sufficiently large.

1.7.3 The quasi-split group G

Let $e \in \{1, 2, 3\}$. Assume $p \neq e$ and that k contains all e-th roots of unity. Fix an injective homomorphism $\theta_{\mathrm{Out}} : \mu_e(k) \to \mathrm{Aut}^\dagger(\mathbb{G})$.

Let G be the quasi-split form of $\mathbb{G}$ over $\mathbb{G}_m = X - \{0, \infty\}$ determined by θ_{Out}. More precisely, let $\widetilde{\mathbb{G}_m} = \mathrm{Spec}\, k[t^{1/e}, t^{-1/e}] \to \mathbb{G}_m = \mathrm{Spec}\, k[t, t^{-1}]$ be the μ_e-torsor, and consider the Weil restriction $\mathrm{Res}_{\widetilde{\mathbb{G}_m}/\mathbb{G}_m}\mathbb{G}$. Then G is the fixed point subgroup of $\mathrm{Res}_{\widetilde{\mathbb{G}_m}/\mathbb{G}_m}\mathbb{G}$ under the diagonal action of μ_e on both $\widetilde{\mathbb{G}_m}$ and on $\mathbb{G}$ via θ_{Out}.

The base changes G_{F_0} and G_{F_∞} are quasi-split forms of $\mathbb{G}$ over the respective local fields determined by the same homomorphism θ_{Out}, viewing $\mu_e(k)$ as the the quotient of $\mathrm{Gal}(F_0^s/F_0)$ and $\mathrm{Gal}(F_\infty^s/F_\infty)$ realized by the tamely ramified extensions $k((t^{1/e}))$ and $k((t^{-1/e}))$.

Let $A = \mathbb{T}^{\mu_e,\circ}$. From the construction of the group scheme G over $\mathbb{G}_m$, the constant torus $A \times \mathbb{G}_m$ is a maximal split torus of G. Similarly, A_{F_0} and A_{F_∞} are maximal split tori of G_{F_0} and G_{F_∞}.

1.7.4 Coefficient field

We fix a prime $\ell \neq \mathrm{char}(k)$. The representations of p-adic groups and adèlic groups will be on $\overline{\mathbb{Q}}_\ell$-vector spaces. Sheaves considered in this paper are étale complexes with

$\overline{\mathbb{Q}}_\ell$-coefficients over k-stacks or $\overline{k}$-stacks. All sheaf-theoretic functors are understood to be derived.

2 Euphotic representations

In this section we introduce a class of representations of the p-adic group $G(F_\infty)$ that generalize the *epipelagic representations* introduced by Reeder and Yu in [21].

This section concerns only the quasi-split group G_{F_∞} over F_∞. We denote G_{F_∞} simply by G in this section. Using the affine coordinate t on $X \setminus \{\infty\}$, we identify F_∞ with $K = k((t^{-1}))$ and write $K_e = k((t^{-1/e}))$.

2.1 Parahorics and gradings

2.1.1 Affine roots and root subgroups

The Lie algebra Lie G is the μ_e-invariants on $\mathfrak{g} \otimes K_e$ (where μ_e acts both on K_e by Galois action and on $\mathfrak{g}$ via θ_{Out}). The torus $A = \mathbb{T}^{\mu_e,\circ}$ (see Sect. 1.7.3) acts on Lie G by the adjoint action and additionally this algebra carries a $\mathbb{G}_m$-action given by scaling the uniformizer $t^{-1/e}$ of K_e. The set of affine roots Ψ_{aff} with respect to A can also be identified with the weights of $A \times \mathbb{G}_m$ for this action on Lie G. The set of real roots $\Psi_{\mathrm{re}} \subset \Psi_{\mathrm{aff}}$ consists of those affine roots which are non-trivial on the torus A.

Denote by LG the loop group of G. For any real root $\alpha \in \Psi_{\mathrm{re}}$ there is a subgroup $U_\alpha \subset LG$ which is isomorphic to $\mathbb{G}_a$ over k and whose Lie algebra is the α-eigenspace under the action of $A \times \mathbb{G}_m$ on Lie G, cf. [27, Sections 2.1 & 2.2].

The Borel subgroup $\mathbb{B}$ fixed in Sect. 1.7.3 gives a set of simple affine roots $\Delta_{\mathrm{aff}} \subset \Psi_{\mathrm{aff}}$ and positive affine roots $\Psi^+_{\mathrm{aff}} \subset \Psi_{\mathrm{aff}}$.

2.1.2 Parahoric subgroups

The maximal split torus $A \otimes K$ of G fixed in Sect. 1.7.3 defines an apartment $\mathfrak{A}$ of the building of $G(K)$. Then Ψ_{aff} can be identified with a set of affine functions on $\mathfrak{A}$, whose vanishing affine hyperplanes give a stratification of $\mathfrak{A}$ into facets. There is a unique set of positive integers $\{n_\alpha\}_{\alpha \in \Delta_{\mathrm{aff}}}$ such that $\sum_{\alpha \in \Delta_{\mathrm{aff}}} n_\alpha \alpha = 1$ as functions on $\mathfrak{A}$.

The fundamental alcove $C \subset \mathfrak{A}$ is cut out by the inequalities $\alpha > 0$ for all $\alpha \in \Delta_{\mathrm{aff}}$. Let $\mathfrak{F} \subset \overline{C}$ be a facet in the closure of C. Let $J = \{\alpha \in \Delta_{\mathrm{aff}} | \alpha|_{\mathfrak{F}} = 0\}$. Let

$$m = m_{\mathfrak{F}} = \sum_{\alpha \in \Delta_{\mathrm{aff}} \setminus J} n_\alpha \in \mathbb{N}.$$

In the rest of the paper we also assume

$$\text{The characteristic } p = \mathrm{char}(k) \text{ does not divide } m.$$

Let $\mathbf{P} \subset G(K)$ be the (standard) parahoric subgroup corresponding to $\mathfrak{F}$. Let $\mathbf{P} \supset \mathbf{P}^+ \supset \mathbf{P}^{++}$ be the first three steps in the Moy-Prasad filtration of $\mathbf{P}$ with respect to $x_{\mathbf{P}}$, the barycenter of $\mathfrak{F}$. In other words

$$\mathbf{P}^+ = G(K)_{x_{\mathbf{P}},1/m}, \quad \mathbf{P}^{++} = G(K)_{x_{\mathbf{P}},2/m}.$$

Let $L_{\mathbf{P}} = \mathbf{P}/\mathbf{P}^+$ be the Levi factor of $\mathbf{P}$ (a connected reductive group over k). There is a canonical section $L_{\mathbf{P}} \hookrightarrow \mathbf{P}$ whose image contains A; we identify $L_{\mathbf{P}}$ with the image of this section. The quotient $V_{\mathbf{P}} = \mathbf{P}^+/\mathbf{P}^{++}$ is a representation of $L_{\mathbf{P}}$ over k.

Let $\Psi(\mathbf{P}) \subset \Psi_{\mathrm{aff}}$ be the affine roots that appear as the weights of $A \times \mathbb{G}_m$ on Lie $\mathbf{P}$; similarly define $\Psi(\mathbf{P}^+)$, $\Psi(\mathbf{P}^{++})$, $\Psi(L_{\mathbf{P}})$ and $\Psi(V_{\mathbf{P}})$. Then

$$\Psi(L_{\mathbf{P}}) = \{\alpha \in \Psi_{\mathrm{aff}} | \alpha(x_{\mathbf{P}}) = 0\}; \quad \Psi(V_{\mathbf{P}}) = \left\{\alpha \in \Psi_{\mathrm{aff}} | \alpha(x_{\mathbf{P}}) = \frac{1}{m}\right\}.$$

2.1.3 Cyclic grading on the Lie algebra

Let $\mathfrak{g} = \mathrm{Lie}\, \mathbb{G}$. Then the barycenter $x_{\mathbf{P}}$ gives a $\mathbb{Z}/m\mathbb{Z}$-grading on $\mathfrak{g}$

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{g}(i)$$

compatible with the $\mathbb{Z}/e\mathbb{Z}$-grading on $\mathfrak{g}$ obtained from θ_{Out} (under the reduction map $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/e\mathbb{Z}$) and such that $\mathfrak{g}(0)$ can be canonically identified with Lie $(L_{\mathbf{P}})$, and $\mathfrak{g}(i)$ can be canonically identified with $G(K)_{x_{\mathbf{P}},\frac{i}{m}}/G(K)_{x_{\mathbf{P}},\frac{i+1}{m}}$ for $i \neq 0$. In particular we have $V_{\mathbf{P}} \cong \mathfrak{g}(1)$ as $L_{\mathbf{P}}$-modules. For more details, see [21, Theorem 4.1].

2.2 Euphotic representations

Let $\mathbf{P}$ be a standard parahoric subgroup of $G(K)$. Let $\psi \in V_{\mathbf{P}}^*$ be a vector whose $L_{\mathbf{P}}$-orbit is closed. Using an $\mathrm{Ad}(\mathbb{G})$-invariant form on $\mathfrak{g}$, which exists by our assumption in Sect. 1.7.2, the $L_{\mathbf{P}}$-module $V_{\mathbf{P}}^*$ may be identified with $\mathfrak{g}(-1)$. Then the $L_{\mathbf{P}}$-orbit of ψ is closed if and only if $\psi \in \mathfrak{g}(-1)$ is semisimple as an element of $\mathfrak{g}$.

Fix an additive character $\psi_k : k \to \overline{\mathbb{Q}}_\ell^\times$. For an admissible representation $(\pi, \mathcal{V})$ of $G(K)$ over $\overline{\mathbb{Q}}_\ell$, let

$$\mathcal{V}^{(\mathbf{P}^+,\psi)} = \{v \in \mathcal{V} | \pi(g)v = \psi_k(\psi(g))v, \forall g \in \mathbf{P}^+\}.$$

2.2.1 Definition Let $\psi \in V_{\mathbf{P}}^*$ be a vector whose $L_{\mathbf{P}}$-orbit is closed. Let $(\pi, \mathcal{V})$ be an irreducible admissible representation of $G(K)$. We say that it is *euphotic* with respect to $(\mathbf{P}, \psi)$ if $\mathcal{V}^{(\mathbf{P}^+,\psi)} \neq 0$.

2.2.2 Action of L_ψ

Let L_ψ be the stabilizer of ψ under $L_{\mathbf{P}}$. Then L_ψ is a (not necessarily connected) reductive group over k. In fact, since ψ can be identified with a semisimple element of

$\mathfrak{g}$ lying in $\mathfrak{g}(-1)$, its centralizer $\mathbb{G}_\psi$ in $\mathbb{G}$ is a reductive group whose Lie algebra $\mathfrak{g}_\psi$ is stable under the $\mathbb{Z}/m\mathbb{Z}$-grading. Viewing the $\mathbb{Z}/m\mathbb{Z}$-grading on $\mathfrak{g}$ as an action of μ_m on $\mathbb{G}$, $L_{\mathbf{P}}$ is the neutral component of $\mathbb{G}^{\mu_m}$. Then $\mathbb{G}_\psi$ is stable under the μ_m-action, and $L_\psi \subset (\mathbb{G}_\psi)^{\mu_m}$ is the union of components that lie in $L_{\mathbf{P}}$.

Let $(\pi, \mathcal{V})$ be a euphotic representation of $G(K)$. There is an action of $L_\psi(k)$ on $\mathcal{V}^{(\mathbf{P}^+,\psi)}$. We will be interested in those $(\pi, \mathcal{V})$ such that $\mathcal{V}^{(\mathbf{P}^+,\psi)}$ contains a principal series representation of $L_\psi(k)$. More precisely, let $B_\psi \subset L_\psi^\circ$ be a Borel subgroup of the neutral component L_ψ° of L_ψ. Let T_ψ be the quotient torus of B_ψ.

2.2.3 Definition Let $\psi \in \mathfrak{g}(-1)$ be semisimple and let $\chi : T_\psi(k) \to \overline{\mathbb{Q}}_\ell^\times$ be a character. Let $(\pi, \mathcal{V})$ be an irreducible admissible representation of $G(K)$. We say that it is *euphotic* with respect to $(\mathbf{P}, \psi, \chi)$ if the action of $B_\psi(k)$ on $\mathcal{V}^{(\mathbf{P}^+,\psi)}$ contains a nonzero eigenvector on which $B_\psi(k)$ acts via the character $B_\psi(k) \to T_\psi(k) \xrightarrow{\chi} \overline{\mathbb{Q}}_\ell^\times$.

By Frobenius reciprocity, an irreducible admissible $(\pi, \mathcal{V})$ is euphotic with respect to $(\mathbf{P}, \psi, \chi)$ if and only if it is a quotient of the compact induction

$$c - \mathrm{ind}_{\mathbf{P}^+ \cdot B_\psi(k)}^{G(K)}((\psi_k \circ \psi) \boxtimes \chi).$$

2.3 Relation with epipelagic representations

For simplicity in this subsection we assume G is split over F (i.e., $e = 1$) and L_ψ° is split over k. We lift T_ψ to a maximal split torus $T_\psi \subset L_\psi$ over k. Up to changing ψ by an element in the same $L_{\mathbf{P}}$-orbit, we may assume $T_\psi \subset A$ (A is a maximal split torus of $L_{\mathbf{P}}$). Then $\mathbb{H} = C_{\mathbb{G}}(T_\psi)$ is a Levi subgroup of $\mathbb{G}$ containing $\mathbb{T}$. From the construction, $\mathbb{H}$ is stable under the μ_m-action on $\mathbb{G}$ which gives the $\mathbb{Z}/m\mathbb{Z}$-grading.

We claim that the induced $\mathbb{Z}/m\mathbb{Z}$-grading on $\mathfrak{h} = \mathrm{Lie}\,\mathbb{H}$ is *stable* in the sense of [20, §5.3]. Indeed, it suffices to show that the stabilizer $(\mathbb{H}_\psi)^{\mu_m}$ is finite modulo T_ψ, but this is true because $(\mathbb{H}_\psi)^{\mu_m}$ is a reductive group containing T_ψ as a maximal torus which at the same time is central, hence $(\mathbb{H}_\psi)^{\mu_m}/T_\psi$ is finite. The theory developed in [20, Corollary 15] then attaches a *regular elliptic conjugacy class* $[w]$ in the extended Weyl group $W_{\mathrm{ext}}(\mathbb{H}, \mathbb{T})$ (component group of the normalizer of $\mathbb{T}$ in $\mathrm{Aut}(\mathbb{H})$). One checks that $[w]$ indeed is a well-defined conjugacy class in $W(\mathbb{G}, \mathbb{H}, \mathbb{T}) = (N_{\mathbb{G}}(\mathbb{H}) \cap N_{\mathbb{G}}(\mathbb{T}))/\mathbb{T}$.

Let $H = \mathbb{H} \otimes_k K$, a Levi subgroup of G. The $\mathbb{Z}/m\mathbb{Z}$-grading on $\mathfrak{h}$ gives a standard parahoric subgroup $\mathbf{P}_H \subset H$ and $\psi \in \mathfrak{h}(-1)$ can be viewed as a linear function on $\mathbf{P}_H^+/\mathbf{P}_H^{++}$. We expect that a euphotic representation of $G(K)$ with respect to $(\mathbf{P}, \psi, \chi)$ should be a composition factor of a parabolic induction from an epipelagic representation of $H(K)$ with respect to $(\mathbf{P}_H, \psi)$ in the sense of [21], whose central character restricts to χ on $T_\psi \subset ZH$.

2.4 Predictions on the Langlands parameter

Again for simplicity we assume $e = 1$, i.e., G is split, and that L_ψ is split over k.

2.4.1. Let $\widehat{G}$ be the Langlands dual of $\mathbb{G}$ (over $\overline{\mathbb{Q}}_\ell$) equipped with a maximal torus $\widehat{T}$ and an isomorphism $\mathbb{X}^*(\widehat{T}) \cong \mathbb{X}_*(\mathbb{T})$. Then the roots $\Phi(\widehat{G}, \widehat{T})$ are identified with the coroots $\Phi^\vee(\mathbb{G}, \mathbb{T})$. Recall the Levi subgroup $\mathbb{H}$ introduced in Sect. 2.3. Let $\widehat{H} \subset \widehat{G}$ be the Levi subgroup containing $\widehat{T}$ such that $\Phi(\widehat{H}, \widehat{T}) \subset \Phi(\widehat{G}, \widehat{T})$ is identified with the set $\Phi^\vee(\mathbb{H}, \mathbb{T}) \subset \Phi^\vee(\mathbb{G}, \mathbb{T})$.

2.4.2. Let $W_K \supset I_K \supset I_K^+$ be the Weil group, inertia group and wild inertia of the local field K. Let π be a euphotic representation of $G(K)$ with respect to $(\mathbf{P}, \psi, \chi)$. Let $\rho_\pi : W_K \to \widehat{G}$ be the Langlands parameter of π.

For $p = \mathrm{char}(k)$ large, we make the following predictions on ρ_π:

(1) Consider the torus $\widehat{S} = [\widehat{H}, \widehat{H}] \cap \widehat{T}$. Then, up to $\widehat{G}$-conjugacy, one should be able to arrange that $\rho_\pi(I_K^+) \subset \widehat{S}$. We also expect that $\rho_\pi(I_K^+)$ is regular in $\widehat{S}$, i.e., its centralizer in $\widehat{G}$ is $C_{\widehat{G}}(\widehat{S})$. This implies that $\rho_\pi(I_K)$ lies in the normalizer $N_{\widehat{G}}(\widehat{S})$ of $\widehat{S}$ in $\widehat{G}$.

(2) By (1), ρ_π induces a homomorphism

$$\rho_\pi^{\mathrm{tame}} : I_K^{\mathrm{tame}} = I_K / I_K^+ \to N_{\widehat{G}}(\widehat{S})/\widehat{S}.$$

The group $N_{\widehat{G}}(\widehat{S})/\widehat{S}$ is a possibly disconnected reductive group with $\widehat{T}/\widehat{S} = \widehat{H}^{ab}$ as a maximal torus. Let $\rho_\pi^{\mathrm{tame},ss} : I_K^{\mathrm{tame}} \to N_{\widehat{S}}(\widehat{S})/\widehat{S}$ be the semisimplification of ρ_π^{tame}.

There is an inclusion $N_{\widehat{G}}(\widehat{H}, \widehat{T}) := N_{\widehat{G}}(\widehat{H}) \cap N_{\widehat{G}}(\widehat{T}) \subset N_{\widehat{G}}(\widehat{S})$. Then up to conjugacy $\rho_\pi^{\mathrm{tame},ss}$ should have image in $N_{\widehat{G}}(\widehat{H}, \widehat{T})/\widehat{S}$. The composition

$$I_K^{\mathrm{tame}} \xrightarrow{\rho_\pi^{\mathrm{tame},ss}} N_{\widehat{G}}(\widehat{H}, \widehat{T})/\widehat{S} \to N_{\widehat{G}}(\widehat{H}, \widehat{T})/\widehat{T} =: W(\widehat{G}, \widehat{H}, \widehat{T})$$

should map a topological generator of I_K^{tame} to a regular elliptic element $w \in W(\widehat{G}, \widehat{H}, \widehat{T}) = W(\mathbb{G}, \mathbb{H}, \mathbb{T})$ which is in the regular elliptic conjugacy class attached to $\mathbf{P}$ in Sect. 2.3.

(3) Let $N_{\widehat{G}}(\widehat{H}, \widehat{T})' \subset N_{\widehat{G}}(\widehat{H}, \widehat{T})$ be the preimage of the cyclic group $\langle w \rangle \subset W(\widehat{G}, \widehat{H}, \widehat{T})$. Then by (2), up to conjugacy $\rho_\pi^{\mathrm{tame},ss}$ should have image in $N_{\widehat{G}}(\widehat{H}, \widehat{T})'/\widehat{S}$, which is an extension of $\langle w \rangle$ by $\widehat{T}/\widehat{S}$. Since w acts trivially on T_ψ, the projection $\widehat{T} \to \widehat{T}/\widehat{S} \to \widehat{T}_\psi$ dual to the inclusion $T_\psi \subset \mathbb{T}$ extends to a homomorphism $N_{\widehat{G}}(\widehat{H}, \widehat{T})'/\widehat{S} \to \widehat{T}_\psi$. Then the composition

$$I_K^{\mathrm{tame}} \xrightarrow{\rho_\pi^{\mathrm{tame},ss}} N_{\widehat{G}}(\widehat{H}, \widehat{T})'/\widehat{S} \to \widehat{T}_\psi$$

should correspond to the character χ of $T_\psi(k)$ under local class field theory.

(4) The slopes of the adjoint representation $\mathrm{Ad}(\rho_\pi) : W_K \to \mathrm{Aut}(\widehat{\mathfrak{g}})$ are either 0 or $1/m$, and

$$\mathrm{Swan}(\mathrm{Ad}(\rho_\pi)) = \dim L - \dim L_\psi. \tag{2.1}$$

The prediction on Swan conductors (2.1) is based on the following heuristics. The Swan conductor should be $1/m$ for each root $\alpha^\vee \in \Phi(\widehat{G}, \widehat{T}) = \Phi^\vee(\mathbb{G}, \mathbb{T})$ which is

nontrivial on the image $\rho(I_K^+)$, and should be zero on other root spaces and on $\widehat{\mathfrak{t}}$. Those $\alpha^\vee$ such that $\alpha^\vee|_{\rho(I_K^+)} = 1$ correspond exactly to the coroots of the centralizer $\mathbb{G}_\psi$ with respect to $\mathbb{T}$. Let $R' = \{\alpha \in \Phi(\mathbb{G}, \mathbb{T}) | \alpha(\psi) \neq 0\}$, then $\mathrm{Swan}(\mathrm{Ad}(\rho_\pi)) = \#R'/m$. On the other hand, the μ_m-action on $\mathfrak{g}$ preserves R' and freely permutes R'. We get that $\mathfrak{g}(0) = \mathfrak{g}^{\mu_m}$ is the direct sum of $\mathfrak{g}_\psi \cap \mathfrak{g}(0) = \mathrm{Lie}\, L_\psi$ and 1-dimension from the sum of root spaces for each μ_m-orbit of R'. Therefore the number of μ_m-orbits on R', which is $\#R'/m$, is the same as $\dim L - \dim L_\psi$, hence the prediction (2.1).

Let $I_K^{\mathrm{tame}}(m) \subset I_K^{\mathrm{tame}}$ be the unique subgroup of index m. Assume the character χ is sufficiently generic, then ρ_π^{tame} will be semisimple and by (3) above, $\rho_\pi^{\mathrm{tame}}(I_K^{\mathrm{tame}}(m))$ should be conjugated into $\widehat{T}/\widehat{S}$. The genericity of χ should imply that the centralizer of $\rho_\pi^{\mathrm{tame}}(I_K^{\mathrm{tame}}(m))$ in $C_{\widehat{G}}(\widehat{S})/\widehat{S}$ is $\widehat{T}/\widehat{S}$. Therefore, for χ sufficiently generic, we should have $\widehat{G}^{\rho_\pi(I_K)} = C_{\widehat{G}}(\widehat{S})^{\rho_\pi^{\mathrm{tame}}(I_K^{\mathrm{tame}})} = \widehat{T}^w$. Therefore we expect to have

$$\dim \widehat{\mathfrak{g}}^{\rho_\pi(I_K)} = \dim \widehat{\mathfrak{t}}^w = \dim T_\psi = \mathrm{rk} L_\psi. \tag{2.2}$$

3 Euphotic automorphic data

In this section we define euphotic automorphic data, and give a criterion for them to be rigid.

3.1 Pre-euphotic automorphic data

3.1.1 Opposite parahorics

Let $\mathbf{P}_\infty$ be a standard parahoric subgroup of $G(F_\infty)$ corresponding to a facet $\mathfrak{F} \subset \overline{C}$ in the apartment $\mathfrak{A}_\infty$ of the building of $G(F_\infty)$ corresponding to the split torus A_{F_∞}. Let $\mathfrak{A}_0$ be the apartment in the building of $G(F_0)$ corresponding to the split torus A_{F_0}. Then there is a unique isomorphism $\mathfrak{A}_\infty \cong \mathfrak{A}_0$ characterized by the following two conditions

(1) The natural action of $\mathbb{X}_*(A)_\mathbb{R}$ on $\mathfrak{A}_\infty$ corresponds to the *opposite* of the natural action of $\mathbb{X}_*(A)_\mathbb{R}$ on $\mathfrak{A}_0$.
(2) The special vertex in $\mathfrak{A}_\infty$ corresponding to the parahoric $G(\mathcal{O}_\infty)$ maps to the special vertex in $\mathfrak{A}_0$ corresponding to the parahoric $G(\mathcal{O}_0)$.

Let us denote both $\mathfrak{A}_\infty$ and $\mathfrak{A}_0$ by $\mathfrak{A}$ under this identification. Let $\mathbf{P}_0$ be the parahoric subgroup of $G(F_0)$ corresponding to the same facet $\mathfrak{F}$ that we used to define $\mathbf{P}_\infty$. We say $\mathbf{P}_0$ thus constructed is *opposite* to $\mathbf{P}_\infty$. Then the Levi factors $L_{\mathbf{P}_\infty}$ and $L_{\mathbf{P}_0}$ can be canonically identified, which we denote by $L_\mathbf{P}$, or simply L.

3.1.2 Definition A *pre-euphotic automorphic datum* is a tuple $(\mathbf{P}_\infty, \psi, \chi, \mathbf{Q}_0)$ where

- $\mathbf{P}_\infty$ is a standard parahoric subgroup of $G(F_\infty)$;
- $\psi \in V^*_{\mathbf{P}_\infty}$ with closed orbit under L;
- $\chi : T_\psi(k) \to \overline{\mathbb{Q}}_\ell^\times$ is a character.

- Let $\mathbf{P}_0 \subset G(F_0)$ be the parahoric subgroup opposite to $\mathbf{P}_\infty$. Then $\mathbf{Q}_0$ is a parahoric subgroup of $G(F_0)$ which is contained in $\mathbf{P}_0$ and contains the torus A.

The parahoric $\mathbf{Q}_0 \subset \mathbf{P}_0$ corresponds to a facet $\mathfrak{F}'$ in $\mathfrak{A}_0$ whose closure contains $\mathfrak{F}$. Let $x_{\mathbf{Q}}$ be the barycenter of $\mathfrak{F}'$. The parahoric $\mathbf{Q}_0$ also determines a parabolic subgroup Q of $L = L_{\mathbf{P}_0}$ (containing A) such that $\mathbf{Q}_0$ is the preimage of Q under the projection $\mathbf{P}_0 \to L$.

3.1.3 Weyl groups

Let $\mathbb{W}$ be the Weyl group of $\mathbb{G}$ with respect to $\mathbb{T}$. The Weyl group W of G with respect to A can be identified with the fixed points $\mathbb{W}^{\mu_e}$. The Iwahori-Weyl groups of $G(F_\infty)$ and $G(F_0)$ with respect to A can be identified under the identification $\mathfrak{A}_\infty = \mathfrak{A}_0$; we denote it by $\widetilde{W} = \mathbb{X}_*(A) \rtimes W$. For $w \in W$, choose a lifting of it in $N_{\mathbb{G}}(\mathbb{T})(k)^{\mu_e}$; for an arbitrary element $w = (\lambda, w_1) \in \mathbb{X}_*(A) \rtimes W = \widetilde{W}$, we have its lifting $\dot{w} = t^\lambda \dot{w}_1$.

Let $W_{\mathrm{aff}} \subset \widetilde{W}$ be the affine Weyl group generated by affine simple reflections. Let $\Omega = \mathrm{Stab}_{\widetilde{W}}(C)$, the stabilizer of the fundamental alcove under $\widetilde{W}$. Then the projection induces an isomorphism $\Omega \xrightarrow{\sim} \widetilde{W}/W_{\mathrm{aff}}$, and Ω is a finite abelian group.

Let $W_{\mathbf{P}}$ (resp. $W_{\mathbf{Q}}$) denote the Weyl group of L (resp. $L_{\mathbf{Q}}$, the Levi of $\mathbf{Q}_0$ or Q), both as subgroups of W_{aff}. We have $W_{\mathbf{Q}} \subset W_{\mathbf{P}}$.

3.2 The space of automorphic forms

3.2.1 A space of automorphic forms

We will impose more conditions on the data $(\mathbf{P}_\infty, \psi, \chi, \mathbf{Q}_0)$. To motivate these conditions, we consider automorphic representations $\pi = \otimes'_{x \in |X|} \pi_x$ of $G(\mathbb{A}_F)$ for which

(1) π_x is unramified for $x \neq 0, \infty$;
(2) $\pi_0^{\mathbf{Q}_0} \neq 0$;
(3) π_∞ is euphotic with respect to $(\mathbf{P}_\infty, \psi, \chi)$ in the sense of Definition 2.2.3.

Any such automorphic representation π contains a nonzero vector in the following space

$$\mathcal{F} = \mathrm{Fun}\left(G(F)\backslash G(\mathbb{A}_F)/(\mathbf{Q}_0 \times \prod_{x \neq 0,\infty} G(\mathcal{O}_x)) \right)^{(B_\psi \mathbf{P}_\infty^+, \mu)}$$

of $\overline{\mathbb{Q}}_\ell$-valued functions on which $B_\psi \mathbf{P}_\infty^+ \subset \mathbf{P}_\infty$ acts via the character μ defined by $\mu|_{B_\psi} = \chi$ and $\mu|_{\mathbf{P}_\infty^+} = \psi_k \circ \psi$. Let $\Gamma_0 = G(k[t, t^{-1}]) \cap \mathbf{Q}_0$. Through the equality

$$G(F)\backslash G(\mathbb{A}_F)/(\mathbf{Q}_0 \times \prod_{x \neq 0,\infty} G(\mathcal{O}_x) \times \mathbf{P}_\infty^{++}) = \mathbf{P}_\infty^{++}\backslash G(F_\infty)/\Gamma_0 \tag{3.1}$$

of double cosets, cf. [27, 2.12] we identify

$$\mathcal{F} = \mathrm{Fun}(G(F_\infty)/\Gamma_0)^{(B_\psi \mathbf{P}_\infty^+, \mu)}.$$

We have the Birkhoff decomposition [27, 3.2]

$$G(F_\infty) = \coprod_{[w] \in W_{\mathbf{P}} \backslash \widetilde{W} / W_{\mathbf{Q}}} \mathbf{P}_\infty \dot{w} \Gamma_0. \tag{3.2}$$

From this we get a decomposition

$$\mathcal{F} = \bigoplus_{[w] \in W_{\mathbf{P}} \backslash \widetilde{W} / W_{\mathbf{Q}}} \mathcal{F}_{[w]}$$

where $\mathcal{F}_{[w]}$ is the subspace of functions supported on $\mathbf{P}_\infty \dot{w} \Gamma_0$.

3.2.2 The space $\mathcal{F}_{[w]}$

Let $w \in \widetilde{W}$. We have

$$\mathbf{P}_\infty^+ \backslash \mathbf{P}_\infty \dot{w} \Gamma_0 / \Gamma_0 = L(k)/Q_w(k) = (L/Q_w)(k) \tag{3.3}$$

where $Q_w = \mathrm{Ad}(w)\Gamma_0 \cap L = \mathrm{Ad}(w)\mathbf{Q}_0 \cap L$ is the parabolic subgroup of L containing A with roots

$$\Psi(Q_w) = \{\alpha \in \Psi_{\mathrm{re}} | \alpha(x_{\mathbf{P}}) = 0, \alpha(w x_{\mathbf{Q}}) \leq 0\}.$$

Let $f \in \mathcal{F}_{[w]}$. Assume f is not identically zero on the double coset $\mathbf{P}_\infty^+ \ell \dot{w} \Gamma_0$ for some $\ell \in L$. For any $\alpha \in \Psi(V_{\mathbf{P}})$ (i.e., $\alpha(x_{\mathbf{P}}) = 1/m$) and $u \in U_\alpha$ we have

$$f(\ell u \dot{w}) = {}^{\ell^{-1}}\psi(u) f(\ell \dot{w}).$$

If we furthermore assume that $\alpha(w x_{\mathbf{Q}}) \leq 0$, then $U_{w^{-1}\alpha} \subset \Gamma_0$ and we find that

$$f(\ell \dot{w}) = f(\ell \dot{w} \mathrm{Ad}(\dot{w}^{-1})(u)) = f(\ell u \dot{w}) = {}^{\ell^{-1}}\psi(u) f(\ell \dot{w}).$$

This implies that ${}^{\ell^{-1}}\psi$ vanishes on the space

$$V_w := \bigoplus_{\substack{\alpha(x_{\mathbf{P}}) = 1/m \\ \alpha(w x_{\mathbf{Q}}) \leq 0}} V_{\mathbf{P}}(\alpha).$$

In other words

$${}^{\ell^{-1}}\psi \in V_w^\perp = \bigoplus_{\substack{\alpha(x_{\mathbf{P}}) = -1/m \\ \alpha(w x_{\mathbf{Q}}) < 0}} V_{\mathbf{P}}^*(\alpha) \subset V_{\mathbf{P}}^*$$

We may alternatively write $V_w^\perp$ into a sum of A-weight spaces

$$V_w^\perp = \bigoplus_{\langle \overline{\alpha}, wx_{\mathbf{Q}} - x_{\mathbf{P}} \rangle < \frac{1}{m}} V_{\mathbf{P}}^*(\overline{\alpha}).$$

Note that $V_w^\perp$ is stable under Q_w.

3.2.3 Definition Let Y_w be the closed subscheme of the partial flag variety L/Q_w defined by

$$Y_w = \{\ell \cdot Q_w \in L/Q_w \mid {}^{\ell^{-1}}\psi \in V_w^\perp\}.$$

3.2.4 Remark (1) The variety Y_w is a Hessenberg variety in the sense of [8], attached to the L-module $V_{\mathbf{P}}^*$ and the Q_w-stable subspace $V_w^\perp$. It carries an action of the stabilizer L_ψ by right translation.

(2) If $w = 1$, or more generally if $\mathfrak{F}_{\mathbf{P}}$ is in the closure of $w\mathfrak{F}_{\mathbf{Q}}$, then $Y_w = L/Q_w$.

(3) If we change w to ww_1 for some $w_1 \in W_{\mathbf{Q}}$, then both Q_w and $V_w^\perp$ are unchanged, hence $Y_w = Y_{ww_1}$.

(4) If we change w to $w_2 w$ for some $w_2 \in W_{\mathbf{P}}$, then we may lift w_2 to $\ddot{w}_2 \in L$, and right multiplication by $\ddot{w}_2$ induces an isomorphism $L/Q_w \xrightarrow{\sim} L/Q_{w_2 w}$ which restricts to an isomorphism $Y_w \xrightarrow{\sim} Y_{w_2 w}$.

The above discussion implies that any function $f \in \mathcal{F}_{[w]}$ must be supported on $Y_w(k)$ (as a function on $(L/Q_w)(k)$ under (3.3)). In view of the eigen property under B_ψ, we get the following description of $\mathcal{F}_{[w]}$.

3.2.5 Lemma *For any $w \in \widetilde{W}$, let $[w]$ be its $(W_{\mathbf{P}}, W_{\mathbf{Q}})$ coset in $\widetilde{W}$. Then there is a canonical isomorphism*

$$\mathcal{F}_{[w]} \cong \mathrm{Fun}(Y_w(k))^{(B_\psi(k), \chi)}$$

where the right side is the space of eigenfunctions on $Y_w(k)$ under the left translation of $B_\psi(k)$ with eigen-character χ. The isomorphism is given by $\mathcal{F}_w \ni f \mapsto f_w$ where $f_w(\ell Q_w) = f(\ell \dot{w} \Gamma_0)$ for all $\ell Q_w \in Y_w(k)$.

3.3 Spectrally meager varieties

In examining when the space $\mathcal{F}_{[w]}$ is zero we arrive at the following notion.

3.3.1 Definition (1) Let H be a connected reductive group over k with Borel subgroup B_H. Let Y be a scheme of finite type over k with an H-action. We say that Y is *spectrally meager* if for any geometric point $y \in Y(\overline{k})$ the stabilizer $\mathrm{Stab}_{B_H}(y)$ contains a nontrivial torus.

(2) If Y is a spectrally meager H-scheme, let $\mathcal{S}(Y)$ be the collection of (nontrivial) subtori of $T_{H,\overline{k}}$ (the universal Cartan of H) given by the images of $\mathrm{Stab}_{B_H}(y)^\circ \to T_H$ for all $y \in Y(\overline{k})$.

3.3.2 Remark (1) The definition of spectrally meager H-scheme does not depend on the choice of the Borel subgroup B_H. It is therefore intrinsic to the H-scheme Y.

(2) If Y is a spectrally meager H-scheme, the collection $\mathcal{S}(Y)$ of subtori is finite. Indeed, we may partition Y into finitely many locally closed connected B_H-stable subschemes $\{Y_\alpha\}$ such that the torus part of the stabilizer of B_H on each point of Y_α has the same dimension, then each Y_α contributes a single torus in $\mathcal{S}(Y)$.

(3) The terminology "spectrally meager" may be justified as follows.
Let $\chi : T_H(k) \to \overline{\mathbb{Q}}_\ell^\times$ be a character with the property that $\chi|_{S(k)} \neq 1$ for any $S \in \mathcal{S}(Y)$ defined over k, then the $H(k)$-module $\mathrm{Fun}(Y(k))$ does not contain any simple constituent of the principal series representation $\mathrm{Ind}_{B_H(k)}^{H(k)}(\chi)$. The same property holds after any finite base change k'/k.
On the other hand, making the obvious definition over $\mathbb{C}$, if Y is an affine H-variety over $\mathbb{C}$ which is spectrally meager, then $\mathcal{O}(Y)$ as an algebraic H-module contains the irreducible H-module V_λ with highest weight λ only if $\lambda \in \mathbb{X}_*(S)^\perp$ for some $S \in \mathcal{S}(Y)$, i.e., λ lies in the union of finitely many proper sublattices in $\mathbb{X}^*(T)$.

3.3.3 Corollary (of Lemma 3.2.5) *Let $w \in \widetilde{W}$. If Y_w is spectrally meager as an L_ψ-scheme, and χ is nontrivial on $S(k)$ for any torus $S \in \mathcal{S}(Y_w)$ that is defined over k, then $\mathcal{F}_{[w]} = 0$.*

Proof For any $y \in Y_w(k)$, the stabilizer $\mathrm{Stab}_{B_\psi}(y)$ maps to a nontrivial torus $S \in \mathcal{S}(Y_w)$. Since $\chi|_{S(k)} \neq 1$, all $(B_\psi(k), \chi)$-eigenfunctions on $Y_w(k)$ must vanish at y. We conclude that $\mathcal{F}_{[w]} = 0$ by Lemma 3.2.5. □

3.4 Euphotic automorphic data

3.4.1 Definition A pre-euphotic automorphic datum $(\mathbf{P}_\infty, \psi, \chi, \mathbf{Q}_0)$ is called a *euphotic automorphic datum* if it satisfies the following conditions:

(1) For $w \in \Omega$, any Borel subgroup B_ψ of L_ψ acts on Y_w with an open orbit with finite stabilizers.
(2) For any $w \in \widetilde{W} - W_{\mathbf{P}}\Omega W_{\mathbf{Q}}$, the L_ψ-scheme Y_w is spectrally meager.
(3) Let $\mathcal{K}_\chi$ be the Kummer local system on T_ψ attached to χ (see [26, Appendix A.3.5]). Then for any $S \in \cup_{w \notin W_{\mathbf{P}}\Omega W_{\mathbf{Q}}} \mathcal{S}(Y_w)$ (this is a subtorus of $T_{\psi,\bar{k}}$ over $\bar{k}$), the restriction $\mathcal{K}_\chi|_S$ is a nontrivial local system.

We call the euphotic automorphic datum $(\mathbf{P}_\infty, \psi, \chi, \mathbf{Q}_0)$ *strict* if moreover the following holds:

(1) For any $w \in \Omega$ and any $x \in (L/Q_w)(\bar{k})$ outside the open B_ψ-orbit let S_x be the image of $\mathrm{Stab}_{B_\psi}(x)^\circ \to T_{\psi,\bar{k}}$ and assume that $\mathcal{K}_\chi|_{S_x}$ is a nontrivial local system (in particular, S_x is a nontrivial torus).
(2) The stabilizer on the open B_ψ-orbit is ZG.

3.4.2 Remark (1) The conditions for a pre-euphotic automorphic datum to be a euphotic automorphic datum can be checked after base changing the situation to $\bar{k}$.

(2) The open B_ψ-orbit condition in Definition 3.4.1 is saying that Y_w is a spherical L_ψ-variety. This implies that B_ψ has finitely many orbits on Y_w by [2].
(3) As w varies in $\widetilde{W}$, there are only finitely many different L_ψ-equivariant isomorphism types of the schemes Y_w. Indeed, there are only finitely many possibilities for Q_w and V_w. Therefore the union $\cup_{w\notin W_{\mathbf{P}}\Omega W_{\mathbf{Q}}}\mathcal{S}(Y_w)$ is a finite set.

3.4.3 Proposition *Let $(\mathbf{P}_\infty, \psi, \chi, \mathbf{Q}_0)$ be a euphotic automorphic datum. Then*

(1) *The space $\mathcal{F}$ is finite-dimensional and consists of cusp forms. In particular, any automorphic representation π satisfying the conditions in Sect. 3.2.1 is cuspidal.*
(2) *For any finite field extension k'/k, consider the similarly defined space $\mathcal{F}^{k'}$ using the base change automorphic data $(\mathbf{P}_\infty, \psi, \chi, \mathbf{Q}_0)$. Then $\dim_{\overline{\mathbb{Q}}_\ell}\mathcal{F}^{k'}$ is bounded independent of k'.*

Proof (1) For any $w \in \widetilde{W} - W_{\mathbf{P}}\Omega W_{\mathbf{Q}}$, the assumptions in Corollary 3.3.3 are satisfied (for any $S \in \mathcal{S}(Y_w)$ defined over k, $\chi|_{S(k)}$ is nontrivial if and only if $\mathcal{K}_\chi|_{S_{\overline{k}}}$ is nontrivial). Therefore $\mathcal{F}_{[w]} = 0$.
If $w \in \Omega$, the space $\mathcal{F}_{[w]} = \mathrm{Fun}(Y_w(k))^{(B_\psi(k),\chi)}$ has finite dimension because there are finitely many B_ψ-orbits on Y_w over $\overline{k}$ (see Remark 3.4.2(2)), hence finitely many rational orbits as well. we conclude that $\mathcal{F}$ is finite-dimensional and stable under the spherical Hecke operators at all places $x \notin \{0, \infty\}$. By [15, Lemme 8.24], $\mathcal{F}$ consists of cusp forms.
(2) The universal bound for $\dim \mathcal{F}^{k'}$ comes from bounding the number of $B_\psi(k')$-orbits on $Y_w(k')$ for $w \in \Omega$. The number of such orbits is bounded by $\sum_x \#\pi_0(\mathrm{Stab}_{B_\psi}(x))$ where x runs over a set of representatives of the finite set $Y_w(\overline{k})/B_\psi(\overline{k})$. □

3.4.4 Remark One may generalize the notion of a (pre-)euphotic automorphic datum by adding a character η of $L_{\mathbf{Q}}(k)$ (or a rank one character local system $\mathcal{K}_\eta$ on $L_{\mathbf{Q}}$). We leave it to the reader to modify the third condition in Definition 3.4.1 in this situation (which should involve $\mathcal{K}_\chi$ and $\mathcal{K}_\eta$).

4 Hecke eigencategory and local systems

Unless otherwise stated, in this subsection all ℓ-adic sheaves are over the relevant spaces base-changed to $\overline{k}$.

4.1 Automorphic sheaves

Let $(\mathbf{P}_\infty, \psi, \chi, \mathbf{Q}_0)$ be a euphotic automorphic datum.

4.1.1 A category of automorphic sheaves

We denote by

$$\mathrm{Bun} := \mathrm{Bun}_G(\mathbf{Q}_0, \mathbf{P}_\infty^{++})$$

the moduli stack of G-bundles on $\mathbb{P}^1$ with level structure $\mathbf{Q}_0$ at 0 and $\mathbf{P}_\infty^{++}$ at ∞. It carries an action of $B_\psi \ltimes V_{\mathbf{P}}$ by changing the level structure at ∞. The character $\psi_k \circ \psi : V_{\mathbf{P}} = \mathbf{P}_\infty^{+}/\mathbf{P}_\infty^{++} \to \overline{\mathbb{Q}}_\ell^*$ determines an Artin-Schreier sheaf AS_ψ on $V_{\mathbf{P}}$. Similarly we get a Kummer sheaf on T_ψ whose pullback along $B_\psi \to T_\psi$ we denote by $\mathcal{K}_\chi$.

Let $\mathcal{D}(\psi, \chi)$ be the derived category of $\overline{\mathbb{Q}}_\ell$-complexes on $\mathrm{Bun}_{\overline{k}}$ with $(B_{\psi,\overline{k}} \ltimes V_{\mathbf{P},\overline{k}}, \mathcal{K}_\chi \boxtimes \mathrm{AS}_\psi)$-equivariant structures. In [26], a more elaborate notion of geometric automorphic data is defined, including the data of a character sheaf on the center ZG. In our case we simply take the trivial local system on ZG. For the details we refer to [26, Section 2.6].

More generally, for any scheme S over k, we define $\mathcal{D}(S, \psi, \chi)$ to be the derived category of $\overline{\mathbb{Q}}_\ell$-complexes on $S_{\overline{k}} \times_{\overline{k}} \mathrm{Bun}_{\overline{k}}$ with $(B_{\psi,\overline{k}} \ltimes V_{\mathbf{P},\overline{k}}, \mathcal{K}_\chi \boxtimes \mathrm{AS}_\psi)$-equivariant structures.

Let $\mathcal{P}(\psi, \chi) \subset \mathcal{D}(\psi, \chi)$ and $\mathcal{P}(S, \psi, \chi) \subset \mathcal{D}(S, \psi, \chi)$ be the full abelian subcategory of perverse sheaves.

4.1.2 Lemma *(1) The category $\mathcal{P}(\psi, \chi)$ has finitely many simple objects up to isomorphism.*

(2) Let S be a scheme of finite type over k. Any simple perverse sheaf in $\mathcal{P}(S, \psi, \chi)$ is of the form $\mathcal{F}_S \boxtimes \mathcal{A}$, where $\mathcal{F}_S$ is a simple perverse sheaf on $S_{\overline{k}}$, and $\mathcal{A} \in \mathcal{P}(\psi, \chi)$ is a simple object.

Proof All stacks in the proof are understood to be over $\overline{k}$; we omit the base change $(-)_{\overline{k}}$ from the notations.

(1) Stratify Bun into locally closed substacks $\mathrm{Bun}_{[w]}$ indexed by classes $[w] \in W_{\mathbf{P}} \backslash \widetilde{W} / W_{\mathbf{Q}}$ using the Birkhoff decomposition (3.1) and (3.2). By the discussion in Sect. 3.2.2, the restriction of any $\mathcal{A} \in \mathcal{D}(\psi, \chi)$ to $\mathrm{Bun}_{[w]}$ can be identified with an object $\mathcal{A}_{[w]} \in D_{(B_\psi, \mathcal{K}_\chi)}(Y_w)$. By the genericity condition on the Kummer sheaf $\mathcal{K}_\chi$ (see Definition 3.4.1), for any $w \in \widetilde{W} - W_{\mathbf{P}} \Omega W_{\mathbf{Q}}$, any geometric point $y \in Y_w$, the restriction of $\mathcal{K}_\chi$ to $\mathrm{Stab}_{B_\psi}(y)$ is nontrivial. Therefore $\mathcal{A}_{[w]} = 0$. This implies that any object $\mathcal{A} \in \mathcal{D}(\psi, \chi)$ has vanishing stalks and costalks outside the open strata $\sqcup_{w \in \Omega} \mathrm{Bun}_{[w]}$ (one for each connected component of Bun). For $w \in \Omega$ let $j_w : Y_w \times V_{\mathbf{P}} \hookrightarrow L/Q_w \times V_{\mathbf{P}} \cong \mathrm{Bun}_{[w]} \hookrightarrow \mathrm{Bun}$ be the embedding. Here the first arrow is a closed embedding and the second is the open inclusion of the open stratum. Then the sum of the functors j_w^* gives a t-exact equivalence

$$j^* : \mathcal{D}(\psi, \chi) \xrightarrow{\sim} \bigoplus_{w \in \Omega} D_{(B_\psi \ltimes V_{\mathbf{P}}, \mathcal{K}_\chi \boxtimes \mathrm{AS}_\psi)}(Y_w \times V_{\mathbf{P}}) \cong \bigoplus_{w \in \Omega} D_{(B_\psi, \mathcal{K}_\chi)}(Y_w).$$

By Remark 3.4.2(2), Y_w has finitely many B_ψ-orbits. This implies that the category $\mathrm{Perv}_{(B_\psi, \mathcal{K}_\chi)}(Y_w)$ has finitely many simple objects. Therefore the same is true for $\mathcal{P}(\psi, \chi)$.

(2) The above argument shows that the restriction map along the embedding $\mathrm{id}_S \times j : S \times \sqcup_{w \in \Omega} Y_w \times V_{\mathbf{P}} \hookrightarrow S \times \mathrm{Bun}$ induces an equivalence

$$\mathcal{D}(S, \psi, \chi) \xrightarrow{\sim} \bigoplus_{w \in \Omega} D_{(B_\psi, \mathcal{K}_\chi)}(S \times Y_w).$$

Let $\mathcal{B} \in \mathrm{Perv}_{(B_\psi, \mathcal{K}_\chi)}(S \times Y_w)$ be a simple object, for some $w \in \Omega$. Let $Y_w = \sqcup_{\alpha \in \Sigma} Z_\alpha$ be the stratification into B_ψ-orbits. Then $\mathcal{B}$ is the middle extension of a local system $\mathcal{B}_0$ on a locally closed B_ψ-stable substack of $S \times Y_w$, which is necessarily of the form $S' \times Z_\alpha$ for some $\alpha \in \Sigma$ and $S' \subset S$ locally closed irreducible. Choose a point $z \in Z_\alpha$ and let Γ_α be the stabilizer of B_ψ at z. Then via restriction to $S' \times \{z\}$, $(B_\psi, \mathcal{K}_\chi)$-equivariant local systems on $S' \times Z_\alpha$ are the same as $(\Gamma_\alpha, \mathcal{K}_\chi|\Gamma_\alpha)$-equivariant local systems on S', with the trivial action of Γ_α on S'. Let Γ_α° be the neutral component of Γ_α. If $\mathcal{K}_\chi|\Gamma_\alpha^\circ$ is nontrivial, there are no such local systems. If $\mathcal{K}_\chi|\Gamma_\alpha^\circ$ is trivial, then $\mathcal{K}_\chi|\Gamma_\alpha$ descends to $\pi_0(\Gamma_\alpha)$ and gives a class $\xi \in \mathrm{H}^2(\pi_0(\Gamma_\alpha), \overline{\mathbb{Q}}_\ell^\times)$ by [26, A.4.1]. The class ξ gives a central extension $1 \to \overline{\mathbb{Q}}_\ell^\times \to \Xi \to \pi_0(\Gamma_\alpha) \to 1$. By [26, A.4.4], $(\Gamma_\alpha, \mathcal{K}_\chi|\Gamma_\alpha)$-equivariant local systems on S' (with the trivial Γ_α-action) are the same as local systems on S' with an action of the group Ξ such that $\overline{\mathbb{Q}}_\ell^\times \subset \Xi$ acts by scaling on the local system. Thus any irreducible $(B_\psi, \mathcal{K}_\chi)$-equivariant local system $\mathcal{B}_0$ on $S' \times Z_\alpha$ must be of the form $\mathcal{F}_0 \boxtimes \rho$ for an irreducible local system $\mathcal{F}_0$ on S' and an irreducible representation ρ of Ξ on which $\overline{\mathbb{Q}}_\ell^\times$ acts by scaling. View ρ as a $(B_\psi, \mathcal{K}_\chi)$-equivariant local system $\mathcal{A}_0$ on Z_α. We have $\mathcal{B}_0 \cong \mathcal{F}_0 \boxtimes \mathcal{A}_0$. Let $\mathcal{F}_S$ and $\mathcal{A}$ be the middle extensions of $\mathcal{F}_0$ and $\mathcal{A}_0$ respectively to $\overline{S}'$ and $\overline{Z}_\alpha$, then $\mathcal{B} \cong \mathcal{F}_S \boxtimes \mathcal{A}$.

□

4.1.3 Geometric Hecke operators

We briefly review the construction of geometric Hecke operators. For details we refer to [26, §4.2]. First consider the case G is split. Let Hk be the Hecke correspondence

$$\mathrm{Bun} \xleftarrow{\overleftarrow{h}} \mathrm{Hk} \xrightarrow{\overrightarrow{h}} \mathbb{G}_m \times \mathrm{Bun}$$

which classifies $(x, \mathcal{E}, \mathcal{E}', \alpha)$ where $x \in \mathbb{G}_m$, $\mathcal{E}, \mathcal{E}' \in \mathrm{Bun}$ and α is an isomorphism of G-torsors $\mathcal{E}|_{X\setminus\{x\}} \cong \mathcal{E}'|_{X\setminus\{x\}}$ respecting the level structures. The maps $\overleftarrow{h}$ and $\overrightarrow{h}$ send $(x, \mathcal{E}, \mathcal{E}', \alpha)$ to $\mathcal{E}$ and $(x, \mathcal{E}')$ respectively.

By the geometric Satake equivalence, for each representation $V \in \mathrm{Rep}(\widehat{G})$ there is a $G[\![t]\!] \rtimes \mathrm{Aut}(k[\![t]\!])$-equivariant perverse sheaf on the affine Grassmannian $\mathrm{Gr}_G = G(\!(t)\!)/G[\![t]\!]$ denoted IC_V. This perverse sheaf can be "spreadout" over Hk (still denoted IC_V) such that its $*$-restriction to every fiber of $\overrightarrow{h}$ is isomorphic to IC_V. The geometric Hecke operator is the functor

$$\mathbf{T}_V = \overrightarrow{h}_!(\overleftarrow{h}^*(-) \otimes \mathrm{IC}_V) : \mathcal{D}(\psi, \chi) \to \mathcal{D}(\mathbb{G}_m, \psi, \chi).$$

The formation of $\mathbf{T}_V$ is additive in V.

More generally, for any finite set I, any scheme S, $V \in \mathrm{Rep}(\widehat{G}^I)$, there is a functor

$$\mathbf{T}^I_{S,V} : \mathcal{D}(S, \psi, \chi) \to \mathcal{D}(\mathbb{G}_m^I \times S, \psi, \chi)$$

defined using the version of the Hecke stack that modifies the bundle simultaneously at a collection of points indexed by I. These functors have factorization structures: for $I = I_1 \sqcup I_2$, $V_i \in \mathrm{Rep}(\widehat{G}^{I_i})$ and $V = V_1 \boxtimes V_2$, there is a canonical isomorphism

$$\mathbf{T}^I_{S,V} \cong \mathbf{T}^{I_1}_{\mathbb{G}_m^{I_2}\times S, V_1} \circ \mathbf{T}^{I_2}_{S,V_2}. \tag{4.1}$$

Moreover, for any surjection $I \twoheadrightarrow J$ with the diagonal map $\widehat{G}^J \hookrightarrow \widehat{G}^I$ and $\mathbb{G}_m^J \hookrightarrow \mathbb{G}_m^I$, there is a canonical isomorphism

$$\mathbf{T}^I_{S,V}|_{\mathbb{G}_m^J \times S \times \mathrm{Bun}} \cong \mathbf{T}^J_{S,V|_{\widehat{G}^J}}.$$

When G is quasi-split, the only modification to the above discussions is that $\mathbb{G}_m$ should be replaced with the μ_e-covering $\widetilde{\mathbb{G}_m}$ over which G is split.

4.1.4 Proposition *Let $(\mathbf{P}_\infty, \psi, \chi, \mathbf{Q}_0)$ be a euphotic automorphic datum.*

(1) *The functor $\mathbf{T}_V[1]$ is exact for the perverse t-structures.*

(2) *For any simple perverse sheaf $\mathcal{A} \in \mathcal{P}(\psi, \chi)$ and $V \in \mathrm{Rep}(\widehat{G})$, $\mathbf{T}_V(\mathcal{A})$ is isomorphic to a finite direct sum*

$$\mathbf{T}_V(\mathcal{A}) \cong \bigoplus_{\mathcal{A}'} E(V)_{\mathcal{A},\mathcal{A}'} \boxtimes \mathcal{A}'.$$

Here $\mathcal{A}'$ runs over simple objects in $\mathcal{P}(\psi, \chi)$ and $E(V)_{\mathcal{A},\mathcal{A}'}$ is a semisimple local system on $\mathbb{G}_{m,\bar{k}}$.

(3) *For simple perverse sheaves $\mathcal{A}, \mathcal{A}' \in \mathcal{P}(\psi, \chi)$ and $V \in \mathrm{Rep}(\widehat{G})$ there is a canonical isomorphism*

$$E(V)_{\mathcal{A},\mathcal{A}'} \cong E(V^\vee)^\vee_{\mathcal{A}',\mathcal{A}}. \tag{4.2}$$

(4) *More generally, for any finite set I, $V \in \mathrm{Rep}(\widehat{G}^I)$ and any simple perverse sheaf $\mathcal{A} \in \mathcal{P}(\psi, \chi)$, $\mathbf{T}^I_V(\mathcal{A})$ takes the form*

$$\mathbf{T}^I_V(\mathcal{A}) \cong \bigoplus_{\mathcal{A}'} E^I(V)_{\mathcal{A},\mathcal{A}'} \boxtimes \mathcal{A}'. \tag{4.3}$$

Here $E^I(V)$ is a finite direct sum of local systems on $\mathbb{G}_m^I$ of the form $\boxtimes_{i\in I} E_i$, where E_i are semisimple local systems on $\mathbb{G}_{m,\bar{k}}$.

(5) *Note that μ_e acts on $\widehat{G}$ by pinned automorphisms and on $\widetilde{\mathbb{G}_{m,\bar{k}}}$ by deck transformations. For $\zeta \in \mu_e$, $V \in \mathrm{Rep}(\widehat{G}^I)$, let V^ζ be the representation given by the composition $\widehat{G}^I \xrightarrow{\zeta^I} \widehat{G}^I \to \mathrm{GL}(V)$; let ζ^I also denote the diagonal action of ζ on $\widetilde{\mathbb{G}_{m,\bar{k}}}^I$. Then there is an isomorphism functorial in $\mathcal{A}, \mathcal{A}'$ and V, and compatible with the group structure of μ_e:*

$$(\zeta^I)^* E^I(V)_{\mathcal{A},\mathcal{A}'} \cong E^I(V^\zeta)_{\mathcal{A},\mathcal{A}'}, \quad \forall \zeta \in \mu_e.$$

Proof (1)(2) We first show a weak version of (1): suppose $\mathcal{A}$ is supported on the neutral component, i.e., it is a clean extension from $\mathrm{Bun}_{[1]}$, then $\mathbf{T}_V(\mathcal{A})[1]$ is perverse for any $V \in \mathrm{Rep}(\widehat{G})$. The proof is similar to the argument in [27, third paragraph in the proof of Theorem 3.8], and we only give a sketch. The key point being that the map $\overrightarrow{h} : {}_{[1]}\mathrm{Hk} \to \mathbb{G}_m \times \mathrm{Bun}$ is ind-affine, where ${}_{[1]}\mathrm{Hk} \subset \mathrm{Hk}$ is the preimage of $\mathrm{Bun}_{[1]}$ under $\overleftarrow{h}$. This boils down to the fact that $\mathrm{Bun}_{[1]} \subset \mathrm{Bun}^0$ (the neutral component of Bun) is the non-vanishing locus of a section of a certain determinant line bundle, which pulls back to an ample line bundle on the affine Grassmannian. Therefore ${}_{[1]}\mathrm{Hk} \subset \mathrm{Hk}$ is the nonvanishing locus of a section of a line bundle relative ample with respect to $\overrightarrow{h}$, hence the ind-affineness of $\overrightarrow{h}|_{[1]}\mathrm{Hk}$.

Then we prove a weak version of (2): $\mathbf{T}_V(\mathcal{A}) \cong \oplus_{\mathcal{A}'} E(V)_{\mathcal{A},\mathcal{A}'} \boxtimes \mathcal{A}'$ for some $E(V)_{\mathcal{A},\mathcal{A}'}$ that is a direct sum of *shifted* semisimple local systems on $\mathbb{G}_{m,\bar{k}}$. By the decomposition theorem ($\overleftarrow{h}$ is locally a fibration and $\overrightarrow{h}$ is ind-proper), $\mathbf{T}_V(\mathcal{A})$ is a semisimple complex on $\mathbb{G}_m \times \mathrm{Bun}$. By Lemma 4.1.2, we conclude that $\mathbf{T}_V(\mathcal{A})$ has the required form for semisimple complexes $E(V)_{\mathcal{A},\mathcal{A}'} \in D(\mathbb{G}_{m,\bar{k}})$. The fact that $\mathbf{T}_V(\mathcal{A})$ is locally constant along $\mathbb{G}_m$ follows from the ULA property of both $\overleftarrow{h}^*\mathcal{A}$ and IC_V with respect to the projection to $\mathbb{G}_m$. For details we refer to the argument in [26, Lemma 4.4.6].

Now we prove (1) in general. Note that the same argument for the weaker version may fail for $\mathcal{A}$ supported on other components of Bun: it may happen for some $\omega \in \Omega$ that the boundary of $\mathrm{Bun}_{[\omega]}$ has codimension > 1, so it cannot be the non-vanishing locus of a section of a line bundle.

For $\omega \in \Omega$, let Bun^ω be the corresponding component of Bun (so that $\mathrm{Bun}_{[\omega]} \subset \mathrm{Bun}^\omega$ is open), and $\mathcal{D}_\omega(\psi, \chi)$ be the subcategory of $\mathcal{D}(\psi, \chi)$ supported on Bun^ω. Similarly define $\mathcal{P}_\omega(\psi, \chi)$. Let $\mathcal{B} \in \mathcal{P}_\omega(\psi, \chi)$. Since $\mathbf{T}_V(\mathcal{B})$ is locally constant along $\mathbb{G}_m$, it suffices to fix any geometric point $x \in \mathbb{G}_m$, and check that $\mathbf{T}_V(\mathcal{B})|_{\{x\}\times\mathrm{Bun}}$ is perverse. Let us denote the geometric Hecke operator at x by

$$V\star_x(-) = \mathbf{T}_V(-)|_{\{x\}\times\mathrm{Bun}} : \mathcal{D}(\psi, \chi) \to \mathcal{D}(\psi, \chi).$$

The weak version of (1) that is already proven says that:

$$\text{For any } V \in \mathrm{Rep}(\widehat{G}),\ \ V\star_x(-) \text{ sends } \mathcal{P}_0(\psi, \chi) \text{ to } \mathcal{P}(\psi, \chi). \tag{4.4}$$

Choose $V \in \mathrm{Rep}(\widehat{G})$ whose central character corresponds to $-\omega$ under the canonical isomorphism $\mathbb{X}^*(Z\widehat{G}) \cong \Omega$. Then $\mathcal{K} = V\star_x\mathcal{B} \in \mathcal{D}_0(\psi, \chi)$. Now $\mathcal{B}$ is a direct summand of $(V^\vee \otimes V)\star_x\mathcal{B} = V^\vee\star_x(\mathcal{K})$. By (4.4), $\mathcal{B}$ is a direct summand of $V^\vee\star_x {}^p\mathrm{H}^0\mathcal{K}$, therefore a direct summand of $V^\vee\star_x\mathcal{A}$ for some simple object $\mathcal{A} \in \mathcal{P}_0(\psi, \chi)$. Now for any $V' \in \mathrm{Rep}(\widehat{G})$, $V'\star_x\mathcal{B}$ is a direct summand of $V'\star_x(V^\vee\star_x\mathcal{A}) \cong (V' \otimes V^\vee)\star_x\mathcal{A}$, which is perverse by (4.4).

Finally (1) together with the weak version of (2) implies the full version of (2).

(3) Let σ be the involution on Hk that switches the two G-bundles. Then there is a well-known isomorphism between IC_V and $\sigma^*\mathbb{D}_{\overrightarrow{h}}(\mathrm{IC}_{V^\vee})$ (both on Hk, where $\mathbb{D}_{\overrightarrow{h}}$ denotes the relative Verdier duality with respect to the map $\overrightarrow{h}$). This follows from

the similar statement for the Satake category. From this and standard sheaf-theoretic functor manipulations we obtain an adjunction

$$\underline{\mathrm{Hom}}(\mathbf{T}_{V^\vee}(\mathcal{A}'), \overline{\mathbb{Q}}_\ell \boxtimes \mathcal{A}) \cong \underline{\mathrm{Hom}}(\overline{\mathbb{Q}}_\ell \boxtimes \mathcal{A}', \mathbf{T}_V(\mathcal{A}))$$

as complexes on $\mathbb{G}_m \times \mathrm{Bun}$, functorial in $\mathcal{A}, \mathcal{A}' \in \mathcal{D}(\psi, \chi)$. In view of the decomposition in (2), we get (4.2).

(4) Same argument as above shows that $\mathbf{T}^I_V(\mathcal{A})$ is a semisimple perverse sheaf on $\mathbb{G}_m^I \times \mathrm{Bun}$ locally constant along $\mathbb{G}_m^I$. By Lemma 4.1.2, we can write $\mathbf{T}^I_V(\mathcal{A})$ in the form (4.3), with $E^I(V)_{\mathcal{A},\mathcal{A}'}$ a local system on $\mathbb{G}_m^I$ upon shifting by $|I|$. Now the factorization structure of $\mathbf{T}^I_V$ allows us to conclude that each $E^I(V)_{\mathcal{A},\mathcal{A}'}$ is indeed an external tensor product. For example, take $I = \{1, 2\}$, and $V = V_1 \boxtimes V_2 \in \mathrm{Rep}(\widehat{G}^I)$, then $\mathbf{T}_{V_2}(\mathcal{A}) \cong \oplus E(V_2)_{\mathcal{A},\mathcal{A}'} \boxtimes \mathcal{A}'$. Acting on both sides by $\mathbf{T}_{\mathbb{G}_m, V_1}$ again, the left side becomes $\mathbf{T}^I_V(\mathcal{A})$ by the factorization isomorphism (4.1), and the right side becomes

$$\bigoplus_{\mathcal{A}',\mathcal{A}''} E(V_1)_{\mathcal{A}',\mathcal{A}''} \boxtimes E(V_2)_{\mathcal{A},\mathcal{A}'} \boxtimes \mathcal{A}''$$

with both $\mathcal{A}'$ and $\mathcal{A}''$ running through simple objects in $\mathcal{P}(\psi, \chi)$. We conclude

$$E^{\{1,2\}}(V_1 \boxtimes V_2)_{\mathcal{A},\mathcal{A}''} \cong \bigoplus_{\mathcal{A}'} E(V_1)_{\mathcal{A}',\mathcal{A}''} \boxtimes E(V_2)_{\mathcal{A},\mathcal{A}'}. \tag{4.5}$$

The case of general I follows by an iteration of the same argument.

(5) follows from the $\mathrm{Out}(\mathbb{G}) = \mathrm{Out}(\widehat{G})$-equivariance of the geometric Satake equivalence. See [26, §4.2.3], and [10, Appendix B]. □

4.2 Eigen local systems

Next we will extract ${}^L G$-local systems from the category $\mathcal{P}(\psi, \chi)$.

4.2.1. Let $\mathcal{P}^{ss}(\psi, \chi) \subset \mathcal{P}(\psi, \chi)$ be the subcategory of semisimple objects. Let $\mathrm{Loc}(\widetilde{\mathbb{G}_{m,\bar{k}}})$ be the tensor category of $\overline{\mathbb{Q}}_\ell$-local systems (of finite rank) over $\widetilde{\mathbb{G}_{m,\bar{k}}}$. Let $\mathcal{C}$ be the full subcategory of $\mathrm{Loc}(\widetilde{\mathbb{G}_{m,\bar{k}}})$ consisting of finite direct sums of simple factors of $E(V)_{\mathcal{A},\mathcal{A}'}$ when V runs over $\mathrm{Rep}(\widehat{G})$ and $\mathcal{A}, \mathcal{A}'$ simple objects in $\mathcal{P}(\psi, \chi)$. Since $E(V_1 \otimes V_2)_{\mathcal{A},\mathcal{A}''} \cong \bigoplus_{\mathcal{A}'} E(V_1)_{\mathcal{A}',\mathcal{A}''} \otimes E(V_2)_{\mathcal{A},\mathcal{A}'}$ (restricting (4.5) to the diagonal), we see that $\mathcal{C}$ is stable under tensor product. Proposition 4.1.4(3) shows that $\mathcal{C}$ is closed under duality. Therefore $\mathcal{C}$ is a semisimple rigid tensor subcategory of $\mathrm{Loc}(\widetilde{\mathbb{G}_{m,\bar{k}}})$, hence neutral Tannakian.

By Proposition 4.1.4, $\mathcal{P} = \mathcal{P}^{ss}(\psi, \chi)$ is a factorizable $\mathcal{R} = \mathrm{Rep}(\widehat{G})$-module category with coefficients in $\mathcal{C}$, in the sense of Sect. A.2 Applying Corollary A.4.2 (and Remark A.4.3 for nonsplit G), we get the following result.

4.2.2 Theorem *Let $(\mathbf{P}_\infty, \psi, \chi, \mathbf{Q}_0)$ be a euphotic automorphic datum. Then there are finitely many semisimple μ_e-equivariant $\widehat{G}$-local systems $\{E_\sigma\}_{\sigma\in\Sigma}$ over $\widetilde{\mathbb{G}_{m,\bar{k}}}$ (for*

some index set Σ), and a decomposition

$$\mathcal{P}^{ss}(\psi, \chi) = \bigoplus_{\sigma \in \Sigma} \mathcal{P}_\sigma$$

such that

(1) *Let $\widehat{G}_\sigma = \mathrm{Aut}_{\widehat{G}}(E_\sigma)$ (a reductive group over $\overline{\mathbb{Q}}_\ell$ which can be identified with a subgroup of $\widehat{G}$ up to conjugacy; it is equipped with a μ_e-action). Each $\mathcal{P}_\sigma$ is an E_2-module category under $\mathrm{Rep}(\widehat{G}_\sigma)$ (denote the action of $W \in \mathrm{Rep}(\widehat{G}_\sigma)$ on $\mathcal{A} \in \mathcal{P}_\sigma$ by $W \bullet \mathcal{A}$) with functorial isomorphisms $W \bullet \mathcal{A} \cong W^\zeta \bullet \mathcal{A}$ for $\zeta \in \mu_e$.*
(2) *For any $\mathcal{A} \in \mathcal{P}_\sigma$ and $V \in \mathrm{Rep}(\widehat{G})$, there is an isomorphism functorial in V and $\mathcal{A}$*

$$\mathbf{T}_V(\mathcal{A}) \cong \bigoplus_E E \boxtimes ([E_\sigma(V) : E] \bullet \mathcal{A}) \in \mathcal{D}(\mathbb{G}_m, \psi, \chi)$$

Here the direct sum is over all irreducible local systems E over $\widetilde{\mathbb{G}_{m,\bar{k}}}$, $E_\sigma(V) \in \mathrm{Loc}(\widetilde{\mathbb{G}_{m,\bar{k}}})$ is the (semisimple) local system on $\widetilde{\mathbb{G}_{m,\bar{k}}}$ associated to E_σ and V, and $[E_\sigma(V) : E]$ is the multiplicity space of E in $E_\sigma(V)$, viewed as an object in $\mathrm{Rep}(\widehat{G}_\sigma)$.
Moreover, there is a version of the above isomorphism for any finite set I and $V \in \mathrm{Rep}(\widehat{G}^I)$, and these isomorphisms are compatible with the factorization structures.

We recall that a $\widehat{G}$-local system on $\widetilde{\mathbb{G}_{m,\bar{k}}}$ is called semisimple if the Zariski closure of the image of $\pi_1(\widetilde{\mathbb{G}_{m,\bar{k}}})$ in $\widehat{G}$ is reductive.

4.2.3 Descent to *k*

After a finite extension of k, we may assume that each simple object $\mathcal{A}$ in $\mathcal{P}(\psi, \chi)$ has the property $\mathrm{Fr}^*\mathcal{A} \cong \mathcal{A}$, where $\mathrm{Fr} : \mathrm{Bun} \to \mathrm{Bun}$ is the Frobenius morphism with respect to k. In particular, each summand $\mathcal{P}_\sigma$ in Theorem 4.2.2 is stable under Fr^*. Fix a Weil structure $\mathrm{Fr}^*\mathcal{A} \cong \mathcal{A}$ for each simple $\mathcal{A} \in \mathcal{P}(\psi, \chi)$. Applying Remark A.4.3 to the $\Gamma = \mathrm{Fr}^{\mathbb{Z}}$-equivariant structure on the factorizable $\mathrm{Rep}(\widehat{G})$-module structure on $\mathcal{P}^{ss}(\psi, \chi)$ with coefficients in $\mathcal{C}$ (where Fr acts trivially on $\mathrm{Rep}(\widehat{G})$ and on $\mathcal{P}^{ss}(\psi, \chi)$ by the fixed Weil structures, and it acts by Fr^* on $\mathcal{C}$), we conclude that each E_σ viewed as a tensor functor $\mathrm{Rep}(\widehat{G}) \to \mathcal{C}$ carries a Fr-equivariant structure. In other words, E_σ carries a Weil structure (depending on the choice of Weil structures of simple objects in $\mathcal{P}(\psi, \chi)$).

4.2.4 Remark (1) A μ_e-equivariant $\widehat{G}$-local system on $\widetilde{\mathbb{G}_{m,\bar{k}}}$ is the same thing as a ${}^L G = \widehat{G} \rtimes \mu_e$-local system on $\mathbb{G}_{m,\bar{k}}$, such that the induced μ_e-cover of $\mathbb{G}_{m,\bar{k}}$ is $\widetilde{\mathbb{G}_{m,\bar{k}}}$. Therefore, after choosing a base point $x \in \mathbb{G}_{m,\bar{k}}$, the E_σ in the above theorem is the same data as continuous homomorphisms $\rho_\sigma : \pi_1(\mathbb{G}_{m,\bar{k}}, x) \to {}^L G(\overline{\mathbb{Q}}_\ell)$.

The discussion in Sect. 4.2.3 gives an extension of ρ_σ to the Weil group of $\mathbb{G}_m$ so it is a Langlands parameter in the usual sense for the quasi-split group G over $F = k(t)$.

(2) The upshot of the above theorem is that, we not only can extract a ${}^L G$-local system E_σ from each indecomposable summand $\mathcal{P}_\sigma$ of the category $\mathcal{P}^{ss}(\psi, \chi)$, but there is a *residual action* of $\mathrm{Rep}(\mathrm{Aut}(E_\sigma))$ on $\mathcal{P}_\sigma$. The $\mathrm{Rep}(\mathrm{Aut}(E_\sigma))$-module category $\mathcal{P}_\sigma$ may be viewed as a secondary invariant attached to the automorphic datum in question that is not covered by the usual Langlands parameter E_σ. The relationship between this secondary invariant and global L-packets deserves further study. A closely related phenomenon is discussed in [6] under the name *fractional Hecke eigensheaves*.

(3) The number of simple objects in $\mathcal{P}(\psi, \chi)$ can be large in some examples. We will see in an example for $G = \mathrm{Sp}_{2n}$ (see Sect. 6.4, case (1)) that $\mathcal{P}(\psi, \chi)$ has 2^n simple objects all supported on the open B_ψ-orbit of L/Q. It remains unclear how $\mathcal{P}^{ss}(\psi, \chi)$ decomposes into indecomposables in this case.

(4) The above theorem (or rather Corollary A.4.2) improves [26, Theorem 4.4.2]. In *loc.cit*, we consider the situation where the relevant category of perverse sheaves $\mathcal{P}$ has a unique simple object on each connected component of Bun (indexed by Ω, and $\Omega \cong \mathbb{X}^*(Z\widehat{G})$ when G is split). In this case, $\mathcal{P}$ is necessarily indecomposable because the Hecke operators will mix up the components transitively. Corollary A.4.2 then gives a $\widehat{G}$-local system E (rather than a weak $\widehat{G}$-local system in the sense of [26, Def. 4.3.2]), together with an E_2-action of $\mathrm{Rep}(\mathrm{Aut}(E))$ on $\mathcal{P}$. Since $Z\widehat{G} \subset \mathrm{Aut}(E)$, $\mathrm{Rep}(\mathrm{Aut}(E))$ is graded by $\mathbb{X}^*(Z\widehat{G})$, and this grading is compatible with the decomposition of $\mathcal{P}$ according the connected components of Bun.

(5) In practice, to calculate these local systems, we need to calculate the local systems $E(V)_{\mathcal{A},\mathcal{A}'}$ that appear in Proposition 4.1.4. These can be calculated in the same way as described in [26], as part of the derived direct image of a family of varieties over $\mathbb{G}_m$ closely related to the Beilinson-Drinfeld affine Grassmannian.

4.2.5 Corollary *Assume that G is split and simply-connected. Let $(\mathbf{P}_\infty, \psi, \chi, \mathbf{Q}_0)$ be a strict euphotic automorphic datum. Denote by ZG^* the set of characters $\sigma : ZG(k) \to \overline{\mathbb{Q}}_\ell^\times$. Then there is a decomposition*

$$\mathcal{P}(\psi, \chi) = \bigoplus_{\sigma \in ZG^*} \mathcal{P}_\sigma$$

such that each $\mathcal{P}_\sigma$ contains a unique simple perverse sheaf $\mathcal{A}_\sigma$ which is a Hecke eigensheaf with semisimple eigen $\widehat{G}$-local system E_σ.

Proof Denote by O the open B_ψ-orbit of L/Q. Using strictness the proof of Lemma 4.1.2 implies an equivalence of categories

$$\mathcal{D}(\psi, \chi) \xrightarrow{\cong} D_{(B_\psi, \mathcal{K}_\chi)}(L/Q) \xrightarrow{\cong} D_{(B_\psi, \mathcal{K}_\chi)}(O).$$

Fixing a point $y \in O(\overline{k})$ identifies $\mathcal{P}(\psi, \chi)$ with $\mathrm{Rep}(ZG, \overline{\mathbb{Q}}_\ell)$ and hence we obtain the desired decomposition and uniqueness of the simple perverse sheaf $\mathcal{A}_\sigma$. More

explicitly $\mathcal{A}_\sigma = j_!\mathcal{F}_\sigma$ is the perverse sheaf whose restriction to O corresponds to the character σ. For any $\mathcal{H} \in \mathcal{D}(S, \psi, \chi)$ we may decompose its restriction to $S \times y$ as follows

$$(\mathrm{id}_S \times y)^*\mathcal{H} \cong \bigoplus_{\sigma \in ZG^*} (\mathrm{id}_S \times y)^*\mathcal{H}_\sigma,$$

according to the action of ZG. For $\mathcal{H} \in \mathcal{P}(S, \psi, \chi)$ (or any shifted perverse sheaf) we obtain a decomposition

$$\mathcal{H} \cong \bigoplus_{\sigma \in ZG^*} (\sigma^{-1} \otimes (\mathrm{id}_S \times y)^*\mathcal{H}_\sigma)^{ZG} \boxtimes j_!\mathcal{F}_\sigma,$$

cf. [26]. Note that the automorphism group of any point of $\mathrm{Bun}_G(\mathbf{Q}_0, B_\psi \mathbf{P}^+_\infty)$ contains ZG. Therefore we may speak of the subcategory $\mathcal{D}(\psi, \chi)_\sigma$ on which ZG acts through σ. By [26, §4.4.1] the geometric Hecke operator $\mathbf{T}_V$ sends $\mathcal{D}(\psi, \chi)_\sigma$ to $\mathcal{D}(\mathbb{G}_m, \psi, \chi)_\sigma$ and by Proposition 4.1.4 $\mathbf{T}_V(\mathcal{A}_\sigma)[1]$ is perverse. Therefore we have

$$\mathbf{T}_V(\mathcal{A}_\sigma) = \mathbf{T}_V(\mathcal{A}_\sigma)_\sigma \cong (\sigma^{-1} \otimes (\mathrm{id}_S \times y)^*\mathbf{T}_V(\mathcal{A}_\sigma))^{ZG} \boxtimes j_!\mathcal{F}_\sigma$$

using the decomposition above. Again by Proposition 4.1.4 $\mathbf{T}_V(\mathcal{A}_\sigma)$ is locally constant along $\mathbb{G}_m$ and the claim follows. □

4.3 Local monodromy and rigidity

In this subsection we assume that G is split, so that ${}^L G = \widehat{G}$. Let $(\mathbf{P}_\infty, \psi, \chi, \mathbf{Q}_0)$ be a euphotic automorphic datum. By Theorem 4.2.2, we have a $\widehat{G}$-local system E_σ over $\mathbb{G}_{m,\bar{k}}$ for each indecomposable summand $\mathcal{P}_\sigma$ of $\mathcal{P}^{ss}(\psi, \chi)$. Let $\rho_\sigma : \pi_1(\mathbb{G}_{m,\bar{k}}, \overline{\eta}) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ be the geometric monodromy representation attached to E_σ.

4.3.1 Lusztig's bijection

To describe the local monodromy of E_σ at 0, recall Lusztig's bijection [17, Theorem 4.8]:

$$\{\text{two-sided cells in } W_{\mathrm{aff}}\} \longleftrightarrow \{\text{unipotent classes in } \widehat{G}\} \tag{4.6}$$

4.3.2 Proposition *For any local system E_σ attached to the euphotic automorphic datum*

$$(\mathbf{P}_\infty, \psi, \chi, \mathbf{Q}_0),$$

the local monodromy $\rho_\sigma|_{I_{F_0}}$ (I_{F_0} is the inertia group at 0) is tame, and maps a topological generator of $I^{\mathrm{tame}}_{F_0}$ into the unipotent class of $\widehat{G}$ which corresponds to the two-sided cell $c_{\mathbf{Q}}$ of W_{aff} containing the longest element of $W_{\mathbf{Q}}$ under Lusztig's bijection (4.6).

Proof It suffices to prove the case where G is simply-connected. The proof is almost the same as in the epipelagic case, cf. [27, §4.11–4.18], replacing $\mathbf{P}_0$ in *loc.cit.* by $\mathbf{Q}_0$. The only thing that needs to be adapted in our situation is [27, Lemma 4.12]. Here the analogous statement should be: consider the action of $D_{\mathbf{Q}_0}(G(\!(t)\!)/\mathbf{Q}_0)$ on $\mathcal{D}(\psi, \chi)$, then any perverse sheaf $\mathcal{K} \in \mathrm{Perv}_{\mathbf{Q}_0}(G(\!(t)\!)/\mathbf{Q}_0)$ acts as a t-exact endo-functor of $\mathcal{D}(\psi, \chi)$. Consider the Hecke correspondence Hk_0 that classifies modifications of Bun at 0, with two maps $\overleftarrow{h}, \overrightarrow{h} : \mathrm{Hk}_0 \to \mathrm{Bun}$. Let $\mathrm{Hk}_0' = \overleftarrow{h}^{-1}(\mathrm{Bun}_{[1]}) \cap \overrightarrow{h}^{-1}(\mathrm{Bun}_{[1]})$. Since $\mathrm{Bun}_{[1]}$ is the preimage of the open stratum $[\mathrm{pt}/L] \subset \mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty)$, we may identify fibers of $\overrightarrow{h}' = \overrightarrow{h}|\mathrm{Hk}_0' : \mathrm{Hk}_0' \to \mathrm{Bun}_{[1]}$ with $\mathrm{Aut}_{X-\{0\}}(\mathcal{E}_{\mathbf{P}_0,\mathbf{P}_\infty})/L$ where $\mathcal{E}_{\mathbf{P}_0,\mathbf{P}_\infty}$ is the open point in $\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty)$ with automorphism L. Equivalently, we may identify $\mathrm{Aut}_{X-\{0\}}(\mathcal{E}_{\mathbf{P}_0,\mathbf{P}_\infty})/L$ with $\mathrm{Aut}_{X-\{0\}}(\mathcal{E}_{\mathbf{P}_0,\mathbf{P}_\infty^+})$ by choosing a $\mathbf{P}_\infty^+$-reduction of $\mathcal{E}_{\mathbf{P}_0,\mathbf{P}_\infty^+}$. From this we see that the fibers of $\overrightarrow{h}'$ are ind-affine. Since $\overrightarrow{h}'$ is a Zariski locally trivial fibration, it is ind-affine. □

4.3.3 Proposition *Under the assumptions in the beginning of Sect. 4.3, assume further*

- *The restriction $\rho_\sigma|_{I_{F_\infty}}$ is as predicted in Sect. 2.4.2.*
- *The image of ρ_σ does not lie in any proper Levi subgroup of $\widehat{G}$ (equivalently, $\mathrm{Aut}(E_\sigma)$ is finite).*

Then E_σ is cohomologically rigid in the sense that

$$\mathrm{H}^*(X_{\bar{k}}, j_{!*}\mathrm{Ad}(E_\sigma)) = 0.$$

Here $j : \mathbb{G}_m \hookrightarrow X$ is the open embedding, and $\mathrm{Ad}(E_\sigma)$ is the adjoint local system attached to E_σ.

Proof We have the exact sequence

$$\begin{aligned} 0 \to \widehat{\mathfrak{g}}^{\rho_\sigma(\pi_1(\mathbb{G}_{m,\bar{k}}))} &\to \widehat{\mathfrak{g}}^{\rho_\sigma(I_{F_0})} \oplus \widehat{\mathfrak{g}}^{\rho_\sigma(I_{F_\infty})} \\ &\to \mathrm{H}^1_c(\mathbb{G}_{m,\bar{k}}, \mathrm{Ad}(E_\sigma)) \to \mathrm{H}^1(X_{\bar{k}}, j_{!*}\mathrm{Ad}(E_\sigma)) \to 0. \end{aligned}$$

So the cohomological rigidity condition boils down to

$$\dim \mathrm{H}^1_c(\mathbb{G}_{m,\bar{k}}, \mathrm{Ad}(E_\sigma)) - \dim \widehat{\mathfrak{g}}^{\rho_\sigma(I_{F_0})} - \dim \widehat{\mathfrak{g}}^{\rho_\sigma(I_{F_\infty})} + \widehat{\mathfrak{g}}^{\rho_\sigma(\pi_1(\mathbb{G}_{m,\bar{k}}))} = 0. \quad (4.7)$$

By the second assumption, $\dim \widehat{\mathfrak{g}}^{\rho_\sigma(\pi_1(\mathbb{G}_{m,\bar{k}}))} = 0$. Let $u \in \widehat{G}$ be a unipotent element in the class corresponding to the longest element $w_{\mathbf{Q},0}$ of $W_{\mathbf{Q}}$. By Proposition 4.3.2, $\dim \widehat{\mathfrak{g}}^{\rho_\sigma(I_{F_0})} = \dim \widehat{G}_u = \mathrm{rk}\widehat{G} + 2\dim \mathcal{B}_u$ ($\mathcal{B}_u$ is the Springer fiber of u). By [27, Lemma 4.6], we have $\dim \mathcal{B}_u = \ell(w_{\mathbf{Q},0})$. Hence

$$\dim \widehat{\mathfrak{g}}^{\rho_\sigma(I_{F_0})} = \mathrm{rk}\widehat{G} + 2\ell(w_{\mathbf{Q},0}) = \dim L_{\mathbf{Q}}.$$

By (2.2), $\dim \widehat{\mathfrak{g}}^{\rho_\sigma(I_{F_\infty})} = \dim T_\psi$.

By the Grothendieck-Ogg-Shafarevich formula

$$\dim \mathrm{H}^1_c(\mathbb{G}_{m,\bar{k}}, \mathrm{Ad}(E_\sigma)) = \mathrm{Swan}(\mathrm{Ad}(E_\sigma)) = \dim L - \dim L_\psi$$

as predicted in Sect. 2.4.2. Using these calculations, (4.7) is equivalent to

$$\dim L - \dim L_{\mathbf{Q}} - \dim T_\psi = \dim L_\psi.$$

Since $\dim B_\psi = (\dim T_\psi + \dim L_\psi)/2$, $\dim L - \dim L_{\mathbf{Q}} = 2\dim L/Q$, the above identity is equivalent to

$$\dim L/Q = \dim B_\psi,$$

which is guaranteed by the condition (1) in Definition 3.4.1. □

5 An example in type G_2

From this section on, we will give several families of examples of (strict) euphotic automorphic data.

5.1 The rigid connection from [11]

The motivating example for our construction of rigid automorphic data is a certain rigid irregular G_2-connection discovered by the first-named author. By [11, Theorem 1.1.] there is a rigid irregular connection $\mathcal{E}$ on $\mathbb{G}_{m,\mathbb{C}}$ with differential Galois group G_2 and with the following local data. At $z = 0$ the connection is regular singular and has subregular unipotent monodromy. On the punctured formal disc D° at $z = \infty$ the connection $\mathcal{E}$ is isomorphic to

$$\mathrm{El}(z^2, \alpha, (\lambda, \lambda^{-1})) \oplus \mathrm{El}(z^2, 2\alpha, 1) \oplus (-1)$$

where by (λ, λ^{-1}) we denote a regular singular formal connection of rank two with monodromy λ and λ^{-1} and similarly for (-1). Further, $\mathrm{El}(z^2, \alpha, (\lambda, \lambda^{-1}))$ is an elementary formal connection in the sense of [22, §2]. It is the direct image of a formal exponential connection twisted by a regular singular connection along a twofold covering of the formal disc.

The local data of this connection dictates our guess for the local representations in Sect. 3.2.1. The parahoric subgroup $\mathbf{Q}_0$ should correspond to the unipotent conjugacy class of the connection at $z = 0$ as in Sect. 4.3, (1).

The choice of the character ψ corresponds to the occurence of the formal exponential connection, an additive parameter, and the character χ reflects the multiplicative parameter at $z = \infty$ of the formal connection, given by the regular singular connection (λ, λ^{-1}). Note that in addition the formal connection at $z = \infty$ becomes diagonalisable after pullback to a two-fold cover.

5.2 Constructing the automorphic form

Assume G is split of type G_2 and denote by $\Delta = \{\alpha_1, \alpha_2\}$ the simple roots of G_2 where α_1 is the long root. Consider the parahoric $\mathbf{P}_\infty$ with $L \cong \mathrm{SO}_4$ with roots α_2 and the highest root η. We have $V = \mathrm{Sym}^3(\mathrm{St}) \otimes \mathrm{St}'$, where St is the standard representation of the short root $\mathrm{SL}_2 \hookrightarrow \mathrm{SO}_4$, and St' is the standard representation of the long root SL_2. In this case $m = 2$ and $V \cong V^*$. We may identify V^* with the space of bihomogeneous polynomials in two sets of variables (x, y) and (u, v) that are cubic in (x, y) and linear in (u, v). Then take $\psi = x^3u + y^3v$. We have $L_\psi \cong \mathbb{G}_m \rtimes \mu_2$: the projection $L_\psi \to \mathrm{PGL}_2$ to the short root factor is an isomorphism onto the normalizer of a maximal torus A in PGL_2; the other projection $L_\psi \to \mathrm{PGL}_2$ onto the long root factor has image $N_{\mathrm{PGL}_2}(A)$ with kernel μ_3. We then have $T_\psi = B_\psi = \mathbb{G}_m \subset L_\psi$ (with index 2) acting as $t \cdot (x, y, u, v) = (tx, t^{-1}y, t^{-3}u, t^3v)$. Take $Q \subset L$ such that $L/Q \cong \mathbb{P}^1$ is the flag variety of the short root factor. The choice of Q determines the parahoric $\mathbf{Q}_0$. Then T_ψ acts on L/Q with an open free orbit.

5.2.1 Proposition *Let $\chi : T_\psi(k) \to \overline{\mathbb{Q}}_\ell^\times$ be a non-trivial character. Then the automorphic datum $(\mathbf{P}_\infty, \psi, \chi, \mathbf{Q}_0)$ is euphotic and strict.*

Proof Note that for dimension reasons any point in L/Q outside the open B_ψ-orbit will have a positive dimensional stabilizer. Therefore since L_ψ is a torus this immediately implies that $(\mathbf{P}_\infty, \psi, \chi, \mathbf{Q}_0)$ is strict if it is euphotic.

For $w \neq \mathrm{id}$ we will prove that if $Y_w \neq \emptyset$ then Y_w is finite. The first step is to single out the cases in which Y_w is empty.

Suppose w is given such that all weights of $V_w^\perp$ lie in a half-space in $\mathbb{X}^*(A)_\mathbb{R}$ not containing 0. For ${}^{\ell^{-1}}\psi \in V_w^\perp$ we can then find a torus T' such that $0 \in \overline{T'.{}^{\ell^{-1}}\psi}$ and since the orbit of ψ is closed this implies $\psi = 0$, a contradiction. In this case we therefore get $Y_w = \emptyset$.

Let $\beta = \alpha_1 + 3\alpha_2$ and suppose w is given such that $-\alpha_1$ and β (resp. α_1 and $-\beta$) are not weights of $V_w^\perp$. In this case every $v \in V_w^\perp$ is a reducible polynomial contradicting the irreducibility of ψ. Again this implies $Y_w = \emptyset$.

These observations determine a region $\mathcal{U} \subset \mathbb{X}_*(A)_\mathbb{R}$ such that $Y_w = \emptyset$ if $wx_\mathbf{Q} \notin \mathcal{U}$ in the following way. Recall that for example $-\beta$ is a weight of $V_w^\perp$ if and only if

$$\langle -\beta, wx_\mathbf{Q} - x_\mathbf{P} \rangle < \frac{1}{2}.$$

This is equivalent to

$$\langle -\beta, wx_\mathbf{Q} - x_\mathbf{P} \rangle \leq \frac{1}{2} - \frac{1}{5}$$

and noting that $\beta(x_\mathbf{P}) = 1/2$ this is furthermore equivalent to

$$\langle \beta, wx_\mathbf{Q} \rangle \geq \frac{1}{5}.$$

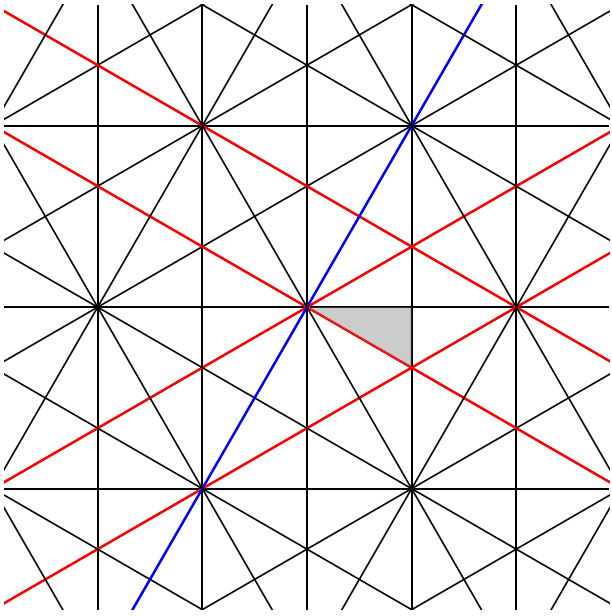


Fig. 1 Affine root system of type G_2 with the region $\mathcal{U}$, the line $\alpha_1 + 2\alpha_2 = 0$ and the fundamental alcove

Combining all cases in which $Y_w = \emptyset$ we find that if $Y_w \neq \emptyset$ then $wx_{\mathbf{Q}} \in \mathcal{U}$ with

$$\mathcal{U} = \{x \in \mathbb{X}_*(A)_{\mathbb{Q}} \mid 0 < \langle \alpha_1, x \rangle < 1 \text{ or } 0 < \langle \beta, x \rangle < 1\},$$

a union of two strips in the plane (Fig. 1).

It remains to prove that Y_w is finite whenever $wx_{\mathbf{Q}} \in \mathcal{U}$. By W_L-symmetry it suffices to consider just one quadrant, e.g. the one defined by $\langle \eta, x \rangle > 0$ and $\langle \alpha_2, x \rangle > 0$. This leaves us with the case where $wx_{\mathbf{Q}}$ lies in the above quadrant and additionally satisifes $\langle \alpha_1 + 2\alpha_2, wx_{\mathbf{Q}} \rangle < 0$ and the case where $w = s_{\alpha_1+\alpha_2}$ is a simple reflection across the hyperplane perpendicular to $\alpha_1 + \alpha_2$. Note that $Y_{w'} \subset Y_w$ whenever w' is in the first case and $w = s_{\alpha_1+\alpha_2}$, so actually it suffices to prove that Y_w is finite in the second case.

In this case we have that $L/Q_w = \mathbb{P}^1 \times \mathbb{P}^1$ and V_w^* is the direct sum of weight spaces corresponding to

$$\{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \beta, -(\alpha_1 + 2\alpha_2), -\beta\}.$$

Let $C \subset \mathbb{P}^1_{(x:y)} \times \mathbb{P}^1_{(u:v)} = L/Q_w$ be the curve defined by $\psi = x^3u + y^3v = 0$. If ${}^{\ell^{-1}}\psi \in V_w^*$ then $\psi(\ell \mod Q_w) = 0$. Moreover in this case the projection $\pi : C \to \mathbb{P}^1$ onto the second factor (which is a finite map of degree 3) is ramified in $\ell \mod Q_w$. Thus Y_w is contained in the ramification locus of π which is finite. □

5.2.2 Corollary *There is a unique cuspidal automorphic representation satisfying*

(1) π_x is unramified for $x \neq 0, \infty$;

(2) $\pi_0^{\mathbf{Q}_0} \neq 0$*;*
(3) π_∞ *is euphotic with respect to* $(\mathbf{P}_\infty, \psi, \chi)$.

In addition $\dim \pi_0^{\mathbf{Q}_0} = \dim \pi_\infty^{(\mathbf{P}_\infty^+, \psi)} = 1$ *and* π *appears with multiplicity one in the automorphic spectrum of* G.

Proof This is immediate from the proof of Proposition 3.4.3. In this case the space of functions $\mathcal{F}$ is one-dimensional and the statement follows. □

Corollary 4.2.5 implies the following geometric version of the above statement.

5.2.3 Corollary *There is a Hecke eigensheaf* A_π *on* $\mathrm{Bun}_{G_2}(\mathbf{Q}_0, \mathbf{P}_\infty^{++})$ *with semisimple eigen* G_2*-local system* E_π*. Under the assumptions in Sect. 4.3 the local system* E_π *is cohomologically rigid.*

6 The hyperspecial cases

6.1 The setup

6.1.1. In this section and the next, we assume that G over F is split and simply-connected. We consider the special case where $\mathbf{P}_\infty = G(\mathcal{O}_\infty)$. The reductive quotient of $\mathbf{P}_\infty$ over k is $\mathbb{G}$; by abuse of notation we will also denote $\mathbb{G}$ by G, $\mathbb{T}$ by T, etc.

In this case, the grading on $\mathfrak{g}$ is trivial, and $\psi \in \mathfrak{g} = \mathfrak{g}(-1)$. Extending k if necessary, we may assume $\psi \in \mathrm{Lie}\, T$. Then $L_\psi = G_\psi$ is a Levi subgroup of G. We will use P_ψ to denote a parabolic subgroup of G containing G_ψ as a Levi subgroup. Note that only the associate class of P_ψ is well-defined.

Recall that Q denotes another parabolic subgroup of G, the level at 0, chosen in such a way that any Borel subgroup $B_\psi \subset G_\psi$ acts on the partial flag variety G/Q with an open almost free orbit. This is equivalent to requiring that $G/P_\psi \times G/Q$ is a spherical G-variety and

$$\dim G_\psi + \dim L_Q = \#\Phi_G \tag{6.1}$$

where Φ_G is the set of roots of G and L_Q is the Levi quotient of Q.

6.1.2. Stembridge [24] has classified pairs of parabolic subgroups (P_ψ, Q) such that $G/P_\psi \times G/Q$ is G-spherical. In type A and C this was preceded by work of Magyar–Weymann–Zelevinsky, see [18, 19]. In this classification, the following are the ones that satisfy the dimension equality (6.1). There are no examples of exceptional types.

6.1.3 Notation

In the sequel we will concentrate on the case where G is one of the groups $\mathrm{SL}(V)$, $\mathrm{Spin}(V)$ or $\mathrm{Sp}(V)$, for some finite-dimensional vector space V over k ($\mathrm{char}(k) \neq 2$) equipped with a quadratic form in the case $G = \mathrm{Spin}(V)$ or a symplectic form in the case $G = \mathrm{Sp}(V)$.

For $d \geq 1$, write $P_d \subset G$ for the stabilizer of a d-dimensional subspace, isotropic in the case outside type A. Similarly, for $1 \leq d < d'$, let $P_{d,d'}$ denote the stabilizer of a d-dimensional subspace inside a d'-dimensional subspace, both being isotropic outside type A.

For parabolic subgroups P' and P'' of G, we write $(P_\psi, Q) \sim (P', P'')$ to denote that P_ψ is conjugate to P' and Q is conjugate to P''.

Below we often base change to $\overline{k}$ without changing the notation.

6.2 Type A_{n-1}, $n \geq 2$, [18, Theorem 2.4]

Let $\mathbb{G} = \mathrm{SL}_n$ and let G be the split form of $\mathbb{G}$ over F. Let λ_Q be the partition of n determined by the blocks of a Levi factor of Q. Let λ_ψ be the partition of n determined by the multiplicities of the eigenvalues of ψ (for example, $\lambda_\psi = (223)$ means that ψ has three distinct eigenvalues, with multiplicities 2, 2 and 3). The following are the only cases where B_ψ has an open orbit on L/Q with finite stabilizers.

(1) $n \geq 2, \lambda_Q = (1, n-1), \lambda_\psi = (1^n)$;
(2) $n \geq 2, \lambda_Q = (1^n), \lambda_\psi = (1, n-1)$;
(3) $n = 2m, m \geq 2, \lambda_Q = (m, m), \lambda_\psi = (m, m-1, 1)$;
(4) $n = 2m, m \geq 2, \lambda_Q = (m, m-1, 1), \lambda_\psi = (m, m)$;
(5) $n = 2m+1, m \geq 2, \lambda_Q = (m+1, m), \lambda_\psi = (m, m, 1)$;
(6) $n = 2m+1, m \geq 2, \lambda_Q = (m, m, 1), \lambda_\psi = (m+1, m)$;
(7) $n = 6, \lambda_Q = (2, 2, 2), \lambda_\psi = (4, 2)$;
(8) $n = 6, \lambda_Q = (4, 2), \lambda_\psi = (2, 2, 2)$.

6.2.1 Remark Any tame rigid local system on $\mathbb{P}^1 - \{0, 1, \infty\}$ is determined by the collection of conjugacy classes of its local monodromies around the punctures. For tame rigid local systems on $\mathbb{P}^1 - \{0, 1, \infty\}$ of rank n with generic semisimple regular monodromy at one puncture Simpson classifies the possible Jordan types of local monodromies in [23, Theorem 4]. They are in canonical bijection with the above list (up to interchanging λ_Q and λ_ψ) in the sense that the collections $(\lambda_Q, \lambda_\psi, (1^n))$ exhaust Simpson's list.

6.3 Type B_n, $n \geq 2$, [24, Corollary 1.3.B.]

Let $\mathbb{G} = \mathrm{Spin}(2n+1)$ and let G be the split form of $\mathbb{G}$ over F. The action of B_ψ on G/Q has an open orbit with finite stabilizers if we have one of the following.

(1) Any n, $(P_\psi, Q) \sim (P_n, P_n)$ (Siegel parabolic);
(2) $n = 2$, $(P_\psi, Q) \sim (P_1, P_2)$;
(3) $n = 2$, $(P_\psi, Q) \sim (P_2, P_1)$;
(4) $n = 3$, $(P_\psi, Q) \sim (P_1, P_2)$;
(5) $n = 3$, $(P_\psi, Q) \sim (P_2, P_1)$.

6.4 Type C_n, $n \geq 3$, [24, Corollary 1.3.C.]

Let $\mathbb{G} = \mathrm{Sp}(2n)$ and let G be the split form of $\mathbb{G}$ over F. The action of B_ψ on G/Q has an open orbit with finite stabilizers if we have one of the following.

(1) Any n, $(P_\psi, Q) \sim (P_n, P_n)$ (Siegel parabolic);
(2) $n = 3$, $(P_\psi, Q) \sim (P_1, P_2)$;
(3) $n = 3$, $(P_\psi, Q) \sim (P_2, P_1)$.

6.5 Type D_n, [24, Corollary 1.3.D.]

Let $\mathbb{G} = \mathrm{Spin}(2n)$ and let G be the split form of $\mathbb{G}$ over F. Note that there are two conjugacy classes of n-dimensional isotropic subspaces (permuted by $O(2n)$) whose stabilizers we simply denote by P_n (two conjugacy classes of maximal parabolics of G). The action of B_ψ on G/Q has an open orbit with finite stabilizers if we have one of the following.

(1) $n = 4$, $(P_\psi, Q) \sim (P_4, P_{1,2})$, or anything in the same orbit under the outer automorphisms of G;
(2) $n = 4$, $(P_\psi, Q) \sim (P_{1,2}, P_4)$, or anything in the same orbit under the outer automorphisms of G;
(3) $n = 5$, $(P_\psi, Q) \sim (P_5, P_3)$;
(4) $n = 5$, $(P_\psi, Q) \sim (P_3, P_5)$;
(5) $n = 6$, $(P_\psi, Q) \sim (P_6, P_3)$;
(6) $n = 6$, $(P_\psi, Q) \sim (P_3, P_6)$.

6.5.1 Theorem *Assume ψ and Q are in any of the above cases. Recall that $\psi \in \mathrm{Lie}\, T$ so that $T_\psi = T$.*

(1) *Assume that $\chi : T(k) \to \overline{\mathbb{Q}}_\ell^\times$ is non-trivial on the connected center of any maximal Levi subgroup of G containing T. Then the automorphic datum $(G(\mathcal{O}_\infty), \psi, \chi, \mathbf{Q}_0)$ is euphotic.*
(2) *All cases in type A and D with χ as in (1) are strict euphotic automorphic data. In type B and C no case is strict.*

The proof of Theorem 6.5.1 is carried out in the following section.

6.5.2 Remark The condition on χ may be described more explicitly in each type.

Type A_n. The character χ on the diagonal torus T of $\mathrm{SL}(V)$ is given by a collection of characters $\chi_1, \ldots, \chi_n$ of $k^\times$ modulo simultaneous multiplication by the same character of $k^\times$. Any maximal Levi subgroup is isomorphic to $S(\mathrm{GL}_a \times \mathrm{GL}_b)$ with $a+b = n$ and $a, b > 0$. Its center is the subtorus given by the image of the embedding $\mathbb{G}_m \hookrightarrow T$, $z \mapsto (z^{b'}, \ldots, z^{b'}, z^{-a'}, \ldots, z^{-a'})$, where $a' = a/\gcd(a,b)$, $b' = b/\gcd(a,b)$, and $z^{b'}$ appears a times, $z^{-a'}$ appears b times. We therefore require for any non-empty subset $I \subset \{1, \ldots, n\}$ of cardinality a and with non-empty complement J of cardinality b that

$$\left(\prod_{i \in I} \chi_i\right)^{b/\gcd(a,b)} \neq \left(\prod_{j \in J} \chi_j\right)^{a/\gcd(a,b)}.$$

Types B_n, C_n, D_n. Identify $T \cong \mathbb{G}_m^n$ in the usual way, and write $\chi = (\chi_1, \dots, \chi_n)$. The maximal Levi subgroups are of the form

$$\mathrm{GL}_a \times G'$$

where G' is a classical group of rank $n - a$ of the same type as G (in the case of type D_n, $a \neq n - 1$). The connected centers of maximal Levi subgroups are the images of maps $\mathbb{G}_m \to \mathbb{G}_m^n$, $z \mapsto (\varphi_1(z), \dots, \varphi_n(z))$ where $\varphi_i(z)$ is either 1 or z or z^{-1}. Therefore the condition on χ is that, for any disjoint subsets $I \coprod J \subset \{1, 2, \dots, n\}$ such that $I \cup J \neq \varnothing$, we have

$$\prod_{i \in I} \chi_i \neq \prod_{j \in J} \chi_j.$$

Here, when I or J is empty, the corresponding product is 1.

6.6 Stabilizers on Hessenberg varieties

Recall that $V_w^{\perp} = \bigoplus_{\langle \alpha, w x_{\mathbf{Q}} \rangle < 1} \mathfrak{g}_\alpha$ which we will denote by $\mathfrak{g}_w$. Then

$$Y_w = \{g Q_w \in G/Q_w \mid \psi \in {}^g\mathfrak{g}_w\}.$$

By definition these spaces are Hessenberg varieties as defined for example in [3]. The subvector space $\mathfrak{g}_w$ of $\mathfrak{g}$ is automatically a Hessenberg space, i.e. it is stable under the adjoint action of the parabolic subgroup Q_w and it contains its Lie algebra $\mathfrak{q}_w$. For classical groups, Hessenberg varieties may be described concretely in terms of (isotropic) flags.

Let V be a finite-dimensional vector space over k and let G be $\mathrm{SL}(V)$, $\mathrm{Spin}(V)$ or $\mathrm{Sp}(V)$ where we endow V with a symmetric bilinear (resp. symplectic) form $\langle -, - \rangle$. Then $\psi \in \mathfrak{g}$ is an anti-self adjoint endomorphism of the vector space V and the condition $\psi \in {}^g\mathfrak{g}_w$ may be translated into the condition that if

$$0 \subset F_1 \subset \cdots \subset F_r \subset V$$

is the flag corresponding to g we have $\psi(F_i) \subset F_{h(i)}$ for a non-decreasing function h satisfying $h(i) \geq i$ associated to the space $\mathfrak{g}_w$, cf. [25, §2].

In this section we study the stabilizers of a Borel subgroup B_ψ of G_ψ acting on the Hessenberg variety Y_w. The goal is prove that for $w \neq \mathrm{id}$ the Hessenberg variety Y_w is a spectrally meager G_ψ-variety.

6.6.1 Lemma *Let $Q \subset G$ be a parabolic subgroup with corresponding Lie algebra $\mathfrak{q}$ and $U \subset \mathfrak{g}$ a subspace containing $\mathfrak{q}$ and which is stable under the adjoint action of Q. Assume that there is a parabolic subgroup P of G with Lie algebra $\mathfrak{p}$ such that $Q \subset P$ and $U \subset \mathfrak{p}$. Let $\psi \in U$ be semisimple and let G_ψ be the centralizer of ψ in G. Then the G_ψ-variety $Y_\psi(Q, U) = \{g \in G/Q \mid \psi \in {}^gU\}$ is spectrally meager.*

Proof Let $B_\psi \subset G_\psi$ be a Borel subgroup. Denote by $e \in G$ the identity element. It suffices to prove that the stabilizer of eQ contains a non-trivial torus. Indeed, if $gQ \in Y_\psi(Q, U)$ we have $\psi \in {}^gU \subset {}^g\mathfrak{p}$ and the assumptions of the Lemma are satisfied for ${}^gQ$, ${}^gU$ and ${}^gP$. Since $\mathrm{Stab}_{B_\psi}(gQ) = B_\psi \cap {}^gQ = \mathrm{Stab}_{B_\psi}(e \cdot {}^gQ)$ we may conclude.

We will argue in two steps. Let M be the Levi quotient of P and $\mathfrak{m}$ its Lie algebra. Consider the map

$$\pi : Y_\psi(Q, U) \to Y_\psi(P, \mathfrak{p})$$

induced by the projection $G/Q \to G/P$ whose fiber above eP is identified with M/Q_M where Q_M is the image of Q in M. Since ψ is semisimple and $\psi \in \mathfrak{p}$, the stabilizer P_ψ of ψ in P is a parabolic subgroup of G_ψ and its Levi quotient P_ψ/U_ψ coincides with the stabilizer $M_{\bar{\psi}}$ of the image of ψ in $\mathfrak{m}$. Write $Y' = \pi^{-1}(eP) \subset Y_\psi(Q, U)$. The parabolic P_ψ acts on Y' and the action factors through $M_{\bar{\psi}}$. Since $Y' \subset M/Q_M$ we know that the center $Z(M)$ stabilizes every point of Y'. Thus $H := G_\psi \cap Q$ contains a group of the form $Z \cdot U_\psi$ where Z is a torus surjecting onto $Z(M)$.

Let $B_\psi \subset G_\psi$ be a Borel subgroup. The second step is to analyze the action of H on G_ψ/B_ψ. Using the Bruhat decomposition we write

$$G_\psi/B_\psi = \coprod_{w \in W_\psi \backslash W} P_\psi w B_\psi/B_\psi$$

where W_ψ is the Weyl group of $M_{\bar{\psi}}$. Since H acts on each cell we only need to consider one such cell $P_\psi w B_\psi/B_\psi$. Note that $P_\psi w B_\psi/B_\psi \cong P_\psi/(P_\psi \cap {}^wB_\psi)$ and $\mathrm{Stab}_H(pwB_\psi) = \mathrm{Stab}_H(pP_\psi \cap {}^wB_\psi)$. Let $\bar{H} = H/U_\psi$. This acts on $M_{\bar{\psi}}$ and we have that

$$\mathrm{Stab}_H(p(P_\psi \cap {}^wB_\psi)) \to \mathrm{Stab}_{\bar{H}}(\bar{p})$$

is surjective (for the action of $\bar{H}$ on $M_{\bar{\psi}}/\bar{B}$). Here $\bar{B}$ denotes the image of $P_\psi \cap {}^wB$ in $M_{\bar{\psi}}$. Finally, since $M_{\bar{\psi}}/\bar{B}$ is the full flag variety of $M_{\bar{\psi}}$ every point is stabilized by the center $Z(M) \subset \bar{H}$ of M. □

6.6.2 Corollary *Let Q, U and ψ be as in the previous Lemma. Assume that there is a maximal parabolic P with Lie algebra $\mathfrak{p}$ such that $Q' = Q \cap P$ is a parabolic subgroup of G and $\psi \in \mathfrak{p}$. Then $Y_\psi(Q, U)$ is spectrally meager for the G_ψ-action.*

Proof Define $U' = U \cap \mathfrak{p}$. Note that since $\psi \in \mathfrak{p}$, actually $eQ \in Y_\psi(Q, U')$. We have a surjective morphism $Y_\psi(Q', U') \to Y_\psi(Q, U')$. For any point $y \in Y_\psi(Q', U')$ denote by $\bar{y}$ the image under this morphism. We have that $\mathrm{Stab}_{B_\psi}(y) \subset \mathrm{Stab}_{B_\psi}(\bar{y})$, so it suffices to prove the claim for $eQ' \in Y_\psi(Q', U')$. This immediately follows from the previous Lemma 6.6.1. □

6.6.3 Remark In particular for flag varieties of classical groups to prove that Y_w is spectrally meager it suffices to prove that for any (isotropic) flag $F_\bullet \in Y_w$ we can refine it to a flag such that one of the (isotropic) subspaces in the resulting flag is ψ-stable. From the proof of Lemma 6.6.1 it follows that in this case the stabilizer of any point in Y_w contains the center of the Levi subgroup of the maximal parabolic subgroup stabilizing the ψ-stable space.

Denote by Φ the set of roots of G and let $\Delta = \{\alpha_1, \dots, \alpha_r\}$ be the positive simple roots. By finite Weyl group symmetry we may and will assume that $wx_{\mathbf{Q}}$ is dominant.

6.6.4 Corollary *Assume that there is a simple root α_i such that $\langle \alpha, wx_{\mathbf{Q}} \rangle \geq 1$. Then Y_w is spectrally meager.*

Proof Let P be the maximal proper parabolic subgroup of G containing Q_w but not the root subgroup corresponding to α_i. If $\mathfrak{g}_w \not\subset \mathfrak{p} = \text{Lie } P$, then $\mathfrak{g}_{\alpha_i} \subset \mathfrak{g}_w$ and $\langle \alpha_i, wx_{\mathbf{Q}} \rangle < 1$. Vice versa if $\langle \alpha_i, wx_{\mathbf{Q}} \rangle \geq 1$, then $\mathfrak{g}_w \subset \mathfrak{p}$. Therefore Lemma 6.6.1 proves the claim. □

This shows that we can restrict ourselves to the study of Y_w for w such that $0 \leq \langle \alpha_i, wx_{\mathbf{Q}} \rangle < 1$ for all simple roots $\alpha_i \in \Delta$.

7 Detailed analysis of stabilizers

In the following we will carry out a case-by-case analysis of the cases with an almost free open orbit listed in Sect. 6. We will show that any flag in any Hessenberg variety Y_w for $w \neq \text{id}$ may be refined to a flag s.th. one of the spaces appearing in it is ψ-stable. This will prove the first part of Theorem 6.5.1. Strictness in types A and D is proved in Sect. 7.5.

7.0.1 Remark Let V be a vector space over k. In types B, C and D we will denote the bilinear form by $\langle -, - \rangle$. We will often make use of the fact that $\mathfrak{g}$ is the Lie algebra of endomorphisms of V which are anti-selfadjoint with respect to the given bilinear form. In particular the non-zero eigenvalues of ψ occur in pairs $x, -x$ and the eigenspaces of x and $-x$ are dual with respect to the given bilinear form. For any non-zero x the eigenspace $V(x)$ is isotropic and orthogonal to all other eigenspaces apart from $V(-x)$. In types B,C and D we will always denote the eigenvalues by $\pm x$ (and $\pm y$ in a single case in type D). Since ψ is semisimple we can decompose any vector v according to the eigenvalues of ψ and if x is an eigenvalue, v_x is the summand of v in the x-eigenspace. When working with flags $F_\bullet$ of a vector space V we will denote flags and the condition on them imposed by ψ in the following way

$$0 \subset \cdots \subset F_k \overset{\curvearrowright}{\subset} F_l \overset{\curvearrowright}{\subset} F_m \subset \cdots \subset V$$

By this we mean that $\psi(F_k) \subset F_l$ and the indices denote the dimensions of the spaces.

7.1 Type A_{n-1}

Let V be a finite-dimensional k-vector space and $\mathbb{G} = \mathrm{SL}(V)$. We identify $\mathbb{X}_*(S) \cong \{k = (k_i) \in \mathbb{Z}^n \mid \sum_{i=1}^n k_i = 0\}$ such that for the standard basis $e_1^*, \ldots, e_n^*$ of $\mathbb{Z}^n$ we have $\alpha_i^\vee = e_i^* - e_{i+1}^*$ for the simple roots α_i. We write $wx_{\mathbf{Q}} = (x_1, \ldots, x_n)$ in coordinates for that basis. Recall that

$$Y_w = \{gQ_w \in G/Q_w \mid \psi \in {}^g\mathfrak{g}_w\}.$$

The parabolic Q_w is determined by the roots that evaluate non-positively on $wx_{\mathbf{Q}}$. The space $\mathfrak{g}_w$ contains the root space $\mathfrak{g}_\alpha$ for any root α if and only if $\langle \alpha, wx_{\mathbf{Q}} \rangle < 1$. Let Q' be the opposite parabolic to Q, so that $-x_{\mathbf{Q}} = x_{\mathbf{Q}'}$. To describe the Hessenberg varieties explicitly in terms of flags we will instead work with the variety Y_{-w} defined by the point $-wx_{\mathbf{Q}}$. This will only switch around the type of Q.

Similarly we define $\mathfrak{g}_{-w}$ so that $\mathfrak{g}_\alpha \subset \mathfrak{g}_{-w}$ if and only if and only if $\langle \alpha, wx_{\mathbf{Q}} \rangle > -1$. Write $\alpha_{ij} = \sum_{k=i}^{j-1} \alpha_k$ for and $\alpha_{ji} = -\alpha_{ij}$ for $i < j$. We have $\langle \alpha_{ji}, wx_{\mathbf{Q}} \rangle = x_j - x_i$ and $\mathfrak{g}_{\alpha_{ji}} \subset \mathfrak{g}_{-w}$ if and only if

$$\langle \alpha_{ji}, wx_{\mathbf{Q}} \rangle > -1$$

or in other words if and only if $x_i - x_j < 1$. In terms of ψ and a flag

$$0 \subset F_k \subset \cdots \subset F_r \subset V$$

the condition $\mathfrak{g}_{\alpha_{ji}} \subset \mathfrak{g}_{-w}$ is satisfied if and only if $\psi(F_i) \subset F_j$. We can therefore read off the condition imposed on any flag $F_\bullet \in Y_w$ from the coordinates of $wx_{\mathbf{Q}}$. The number of different entries in $wx_{\mathbf{Q}}$ determines the number of different spaces in the flag. The numbers of entries that coincide determine the dimensions of the associated graded spaces. We label the associated graded by the entries of $wx_{\mathbf{Q}}$. For example if $n = 5$ and $wx_{\mathbf{Q}} = (x_1, x_2, x_2, x_3, x_4)$ we have

$$0 \subset_{x_1} F_1 \subset_{x_2} F_3 \subset_{x_3} F_4 \subset_{x_4} V$$

and ψ is allowed to map F_1 to any space whose associated graded is labelled by a number y such that $x_1 - y < 1$.

The Weyl group acts on $\mathbb{X}_*(\mathbb{T}) \otimes \mathbb{Q}$ by permutation of the coordinates and the coroot lattice by integer translations with vectors $(k_1, \ldots, k_n)$ such that $\sum_{i=1}^n k_i = 0$. The arguments in each case work for any parabolic subgroup Q corresponding to the partition λ_Q. Therefore we only argue for one of these.

7.1.1 Case (1), 6.2

The barycenter of $\mathbf{Q}_0$ is $x_{\mathbf{Q}} = (\frac{n-1}{2n}, -\frac{1}{2n}, \ldots, -\frac{1}{2n})$. If $wx_{\mathbf{Q}} \neq x_{\mathbf{Q}}$ we find that $wx_{\mathbf{Q}}$ has at least four different entries. Mod $\mathbb{Z}$ only one entry may be congruent to $\frac{n-1}{2n}$, all others are congruent to $-\frac{1}{2n}$. Four of them being pairwise different forces two

consecutive entries to have a difference of 1. Therefore the corresponding space in the flag has to be ψ-stable.

7.1.2 Case (2), 6.2

The barycenter is

$$x_{\mathbf{Q}} = \left(\frac{n-1}{2n}, \frac{n-3}{2n}, \ldots, \frac{3-n}{2n}, \frac{1-n}{2n}\right).$$

Let $wx_{\mathbf{Q}} = (x_1, \ldots, x_n)$ be dominant and not equal to $x_{\mathbf{Q}}$. Assume that there is an index i_0 such that $x_{i_0} - x_{i_0+1} \geq 2/n$. Then we have

$$x_1 - x_n = \sum_{i=1}^{n-1} x_i - x_{i+1} \geq (n-2)\frac{1}{n} + \frac{2}{n} = 1.$$

Since any flag $F_\bullet \in Y_w$ is a full flag this implies that $\psi(F_1) \subset F_j$ for some $j \leq n-1$. Let $v \in F_1 \setminus \{0\}$ and write $v = v_x + v_y$ according to the eigenvalues of ψ. Assume the x-eigenspace $V(x)$ is the one-dimensional eigenspace. If $v_x = 0$, F_1 is ψ-stable and we are done. If $v_x \neq 0$, $V(x) \subset F_1 + \psi(F_1) \subset F_j$ and F_j is ψ-stable.

We may therefore assume that $x_i - x_{i+1} \leq 1/n$. Write $x_i = q_i + k_i$ with $q_i \in \{\frac{n-1}{2n}, \frac{n-3}{2n}, \ldots, \frac{3-n}{2n}, \frac{1-n}{2n}\}$ and $k_i \in \mathbb{Z}$. We have $\sum_{i=1}^n k_i = 0$. Since

$$|q_i - q_{i+1}| \leq \frac{n-1}{n}$$

we have the following three cases

$$q_i - q_{i+1} = -\frac{n-1}{n}, k_i - k_{i+1} = 1,$$
$$q_i - q_{i+1} = \frac{n-1}{n}, k_i - k_{i+1} = -1,$$
$$q_i - q_{i+1} = \frac{1}{n}, k_i - k_{i+1} = 0.$$

Now $q_i - q_{i+1} = -\frac{n-1}{n}$ implies that $q_i = \frac{1-n}{2n}$ and $q_{i+1} = \frac{n-1}{2n}$ and vice versa in the second case. In particular, since the q_i are pairwise different, only one of the first two cases may appear and it can only happen for a single index. This implies that the integer vector $(k_1, \ldots, k_n)$ has only two distinct entries, but this contradicts the assumption $\sum_{i=1}^n k_i = 0$. Therefore the only case in which $x_i - x_{i+1} \leq 1/n$ is the case in which $wx_{\mathbf{Q}} = x_{\mathbf{Q}}$.

7.1.3 Case (3), 6.2

We have $x_{\mathbf{Q}} = (\frac{m}{2n}, \ldots, \frac{m}{2n}, -\frac{m}{2n}, \ldots, -\frac{m}{2n})$ with both entries occuring m-times. If $wx_{\mathbf{Q}} \neq x_{\mathbf{Q}}$ and $wx_{\mathbf{Q}} = (x_1, \ldots, x_n)$ is dominant, then it has at least four different entries. The condition $0 \leq \langle \alpha_i, wx_{\mathbf{Q}} \rangle \leq 1/2$ implies that any flag in Y_w contains a part of the form

$$\ldots \subset F_1 \overset{\frown}{\subset} F_2 \overset{\frown}{\subset} F_3 \subset \ldots.$$

Then $W = F_2 + \psi(F_2) + \psi^2(F_1)$ is ψ-stable. Indeed, let x, y, z be the eigenvalues of ψ with eigenspaces $V(x), V(y), V(z)$ of dimensions $m, m-1, 1$ respectively. If every vector $V \in F_1$ is of the form $v = v_x + v_y$, then $F_1 + \psi(F_1)$ is ψ-stable. Otherwise $W \supset V(z)$, so for any $w = w_x + w_y + w_z \in F_2$ we get $w_x + w_y \in W$ and $xw_x + yw_y \in W$, i.e. $w_x, w_y \in W$, proving the claim.

7.1.4 Case (4), 6.2

Here we have

$$x_{\mathbf{Q}} = \left(\frac{m+1}{3n}, \ldots, \frac{m+1}{3n}, -\frac{m-1}{3n}, \ldots, -\frac{m-1}{3n}, -\frac{3m-1}{3n} \right)$$

with entries occurring m-times, $m-1$ times and one time in this order. We may assume that for $x_{\mathbf{Q}} \neq wx_{\mathbf{Q}} = (x_1, \ldots, x_n)$ we have $0 \leq x_i - x_{i+1} \leq 1/3$, because otherwise we can insert a space of the form $F + \psi(F)$.

In addition we know that $wx_{\mathbf{Q}}$ must have at least five different entries, say $y_1, \ldots y_5$ and their classes mod $\mathbb{Z}$ need to be ordered as follows

$$\left(\frac{m+1}{3n}, -\frac{m-1}{3n}, -\frac{3m-1}{3n}, \frac{m+1}{3n}, -\frac{m-1}{3n} \right),$$

since $-\frac{3m-1}{3n}$ may only appear once. Write $y_1 = \frac{m+1}{3n} + k_1$ and so on. The condition $0 \leq y_i - y_{i+1} \leq 1/3$ implies that $k_1 = k_2 = k_3$ and $k_4 = k_5 = k_3 - 1$. Since all other entries are congruent to $\frac{m+1}{3n}$ or $-\frac{m-1}{3n}$ mod $\mathbb{Z}$ we conclude that the integer vector by which we translate has the shape

$$(k, \ldots, k, k-1, \ldots, k-1),$$

a contradiction to the sum of these entries vanishing.

7.1.5 Case (5) & (6), 6.2

The arguments in these cases are the same as in the previous two.

7.1.6 Case (7), 6.2

The barycenter is $x_{\mathbf{Q}} = (1/3, 1/3, 0, 0, -1/3, -1/3)$. We may assume that for $wx_{\mathbf{Q}} = (x_1, \ldots, x_6)$ we have $0 \leq x_i - x_{i+1} \leq 1/3$. Otherwise there is an index i such that $x_i - x_{i+2} = 1$ and $\psi(F_i) \subset F_{i+1}$. In that case since ψ has only two eigenvalues we may insert the stable space $F_i + \psi(F_i)$.

Now assume that all entries of $wx_{\mathbf{Q}} = (x_1, \ldots, x_6)$ are pairwise different. In that case the classes of the entries must be ordered as follows

$$\left(\frac{2}{3}, \frac{1}{3}, 0, \frac{2}{3}, \frac{1}{3}, 0\right),$$

because otherwise we will find two successive entries whose difference is at least $2/3$. If $x_1 = 2/3 + k_1$ and so on where $k_i \in \mathbb{Z}$ then we find that $k_1 = k_2 = \cdots = k_6$. This contradicts the condition $\sum_{i=1}^{6} k_i = 0$.

We are therefore reduced to the following situation. For $wx_{\mathbf{Q}} \neq x_{\mathbf{Q}}$ we always have exactly four different proper subspaces and ψ maps as follows

$$\ldots \subset F_1 \subset F_2 \subset F_3 \subset F_4 \subset \ldots.$$

We are left to consider two cases, either $\dim(F_1) = 1$ or $\dim(F_1) = 2$.

In the first case we assume that F_1 is not already stable. Choose a non-zero vector $v \in F_1$ and extend it to a basis v, w of F_2. Write $v = v_x + v_y$ according to the eigenvalues of ψ (where the y-eigenspace $V(y)$ has dimension 2) and similarly for w. The space $F_2 + \psi(F_2)$ is spanned by v_x, w_x, v_y and w_y. If v_y and w_y are linearly independent, $V(y) \subset F_2 + \psi(F_2) \subset F_4$ and hence F_4 is stable. If they are linearly dependent, $F_2 + \psi(F_1)$ is ψ-stable and we may insert that.

In the second case choose a basis v, w of F_2. Then with the same notation as before $F_2 + \psi F_2$ contains v_x, w_x, v_y and w_y. If v_y and w_y are linearly dependent F_2 contains a non-zero eigenvector and we may insert the line spanned by it. If not, $F_2 + \psi(F_2)$ contains the whole y-eigenspace $V(y)$ and we are done.

7.1.7 Case (8), 6.2

The barycenter is $x_{\mathbf{Q}} = (1/6, 1/6, 1/6, 1/6, -1/3, -1/3)$. For $wx_{\mathbf{Q}} \neq x_{\mathbf{Q}}$ we will find at least four different entries $x_1, \ldots, x_4$. We may assume as before that $0 \leq x_i - x_{i+1} \leq 1/2$, because otherwise a space in the flag will already be ψ-stable. Therefore $wx_{\mathbf{Q}}$ will have exactly four different entries, say y_1, y_2, y_3, y_4 and the classes mod $\mathbb{Z}$ will be ordered either as $(-1/3, 1/6, -1/3, 1/6)$ or $(1/6, -1/3, 1/6, -1/3)$. The associated graded spaces that are labelled by $-1/3$ have to be one-dimensional and cannot occur consecutively. Therefore we get the following list of possible flags that we need to consider:

$$0 \subset F_1 \subset F_2 \subset F_3 \subset V,$$
$$0 \subset F_2 \subset F_3 \subset F_5 \subset V,$$

$$0 \subset F_3 \subset F_4 \subset F_5 \subset V,$$
$$0 \subset F_1 \subset F_4 \subset F_5 \subset V,$$
$$0 \subset F_1 \subset F_3 \subset F_4 \subset V.$$

The indices indicate the dimensions of the spaces and the endomorphism ψ always maps a space to the next one. In the first three cases we may insert the spaces $F_1 + \psi(F_1) + \psi^2(F_1)$, $F_2 + \psi(F_2) + \psi^2(F_2)$ and $F_3 + \psi(F_3) + \psi^2(F_3)$ respectively.

The last two cases are similar to each other. We will present the argument only for the fourth case. First note that we may assume that $F_4 + \psi(F_4) = F_5$ since otherwise F_4 is already ψ-stable. Thus $F_1 + \psi(F_1) + \psi^2(F_1) \subset F_4 + \psi(F_4)$ and for $v = v_x + v_y + v_z \in F_1 \setminus \{0\}$ we get $v_x, v_y, v_z \in F_4 + \psi(F_4)$. We may assume that they are all non-zero because otherwise $F_1 + \psi(F_1)$ is ψ-stable. Since $F_4 \cap \psi(F_4) \neq 0$ there is a vector $w \in F_4$ such that $\psi(w) \in F_4$. Thus $F_4 + \psi(F_4)$ contains w_x, w_y, w_z. If any pair (w_x, v_x), (w_y, v_y) or (w_z, v_z) is linearly independent, the corresponding eigenspace lies in $F_4 + \psi(F_4)$ and this would imply that $F_4 + \psi(F_4)$ is ψ-stable.

We therefore assume that the above pairs are all linearly dependent. Since $w \neq 0$ we may additionally assume that $w_x \neq 0$ and hence we can write $v_x = \lambda w_x$ for some non-zero λ. Since $F_4 \supset F_1 + \psi(F_1)$ it contains $(x - y)v_x + (z - y)v_z$ and also $(x - y)w_x + (z - y)w_z$. In particular $\lambda w_z - v_z \in F_4$ and therefore if $\lambda w_z - v_z \neq 0$ we get $v_z \in F_4$ and $F_1 + \psi(F_1) + \psi^2(F_1) \in F_4$. If $\lambda w_z - v_z = 0$ we can do the same for v_y and find that actually $v = \lambda w$. Since $w \in F_4 \cap \psi(F_4)$ is arbitrary this implies $\dim(F_4 \cap \psi(F_4)) = 1$ and hence $\ker(\psi) \cap F_4 \neq 0$, i.e. say $z = 0$ and F_4 contains a 0-eigenvector u_0. Again if u_0 and v_0 are linearly independent, then $V(0) \subset F_4 + \psi(F_4)$ and if they are dependent, then $F_1 + \psi(F_1) + \psi^2(F_1) \subset F_4$, so we are done.

7.2 Type B_n

Let V be a finite dimensional k-vector space of odd dimension equipped with a non-degenerate symmetric bilinear form $\langle -, - \rangle$ and $\mathbb{G} = \mathrm{Spin}(V)$. As in type A we use the evident identification $\mathbb{X}^*(\mathbb{T}) \cong \mathbb{Z}^n$ to write $wx_{\mathbf{Q}} \in \mathbb{X}_*(\mathbb{T})$ in coordinates. A (partial) flag in type B stabilized by a parabolic Q_w can be thought of as a flag of isotropic spaces together with their complements

$$0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n \subset F_{n-1}^{\perp} \subset F_1^{\perp} \subset V.$$

The condition that $\psi \in \mathfrak{g}_w$ imposes the same conditions as in type A for the flag

$$0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n.$$

In addition, mapping from a space F_i to $F_j^{\perp}$ is determined by the value of the root $e_i + e_j$ (or e_i for $j = i$) and hence for $wx_{\mathbf{Q}} = (x_1, \ldots, x_n)$ we may label the flag above as follows

$$0 \subset_{x_1} F_1 \subset \cdots \subset_{x_n} F_n \subset_0 F_n^{\perp} \subset_{-x_n} F_{n-1}^{\perp} \subset F_1^{\perp} \subset_{-x_1} V.$$

Again ψ is allowed to map F_i to any space whose associated graded is labelled by y such that $x_i - y < 1$.

The finite Weyl group acts on $\mathbb{X}_*(\mathbb{T}) \otimes \mathbb{Q}$ by arbitrary permutations and sign changes of the coordinates. We may translate using the coroot lattice, i.e. by integer vectors whose coordinates sum to an even number. We can therefore produce a finite list of possible cases for $wx_{\mathbf{Q}}$ (using the condition that $0 \leq \langle \alpha_i, wx_{\mathbf{Q}} \rangle < 1$ for all simple roots).

7.2.1 Case (1), 6.3, $(P_{\psi}, Q) \sim (P_n, P_n)$

In this case the barycenter is $x_{\mathbf{Q}} = (1/2, \ldots, 1/2)$. Therefore $wx_{\mathbf{Q}}$ will have coordinates in $1/2 + \mathbb{Z}$. The condition $0 \leq \langle \alpha_i, wx_{\mathbf{Q}} \rangle < 1$ immediately implies that all coordinates of $wx_{\mathbf{Q}}$ have to be equal to $1/2$, i.e. $wx_{\mathbf{Q}} = x_{\mathbf{Q}}$ is the only possibility and there is nothing to prove.

7.2.2 Case (2), 6.3

This case is similar to Case (1). Since $x_{\mathbf{Q}} = (1/2, 1/2)$, there is no other possibility than $wx_{\mathbf{Q}} = x_{\mathbf{Q}}$.

7.2.3 Case (3), 6.3

We have $x_{\mathbf{Q}} = (1/2, 0)$ and the only non-trivial possibility for $wx_{\mathbf{Q}}$ is $wx_{\mathbf{Q}} = (1, 1/2)$. In this case the flag is

$$0 \subset \overset{\frown}{F_1} \subset \overset{\frown}{F_2} \subset \overset{\frown}{F_2^{\perp}} \subset F_1^{\perp} \subset V.$$

and we may insert $F_1 + \psi(F_1)$.

7.2.4 Case (4), 6.3

We have $x_{\mathbf{Q}} = (1/2, 1/2, 0)$. The possible non-trivial cases for $wx_{\mathbf{Q}}$ are

(i) $wx_{\mathbf{Q}} = (1, 1/2, 1/2)$,
(ii) $wx_{\mathbf{Q}} = (2, 3/2, 1/2)$.

In case (i) the corresponding flag is

$$0 \subset \overset{\frown}{F_1} \subset \overset{\frown}{F_3} \subset \overset{\frown}{F_3^{\perp}} \subset F_1^{\perp} \subset V.$$

Choose a non-zero vector $v \in F_1$. Denote the non-zero eigenvalues of ψ by x and $-x$ and write $v = v_x + v_0 + v_{-x}$ according to the eigenspace decomposition of V. We get that

$$\langle v, \psi^2 v \rangle = 0$$

and from that it follows that $\langle v_x, v_{-x}\rangle = 0$. Since $P_\psi \sim P_1$, we have $\dim V(x) = \dim V(-x) = 1$ and either $v_x = 0$ or $v_{-x} = 0$. Assume that $v_{-x} = 0$. In that case $W = F_1 + \psi F_1$ contains $v = v_0 + v_x$ and v_x. This space therefore has a basis of eigenvectors and is ψ-stable. Since $W \subset F_3$ it is automatically isotropic and we may insert this space into the given flag.
In case (ii) the flag is

$$0 \subset F_1 \subset F_2 \subset F_3 \subset F_3^\perp \subset F_2^\perp \subset F_1^\perp \subset V.$$

Because $P_\psi \sim P_1$, ψ has three eigenvalues and if neither F_1 nor F_2 are ψ-stable, then $F_3 = F_1 + \psi F_1 + \psi^2 F_1$ which is stable.

7.2.5 Case (5), 6.3

We have $x_{\mathbf{Q}} = (1/2, 0, 0)$. The possible non-trivial cases for $wx_{\mathbf{Q}}$ are

(i) $wx_{\mathbf{Q}} = (1, 1/2, 0)$,
(ii) $wx_{\mathbf{Q}} = (1, 1, 1/2)$.

In case (i) the flag is

$$0 \subset F_1 \subset F_2 \subset F_2^\perp \subset F_1^\perp \subset V.$$

Let $v \in F_1$ be non-zero and write $v = v_x + v_0 + v_{-x}$ according to the eigenspace decomposition of V. Since $\psi(v) \in F_2$, it's isotropic. We have $\psi(v) = xv_x - xv_{-x}$ and

$$0 = \langle \psi(v), \psi(v)\rangle = -2x\langle v_x, v_{-x}\rangle.$$

Therefore $\langle v_x, v_{-x}\rangle = 0$. The space $W = F_1 + \psi F_1 + \psi^2 F_1$ is ψ-stable and we claim it is isotropic. Since $\psi(v) \in F_2$ and $\psi^2(v) \in F_2^\perp$ it is enough to prove that $\langle \psi^2(w), \psi^2(w')\rangle = 0$ for any $w, w' \in F_1$. But F_1 is a line, so $w = \lambda v$ and $w' = \mu v$ for some $\lambda, \mu \in k$. Therefore

$$\langle \psi^2(w), \psi^2(w')\rangle = \lambda\mu\langle x^2 v_x + x^2 v_{-x}, x^2 v_x + x^2 v_{-x}\rangle = 2x^4\langle v_x, v_{-x}\rangle = 0$$

and we can insert W into the above flag.
In case (ii) the flag is

$$0 \subset F_2 \subset F_3 \subset F_3^\perp \subset F_2^\perp \subset V.$$

For $v, w \in F_2$, since $\psi^2 F_2 \subset F_3^\perp \subset F_2^\perp$ we have

$$0 = \langle v, \psi^2(w)\rangle = x^2(\langle v_x, w_{-x}\rangle + \langle v_{-x}, w_x\rangle).$$

The space $W = F_2 + \psi F_2 + \psi^2 F_2$ is ψ-stable. We claim it is isotropic. For this it is enough to show that $\langle \psi^2(v), \psi^2(w)\rangle = 0$ for any $v, w \in F_2$. But

$$\langle \psi^2(v), \psi^2(w)\rangle = \langle x^2(v_x + v_{-x}), x^2(w_x + w_{-x})\rangle = x^4(\langle v_x, w_{-x}\rangle + \langle v_{-x}, w_x\rangle)$$

which vanishes by the previous calculation. Therefore W is isotropic and since $F_3 \subset W$ they have to be equal.

7.3 Type C_n

Let V be a finite dimensional k-vector space of even dimension equipped with a symplectic form $\langle -, -\rangle$ and $\mathbb{G} = \mathrm{Sp}(V)$. As before we may write $wx_{\mathbf{Q}}$ in coordinates using the identification $\mathbb{X}^*(\mathbb{T}) \cong \mathbb{Z}^n$. Similar to type B we consider flags

$$0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n \subset F_{n-1}^{\perp} \subset F_1^{\perp} \subset V$$

and label them the same way, but without the middle step. The endomorphism ψ is allowed to map F_i to any space whose associated graded is labelled by y such that $x_i - y < 1$.

The finite Weyl group acts by arbitrary permutations and sign changes on the coordinates of $wx_{\mathbf{Q}}$ and the coroot lattice by arbitrary integer vector translations.

7.3.1 Case (1), 6.4

The barycenter is $x_{\mathbf{Q}} = (1/4, \ldots, 1/4)$. Thus any coordinate of $wx_{\mathbf{Q}} = (x_1, \ldots, x_n)$ lies in $1/4 + \mathbb{Z}$ or $3/4 + \mathbb{Z}$. Therefore $x_i - x_j = 0$ or $x_i - x_j = 1/2$ for any $i \leq j$. By the condition $0 \leq \langle \alpha_n, wx_{\mathbf{Q}}\rangle < 1$ we conclude that $x_n = 1/4$. Therefore if all coordinates agree, $wx_{\mathbf{Q}} = x_{\mathbf{Q}}$. For flags of the form

$$0 \subset F \subset F' \subset (F')^{\perp} \subset F^{\perp} \subset V$$

we find that $\psi(F) \subset F'$ since the difference of any two distinct coordinates is $1/2$. Since $P_\psi \sim P_n$, the space $W = F + \psi(F)$ is stable and can be inserted into the flag. We can use this argument whenever we have at least three distinct coordinates. In the case that we only have two distinct coordinates the flag is of the form

$$0 \subset F \subset F^{\perp} \subset V$$

with $\psi(F) \subset F^{\perp}$ and we can insert the space $W = F + \psi(F)$. Let $v + \psi(v'), w + \psi(w') \in W$. Then

$$\begin{aligned}\langle v + \psi(v'), w + \psi(w')\rangle &= \langle v, w\rangle + \langle v, \psi(w')\rangle + \langle \psi(v'), w\rangle + \langle \psi(v'), \psi(w')\rangle \\ &= \langle \psi(v'), \psi(w')\rangle\end{aligned}$$

since F is isotropic and $\psi(F) \subset F^{\perp}$. Since ψ is anti-self adjoint, $\langle \psi(v'), \psi(w') \rangle = -\langle v', \psi^2(w') \rangle$. Now since $P_\psi \sim P_n$ is the Siegel parabolic, ψ has eigenvalues say x and $-x$ and it follows that ψ^2 is scalar multiplication by x^2. Therefore

$$-\langle v', \psi^2(w') \rangle = -x^2 \langle v', w' \rangle = 0$$

and we find that W is isotropic.

7.3.2 Case (2), 6.4

We have $x_{\mathbf{Q}} = (1/3, 1/3, 0)$. The possible non-trivial cases for $wx_{\mathbf{Q}}$ are

(i) $wx_{\mathbf{Q}} = (2, 1/3, 0)$,
(ii) $wx_{\mathbf{Q}} = (2/3, 2/3, 0)$,
(iii) $wx_{\mathbf{Q}} = (1, 2/3, 1/3)$,
(vi) $wx_{\mathbf{Q}} = (4/3, 2/3, 0)$,
(v) $wx_{\mathbf{Q}} = (1, 1/3, 1/3)$,
(vi) $wx_{\mathbf{Q}} = (4/3, 1, 1/3)$.

In case (i) we have flags

$$0 \subset F_1 \subset F_2 \subset F_2^{\perp} \subset F_1^{\perp} \subset V.$$

We claim that $W = F_2 + \psi(F_1)$ is isotropic and that if neither F_1 nor F_2 are ψ-stable then $F_2 + \psi(F_1)$ is ψ-stable. Because $\psi(F_1) \subset F_2$ it suffices to prove that $\langle \psi v, \psi v' \rangle = 0$ for any $v, v' \in F_1$. This is clear since F_1 is one-dimensional.

To prove W is stable, let $v \in F_1$ be non-zero. We have

$$0 = \langle v, \psi v \rangle = -2x \langle v_x, v_{-x} \rangle,$$

i.e. $v_x = 0$ or $v_{-x} = 0$. Assume we have $v_{-x} = 0$ and let $w = w_x + w_0 + w_{-x} = F_2$ be any vector. Then

$$0 = \langle v, \psi w \rangle = -x \langle v_x, w_{-x} \rangle.$$

If $v_x = 0$, then F_1 is ψ-stable, so we get that $w_{-x} = 0$ and $F_2 + \psi(F_1)$ contains v_x, v_0 and w_0. If v_0 and w_0 are linearly dependent, then $V(x) \subset F_2$ and by what we said above no vector in F_2 has a component in the $(-x)$-eigenspace. This implies that F_2 is ψ-stable. We may therefore assume that v_0 and w_0 are linearly independent and hence $F_2 + \psi(F_1)$ is spanned by eigenvectors.

In case (ii) the flags are of the form

$$0 \subset F_2 \subset F_2^{\perp} \subset V.$$

Choose a basis v, w for F_2. Since $\dim V(x) = 1$ there are scalars λ, μ such that $\lambda v_x + \mu w_x = 0$ and not both of them are zero. Let $u = \lambda v + \mu w$. Since v, w are a basis, $u \neq 0$. If $u_{-x} = 0$ then $u_0 \neq 0$ and hence F_2 contains the eigenline spanned by u_0. If $u_{-x} \neq 0$ then for any $u' \in F_2$ we have

$$0 = \langle u', \psi u \rangle = \langle u'_x, u_{-x} \rangle,$$

i.e. $u'_x = 0$. In this case $F_2 + \psi(F_2)$ is ψ-stable and isotropic.

In case (iii) we have flags

$$0 \subset F_1 \subset \overset{\frown}{F_2 \subset F_3} \subset F_2^\perp \subset F_1^\perp \subset V.$$

We claim that $W = F_2 + \psi(F_2) + \psi^2(F_2)$ is isotropic. To prove that it suffices to show that $\langle \psi v, \psi^2 w \rangle = 0$ and that $\langle \psi^2 v, \psi^2 w \rangle = 0$ for any $v, w \in F_2$. We have

$$\langle \psi v, \psi^2 w \rangle = \langle \psi v, \psi^2 w \rangle + \langle \psi v, x^2 w_0 \rangle = x^2 \langle \psi v, w \rangle = 0.$$

The same argument works for $\langle \psi^2 v, \psi^2 w \rangle = 0$ and thus W is isotropic. If F_2 is not ψ-stable, then $F_3 = F_2 + \psi(F_2)$ and $F_3 \subset W$. Hence $W = F_3$.

In case (iv) the flags are

$$0 \subset \overset{\frown}{F_1 \subset F_2} \subset \overset{\frown}{F_2^\perp} \subset F_1^\perp \subset V$$

and we may take $F_1 + \psi(F_1) + \psi^2(F_1)$ which is isotropic with the same argument as in case (iii).

In case (v) consider

$$0 \subset \overset{\frown}{F_1 \subset F_3} \subset F_1^\perp \subset V.$$

For $v \in F_1$ non-zero we have $v_x = 0$ or $v_{-x} = 0$ and hence $F_1 + \psi(F_1)$ is ψ-stable.

For case (vi) we get full flags

$$0 \subset \overset{\frown}{F_1 \subset F_2} \subset \overset{\frown}{F_3 \subset F_2^\perp} \subset F_1^\perp \subset V$$

and we may simply insert $F_1 + \psi(F_1) + \psi^2(F_1)$.

7.3.3 Case (3), 6.4

We have $x_{\mathbf{Q}} = (1/3, 0, 0)$. The possible non-trivial cases for $wx_{\mathbf{Q}}$ are

(i) $wx_{\mathbf{Q}} = (2/3, 0, 0)$,

(ii) $wx_{\mathbf{Q}} = (1, 1/3, 0)$,
(iii) $wx_{\mathbf{Q}} = (1, 2/3, 0)$,
(vi) $wx_{\mathbf{Q}} = (1, 1, 1/3)$.

In case (i) the flags are of the form

$$0 \subset \overset{\frown}{F_1} \subset F_1^{\perp} \subset V.$$

Let $v \in F_1$ be non-zero. Then $\langle v, \psi^2 v\rangle = -\langle v, v\rangle = 0$, hence $W = F_1 + \psi(F_1) + \psi^2(F_1) \subset F_1^{\perp}$. Since also $\langle v, \psi v\rangle = 0$ we find that $\langle v_x, v_{-x}\rangle = 0$. This implies that W is isotropic.

In case (ii) we have flags

$$0 \subset \overset{\frown}{F_1} \subset \overset{\frown}{F_2} \subset F_2^{\perp} \subset F_1^{\perp} \subset V.$$

If F_1 is not stable, then $F_2 = F_1 + \psi(F_1)$. Let F_1 be spanned by $v = v_x + v_0 + v_{-x}$. As before we find that $\langle v_x, v_{-x}\rangle = 0$ and hence $\langle \psi v, \psi^2 v\rangle = 0$. Thus $\psi(F_2) \subset F_2^{\perp}$ and we may insert $F_1 + \psi(F_1) + \psi^2(F_1)$.

In case (iii) we consider

$$0 \subset \overset{\frown}{F_1} \subset \overset{\frown}{F_2} \subset \overset{\frown}{F_2^{\perp}} \subset F_1^{\perp} \subset V.$$

This works the same as case (ii).

In case (iv) we have

$$0 \subset \overset{\frown}{F_2} \subset \overset{\frown}{F_3} \subset F_2^{\perp} \subset V.$$

We have $F_2 \cap \psi(F_2) \neq 0$ and there is a $v \in F_2$ such that $\psi(v) \in F_2$. Therefore F_2 is spanned by two vectors of the form $w_x + w_0$ and $u_{-x} + u_0$ and $F_2 + \psi(F_2)$ is ψ-stable.

7.4 Type D_n

Let V be a finite dimensional k-vector space of even dimension equipped with a non-degenerate symmetric bilinear form $\langle -, -\rangle$ and $\mathbb{G} = \mathrm{Spin}(V)$. Using the identification $\mathbb{X}^*(\mathbb{T}) \cong \mathbb{Z}^n$ we write $wx_{\mathbf{Q}}$ in coordinates. A full flag in type D_n is the data of a flag of isotropic spaces

$$0 \subset F_1 \subset F_2 \subset \ldots \subset F_{n-1} \begin{array}{ccc} & F_n & \\ \curvearrowleft & & \curvearrowright \\ & & \\ \curvearrowright & & \curvearrowleft \\ & F'_n & \end{array} F_{n-1}^{\perp} \subset \ldots \subset F_2^{\perp} \subset F_1^{\perp} \subset V.$$

with two Lagrangian subspaces F_n and F'_n such that $F_n \cap F'_n = F_{n-1}$. The labelling is done as before. We denote the non-zero eigenvalues of ψ by $\pm x$ (and $\pm y$ in the case $n = 4$). The finite Weyl group acts by arbitrary permutations and even sign changes on the coordinates of $wx_{\mathbf{Q}}$. The coroot lattice acts by translation with integer vectors whose coordinates sum to an even integer.

7.4.1 Case (1), 6.5

We have $x_{\mathbf{Q}} = (1/2, 1/4, 0, 0)$. Because ψ has only two eigenvalues we may eliminate cases in the orbit of $x_{\mathbf{Q}}$ for which ψ maps a space to the next one. This leaves us with the following possibilities.

(i) $wx_{\mathbf{Q}} = (3/4, 1/2, 0, 0)$,
(ii) $wx_{\mathbf{Q}} = (1, 1/2, 1/4, 0)$,
(iii) $wx_{\mathbf{Q}} = (1, 3/4, 1/2, 0)$,
(iv) $wx_{\mathbf{Q}} = (1, 1, 1/2, 1/4)$.

In case (i) the flags are of the form

$$0 \subset F_1 \subset \overset{\frown}{F_2} \subset F_2^{\perp} \subset F_1^{\perp} \subset V.$$

It's enough to show that $F_2 + \psi(F_2)$ is isotropic. For that it suffices to prove that $\langle \psi v, \psi w \rangle = 0$ for all $v, w \in F_2$. This follows from the fact that $\psi^2(w) = x^2 w$. Indeed we have

$$\langle \psi v, \psi w \rangle = -\langle v, \psi^2 w \rangle = -x^2 \langle v, w \rangle = 0.$$

The same argument works in cases (iii) and (iv).

In case (ii) we have flags

$$0 \subset F_1 \subset F_2 \subset F_3 \begin{array}{ccc} & F_4 & \\ \curvearrowleft & & \curvearrowright \\ & & \\ \curvearrowright & & \curvearrowleft \\ & F'_4 & \end{array} F_3^{\perp} \subset F_2^{\perp} \subset F_1^{\perp} \subset V.$$

Assume that $F_2 + \psi(F_1) \neq F_3$. Then $\psi(F_1) \subset F_2$ and we may take $F_1 + \psi(F_1)$. If $F_2 + \psi(F_1) = F_3$, then $F_3 \subset F_2 + \psi(F_2)$ and this space is isotropic by the same argument as before. Therefore this space is a Lagrangian subspace containing F_3 and as such it is either F_4 or F_4' and we are done.

7.4.2 Case (2), 6.5

We have $x_{\mathbf{Q}} = (1/4, 1/4, 1/4, 1/4)$ and the possible non-trivial cases are

(i) $wx_{\mathbf{Q}} = (3/4, 3/4, 1/4, 1/4)$,
(ii) $wx_{\mathbf{Q}} = (5/4, 3/4, 1/4, -1/4)$.

In case (i) we consider flags

$$0 \subset F_2 \subset F_4 \subset F_2^{\perp} \subset V.$$

Choose a basis $u, v \in F_2$. We first want to prove that $v_x' = 0$ for all $v' \in F_2$. Since $\dim V(x) = 1$ there are scalars λ_x, μ_x which are not both zero such that $\mu_x u_x + \lambda_x v_x = 0$. Define $w = \lambda_x v + \mu_x u$. Then $w \neq 0$ and since $\langle w, \psi^2 w\rangle = 0$ we find that $\langle w_y, w_{-y}\rangle = 0$. This implies that $w_y = 0$ or $w_{-y} = 0$. Without loss of generality we may assume that $w_y = 0$, i.e. $w = w_0 + w_{-y} + w_{-x}$. Now $\psi w, \psi^2 w \in F_2^{\perp}$ implies that $w_{-x} \in F_2^{\perp}$ and $w_{-y} \in F_2^{\perp}$. If $w_{-x} = 0$ then if also $w_{-y} = 0$ the space F_2 contains $w_0 \neq 0$ and in particular the line spanned by it and we may insert that. So either $w_{-x} \neq 0$ or $w_{-y} \neq 0$ and we may assume that $w_{-x} \neq 0$ (otherwise we end up eliminating the y-component in F_2). Now for all $v' \in F_2$ we have $\langle v', w_{-x}\rangle = 0$ and therefore $v' = v_y' + v_0' + v_{-y}' + v_{-x}'$.

In the second step we will prove that also $v_y' = 0$ for all $v' \in F_2$. As before there are scalars λ_x, μ_{-x} such that

$$\lambda_{-x} v_{-x} + \mu_{-x} u_{-x} = 0.$$

We define $w' = \lambda_{-x} v + \mu_{-x} u \neq 0$. The same argument as before shows that $w_{-y}' \in F_2^{\perp}$ and that we may assume it's non-zero. This proves that $v_y' = 0$ for all $v' \in F_2$.

A simple calculation now shows that $W = F_2 + \psi(F_2) + \psi^2(F_2)$ is ψ-stable and isotropic. If neither F_2 nor $F_2 + \psi(F_2)$ are already stable, then W is either F_4 or some other Lagrangian and we may insert that space.

In case (ii) the flags are

$$0 \subset F_1 \subset F_2 \subset F_3 \subset F_4 \subset F_3^{\perp} \subset F_2^{\perp} \subset F_1^{\perp} \subset V.$$

Since $\psi^2(F_2) \subset F_2^{\perp}$ we can insert the space $F_2 + \psi(F_2) + \psi^2(F_2)$ and the argument is the same as in case (i).

7.4.3 Case (3), 6.5

We have $x_{\mathbf{Q}} = (1/4, 1/4, 0, 0, 0)$ and may assume that $x_i - x_{i+1} \leq 1/2$ for $i \leq 3$ (otherwise we may insert a space of the form $F + \psi(F)$ immediately). This leaves us with the following non-trivial cases

(i) $wx_{\mathbf{Q}} = (3/4, 3/4, 1/4, 0, 0)$,
(ii) $wx_{\mathbf{Q}} = (5/4, 3/4, 3/4, 0, 0)$,
(iii) $wx_{\mathbf{Q}} = (5/4, 1, 3/4, 3/4, 0)$.

In case (i) the flags are of the form

$$0 \subset F_2 \subset F_3 \subset F_3^{\perp} \subset F_2^{\perp} \subset V.$$

It's enough to show that actually $\psi(F_3) \subset F_3^{\perp}$. This will imply that $F_3 + \psi(F_3)$ is isotropic and stable. Choose a basis v_1, v_2 of F_2 and extend to the basis v_1, v_2, v_3 of F_3. We need to show that $\langle u, \psi(v)\rangle = 0$ for all $u, v \in F_3$. Write $u = \sum \lambda_i v_i$ and $v = \sum \mu_i v_i$ in the chosen basis. We have

$$\langle u, \psi(v)\rangle = \mu_3 \langle u, \psi(v_3)\rangle$$

since $\psi(F_2) \subset F_3^{\perp}$. Furthermore since also $\psi(F_3) \subset F_2^{\perp}$ we have

$$\mu_3 \langle u, \psi(v_3)\rangle = \lambda_3 \mu_3 \langle v_3, \psi(v_3)\rangle.$$

The last term vanishes because for any $w = w_x + w_{-x}$ we have

$$\langle w, \psi(w)\rangle = -x\langle w_x, w_{-x}\rangle + x\langle w_{-x}, w_x\rangle = 0$$

since the pairing is symmetric.

Cases (ii) and (iii) work the same as case (i) in Sect. 7.4.1.

7.4.4 Case (4), 6.5

We have $x_{\mathbf{Q}} = (1/4, 1/4, 1/4, 1/4, 1/4)$ and the non-trivial possibilities for $wx_{\mathbf{Q}}$ are

(i) $wx_{\mathbf{Q}} = (3/4, 3/4, 1/4, 1/4, 1/4)$,
(ii) $wx_{\mathbf{Q}} = (5/4, 3/4, 1/4, 1/4, -1/4)$,
(iii) $wx_{\mathbf{Q}} = (7/4, 5/4, 3/4, 1/4, 1/4)$,

In case (i) we consider flags

$$0 \subset F_2 \subset F_5 \subset F_2^{\perp} \subset V.$$

We claim that $W = F_2 + \psi(F_2) + \psi^2(F_2)$ is isotropic. Note that we may assume ψ to be injective on F_2, because otherwise we can insert its kernel. It suffices to show that $\langle \psi v, \psi^2 w\rangle = 0$ and that $\langle \psi^2 v, \psi^2 w\rangle = 0$ for all $v, w \in F_2$. We have

$$\langle \psi v, \psi^2 w\rangle = \langle \psi v, \psi^2 w\rangle + \langle \psi v, x^2 w_0\rangle = x^2\langle \psi v, w\rangle = 0$$

and similarly we get $\langle \psi^2 v, \psi^2 w\rangle = 0$. This implies that the dimension of W is at most 5 and hence we have $F_2 \cap \psi(F_2) \neq 0$ or $(F_2 + \psi(F_2)) \cap \psi^2(F_2) \neq 0$. In the first case F_2 contains a non-zero vector w such that $\psi w \in F_2$ and hence it contains $2x w_x + x w_0$ and $-2x w_{-x} - x w_0$. We may assume that $w_x, w_0, w_{-x} \neq 0$ because otherwise F_2 contains an eigenvector. Now F_2 is spanned by $2x w_x + x w_0$ and $-2x w_{-x} - x w_0$ and hence $F_2 + \psi(F_2)$ is ψ-stable.

If $F_2 \cap \psi(F_2) = 0$ and $(F_2 + \psi(F_2)) \cap \psi^2(F_2) \neq 0$ we either have that W is five-dimensional in which case it's Lagrangian and we may insert it into the flag or $\psi^2(F_2) \subset F_2 + \psi(F_2)$ in which case $F_2 + \psi(F_2)$ is ψ-stable.

In cases (ii) and (iii) we may simply insert the space $F_1 + \psi(F_1) + \psi^2(F_1)$ which is automatically stable and isotropic.

7.4.5 Case (5), 6.5

We have $x_{\mathbf{Q}} = (1/4, 1/4, 1/4, 0, 0, 0)$ and as for $n = 5$ we may assume that $x_i - x_{i+1} \leq 1/2$ for $i \leq 4$. This leaves us with the following cases

(i) $wx_{\mathbf{Q}} = (3/4, 3/4, 1/4, 0, 0, 0)$,
(ii) $wx_{\mathbf{Q}} = (1, 1, 3/4, 3/4, 1/4, 0)$,
(iii) $wx_{\mathbf{Q}} = (5/4, 1, 3/4, 3/4, 0, 0)$,

The arguments in these cases are the same as for $n = 5$.

7.4.6 Case (6), 6.5

We have $x_{\mathbf{Q}} = (1/4, 1/4, 1/4, 1/4, 1/4, 1/4)$. The list of possible cases is

(i) $wx_{\mathbf{Q}} = (3/4, 3/4, 1/4, 1/4, 1/4, 1/4)$,
(ii) $wx_{\mathbf{Q}} = (3/4, 3/4, 3/4, 3/4, 1/4, 1/4)$,
(iii) $wx_{\mathbf{Q}} = (5/4, 3/4, 1/4, 1/4, 1/4, -1/4)$,
(iv) $wx_{\mathbf{Q}} = (5/4, 3/4, 3/4, 3/4, 1/4, -1/4)$,
(v) $wx_{\mathbf{Q}} = (5/4, 5/4, 3/4, 3/4, 1/4, 1/4)$,
(vi) $wx_{\mathbf{Q}} = (5/4, 5/4, 5/4, 3/4, 1/4, -1/4))$,
(vii) $wx_{\mathbf{Q}} = (7/4, 5/4, 3/4, 1/4, 1/4, 1/4)$,
(viii) $wx_{\mathbf{Q}} = (7/4, 7/4, 5/4, 3/4, 1/4, -1/4)$,
(ix) $wx_{\mathbf{Q}} = (9/4, 7/4, 5/4, 3/4, 1/4, 1/4)$.

In case (i) we look at flags

$$0 \subset F_2 \subset F_6 \subset F_2^{\perp} \subset V.$$

We may assume that $\psi|_{F_2}$ is injective. As before $W = F_2 + \psi(F_2) + \psi^2(F_2)$ is isotropic and ψ-stable. If $\dim(W) = 6$ it's Lagrangian and either equal to F_6 or we may insert it as a second Lagrangian subspace. If $\dim(W) < 6$ we have $F_2 \cap \psi(F_2) \neq 0$ or $(F_2 + \psi(F_2)) \cap \psi^2(F_2) \neq 0$.

If $F_2 \cap \psi(F_2) \neq 0$ we conclude that $F_2 + \psi(F_2)$ is ψ-stable as in the corresponding case for $n = 5$. If $F_2 \cap \psi(F_2) = 0$ and $(F_2 + \psi(F_2)) \cap \psi^2(F_2) \neq 0$ there is a vector $v + \psi(u) \in \psi^2(F_2)$. This implies $v_0 = 0$ and $\psi(u) = x^2(v + \psi(u)) - \psi^2(v) \in \psi^2(F_2)$. Therefore $\psi(u) = \psi^2(u')$ for some $u' \in F_2$ and hence $u \in F_2 \cap \psi(F_2) = 0$. In the end we find that $v \neq 0$, i.e. F_2 contains the non-zero vector $v = v_x + v_{-x}$. If either component vanishes, then v is an eigenvector and we may insert the line spanned by it into the flag. If both are non-zero, then since $v_x, v_{-x} \in \psi^2(F_2)$ we get $v_x, v_{-x} \in (F_2 + \psi(F_2)) \cap \psi^2(F_2)$ and this intersection has to be two-dimensional. But that implies $\psi^2(F_2) \subset (F_2 + \psi(F_2)$ and hence $F_2 + \psi(F_2)$ is ψ-stable.

In case (ii) we consider flags of the form

$$0 \subset F_4 \subset F_6 \subset F_4^\perp \subset V.$$

The space $F_4 + \psi(F_4) + \psi^2(F_4)$ is isotropic. If F_4 is not stable, then $F_4 + \psi(F_4)$ is at least five dimensional. If this space itself is not already stable, then $F_4 + \psi(F_4) + \psi^2(F_4)$ is Lagrangian and we may use it to refine the flag. In case (iii) we may simply insert the space $F_1 + \psi(F_1) + \psi^2(F_1)$. Case (iv) is similar to case (ii), we may insert the space $F_4 + \psi(F_4) + \psi^2(F_4)$.

In case (v) we have flags

$$0 \subset F_2 \subset F_4 \subset F_6 \subset F_4^\perp \subset F_2^\perp \subset V.$$

We may insert $F_2 + \psi(F_2)$ if it is equal to F_4. Otherwise $F_2 \cap \psi(F_2) \neq 0$ and there is a vector $w \in F_2$ such that $\psi(w) \in F_2$. We then find that $F_2 + \psi(F_2)$ contains w_x, w_{-x} and w_0. If any of these vanishes, F_2 contains an eigenvector. Otherwise $F_2 + \psi(F_2)$ is generated by these three vectors.

Cases (vi)-(ix) are easy, simply use the spaces $F + \psi(F) + \psi^2(F)$ where F always denotes the first-non zero step of the flag.

7.5 Stabilizers on the unit coset

In the previous section we proved the first part of Theorem 6.5.1. In the following we analyze the unit coset in more detail. We prove strictness in types A and D.

7.5.1 Remark The covering $\mathrm{Spin}(V) \to \mathrm{SO}(V)$ induces an isomorphism on flag varieties. The stabilizers only differ by the kernel of the covering. Therefore in types B and D we may (and will) work with $\mathrm{SO}(V)$ in the following.

7.6. Fix a maximal torus T and a Borel B containing it. We may assume $Q \supset B$ and $\psi \in \mathfrak{t}$. Then $G_\psi \supset T$, and we may assume $T \subset B_\psi \subset B$. Let $P_\psi = BG_\psi$; this is a

standard parabolic subgroup of G containing G_ψ as a Levi subgroup. Let N, N^ψ, N_ψ be the unipotent radicals of B, P_ψ and B_ψ respectively.

Let W_Q be the Weyl group of the Levi subgroup of Q containing T. The double cosets $B\backslash G/Q$ are parametrized by W/W_Q. Let $w \in W$, and let $\dot{w}$ be any lift of w to $N_G(T)$. Then we can identify the B-orbit of $\dot{w}Q/Q$ with $N/N \cap {}^wQ$ (write ${}^wQ = \mathrm{Ad}(\dot{w})Q$). The left translation action of $B_\psi = N_\psi T$ on $B\dot{w}Q/Q$ becomes left translation of N_ψ on $N/N \cap {}^wQ$ and the action of T by conjugation. Using the left N_ψ-action, every B_ψ orbit on $N/N\cap {}^wQ$ intersects $N^\psi/N^\psi \cap {}^wQ$. On $N^\psi/N^\psi \cap {}^wQ$, there is the residual action of $B_\psi \cap {}^wQ$ by conjugation. Therefore, it suffices to show that the stabilizers of the action of $B_\psi \cap {}^wQ$ on $N^\psi/N^\psi \cap {}^wQ$ by conjugation contain nontrivial tori, except in one case.

Let $W_\psi \subset W$ be the Weyl group of G_ψ (with respect to T).

7.6.1 Lemma *Let w_0 be the longest element in W with respect to the simple reflections defined by B. If w is not in the double coset $W_\psi w_0 W_Q$, then the action of T on $N^\psi/N^\psi \cap {}^wQ$ has positive dimensional stabilizers.*

Proof Let N_Q^- be the unipotent radical of the parabolic of G opposite to Q and containing T. The inclusion $N^\psi \cap {}^wN_Q^- \subset N^\psi$ induces an isomorphism $N^\psi \cap {}^wN_Q^- \xrightarrow{\sim} N^\psi/N^\psi \cap {}^wQ$. To verify the claim, it suffices to show that the set of roots $\Phi(N^\psi \cap {}^wN_Q^-)$ does not span $\mathbb{X}^*(\mathbb{T})_\mathbb{Q}$ rationally, unless $w \in W_\psi w_0 W_Q$. This can be checked case by case.

For example, consider the case where $G = \mathrm{Sp}(V)$ and P_ψ and Q are both Siegel parabolic subgroups. Let $\{e_1, \ldots, e_n, e_{-n}, \ldots, e_{-1}\}$ be a symplectic basis of V; let T be the diagonal torus with respect to this basis. The above claim is equivalent to the following statement: let L, L' be two Lagrangian subspaces spanned by part of the basis. Let N_L be the unipotent radical of the stabilizer of L, and similarly define $N_{L'}$. Then as long as $L \neq L'$, the action of T on $L \cap L'$ has positive dimensional stabilizers.

This latter statement can be proved as follows. Identify N_L with $\mathrm{Sym}^2(L)$ using the symplectic form. Then $N_L \cap N_{L'} \subset N_L$ is identified with the subspace $\mathrm{Sym}^2(L \cap L')$ of $\mathrm{Sym}^2(L)$. If $L \neq L'$, let T_1 be the subtorus of T corresponding the basis elements $e_{\pm i}$ such that $e_{\pm i} \notin L \cap L'$. Then T_1 acts trivially on $\mathrm{Sym}^2(L \cap L') \cong N_L \cap N_{L'}$. □

7.6.2 Remark More precisely let U denote the rational span of $\Phi(N^\psi \cap {}^wN_Q^-)$. If $U \neq \mathbb{X}^*(T)_\mathbb{Q}$ then the stabilizer contains $\bigcap_{\alpha\in\Phi(G)\cap U} \ker(\alpha)$. The set $\Phi(G) \cap U$ is the root system of a Levi subgroup of G and $\bigcap_{\alpha\in\Phi(G)\cap U} \ker(\alpha)$ is its center.

7.7. It remains to treat the case $w \in W_\psi w_0 W_Q$. Write $w \in vw_0W_Q$ for some $v \in W_\psi$. We need to consider the stabilizers of the conjugation action of $B_\psi \cap {}^{vw_0}Q$ on $N^\psi/N^\psi \cap {}^{vw_0}Q$. Conjugating by a lifting of v in $N_{G_\psi}(T)$, we may as well consider the action of ${}^{v^{-1}}B_\psi \cap {}^{w_0}Q$ on $N^\psi/N^\psi \cap {}^{w_0}Q$. Let $Q' = ({}^{w_0}Q)^-$ be the opposite parabolic of ${}^{w_0}Q$ and let $M = Q' \cap {}^{w_0}Q$ (common Levi of Q' and ${}^{w_0}Q$). Then ${}^{v^{-1}}B_\psi \cap {}^{w_0}Q$ always contains a Borel subgroup B_H of the Levi subgroup $H = G_\psi \cap M$ of G. Identifying $N^\psi/N^\psi \cap {}^{w_0}Q$ with $N^\psi \cap N_{Q'}$, we reduce to considering the action of B_H on $N^\psi \cap N_{Q'}$ by conjugation (the whole H acts on $N^\psi \cap N_{Q'}$ by conjugation). We remark that after this reduction, the roles played by P_ψ and Q' are symmetric.

Below we describe case-by-case the action of H on $N^\psi \cap N_{Q'}$ in linear algebra terms. In the following, V_n, V_n' always denote an n-dimensional space; when $n = 1$, we use $\mathbb{G}_m(V_1), \mathbb{G}_m(V_1')$ to denote the one-dimensional torus that acts on V_1, V_1' by scaling.

In all cases $N^\psi \cap N_{Q'}$ is a vector group, and we describe it as a representation of H. The Borel subgroup B_H of H acts on $N^\psi \cap N_{Q'}$ with an open orbit with finite stabilizer. In the following we analyze the orbits of B_H acting on $N^\psi \cap N_{Q'}$ more precisely. Recall that by P_d we denote the stabilizer of a d-dimensional subspace V_d and by $P_{d,d'}$ the stabilizer of a flag $0 \subset V_d \subset V_{d'} \subset V$ with $V_{d'}$ being d'-dimensional. We list the possible conjugacy classes of Q and the corresponding H-representation $N^\psi \cap N_{Q'}$.

7.8 Type *A*

(1) $H = T$ (in this case there is no need to describe $N^\psi \cap N_{Q'}$).

(2a) $G = \mathrm{SL}_{2m}$, $(P_\psi, Q) \sim (P_m, P_{m,m+1})$ or $(P_m, P_{m,2m-1})$ or $(P_m, P_{1,m})$ or $(P_m, P_{m-1,m})$. Then we have

$$H = S(\mathrm{GL}(V_m) \times \mathrm{GL}(V_{m-1}) \times \mathbb{G}_m(V_1)),$$

and $N^\psi \cap N_{Q'} = \mathrm{Hom}(V_{m-1}, V_m) \oplus \mathrm{Hom}(V_1, V_m)$ or its dual.

(2b) $G = \mathrm{SL}_{2m}$, $(P_\psi, Q) \sim (P_m, P_{1,m})$ or $(P_m, P_{m-1,2m-1})$. Then we have

$$H = S(\mathrm{GL}(V_{m-1}) \times \mathrm{GL}(V_{m-1}') \times \mathbb{G}_m(V_1) \times \mathbb{G}_m(V_1')),$$

and $N^\psi \cap N_{Q'} = \mathrm{Hom}(V_{m-1}, V_{m-1}') \oplus \mathrm{Hom}(V_1, V_1') \oplus \mathrm{Hom}(V_{m-1}, V_1')$ or its dual.

(3a) $G = \mathrm{SL}_{2m+1}$, $(P_\psi, Q) \sim (P_m, P_{m,m+1})$ or $(P_m, P_{1,m+1})$ or $(P_{m+1}, P_{m,2m})$ or $(P_{m+1}, P_{m,m+1})$. Then we have

$$H = S(\mathrm{GL}(V_m) \times \mathrm{GL}(V_m') \times \mathbb{G}_m(V_1)),$$

and $N^\psi \cap N_{Q'} = \mathrm{Hom}(V_m, V_m') \oplus \mathrm{Hom}(V_1, V_m')$ or its dual.

(3b) $G = \mathrm{SL}_{2m+1}$, $(P_\psi, Q) \sim (P_m, P_{m,2m})$ or $(P_{m+1}, P_{1,m+1})$. Then we have

$$H = S(\mathrm{GL}(V_{m-1}) \times \mathrm{GL}(V_m) \times \mathbb{G}_m(V_1) \times \mathbb{G}_m(V_1')),$$

and $N^\psi \cap N_{Q'} = \mathrm{Hom}(V_m, V_{m-1}) \oplus \mathrm{Hom}(V_m, V_1) \oplus \mathrm{Hom}(V_1, V_1')$ or its dual.

(4) $G = \mathrm{SL}_6$. We have

$$H = S(\mathrm{GL}(V_2) \times \mathrm{GL}(V_2') \times \mathrm{GL}(V_2'')),$$

and $N^\psi \cap N_{Q'} = \mathrm{Hom}(V_2, V_2') \oplus \mathrm{Hom}(V_2, V_2'')$ or its dual.

Case (1) is clear for dimension reasons: any non-open orbit in $N^\psi \cap N_{Q'}$ has dimension less than that of T, hence has positive dimensional subtorus in the stabilizers. In

addition it's easy to check that the stabilizer on the open orbit is just the center of SL_n and that outside the open orbit the stabilizer always contains the center of a maximal Levi subgroup.

We reduce the cases (2a)(2b)(3a)(3b) to Case (1). We give the argument only for (2a). Let $a : V_{m-1} \to V_m$, $b : V_1 \to V_m$, and $a \oplus b : V_{m-1} \oplus V_1 \to V_m$. The image $\mathrm{Stab}_{B_H}(a, b) \to \mathrm{GL}(\mathrm{coker}(a \oplus b))$ contains a Borel subgroup of the target. Therefore $\mathrm{Stab}_{B_H}(a, b)$ contains a nontrivial torus if $a \oplus b$ is not surjective (equivalently not an isomorphism). In the remaining case, we may assume $a \oplus b : V_{m-1} \oplus V_1 \xrightarrow{\sim} V_m$ is an isomorphism. Such (a, b) form a single orbit O under H. We have an isomorphism of stacks $B_H \backslash O \cong B(V_{m-1}) \backslash \mathrm{GL}(V_m)/B(V_m)$, where $B_H = B(V_{m-1}) \times B(V_m)$, and $B(V_{m-1})$ is embedded into $\mathrm{GL}(V_m)$ via a. This is exactly the situation of case (1) for $\mathrm{GL}(V_m)$ where B_ψ is the Borel subgroup of a Levi subgroup of of type $(m-1, 1)$. The computation of the generic stabilizer and stabilizer outside the open orbit is also reduced to Case (1).

For Case (4) we can use similar argument as above to reduce to the open H-orbit O where $a : V_2 \to V_2'$ and $b : V_2 \to V_2'$ are both isomorphisms. For this H-orbit O, we use a and b to identify V_2' and V_2'' with V_2, and write $B_H = S(B_2 \times B_2' \times B_2'')$ (where $B_2 \subset \mathrm{GL}(V_2)$ is a Borel etc), then $\mathrm{Stab}_{B_H}(a, b)$ surjects onto $(B_2 \cap B_2' \cap B_2'')/\mathbb{G}_m$ (modulo scalar matrices). If (a, b) is in a non-open B_H-orbit, then two of the Borels B_2, B_2', B_2'' are the same and $(B_2 \cap B_2' \cap B_2'')/\mathbb{G}_m$ contains the center of some maximal Levi subgroup. On the open orbit it's clear that the intersection $B_2 \cap B_2' \cap B_2''$ is the center of GL_2 and hence the generic stabilizer is the center of SL_6.

7.9 Type *B*

(1) $H = \mathrm{GL}(V_n)$, $N^\psi \cap N_{Q'} = \wedge^2(V_n) \oplus V_n$.
(2) $H = T = \mathbb{G}_m(V_1) \times \mathbb{G}_m(V_1')$, $N^\psi \cap N_{Q'} = V_1 \otimes V_1' \oplus V_1 \otimes (V_1')^\vee$
(3) $H = \mathbb{G}_m(V_1) \times \mathbb{G}_m(V_1') \times \mathrm{SO}(V_3)$, $N^\psi \cap N_{Q'} = V_1 \otimes V_3 \oplus V_1 \otimes V_1'$.

Case (1) is not strict as it may have several relevant orbits.

In Case (2) the stabilizer on the open orbit is $\mu_2 = \{\pm \mathrm{id}\}$, so it is not strict.

We consider Case (3). We identify $N^\psi \cap N_{Q'}$ with $V_3 \otimes V_1$ on which $(\lambda, \mu, X) \in \mathbb{G}_m \times \mathbb{G}_m \times \mathrm{SO}(V_3)$ acts as $(v, x) \mapsto (\lambda X v, \lambda \mu x)$. Let $B_3 \in \mathrm{SO}(V_3)$ be the stabilizer of an isotropic line ℓ. Fix a vector (v, x). If v is isotropic or $x = 0$ it's easy to see that the stabilizer of (v, x) in $B_H = B_3 \times \mathbb{G}_m \times \mathbb{G}_m$ contains a torus. Therefore assume that v is anisotropic and $x \neq 0$. Denote by $\langle , \rangle$ the symmetric bilinear form on V_3. Then if $(\lambda, \mu, X) \in \mathrm{Stab}_{B_H}(v, x)$ we find that $\mu = \lambda^{-1}$ and $\langle v, v \rangle = \langle \lambda X v, \lambda X v \rangle = \lambda^2 \langle v, v \rangle$, implying that $\lambda = \pm 1$. The vector v determines a splitting $V = \langle v \rangle^\perp \oplus \langle v \rangle$ and $\mathrm{Stab}_{\mathrm{SO}(V_3)}(v) \cong \mathrm{SO}(\langle v \rangle^\perp) \cong \mathbb{G}_m$. If $v \in \ell^\perp$ the stabilizer $\mathrm{Stab}_{B_H}(v, x)$ contains this $\mathbb{G}_m$. The open orbit is given by those (v, x) where $v \notin \ell^\perp$ and $x \neq 0$. It is easily verified that in this case the stabilizer is $\mu_2 = \{\pm \mathrm{id}\}$ and hence this case is also not strict.

7.10 Type *C*

(1) $H = \mathrm{GL}(V_n)$, $N^\psi \cap N_{Q'} = \mathrm{Sym}^2(V_n)$.
(2) $H = \mathbb{G}_m(V_1) \times \mathbb{G}_m(V_1') \times \mathrm{SL}(V_2)$, $N^\psi \cap N_{Q'} = V_1 \otimes V_2 \oplus V_1 \otimes V_1 \oplus V_1 \otimes V_{1'}$.

We consider Case (1). We may identify $\mathrm{Sym}^2(V_n)$ with the space of quadratic forms on V_n^*. Let $L = \ker(q) \subset V_n^*$, and $P_L \subset \mathrm{GL}(V_n)$ be the parabolic subgroup stabilizing L. Then we have a natural map $\mathrm{Stab}_{B_H}(q) \subset B_H \cap P_L \to \mathrm{GL}(L)$ whose image is a Borel subgroup of $\mathrm{GL}(L)$. Therefore if $L \neq 0$, $\mathrm{Stab}_{B_H}(q)$ contains a nontrivial torus. Now suppose $L = 0$, i.e., q is nondegenerate. We equip V_n with the quadratic form induced from q, and still denote it by q. Let B_H be the stabilizer of a complete flag $F = (0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n)$. Consider the relative position of the flag $F^\perp = (0 \subset V_{n-1}^\perp \subset \cdots V_1^\perp \subset V_n)$ ($(-)^\perp$ is taken under the quadratic form q). If F and $F^\perp$ are not opposite, consider the first $i \geq 1$ such that $V_i \cap V_i^\perp \neq 0$ (i.e., the first i such that $q|_{V_i}$ is degenerate), in which case $\ker(q|V_i)$ is 1-dimensional. Then $\mathrm{Stab}_{B_H}(q) \to \mathbb{G}_m(\ker(q|V_i))$ is surjective, hence $\mathrm{Stab}_{B_H}(q)$ contains a nontrivial torus. If F and $F^\perp$ are opposite, then q is the in the open B_H-orbit of $\mathrm{Sym}^2(V_n)$. The intersection of the stabilizers of F and $F^\perp$ is a maximal torus T of $\mathrm{GL}(V_n)$ and the stabilizer on the open orbit is the 2-torsion $T[2] \cong \mu_2^n$.

We consider Case (2). An element $(\lambda, \mu, X) \in H$ acts on a vector (v, x, y) as $(\lambda X v, \lambda^2 x, \lambda\mu y)$. Let B_2 be the stabilizer of a line $\ell \subset V_2$ and $B_H = \mathbb{G}_m \times \mathbb{G}_m \times B_2$. If $v = 0$ then $\mathrm{Stab}_{B_H}(v, x, y)$ contains B_2. If $x = 0$ then the stabilizer contains the intersection of B_2 and the stabilizer of $\langle v \rangle$ and if $y = 0$ the stabilizer contains a torus coming from the $\mathbb{G}_m$-factors. We therefore assume that $v \neq 0$, $x \neq 0$ and $y \neq 0$. The open orbit is given by those (v, x, y) for which in addition $v \notin \ell$. The stabilizer on the open orbit is $\mu_2 = \{\pm \mathrm{id}\}$. If $v \in \ell$ the stabilizer does not contain a torus but the unipotent radical of B_2.

7.11 Type *D*

(1) $(P_\psi, Q) \sim (P_4, P_{1,2})$. Then $H = \mathbb{G}_m(V_1) \times \mathbb{G}_m(V_1') \times \mathrm{GL}(V_2)$, $N^\psi \cap N_{Q'} = V_1 \otimes V_2 \oplus V_1' \otimes V_2 \oplus V_1 \otimes V_1'$.
(2) $H = \mathrm{GL}(V_2) \times \mathrm{GL}(V_3)$, $N^\psi \cap N_{Q'} = V_3 \otimes V_2 \oplus \wedge^2(V_3)$.
(3) $H = \mathrm{GL}(V_3) \times \mathrm{GL}(V_3')$, $N^\psi \cap N_{Q'} = V_3 \otimes V_3' \oplus \wedge^2(V_3)$.

We consider Case (1). An element $(\lambda, \mu, X) \in H$ acts on a vector (u, v, x) via $(u, v, x) \mapsto (\lambda X u, \mu X v, \lambda\mu x)$. Let $B_2 \in GL(V_2)$ be a Borel subgroup stabilizing a line $\ell \subset V_2$ and $B_H = \mathbb{G}_m \times \mathbb{G}_m \times B_2$. If $x = 0$ the stabilizer of (u, v, x) contains a torus coming from the $\mathbb{G}_m$-factors. If $u = 0$ then the condition on X is equivalent to asking that it stabilizes the line spanned by v and $\mathrm{Stab}_{B_H}(0, v, w)$ contains the intersection of B_2 with the Borel subgroup stabilizing $\langle v \rangle$. The same happens for $v = 0$ and we may therefore assume that $u \neq 0$, $v \neq 0$ and $x \neq 0$. The open orbit is given by those vectors (u, v, x) for which $x \neq 0$, $u \notin \ell$, $v \notin \ell$ and u and v are linearly independent (i.e. X is contained in the intersection of three pairwise different Borel subgroups). It's easy to check that the stabilizer on the open orbit is $\mu_2 \cong \{\pm \mathrm{id}\}$. One may check explicitly that outside the open orbit the stabilizers contain tori which

are contained in centers of maximal Levi subgroups. For $x = 0$ one finds a torus of the form $Z(\mathrm{GL}_2)$ in a Levi subgroup isomorphic to $\mathrm{GL}_2 \times \mathrm{SO}(4)$ and for $x \neq 0$ one obtains the center of a Levi subgroup isomorphic to $\mathbb{G}_m \times \mathrm{SO}(6)$ (as long as we're outside the open orbit).

For Case (2) we identify $V_3 \otimes V_2$ with $\mathrm{Hom}(V_3^*, V_2)$. An element $(Y, X) \in H$ acts on $f \in \mathrm{Hom}(V_3^*, V_2)$ via $f \mapsto YfX^{-1}$. Let $(\omega, f) \in \mathrm{Hom}(V_3^*, V_2) \oplus \wedge^2(V_3)$ and let $B_H = \mathrm{Stab}(\ell) \times \mathrm{Stab}(F)$ for a line $\ell \subset V_2$ and a full flag F given by $0 \subset F_1 \subset F_2 \subset V_3$. If f is not surjective clearly $\mathrm{Stab}_{B_H}(f, \omega)$ contains a torus. We therefore assume f is surjective and hence has a one-dimensional kernel $L \subset V_3^*$. If $(Y, X) \in \mathrm{Stab}_{B_H}(f, \omega)$ then X stabilizes the flag

$$0 \subset L \subset f^{-1}(\ell) \subset V_3^*.$$

Denote by $F^\perp$ the flag in V_3^* orthogonal to F with respect to the canonical pairing for V_3 and V_3^*. The open orbit is given by those (f, ω) for which $F^\perp$ and the above flag are opposite (i.e. X lies in a maximal torus of $\mathrm{GL}(V_3)$) and for which ω (considered as a 2-form on V_3^*) is not contained in any of the duals of the planes $\wedge^2(F_2^\perp \oplus L) \oplus \wedge^2(f^{-1}(\ell))$, $\wedge^2(F_2^\perp \oplus L) \oplus \wedge^2(F_1^\perp)$ or $\wedge^2(F_1^\perp) \oplus \wedge^2(f^{-1}(\ell))$. It is easy to verify that the stabilizer on the open orbit is $\mu_2 = \{\pm \mathrm{id}\}$ and that it contains a torus outside this orbit. More precisely if f is not surjective, the stabilizer contains the center of a Levi subgroup isomorphic to $\mathbb{G}_m \times \mathrm{SO}(8)$. If f is surjective then outside the open orbit the stabilizers contain the center of a Levi subgroup isomorphic to GL_5. Case (3) works the same. This concludes the proof of strictness in type A and D.

8 Potential examples

In this section we give a list of triples $(\mathbf{P}_\infty, \psi, \mathbf{Q}_0)$ in type A and exceptional types as potential examples of euphotic automorphic data with a generic choice of χ. In these examples, we only check that B_ψ acts on L/Q with an open orbit with finite stabilizers (part of condition (1) in Definition 3.4.1).

8.1 Type *A*

8.1.1 Setup

Let $\mathbb{G} = \mathrm{PGL}(V)$ for some vector space V over k of dimension n, and let G be the split form of $\mathbb{G}$ over F. Since all parahoric subgroups of $G(F_\infty)$ can be conjugated to be contained in $G(\mathcal{O}_\infty)$ in this case, we may assume $\mathbf{P}_\infty \subset G(\mathcal{O}_\infty)$. For such $\mathbf{P}_\infty$ the corresponding $\mathbb{Z}/m\mathbb{Z}$-grading on $\mathfrak{g} = \oplus_{i \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{g}(i)$ is induced from a $\mathbb{Z}/m\mathbb{Z}$-grading on the vector space

$$V = \oplus_{i \in \mathbb{Z}/m\mathbb{Z}} V_i$$

such that

$$\mathfrak{g}(i) = \oplus_{j\in\mathbb{Z}/m\mathbb{Z}}\mathrm{Hom}_k(V_j, V_{i+j}), \quad \forall i \in \mathbb{Z}/m\mathbb{Z} - \{0\},$$
$$\mathfrak{g}(0) = \mathrm{Lie}\, L = (\oplus_{j\in\mathbb{Z}/m\mathbb{Z}}\mathrm{End}_k(V_j))/k \cdot \mathrm{id}_V.$$

Conversely, any $\mathbb{Z}/m\mathbb{Z}$-grading on V with $V_i \neq 0$ for all $i \in \mathbb{Z}/m\mathbb{Z}$ arises from a parahoric subgroup $\mathbf{P}_\infty \subset G(\mathcal{O}_\infty)$. Note that

$$L \cong \left(\prod_{j\in\mathbb{Z}/m\mathbb{Z}} \mathrm{GL}(V_j)\right)/\Delta\mathbb{G}_m,$$
$$V_{\mathbf{P}}^* = \oplus_{j\in\mathbb{Z}/m\mathbb{Z}}\mathrm{Hom}_k(V_j, V_{j-1}).$$

We give two classes of potential examples.

8.1.2 Case (1)

Assume the dimensions of V_i satisfy

$$\dim V_0 = d_0, \quad \dim V_i = d \text{ for } i \neq 0, \quad \text{and } d_0 > d > 0.$$

Let $Q_0 \subset \mathrm{GL}(V_0)$ be a parabolic subgroup; let $\psi_0 \in \mathrm{End}(V_0)$ be a semisimple element. We assume

- The pair (Q_0, ψ_0) appears in the list of hyperspecial euphotic data of type A in Sect. 6.2.
- Let V_0^0 be the zero eigenspace of ψ_0; and let V_0' be the sum of nonzero eigenspaces. Then $\dim V_0^0 = d_0 - d$ and $\dim V_0' = d$.

With these preliminary data, we construct Q and ψ as follows. Let $Q \subset L$ be the parabolic subgroup

$$Q = \left(Q_0 \times \prod_{i\in\mathbb{Z}/m\mathbb{Z}-\{0\}} \mathrm{GL}(V_i)\right)/\Delta\mathbb{G}_m.$$

Viewing ψ as a collection of maps $V_i \to V_{i-1}$ for $i \in \mathbb{Z}/m\mathbb{Z}$, we then require it to restrict to isomorphisms $V_0' \xrightarrow{\sim} V_{-1} \xrightarrow{\sim} \cdots \xrightarrow{\sim} V_1 \xrightarrow{\sim} V_0' \subset V_0$, and to restrict to zero on V_0^0. Moreover we require that $\psi^m|V_0 = \psi_0$.

8.1.3. We check that in the above situation, B_ψ acts on L/Q with an open orbit with finite stabilizers. Indeed, $L/Q \cong \mathrm{GL}(V_0)/Q_0$, and $L_\psi \cong \mathrm{PGL}(V_0)_{\psi_0}$ (the centralizer of ψ_0 in $\mathrm{PGL}(V_0)$). Therefore we reduce to the case discussed in Sect. 6.2 for the group $\mathrm{PGL}(V_0)$.

8.1.4 Case (2)

Take $\mathbf{Q}_0$ to be the standard Iwahori subgroup of $G(F_0)$ (i.e., $Q \subset L$ is a Borel subgroup). Fix a decomposition

$$V_i = \ell_i \oplus V_i^0, \quad i \in \mathbb{Z}/m\mathbb{Z}$$

where $\dim \ell_i = 1$. Viewing $\psi \in V_{\mathbf{P}}^*$ as a collection of maps $V_i \to V_{i-1}$, let it restrict to an isomorphism $\ell_i \xrightarrow{\sim} \ell_{i-1}$ and be zero on V_i^0.

8.1.5. We check that in the above situation, B_ψ acts on L/Q with an open orbit with finite stabilizers. We have $L/Q = \prod_{i\in\mathbb{Z}/m\mathbb{Z}} \mathrm{Fl}(V_i)$. We also have $L_\psi \cong \prod_{i\in\mathbb{Z}/m\mathbb{Z}} \mathrm{GL}(V_i^0)$ (an extra factor of $\mathbb{G}_m$ acting on all the lines ℓ_i gets cancelled after dividing by scalar matrices). Therefore $B_\psi \cong \prod_{i\in\mathbb{Z}/m\mathbb{Z}} B_{\psi,i}$ where $B_{\psi,i} \subset \mathrm{GL}(V_i^0)$ is a Borel subgroup. The required property of the B_ψ-action on L/Q follows from the same property for the $B_{\psi,i}$-action on $\mathrm{Fl}(V_i)$, which is checked in case (2) of Sect. 6.2.

8.1.6 Remark We expect case (2) to correspond to hypergeometric local systems with slope $1/m$ at ∞ and unipotent monodromy at 0. Rigid automorphic data corresponding to hypergeometric local systems are constructed in the work of Kamgarpour and Yi [12].

8.2 Convention

In the exceptional cases, we always assume G is of *adjoint type*. We will indicate the type of $\mathbf{P}_\infty$ by coloring the affine Dynkin diagram of $G(F_\infty)$: the white nodes are simple roots of $L = L_{\mathbf{P}}$, and the black nodes are simple roots not contained in L.

When we describe L and $V_{\mathbf{P}}^*$, we will use V_i, V_i', W_i, F_i, etc. to indicate vector spaces of dimension i over k.

8.3 Type ${}^3D_4$

Type of $\mathbf{P}_\infty$:

In this case $m = 3$. We have $L \cong \mathrm{PGL}(V_3)$ acting on $V_{\mathbf{P}}^* \cong \mathrm{Sym}^3(V_3) \otimes \det(V_3)^{-1}$.

Choose a basis $\{e_1, e_2, e_3\}$ for V_3. Take $\psi = (e_1e_2e_3) \otimes (e_1 \wedge e_2 \wedge e_3)^{-1} \in V_{\mathbf{P}}^*$. Then L_ψ is the normalizer of the diagonal torus in L with respect to the basis $\{e_i\}$.

Potential choices of Q: take Q to be a maximal proper parabolic subgroup of L so that $L/Q = \mathbb{P}(V_3)$ or $\mathbb{P}^\vee(V_3)$. Then $B_\psi = L_\psi^\circ$ acts on L/Q with an open free orbit.

8.4 Type F_4

8.4.1. Type of $\mathbf{P}_\infty$:

In this case $m = 3$. We have $L \cong (\mathrm{SL}(V_3) \times \mathrm{SL}(V_3'))/\Delta\mu_3$ (modulo diagonal center) acting on $V_{\mathbf{P}}^* = \mathrm{Sym}^2(V_3) \otimes V_3'$. The factor $\mathrm{SL}(V_3)$ has short roots of G and $\mathrm{SL}(V_3')$ has long roots.

Choose a basis $\{x_1, x_2, x_3\}$ of V_3, and a basis $\{e_1, e_2, e_3\}$ of V_3'. Take $\psi = x_1^2 \otimes e_1 + x_2^2 \otimes e_2 + x_3^2 \otimes e_3$. Then L_ψ° is a 2-dimensional torus. The projection $L_\psi^\circ \to \mathrm{PGL}(V_3)$ is an isomorphism onto the diagonal torus of $\mathrm{PGL}(V_3)$ with respect to the basis $\{x_i\}$. The other projection $L_\psi^\circ \to \mathrm{PGL}(V_3')$ is a μ_2^2-cover of the maximal torus of $\mathrm{PGL}(V_3')$ with respect to the basis $\{e_i\}$.

Potential choices of Q:

(1) $L/Q = \mathbb{P}(V_3)$ or $\mathbb{P}^\vee(V_3)$. In these cases $B_\psi = L_\psi^\circ$ acts on L/Q with an open free orbit.
(2) $L/Q = \mathbb{P}(V_3')$ or $\mathbb{P}^\vee(V_3')$. In these cases $B_\psi = L_\psi^\circ$ acts on L/Q with an open orbit with stabilizer μ_2^2.

8.4.2. Type of $\mathbf{P}_\infty$:

In this case $m = 2$. We have $L \cong (\mathrm{SL}(V_2) \times \mathrm{Sp}(V_6))/\Delta\mu_2$ (V_6 is a symplectic space of dimension 6, μ_2 embeds diagonally into the center of each factor), and $V_{\mathbf{P}}^* \cong V_2 \otimes \overline{\wedge}^3(V_6)$. Here $\overline{\wedge}^3(V_6)$ is the cokernel $V_6 \to \wedge^3 V_6$ given by wedging with the symplectic form on V_6, so $\dim \overline{\wedge}^3(V_6) = 14$.

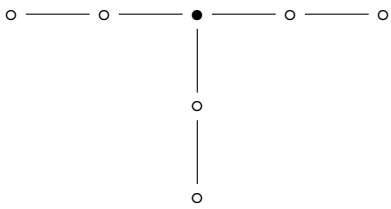
Choose a basis $\{e_1, e_2\}$ for V_2; choose a Lagrangian splitting $V_6 = W_3 \oplus W_3^*$. Let θ be a volume form on W_3 (i.e., $\theta \in \wedge^3 W_3, \theta \neq 0$) and θ^* be the dual volume form on W_3^*. Consider $\psi = e_1 \otimes \theta + e_2 \otimes \theta^*$. Then $L_\psi^\circ \cong \mathrm{GL}(W_3)/\mu_2$, where $\mathrm{GL}(W_3) \hookrightarrow \mathrm{SL}(V_2) \times \mathrm{Sp}(V_6)$ is the following embedding. The projection $\mathrm{GL}(W_3) \to \mathrm{Sp}(V_6)$ identifies $\mathrm{GL}(W_3)$ as the Siegel Levi preserving the splitting $W_3 \oplus W_3^*$. The projection $\mathrm{GL}(W_3) \to \mathrm{SL}(V_2)$ is the composition of $\det : \mathrm{GL}(W_3) \to \mathbb{G}_m$ and $\mathbb{G}_m \to \mathrm{SL}(V_2)$ given by $t \mapsto \mathrm{diag}(t^{-1}, t)$ in the basis $\{e_1, e_2\}$.

Potential choices of Q:

(1) $L/Q \cong \mathbb{P}(V_2) \times \mathbb{P}(V_6)$. Then B_ψ acts on L/Q with an open free orbit.
(2) L/Q is the space of Lagrangians in V_6. The open L_ψ°-orbit of L/Q is isomorphic to the space of non-degenerate quadratic forms on W_3, which has an open B_ψ-orbit with stabilizer μ_2^2.

8.5 Type E_6

8.5.1. Type of $\mathbf{P}_\infty$:

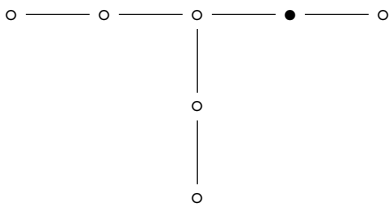


In this case $m = 3$. We have $L \cong \mathrm{SL}(V_3) \times \mathrm{SL}(V_3') \times \mathrm{SL}(V_3'')/(\mu_3 \times \mu_3 \times \mu_3)^{\prod=1}$ acting on $V_{\mathbf{P}}^* \cong V_3 \otimes V_3' \otimes V_3''$. Here $(\mu_3 \times \mu_3 \times \mu_3)^{\prod=1}$ is the subgroup of the central μ_3's with product 1.

Choose bases $\{x_i\}_{1\le i\le 3}$ for V_3, $\{x_i'\}_{1\le i\le 3}$ for V_3' and $\{x_i''\}_{1\le i\le 3}$ for V_3''. Take $\psi = x_1 \otimes x_1' \otimes x_1'' + x_2 \otimes x_2' \otimes x_2'' + x_3 \otimes x_3' \otimes x_3'' \in V_{\mathbf{P}}^*$. Then L_ψ° is a 4-dimensional torus. The projection $L_\psi^\circ \to \mathrm{PGL}(V_3) \times \mathrm{PGL}(V_3')$ is an isomorphism onto the diagonal torus in the target with respect to the chosen bases. Same for the other two projections.

Potential choices of Q: L/Q can be $\mathbb{P}(V_3) \times \mathbb{P}(V_3')$, or changing $\mathbb{P}$ to $\mathbb{P}^\vee$, and changing (V_3, V_3') to other pairs (V_3', V_3'') or (V_3'', V_3). The action of $B_\psi = L_\psi^\circ$ on L/Q has an open free orbit in all cases.

8.5.2. Type of $\mathbf{P}_\infty$:



and two other cases by symmetry.

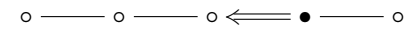
In this case $m = 2$. We have $L \cong \mathrm{SL}(V_6) \times \mathrm{SL}(V_2)/(\mu_3 \times 1)\Delta\mu_2$ acting on $V_{\mathbf{P}}^* \cong \wedge^3(V_6) \otimes V_2$.

Choose a basis $\{e_1, e_2\}$ for V_2; choose a splitting $V_6 = W_3 \oplus W_3'$ into two 3-dimensional spaces. Let θ (resp. θ') be a volume form on W_3 (resp. W_3'). Consider $\psi = \theta \otimes e_1 + \theta' \otimes e_2$. The projection $L_\psi^\circ \to \mathrm{PGL}(V_6)$ identifies L_ψ° with the Levi subgroup of $\mathrm{PGL}(V_6)$ preserving the splitting $W_3 \oplus W_3'$.

Potential choices of Q: L/Q is a partial flag variety of $\mathrm{PGL}(V_6)$ of type $(3, 2, 1)$ (dimensions of associated graded of the partial flag, in any order). Now B_ψ is a Borel of L_ψ°, which projects isomorphically to a Levi of $\mathrm{PGL}(V_6)$ of type $(3, 3)$. The situation $B_\psi \backslash L/Q$ appears as a special case of Sect. 6.2(4), from which we know that B_ψ acts on L/Q with an open free orbit.

8.6 Type ${}^2E_6$

8.6.1. Type of $\mathbf{P}_\infty$:

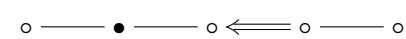


In this case $m = 4$. Then $L = (\mathrm{SL}(V_4) \times \mathrm{SL}(V_2))/\Gamma$ acting on $V_{\mathbf{P}}^* \cong \mathrm{Sym}^2(V_4) \otimes V_2$, where Γ is the central subgroup $\{(x, x^2) \in \mu_4 \times \mu_2\}$.

Choose a basis $\{e_1, e_2\}$ for V_2 and a basis $\{x_1, y_1, x_2, y_2\}$ for V_4. Take $\psi = x_1 y_1 \otimes e_1 + x_2 y_2 \otimes e_2$. Then the projection $L_\psi^\circ \to \mathrm{PGL}(V_4)$ is an isomorphism onto the diagonal torus with respect to the basis $\{x_1, y_1, x_2, y_2\}$.

Potential choices of Q: $L/Q = \mathbb{P}(V_4)$ or $\mathbb{P}^\vee(V_4)$. It is clear that in both cases $B_\psi = L_\psi^\circ$ acts on L/Q with an open free orbit.

8.6.2. Type of $\mathbf{P}_\infty$:



In this case $m = 4$. We have $L = (\mathrm{Spin}(V_7) \times \mathrm{SL}(V_2))/\Delta\mu_2$ acting on $V_{\mathbf{P}}^* = \Delta_8 \otimes V_2$ where Δ_8 is the 8-dimensional spin representation of $\mathrm{Spin}(V_7)$.

We have an embedding $\varphi : \mathrm{Spin}(V_7)/P_3 \hookrightarrow \mathbb{P}\Delta_8$, where $\mathrm{Spin}(V_7)/P_3$ classifies maximal isotropic subspaces in V_7. Choose a splitting $V_7 = W_3 \oplus W_3' \oplus \langle x_0 \rangle$, where W_3 and W_3' are maximal isotropic and paired perfectly to each other and both orthogonal to x_0. Let $\{e_1, e_2\}$ be a basis for V_2. Take $\psi = \widetilde{\varphi}([W_3]) \otimes e_1 + \widetilde{\varphi}([W_3']) \otimes e_2$ (here $\widetilde{\varphi}([W_3]) \in \Delta_8$ is a lifting of $\varphi([W_3])$, unique up to a scalar; same for $\widetilde{\varphi}([W_3'])$). The projection $L_\psi^\circ \to \mathrm{SO}(V_7)$ is an isomorphism onto the Levi subgroup isomorphic to $\mathrm{GL}(W_3)$ that stabilizes the splitting $V_7 = W_3 \oplus W_3' \oplus \langle x_0 \rangle$. The projection $L_\psi^\circ \cong \mathrm{GL}(W_3) \to \mathrm{PGL}(V_2)$ factors through the determinant and maps onto the diagonal torus with respect to the basis $\{e_1, e_2\}$.

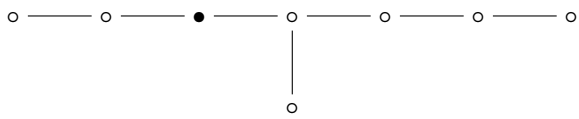
Potential choices of Q:

(1) L/Q is the partial flag variety of $\mathrm{Spin}(V_7)$ classifying maximal isotropic subspaces of V_7. The situation $B_\psi \backslash L/Q$ appears as a special case of Sect. 6.3(1), and we have checked that B_ψ acts on L/Q with an open orbit with finite stabilizers.

(2) $L/Q = \mathbb{Q}(V_7) \times \mathbb{P}(V_2)$, where $\mathbb{Q}(V_7) \subset \mathbb{P}(V_7)$ is the quadric. Identify $B_\psi \backslash L/Q$ with $B(W_3) \backslash \mathbb{Q}(V_7)$, where $B(W_3) \subset \mathrm{SL}(W_3)$ is a Borel subgroup. It is then easy to check that $B(W_3)$ acts on $\mathbb{Q}(V_7)$ with an open free orbit.

8.7 Type E_7

8.7.1. Type of $\mathbf{P}_\infty$:



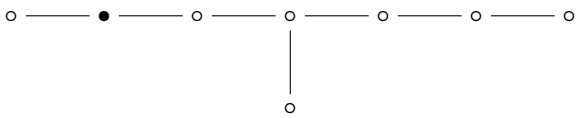
and another case by symmetry.

In this case $m = 3$. We have $L \cong \mathrm{SL}(V_6) \times \mathrm{SL}(V_3)/(\mu_2 \times 1)\Delta\mu_3$ acting on $V_{\mathbf{P}}^* \cong \wedge^2(V_6) \otimes V_3$.

Choose a basis $\{e_1, e_2, e_3\}$ for V_3; choose a splitting $V_6 = W_1 \oplus W_2 \oplus W_3$ into three 2-dimensional subspaces. Let θ_i be a volume form on W_i. Consider the element $\psi = \theta_1 \otimes e_1 + \theta_2 \otimes e_2 + \theta_3 \otimes e_3$. The projection $L_\psi^\circ \to \mathrm{PGL}(V_6)$ is an isomorphism onto the Levi of $\mathrm{SL}(V_6)$ stabilizing the splitting $V_6 = W_1 \oplus W_2 \oplus W_3$, and the projection $L_\psi^\circ \to \mathrm{PGL}(V_3)$ has image equal to the diagonal torus with respect to the basis $\{e_i\}$.

Potential choices of Q: L/Q is the partial flag variety of $\mathrm{PGL}(V_6)$ with associated graded dimensions (4, 2) (in any order). The situation $B_\psi \backslash L/Q$ appears in the example Sect. 6.2(8), from which we know that B_ψ acts on L/Q with an open free orbit.

8.7.2. Type of $\mathbf{P}_\infty$:



and another case by symmetry.

In this case $m = 2$. Then L is isogenous to $(\mathrm{Spin}^+(V_{12}) \times \mathrm{SL}(V_2))/\Delta\mu_2$ acting on $V_{\mathbf{P}}^* \cong \Delta_{32}^+ \otimes V_2$, where $\mathrm{Spin}^+(V_{12})$ is one of the half-spin quotient of $\mathrm{Spin}(V_{12})$ acting on its half-spin representation Δ_{32}^+.

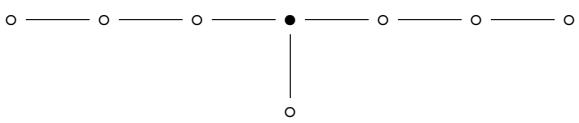
Choose a basis $\{e_1, e_2\}$ for V_2. There is an embedding $\varphi : \mathrm{Spin}(V_{12})/P_6 \hookrightarrow \mathbb{P}\Delta_{32}^+$, where $\mathrm{Spin}(V_{12})/P_6$ is the partial flag variety of one of the two families of Lagrangian subspaces in V_{12}. Fix a splitting $V_{12} = W_6 \oplus W_6'$ into Lagrangians. Take $\psi = \widetilde{\varphi}([W_6]) \otimes e_1 + \widetilde{\varphi}([W_6']) \otimes e_2$ (here $\widetilde{\varphi}([W_6])$ is a lifting of $\varphi([W_6])$ to Δ_{32}, up to scalar; same for $\widetilde{\varphi}([W_6])$). The projection $L_\psi^\circ \to \mathrm{PSO}(V_{12})$ is an isomorphism onto the Siegel Levi stabilizing the splitting $V_{12} = W_6 \oplus W_6'$, so $L_\psi^\circ \cong \mathrm{GL}(W_6)/\mu_2$. The projection $L_\psi^\circ \cong \mathrm{GL}(W_6)/\mu_2 \to \mathrm{PGL}(V_2)$ factors through the determinant and maps onto the diagonal torus of $\mathrm{PGL}(V_2)$ with respect to the basis $\{e_1, e_2\}$. Note that $\dim B_\psi = 21$.

Potential choices of Q:

(1) $L/Q = \mathrm{SO}(V_{12})/P_3$ is the partial flag variety classifying isotropic $F_3 \subset V_{12}$. The situation $B_\psi \backslash L/Q$ has been analyzed in Sect. 6.5(3).
(2) $L/Q = \mathrm{SO}(V_{12})/P_5 \times \mathbb{P}(V_2)$, where the first factor classifies isotropic $F_5 \subset V_{12}$. We check the open orbit condition as follows. We first reduce to study the action of B_ψ^1 (a Borel subgroup of $\mathrm{SL}(W_6)$) on $Y = \mathrm{SO}(V_{12})/P_5$. Note that Y classifies a pair of Lagrangians $U_6, U_6' \subset V_{12}$ such that $\dim(U_6 \cap U_6') = 5$. We may assume U_6 is conjugate to W_6. There is an open subset $Y' \subset Y$ classifying those (U_6, U_6') such that U_6 is the graph of a skew-symmetric map $a : W_6' \to W_6$. We may identify Y' with $\wedge^2(W_6) \times \mathbb{P}(W_6)$ (the choice of U_6' is the same as choosing a hyperplane in U_6, or in W_6'). The situation of B_ψ^1 acting on Y' is essentially the same as case (1) of Sect. 7.9.
(3) $L/Q = \mathrm{SO}(V_{12})/P_{1,6} \times \mathbb{P}(V_2)$, where the first factor classifies isotropic $F_1 \subset F_6 \subset V_{12}$ for F_6 a Lagrangian in the same connected component of W_6. The same

argument as in the previous case reduces to the action of B_ψ^1 on $\wedge^2 W_6 \times \mathbb{P}^\vee(W_6)$, which is essentially the same as case (1) of Sect. 7.9.

8.7.3. Type of $\mathbf{P}_\infty$:



In this case $m = 4$. We have L isogenous to $(\mathrm{SL}(V_4) \times \mathrm{SL}(V_4') \times \mathrm{SL}(V_2))/\Gamma$ acting on $V_{\mathbf{P}}^* \cong V_4^* \otimes V_4' \otimes V_2$, where Γ is the central subgroup $\{(x, y, z) \in \mu_4 \times \mu_4 \times \mu_2 | x^{-1}yz = 1\}$ of L.

Fix splittings $V_4 = X_2 \oplus Y_2$ and $V_4' = X_2' \oplus Y_2'$ into planes. Let $\varphi_X : X_2 \xrightarrow{\sim} X_2'$ and $\varphi_Y : Y_2 \xrightarrow{\sim} Y_2'$ be isomorphisms. Let $\widetilde{\varphi}_X$ be the composition $V_4 \twoheadrightarrow X_2 \xrightarrow{\varphi_X} X_2' \hookrightarrow V_4'$, viewed as an element in $V_4^* \otimes V_4'$. Similarly define $\widetilde{\varphi}_Y$. Let $\{e_1, e_2\}$ be a basis of V_2. Take $\psi = \widetilde{\varphi}_X \otimes e_1 + \widetilde{\varphi}_Y \otimes e_2$. Then L_ψ° is the image of $i : S(\mathrm{GL}(X) \times \mathrm{GL}(Y)) \times \mathbb{G}_m \to \mathrm{SL}(V_4) \times \mathrm{SL}(V_4') \times \mathrm{SL}(V_2) \twoheadrightarrow L$. Here i sends (g_X, g_Y, λ) (with $\det(g_X)\det(g_Y) = 1$) to the triple $(g_X \oplus g_Y, \lambda^{-1}\varphi_X g_X \varphi_X^{-1} \oplus \lambda \varphi_Y g_Y \varphi_Y^{-1}, \mathrm{diag}(\lambda, \lambda^{-1}))$. We have $\dim B_\psi = 6$.

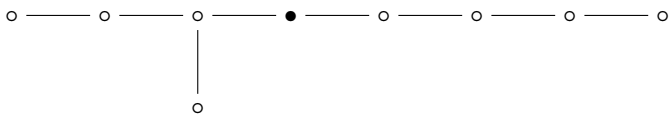
Potential choices of Q:

(1) $L/Q = \mathrm{SL}(V_4)/Q_1 \times \mathbb{P}(V_2)$, where $\mathrm{SL}(V_4)/Q_1$ is a partial flag variety of $\mathrm{SL}(V_4)$ with associated graded dimensions $(2, 1, 1)$ (in any order). The situation $B_\psi \backslash L/Q$ appears as a special case of Sect. 6.2(4), and we have checked that B_ψ acts on L/Q with an open orbit with finite stabilizers.

(2) $L/Q = \mathrm{SL}(V_4')/Q_1' \times \mathbb{P}(V_2)$, where $\mathrm{SL}(V_4')/Q_1'$ is the partial flag variety of $\mathrm{SL}(V_4')$ with associated graded dimensions $(2, 1, 1)$ (in any order). Again the situation $B_\psi \backslash L/Q$ appears as a special case of Sect. 6.2(4), from which we know that B_ψ acts on L/Q with an open orbit with finite stabilizers.

8.8 Type E_8

Type of $\mathbf{P}_\infty$:



In this case $m = 5$. We have $L \cong (\mathrm{SL}(V_5) \times \mathrm{SL}(V_5'))/\mu_5$ acting on $V_{\mathbf{P}}^* = \wedge^2(V_5) \otimes V_5'$, here the embedding $\mu_5 \hookrightarrow \mathrm{SL}(V_5) \times \mathrm{SL}(V_5')$ is $z \mapsto (z^2 \mathrm{id}_{V_5}, z\mathrm{id}_{V_5'})$.

Choose a basis $\{x_i\}_{1 \le i \le 5}$ of V_5, and a basis $\{e_i\}_{1 \le i \le 5}$ of V_5'. Take $\psi = \sum_{i \in \mathbb{Z}/5\mathbb{Z}} x_{i-1} \wedge x_{i+1} \otimes e_i$. Then the projection $L_\psi^\circ \to \mathrm{PGL}(V_5)$ is an isomorphism onto the diagonal torus of $\mathrm{PGL}(V_5)$ with respect to the basis $\{x_i\}$. Same for the other projection $L_\psi^\circ \to \mathrm{PGL}(V_5')$.

Potential choices of Q: $L/Q = \mathbb{P}(V_5), \mathbb{P}^\vee(V_5), \mathbb{P}(V_5')$ or $\mathbb{P}^\vee(V_5')$. In all these cases $B_\psi = L^\circ_\psi$ acts on L/Q with an open free orbit.

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Appendix A: Factorizable module categories

In this appendix we define and classify semisimple factorizable module categories over a neutral Tannakian category with coefficients. We will apply the classification result here to the category of semisimple perverse sheaves in the automorphic category $\mathcal{D}(\psi, \chi)$ in Sect. 4. The materials presented here are an elementary case of the theory of chiral homology that does not involve the language of ∞-categories, so that we give self-contained proofs.

A.1. Notations

The notations used in the appendix differ from the ones in the main body of the paper.

Let L be an algebraically closed field of characteristic zero. All abelian categories in this subsection will be L-linear. Let Vect denote the category of finite-dimensional vector spaces over L.

Let $\mathcal{P}$ be a semisimple L-linear abelian category such that $\mathrm{End}_{\mathcal{P}}(X) = L$ for each simple object $X \in \mathcal{P}$. Let $\mathrm{Irr}(\mathcal{P})$ denote the set of isomorphism classes of simple objects in $\mathcal{P}$. Objects in $\mathcal{P}$ will be denoted $X, Y, \ldots$.

Let $(\mathcal{R}, \otimes)$ and $(\mathcal{C}, \otimes)$ be semisimple rigid tensor category over L. Objects in $\mathcal{R}$ and $\mathcal{C}$ will be denoted $V, W, \ldots$.

A.2. Factorizable module categories with coefficients

We say that $\mathcal{P}$ is a factorizable $\mathcal{R}$-module category with coefficients in $\mathcal{C}$, if for every finite set I there is a bi-exact functor $\mathcal{R}^{\boxtimes I} \times \mathcal{P} \to \mathcal{C}^{\boxtimes I} \boxtimes \mathcal{P}$ (Deligne's tensor product)

$$(V, X) \mapsto V \star_I X, \text{ for } V \in \mathcal{R}^{\boxtimes I}, X \in \mathcal{P}$$

with the following extra structures:

(1) When $I = \varnothing$, we understand that $\mathcal{R}^{\boxtimes\varnothing} \cong \mathrm{Vect}$, and the action of $\mathcal{R}^{\boxtimes\varnothing}$ is the usual action of Vect on $\mathcal{P}$ by tensoring.
(2) Any map of finite sets $\varphi : I \to J$ induces $\varphi_{\mathcal{R}} : \mathcal{R}^{\boxtimes I} \to \mathcal{R}^{\boxtimes J}$ sending $\boxtimes_{i\in I} V_i$ to $\boxtimes_{j\in J}(\otimes_{i\mapsto j} V_i)$. Similarly it induces $\varphi_{\mathcal{C}} : \mathcal{C}^{\boxtimes I} \to \mathcal{C}^{\boxtimes J}$. Then there is a functorial isomorphism $\varphi_{\mathcal{R}}(V)\star_J X \cong (\varphi_{\mathcal{C}} \boxtimes \mathrm{id}_{\mathcal{P}})(V\star_I X) \in \mathcal{C}^{\boxtimes J} \boxtimes \mathcal{P}$, for $V \in \mathcal{R}^{\boxtimes I}$ and $X \in \mathcal{P}$.
(3) If $I = I' \sqcup I''$ is a partition of I then there is a functorial isomorphism $V'\star_{I'}(V''\star_{I''} X) \cong (V' \boxtimes V'')\star_I X$ for $V' \in \mathcal{R}^{\boxtimes I'}$, $V'' \in \mathcal{R}^{\boxtimes I''}$ and $X \in \mathcal{P}$. Here on the left side, when V' acts on $V''\star_{I''} X \in \mathcal{C}^{\boxtimes I''} \boxtimes \mathcal{P}$, it only acts on the $\mathcal{P}$-factor.

These structures have to satisfy the usual compatibilities: composition of maps in (1), refinement of partitions in (2), and the compatibility of (1) and (2) for maps $\varphi' \sqcup \varphi'' : I' \sqcup I'' \to J' \sqcup J''$. We do not spell out the details.

For $X' \in \mathcal{C}^{\boxtimes I} \boxtimes \mathcal{P}$ and $Y \in \mathcal{P}$, let $\underline{\mathrm{Hom}}(X', Y)$ and $\underline{\mathrm{Hom}}(Y, X')$ denote the inner homs taking values in $\mathcal{C}^{\boxtimes I}$. For example, $\underline{\mathrm{Hom}}(Y, X')$ is characterized by having an isomorphism $\mathrm{Hom}_{\mathcal{C}^{\boxtimes I}}(C, \underline{\mathrm{Hom}}(Y, X')) \cong \mathrm{Hom}_{\mathcal{C}^{\boxtimes I}\boxtimes\mathcal{P}}(C \boxtimes Y, X')$ functorial in $C \in \mathcal{C}^{\boxtimes I}$. Then the axioms imply that for $X, Y \in \mathcal{P}$, $V \in \mathcal{R}^{\boxtimes I}$, there is a functorial isomorphism

$$\underline{\mathrm{Hom}}(V\star_I X, Y) \cong \underline{\mathrm{Hom}}(X, V^\vee\star_I Y) \in \mathcal{C}^{\boxtimes I}. \tag{A.1}$$

For I equal to a singleton set, we denote $V\star_I X$ simply by $V\star X$. The factorizable $\mathcal{R}$-module structure on $\mathcal{P}$ in particular gives an E_2-action $V \mapsto V\star(-)$ of $\mathcal{R}$ on $\mathcal{P}$.

A.2.1 Example (1) Let $r : \mathcal{R} \to \mathcal{C}$ be a tensor functor, which extends to $r^I : \mathcal{R}^{\boxtimes I} \to \mathcal{C}^{\boxtimes I}$. Then for any semisimple abelian category $\mathcal{P}$, $V\star_I X := r^I(V) \boxtimes X$ gives $\mathcal{P}$ the structure of a factorizable $\mathcal{R}$-module category with coefficients in $\mathcal{C}$. Such $\mathcal{P}$ are a categorical analogue of an eigenspace under a commutative algebra action, therefore we say that $\mathcal{P}$ is eigen with eigenvalue $r : \mathcal{R} \to \mathcal{C}$.
(2) A factorizable $\mathcal{R}$-module category with coefficients in $\mathcal{C} = \mathrm{Vect}$ is the same as an E_2-module category for $\mathcal{R}$.
(3) We may combine the above two examples. Let $\mathcal{C}'$ be another semisimple rigid tensor category and $r : \mathcal{R} \to \mathcal{C} \boxtimes \mathcal{C}'$ be a tensor functor. Let $\mathcal{P}$ be an E_2-module category for $\mathcal{C}'$. Then $\mathcal{P}$ also carries the structure of a factorizable $\mathcal{R}$-module category with coefficients in $\mathcal{C}$ as follows. For $V \in \mathcal{R}^{\boxtimes I}$ and $X \in \mathcal{P}$, let $V\star_I X$ be the image of $r^I(V) \boxtimes X$ under the functor $\mathrm{id} \boxtimes a'_I : \mathcal{C}^{\boxtimes I} \boxtimes \mathcal{C}'^{\boxtimes I} \boxtimes \mathcal{P} \to \mathcal{C}^{\boxtimes I} \boxtimes \mathcal{P}$, where a'_I is the action map of $\mathcal{C}'^{\boxtimes I}$ on $\mathcal{P}$. We say that $\mathcal{P}$ is *inflated* from the E_2-action of $\mathcal{C}'$ on $\mathcal{P}$.
(4) As a special case of the above example, consider the case $\mathcal{R} = \mathrm{Rep}(H)$ and $\mathcal{C} = \mathrm{Rep}(M)$ for reductive groups H and M over L. Let $\rho : M \to H$ be a homomorphism and H_ρ be the centralizer of $\rho(M)$, and let $\mathcal{C}' = \mathrm{Rep}(H_\rho)$. Then we have the restriction functor $r : \mathcal{R} = \mathrm{Rep}(H) \to \mathrm{Rep}(M \times H_\rho) \cong \mathcal{C} \boxtimes \mathcal{C}'$. For any E_2-module category $\mathcal{P}$ under $\mathrm{Rep}(H_\rho)$, the construction in (3) gives a factorizable $\mathrm{Rep}(H)$-module category with coefficients in $\mathcal{C} = \mathrm{Rep}(M)$, *inflated* from the E_2-action of $\mathrm{Rep}(H_\rho)$ on $\mathcal{P}$.

A.3 Indecomposable module categories

We call a factorizable $\mathcal{R}$-module category $\mathcal{P}$ with coefficients in $\mathcal{C}$ *indecomposable* if $\mathcal{P}$ is not the direct sum of two nonzero factorizable $\mathcal{R}$-module categories with coefficients in $\mathcal{C}$.

A.3.1 Lemma *Let $\mathcal{P}$ be a semisimple abelian category over L with finite set* $\mathrm{Irr}(\mathcal{P})$ *of simple objects up to isomorphism. Suppose $\mathcal{P}$ is equipped with the structure of a factorizable $\mathcal{R}$-module categories with coefficients in $\mathcal{C}$.*

(1) For $X, Y \in \mathrm{Irr}(\mathcal{P})$, define $X \sim Y$ if for some $V \in \mathcal{R}$, $V \star X$ contains $C \boxtimes Y$ as a direct summand, for some nonzero object $C \in \mathcal{C}$. Then $\sim$ is an equivalence relation.

(2) For each equivalence class $s \in \mathrm{Irr}(\mathcal{P})/\sim$, let $\mathcal{P}_s$ be the full subcategory whose objects are direct sums of objects in s. Then $\mathcal{P}_s$ is an indecomposable factorizable $\mathcal{R}$-module categories with coefficients in $\mathcal{C}$, and $\mathcal{P} \cong \oplus_{s \in \mathrm{Irr}(\mathcal{P})/\sim} \mathcal{P}_s$ as factorizable $\mathcal{R}$-module categories.

Proof (1) The transitivity of $\sim$ is clear from the definition of the $\mathcal{R}$-action. Taking $V = \mathbf{1}_{\mathcal{R}}$ the unit in $\mathcal{R}$, we see that $X \sim X$. To show that $\sim$ is reflexive, suppose Y shows up in $V \star X$, then we have a nonzero map $f : V \star X \to C \boxtimes Y$ for some $C \in \mathcal{C}$. Rewrite f as a map $h : C^\vee \to \underline{\mathrm{Hom}}(V \star X, Y)$ in $\mathcal{C}$. Using the adjunction (A.1), h corresponds to a nonzero map $h' : C^\vee \to \underline{\mathrm{Hom}}(X, V^\vee \star Y)$, which corresponds to a nonzero $f' : C^\vee \boxtimes X \to V^\vee \star Y$, showing that $Y \sim X$.

(2) is clear.

□

A.4. Classification

Now assume $\mathcal{R} = \mathrm{Rep}(H), \mathcal{C} = \mathrm{Rep}(M)$ for reductive groups H and M over L. The next result shows that any indecomposable factorizable $\mathrm{Rep}(H)$-module category with coefficients in $\mathrm{Rep}(M)$ must take the form of Example A.2.1(4).

A.4.1 Theorem *Let $\mathcal{P}$ be a semisimple abelian category over L with finitely many simple objects. Suppose $\mathcal{P}$ is equipped with the structure of an indecomposable factorizable* $\mathrm{Rep}(H)$*-module category with coefficients in $\mathcal{C} = \mathrm{Rep}(M)$. Then there is a homomorphism $\rho : M \to H$, unique up to H-conjugation, such that the factorizable* $\mathrm{Rep}(H)$*-module structure on $\mathcal{P}$ with coefficients in* $\mathrm{Rep}(M)$ *is inflated from an E_2-module structure of $\mathcal{P}$ under* $\mathrm{Rep}(H_\rho)$*, where H_ρ is the centralizer of* $\mathrm{Im}(\rho)$ *in H (see Example A.2.1(4)).*

Combine this theorem with Lemma A.3.1, we can speak about the eigen-decomposition of a decomposable $\mathcal{P}$. We give a statement that does not mention the fiber functors of $\mathcal{R}$ and $\mathcal{C}$ explicitly.

A.4.2 Corollary *Let $\mathcal{P}$ be a semisimple abelian category over L with finitely many simple objects. Let $\mathcal{R}$ and $\mathcal{C}$ be semisimple neutral Tannakian categories over L. Suppose $\mathcal{P}$ is equipped with the structure of a factorizable $\mathcal{R}$-module category with*

coefficients in $\mathcal{C}$. Then there is a well-defined finite set of (isomorphism classes of) tensor functors $\{\sigma : \mathcal{R} \to \mathcal{C}\}_{\sigma\in\Sigma}$ and a unique decomposition

$$\mathcal{P} = \bigoplus_{\sigma\in\Sigma} \mathcal{P}_\sigma$$

such that the factorizable $\mathcal{R}$-module structure on $\mathcal{P}_\sigma$ with coefficients in $\mathcal{C}$ is inflated from an E_2-action of $\mathrm{Rep}(\mathrm{Aut}^\otimes(\sigma))$ *on $\mathcal{P}_\sigma$.*

Proof We choose fiber functors of $\mathcal{R}$ and $\mathcal{C}$ to identify them with $\mathrm{Rep}(H)$ and $\mathrm{Rep}(M)$. In the decomposition of $\mathcal{P}$ into indecomposables (see Lemma A.3.1(2)), we apply Theorem A.4.1 to each $\mathcal{P}_s$ to get a homomorphism $\rho_s : M \to H$, such that $\mathcal{P}_s$ is inflated from an E_2-action of $\mathrm{Rep}(H_{\rho_s})$ on $\mathcal{P}_s$. Now let Σ be the set of H-conjugacy classes of $\{\rho_s\}_{s\in\mathrm{Irr}(\mathcal{P})/\sim}$. A homomorphism $\rho : M \to H$ up to H-conjugacy is the same datum as a tensor functor $\sigma : \mathcal{R} \to \mathcal{C}$, so we may identify Σ with a set of tensor functors $\{\sigma : \mathcal{R} \to \mathcal{C}\}$. For $\sigma \in \Sigma$ with the corresponding $\rho : M \to H$, Let $\mathcal{P}_\sigma$ be the direct sum of $\mathcal{P}_s$ for those ρ_s conjugate to ρ under H. Note that $\mathrm{Aut}^\otimes(\sigma) \cong H_\rho$, so $\mathcal{P}_\sigma$ is inflated from an E_2-action of $\mathrm{Rep}(\mathrm{Aut}^\otimes(\sigma))$. □

A.4.3 Remark We state an equivariant version of Corollary A.4.2. Suppose both $\mathcal{R}$ and $\mathcal{C}$ are equipped with actions of a group Γ. The action of $\gamma \in \Gamma$ on $V \in \mathcal{R}^{\boxtimes I}$ and $W \in \mathcal{C}^{\boxtimes I}$ are denoted V^γ and W^γ. Suppose further that the action of $\mathcal{R}$ on $\mathcal{P}$ is equipped with functorial isomorphisms

$$V^\gamma \star_I X \cong (V \star_I X)^\gamma, \quad \forall \gamma \in \Gamma$$

compatible with the group structure on Γ and the factorization structure. Here the action of γ on the right side is only on the $\mathcal{C}^{\boxtimes I}$-factor. Under these assumptions, each functor $\sigma : \mathcal{R} \to \mathcal{C}$ constructed in Corollary A.4.2 is equipped with a Γ-equivariant structure. Therefore $\mathrm{Aut}^\otimes(\sigma)$ also carries an action of Γ. Moreover, the E_2-action of $\mathrm{Rep}(\mathrm{Aut}^\otimes(\sigma))$ on $\mathcal{P}_\sigma$ (denoted $\bullet$) is equipped with a Γ-invariant structure, i.e., functorial isomorphisms

$$U \bullet_I X \cong U^\gamma \bullet_I X, \quad \forall \gamma \in \Gamma, U \in \mathrm{Rep}(\mathrm{Aut}^\otimes(\sigma))^{\boxtimes I}, X \in \mathcal{P}_\sigma$$

compatible with the group structure on Γ and the factorization structure.

The rest of the appendix is devoted to the proof of Theorem A.4.1.

A.5. Proof of Theorem A.4.1

First some notations. Let $\mathrm{Ind}\mathcal{P}$ be the category of ind-objects in $\mathcal{P}$: it is equivalent to $\mathrm{Irr}(\mathcal{P})$-graded vector spaces of possibly infinite dimension. We denote by $\omega : \mathcal{C}^{\boxtimes I} = \mathrm{Rep}(M^I) \to \mathrm{Vect}$ the forgetful functor for various I. We also denote the forgetful functor $\mathcal{C}^{\boxtimes I} \boxtimes \mathcal{P} \to \mathcal{P}$ by ω. Let $\mathrm{Irr}(H)$ and $\mathrm{Irr}(M)$ denote the set of (isomorphism classes of) irreducible representations of H and M.

For any $V \in \mathrm{Irr}(H)$ we have an embedding $m_V : V \otimes V^\vee \cong \mathrm{End}(V) \subset \mathcal{O}_H$ as matrix coefficients. Same for M.

The proof goes in several steps.

A.5.1. The affine scheme $\mathcal{S}$

For each $V \in \mathrm{Irr}(H)$, the action $V \star X \in \mathrm{Rep}(M) \boxtimes \mathcal{P}$ for various $X \in \mathrm{Irr}(\mathcal{P})$ only involves finitely many irreducible representations $W \in \mathrm{Irr}(M)$. We denote this finite set by $\Gamma_V \subset \mathrm{Irr}(M)$. Note that Γ_V always contains the trivial representation $\mathbf{1}_{\mathcal{R}}$ of M, for $V \star (V^\vee \star X)$ contains $\mathbf{1}_{\mathcal{R}} \boxtimes X$ as a direct summand.

We define a moduli problem as follows. For any L-algebra R, let $\mathcal{S}(R)$ be the set of Hopf algebra homomorphisms $\varphi : \mathcal{O}_H \to \mathcal{O}_M \otimes R$ such that, for any $V \in \mathrm{Irr}(H)$, $\varphi(m_V(\mathrm{End}(V)))$ lies in the span of $m_W(\mathrm{End}(W)) \otimes R$ for $W \in \Gamma_V$. Then $\mathcal{S}(R)$ is a subset of homomorphisms of algebraic groups $M_R \to H_R$. Note that $\mathcal{S}(R) \neq \varnothing$ since it contains the trivial homomorphism $M_R \to H_R$ (because Γ_V contains $\mathbf{1}_{\mathcal{R}}$).

We claim that $\mathcal{S}$ is representable by an affine scheme of finite type over L. Indeed, choose a faithful $V_0 \in \mathrm{Irr}(H)$, then $\varphi \in \mathcal{S}(R)$ is determined by the restriction $\varphi|_{m_{V_0}(\mathrm{End}(V_0))} : \mathrm{End}(V_0) \to \oplus_{W \in \Gamma_{V_0}} \mathrm{End}(W) \otimes R$, which is representable by an affine space of finite dimension. This realizes $\mathcal{S}$ as a closed subscheme of an affine space. Moreover, $\mathcal{S}$ carries an action of $M \times H$ by conjugation on $\mathcal{O}_M$ and $\mathcal{O}_H$.

We give generators and relations for the ring of regular functions $\mathcal{O}_{\mathcal{S}}$. This part is inspired by [16, §6]. For any $f \in \mathcal{O}_H$ and any $g \in M(L)$, define a function $\Phi_{f,g} \in \mathcal{O}_{\mathcal{S}}$ that assigns to each R-point $\varphi : \mathcal{O}_H \to \mathcal{O}_M \otimes R$ the value $\Phi_{f,g}(\varphi) = \mathrm{ev}_g \varphi(f) \in R$, where $\mathrm{ev}_g : \mathcal{O}_M \otimes R \to R$ is the evaluation at g.

The functions $\{\Phi_{f,g}\}_{f \in \mathcal{O}_H, g \in M(L)}$ generate $\mathcal{O}_{\mathcal{S}}$ as an L-algebra. Indeed it suffices to run f through a basis of the matrix coefficients for a faithful $V_0 \in \mathrm{Irr}(H)$, and take a finite set of g_i such that their images in $\prod_{W \in \Gamma_{V_0}} \mathrm{End}(W)$ span.

We now give the relations among $\{\Phi_{f,g}\}_{f \in \mathcal{O}_H, g \in M(L)}$. We claim that the relations are generated by the following:

(1) For any $g \in M(L)$, the assignment $f \mapsto \Phi_{f,g}$ is L-linear.
(2) Let $V \in \mathrm{Irr}(H)$. Then for any finite L-linear combination $\sum_i c_i g_i$ of elements in $M(L)$ such that $\sum_i c_i g_i|_W = 0$ for all $W \in \Gamma_V$, then $\sum_i c_i \Phi_{f,g_i} = 0$ for all $f \in m_V(\mathrm{End}(V))$.
(3) For any $f, f' \in \mathcal{O}_H$ and $g \in M(L)$, we have $\Phi_{ff',g} = \Phi_{f,g}\Phi_{f',g}$.
(4) For any $f \in \mathcal{O}_H$ and $g, g' \in M(L)$, we have $\Phi_{f,gg'} = \sum_i \Phi_{f_i,g}\Phi_{f_i',g'}$ if $\Delta(f) = \sum_i f_i \otimes f_i'$ for the comultiplication Δ on $\mathcal{O}_H$.

It is easy to see that these relations indeed hold in $\mathcal{O}_{\mathcal{S}}$. To show they are all the relations, suppose we are given an assignment $\Phi_{f,g} \to \varphi_{f,g} \in R$ satisfying the above relations, we show how to construct a Hopf algebra map $\varphi : \mathcal{O}_H \to \mathcal{O}_M \otimes R$ such that $\Phi_{f,g}$ evaluated at $\varphi \in \mathcal{S}(R)$ is $\varphi_{f,g}$. For any $f \in \mathcal{O}_H$, relation (2) ensures that there exists a unique element $\varphi(f) \in \oplus_{W \in \Gamma_V} m_W(\mathrm{End}(W)) \otimes R \subset \mathcal{O}_M \otimes R$ such that $\mathrm{ev}_g \varphi(f) = \varphi_{f,g}$ for any $g \in M(L)$. Relation (1) says that the assignment $f \mapsto \varphi(f)$ gives a linear map $\varphi : \mathcal{O}_H \to \mathcal{O}_M \otimes R$. Relation (3) shows that φ is an algebra homomorphism. Relation (4) shows that φ is a coalgebra homomorphism. This proves the claim.

A.5.2. Construction of an action of $\mathcal{O}_{\mathcal{S}}$ on each object in $\mathcal{P}$

We view $\mathcal{O}_{\mathcal{S}}$ as an algebra ind-object in $\mathrm{Rep}(H)$ using the conjugation action of H. For each $X \in \mathcal{P}$ we will construct a map $\alpha_X : \omega(\mathcal{O}_{\mathcal{S}} \star X) \to X$ in $\mathrm{Ind}\mathcal{P}$, compatible with the algebra structure on $\mathcal{O}_{\mathcal{S}}$, such that the following diagram is commutative for any $V \in \mathrm{Rep}(H)$ and $X \in \mathcal{P}$

$$\begin{array}{ccc} V \star \omega(\mathcal{O}_{\mathcal{S}} \star X) & \xrightarrow{\sim} & (\omega \boxtimes \mathrm{id}_{\mathcal{C}})(\mathcal{O}_{\mathcal{S}} \star (V \star X)) \\ \downarrow{\scriptstyle \mathrm{id}_V \star \alpha_X} & & \downarrow{\scriptstyle \alpha_{V \star X}} \\ V \star X & = & V \star X \end{array}$$

Here the top row is induced by the commutativity constraint of the action of $\mathcal{R}$ on $\mathcal{P}$; in $(\omega \boxtimes \mathrm{id}_{\mathcal{C}})(\mathcal{O}_{\mathcal{S}} \star (V \star X))$ we emphasize that ω is applied to the first factor of $\mathcal{C}$, so the result is still an object in $\mathcal{C} \boxtimes \mathcal{P}$. Moreover, all maps in $\mathrm{Ind}\mathcal{P}$ are compatible with the $\mathcal{O}_{\mathcal{S}}$-actions. See [7, §22].

In other words, if we define an internal $\underline{\mathrm{Hom}}(X, Y) \in \mathrm{IndRep}(H)$ for $X, Y \in \mathcal{P}$ by the adjunction $\mathrm{Hom}_{\mathcal{P}}(\omega(V \star X), Y) = \mathrm{Hom}_H(V, \underline{\mathrm{Hom}}(X, Y))$ for all $V \in \mathrm{Rep}(H)$, then we need to construct an algebra homomorphism $\alpha'_X : \mathcal{O}_{\mathcal{S}} \to \underline{\mathrm{End}}(X)$ in $\mathrm{IndRep}(H)$, that is compatible with morphisms in $\mathcal{P}$.

The construction of α_X is analogous to V. Lafforgue's excursion operators [15]. For any $g \in M(L)$ we first define a map $\alpha_{g,X} : \omega(\mathcal{O}_H \star X) \to X$ as the composition

$$\begin{aligned} \omega(\mathcal{O}_H \star X) &= \bigoplus_{V \in \mathrm{Irr}(H)} \omega((V \boxtimes V^\vee) \star_{\{1,2\}} X) \\ &\xrightarrow{(g,1)} \bigoplus_{V \in \mathrm{Irr}(H)} \omega((V \boxtimes V^\vee) \star_{\{1,2\}} X) = \omega(\mathcal{O}_H \star X) \xrightarrow{\mathrm{ev}_1} \mathbf{1}_{\mathcal{R}} \star X = X. \quad \text{(A.2)} \end{aligned}$$

Here the second step uses that $(V \boxtimes V^\vee) \star_{\{1,2\}} X \in \mathrm{Rep}(M^2) \boxtimes \mathcal{P}$, hence $(g, 1) \in M^2$ acts on $\omega((V \boxtimes V^\vee) \star_{\{1,2\}} X)$. The last map $\mathrm{ev}_1 : \mathcal{O}_H \to \mathbf{1}_{\mathcal{R}}$ is evaluation at $1 \in H$.

The map $\alpha_{g,X}$ then gives a map $\alpha'_{g,X} : \mathcal{O}_H \to \underline{\mathrm{End}}(X)$ in $\mathrm{IndRep}(H)$. It is supposed to be the composition

$$\mathcal{O}_H \xrightarrow{\Phi_{\bullet,g}} \mathcal{O}_{\mathcal{S}} \xrightarrow{\alpha'_X} \underline{\mathrm{End}}(X)$$

where $\Phi_{\bullet,g}$ denotes the H-equivariant map $\mathcal{O}_H \to \mathcal{O}_{\mathcal{S}}$ that sends $f \in \mathcal{O}_H$ to $\Phi_{f,g}$. We need to check the ring relations (2)–(4) in Sect. A.5.1 hold for $\alpha'_{g,X}$ to ensure that the maps $\{\alpha'_{g,X}\}_{g \in M(L)}$ together give an algebra map $\mathcal{O}_{\mathcal{S}} \to \underline{\mathrm{End}}(X)$.

(2) This follows from the definition of Γ_V: in $V \boxtimes V^\vee \star_{\{1,2\}} X = V^\vee \star_{\{2\}} (V \star_{\{1\}} X) \in \mathcal{C}^{\boxtimes 2} \boxtimes \mathcal{P}$, the first factor only involves $W \in \Gamma_V \subset \mathrm{Irr}(M)$, and $\sum c_i g_i$ acts on these W by zero.

(3) We need to show that the following diagram is commutative

$$\begin{array}{ccccc}
\omega(\mathcal{O}_H \boxtimes \mathcal{O}_H \star_{\{1,2\}} X) & \xrightarrow{\mathrm{id}_{\mathcal{O}_H} \boxtimes \alpha_{g,X}} & \omega(\mathcal{O}_H \star X) & \xrightarrow{\alpha_{g,X}} & X \\
\downarrow{\scriptstyle \mathrm{mult}} & & & & \| \\
\omega(\mathcal{O}_H \star X) & & \xrightarrow{\qquad\qquad\alpha_{g,X}\qquad\qquad} & & X
\end{array}$$

Here the map "mult" is induced by the multiplication $\mathcal{O}_H \otimes \mathcal{O}_H \to \mathcal{O}_H$.Let $V, V', W \in \mathrm{Irr}(H)$. Let $m_W^{V,V'} : \mathrm{End}(V) \otimes \mathrm{End}(V') \to \mathrm{End}(W)$ be the composition of the multiplication of matrix coefficients in $\mathcal{O}_H$ with the projection to $\mathrm{End}(W)$. Since $m_W^{V,V'}$ is equivariant under left and right H-action, it is induced from a pair of maps $\mu_W^{V,V'} : V \otimes V' \to W$ and $\nu^{V,V'} : V^\vee \otimes V'^\vee \to W^\vee$ in $\mathrm{Rep}(H)$ such that $m_W^{V,V'} = \mu_W^{V,V'} \boxtimes \nu_W^{V,V'}$ in $\mathrm{Rep}(H^2)$. Then the required commutativity restricted to $m_V(\mathrm{End}(V)) \boxtimes m_{V'}(\mathrm{End}(V'))$ follows from the following commutative diagram in $\mathrm{Rep}(H^2)$

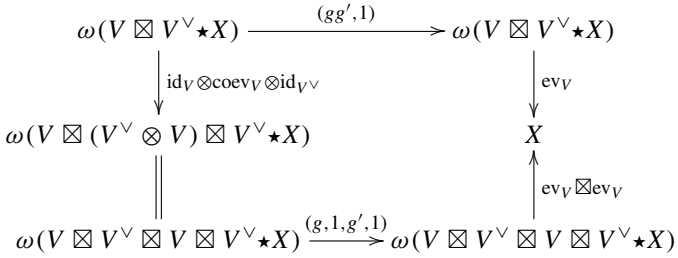
$$\begin{array}{ccc}
\omega(V \boxtimes V^\vee \boxtimes V' \boxtimes V'^\vee \star_{\{1,2,1',2'\}} X) & \xrightarrow{(g,1,g,1)} & \omega(V \boxtimes V^\vee \boxtimes V' \boxtimes V'^\vee \star_{\{1,2,1',2'\}} X) \\
\| & & \| \\
\omega((V \otimes V') \boxtimes (V^\vee \otimes V'^\vee) \star_{\{1,2\}} X) & \xrightarrow{(g,1)} & \omega((V \otimes V') \boxtimes (V^\vee \otimes V'^\vee) \star_{\{1,2\}} X) \\
\downarrow & & \downarrow \\
\omega(W \boxtimes W^\vee \star_{\{1,2\}} X) & \xrightarrow{\qquad(g,1)\qquad} & \omega(W \boxtimes W^\vee \star_{\{1,2\}} X)
\end{array}$$

(4) We need to show, for any $g, g' \in M(L)$, the following diagram is commutative in $\mathrm{IndRep}(H)$

$$\begin{array}{ccc}
\mathcal{O}_H & \xrightarrow{\alpha'_{gg',X}} & \underline{\mathrm{End}}(X) \\
\downarrow{\scriptstyle \Delta} & & \uparrow \\
\mathcal{O}_H \otimes \mathcal{O}_H & \xrightarrow{\alpha'_{g,X} \otimes \alpha'_{g',X}} & \underline{\mathrm{End}}(X) \otimes \underline{\mathrm{End}}(X)
\end{array}$$

Here the right vertical map is given by composition of internal Hom. Restricting to the sub-coalgebra $\mathrm{End}(V) \subset \mathcal{O}_H$ for any $V \in \mathrm{Irr}(H)$, we need to show the

following diagram is commutative



Both compositions $\omega(V \boxtimes V^\vee \star X) \to X$ are adjoint to the following map

$$\omega(V \star X) \xrightarrow{g'} \omega(V \star X) \xrightarrow{g} \omega(V \star X).$$

This finishes the proof of relation (4).

A.5.3. Construction of ρ

The action of $\mathcal{O}_{\mathcal{S}}$ on $X \in \mathcal{P}$ gives the following map in $\mathcal{P}$ (here we are viewing $\mathcal{O}_{\mathcal{S}}^H$ as a trivial H-submodule of $\mathcal{O}_{\mathcal{S}}$)

$$\mathcal{O}_{\mathcal{S}}^H \otimes X \to \omega(\mathcal{O}_{\mathcal{S}}^H \star X) \to \omega(\mathcal{O}_{\mathcal{S}} \star X) \xrightarrow{\alpha_X} X.$$

This gives an action of $\mathcal{O}_{\mathcal{S}}^H$ (as a plain L-algebra) on $X \in \mathrm{Ind}\mathcal{P}$, commuting with all morphisms in $\mathrm{Ind}\mathcal{P}$ (i.e., $\mathcal{O}_{\mathcal{S}}^H$ acts on $\mathrm{id}_{\mathcal{P}}$). For each $X \in \mathrm{Irr}(\mathcal{P})$, Since $\mathrm{End}(X) = L$, this action factors through a homomorphism $\theta_X : \mathcal{O}_{\mathcal{S}}^H \to L$. Since all morphisms in $\mathcal{P}$ commute with the $\mathcal{O}_{\mathcal{S}}^H$-action, all simple objects X with the same θ_X form a union of equivalence classes in the sense of Lemma A.3.1. Since $\mathcal{P}$ is indecomposable, there is only one equivalence class on $\mathrm{Irr}(\mathcal{P})$, hence all simple objects X have the same θ_X, which we denote by θ.

Let $\mathcal{I} \subset \mathcal{O}_{\mathcal{S}}$ be the ideal generated by $\ker(\theta)$. Let $\widetilde{\mathcal{Z}} \subset \mathcal{S}$ be the closed subscheme defined by $\mathcal{I}$, and $\mathcal{Z} := \widetilde{\mathcal{Z}}_{\mathrm{red}} \subset \mathcal{S}$ the reduced scheme. Since M is reductive, conjugacy classes of homomorphisms $M \to H$ form a discrete set, hence $\mathcal{S}_{\mathrm{red}}$ is a disjoint union of H-orbits. Since the only H-invariant functions on $\mathcal{Z}$ are the scalars, it is a single H-orbit, i.e., the H-orbit of some homomorphism $\rho : M \to H$. Therefore $\mathcal{Z} \cong H/H_\rho$.

A.5.4. The action of $\mathcal{O}_{\mathcal{S}}$ on $X \in \mathcal{P}$ factors through $\mathcal{O}_{\mathcal{Z}}$

By the previous step, the action of $\mathcal{O}_{\mathcal{S}}$ on any $X \in \mathcal{P}$ factors through the quotient $\mathcal{O}_{\widetilde{\mathcal{Z}}}$, i.e., a map $\overline{\alpha}_X : \omega(\mathcal{O}_{\widetilde{\mathcal{Z}}} \star X) \to X$. We show that $\overline{\alpha}_X$ further factors through a map $\beta_X : \omega(\mathcal{O}_{\mathcal{Z}} \star X) \to X$.

Let $\mathcal{J} \subset \mathcal{O}_{\widetilde{\mathcal{Z}}}$ be the nilpotent radical. Since $\mathcal{S}$ is of finite type over L, $\mathcal{J}^n = 0$ for some $n > 0$. Let $X \in \mathrm{Irr}(\mathcal{P})$, we show that the restriction of $\overline{\alpha}_X$ to $\omega(\mathcal{J} \star X)$ is zero.

If not, let ℓ be the smallest positive integer such that $\overline{\alpha}_X|\omega(\mathcal{J}^\ell\star X) = 0$. Since $\overline{\alpha}_X : \omega(\mathcal{J}^{\ell-1}\star X) \to X$ is nonzero, it is surjective since X is simple. Applying $\mathcal{J}$-action we still get a surjection $\omega(\mathcal{J}\star\omega(\mathcal{J}^{\ell-1}\star X)) \twoheadrightarrow \omega(\mathcal{J}\star X)$. We have a commutative diagram

$$\begin{array}{ccc} \omega(\mathcal{J}\star\omega(\mathcal{J}^{\ell-1}\star X)) & \longrightarrow & \omega(\mathcal{J}\star X) \\ \downarrow & & \downarrow \\ \omega(\mathcal{J}^{\ell}\star X) & \longrightarrow & X \end{array}$$

where the left arrow is given by the multiplication $\mathcal{J}\star\mathcal{J}^{\ell-1} \to \mathcal{J}^\ell$, and all other maps are the action maps. Now the top arrow is surjective and the bottom one is zero by assumption, which implies the right arrow is zero. This shows that $\overline{\alpha}_X : \omega(\mathcal{J}\star X) \to X$ is zero, i.e., the action of $\mathcal{O}_{\widetilde{\mathcal{Z}}}$ on X factors through $\mathcal{O}_{\mathcal{Z}}$.

In the sequel we denote the action map of $\mathcal{O}_{\mathcal{Z}}$ on X by

$$\beta_X : \omega(\mathcal{O}_{\mathcal{Z}}\star X) \to X.$$

A.5.5. Construction of an E_2-action of $\mathrm{Rep}(H_\rho)$ on $\mathcal{P}$

We will denote this action by $\bullet$. Since the image of the restriction functor $\mathrm{Rep}(H) \to \mathrm{Rep}(H_\rho)$ generate $\mathrm{Rep}(H_\rho)$ under taking direct summands, it suffices to define $V \bullet X := \omega(V\star X)$ for any $V \in \mathrm{Rep}(H)$, and check that any H_ρ-equivariant map $V \to V'$ induces a map $\omega(V\star X) \to \omega(V'\star X)$ in a way functorial in V, V' and X and compatible with compositions.

In the previous step we have shown that $\mathcal{O}_{\mathcal{Z}} = L[H]^{H_\rho}$ (right translation invariants) acts on X. For any $W \in \mathrm{Rep}(H)$, we have a map $m_{W,\mathcal{Z}} : W^{H_\rho} \otimes W^\vee \to \mathcal{O}_{\mathcal{Z}}$ in $\mathrm{Rep}(H)$ given by the taking the matrix coefficient $w \otimes \xi \mapsto (h \mapsto \langle \xi, hw\rangle)$. In particular, taking $W = V^\vee \otimes V'$ for $V, V' \in \mathrm{Rep}(H)$, we get a map $m_{W,\mathcal{Z}} : \mathrm{Hom}_{H_\rho}(V, V') \otimes (V \otimes V'^\vee) \to \mathcal{O}_{\mathcal{Z}}$ in $\mathrm{Rep}(H)$. Composed with the action map β_X we get

$$\widetilde{\gamma}_{V,V',X} : \mathrm{Hom}_{H_\rho}(V, V') \otimes \omega((V \otimes V'^\vee)\star X) \xrightarrow{m_{W,\mathcal{Z}}\star \mathrm{id}_X} \omega(\mathcal{O}_{\mathcal{Z}}\star X) \xrightarrow{\beta_X} X \quad \text{(A.3)}$$

which is the same as a map

$$\begin{aligned} \gamma_{V,V',X} &: \mathrm{Hom}_{H_\rho}(V, V') \to \mathrm{Hom}_{\mathcal{P}}(\omega((V \otimes V'^\vee)\star X), X) \\ &= \mathrm{Hom}_{\mathcal{P}}(\omega(V\star X), \omega(V'\star X)). \end{aligned}$$

One checks these maps are compatible with compositions $V \to V' \to V''$ using the fact that β_X is compatible with the ring structure on $\mathcal{O}_{\mathcal{Z}}$. This finishes the construction of $V \bullet X$.

A.5.6. The action of Rep(H) on $\mathcal{P}$ is inflated from the Rep(H_ρ)-action

To show this, we need to check that for any $g \in M(L)$, $V \in \mathrm{Rep}(H)$ and $X \in \mathcal{P}$, the following two endomorphisms of $\omega(V \star X)$ are the same: the first one, which we denote by a_g, is obtained by the action of g on the forgetful functor ω; the second one, which we denote by b_g, comes by evaluating $\gamma_{V,V,X}$ at $\rho(g) \in \mathrm{Hom}_{H_\rho}(V, V)$.

Consider the map $c_{V,g} : V \otimes V^\vee \to \mathcal{O}_\mathcal{Z}$ given by $\xi \otimes v \mapsto c_{V,g}(h) = \langle \xi, h\rho(g)h^{-1}v\rangle = \langle h^{-1}\xi, \rho(g)h^{-1}v\rangle$. We are using that $\rho(g)$ commutes with H_ρ to conclude that $c_{V,g}$ is right invariant under H_ρ hence giving a function on $\mathcal{Z}$. The map $c_{V,g}$ is H-equivariant for the diagonal action of H on $V \otimes V^\vee$. Consider the following map

$$\delta_{V,g,X} : \omega((V \otimes V^\vee)\star X) \xrightarrow{c_{V,g}\star \mathrm{id}_X} \omega(\mathcal{O}_\mathcal{Z}\star X) \xrightarrow{\beta_X} X.$$

By adjunction it gives an endomorphism d_g of $\omega(V\star X)$. On the one hand, the definition of the $\mathcal{O}_\mathcal{S}$-action, see (A.2), implies that $d_g = a_g$; on the other hand, comparing $\delta_{V,g,X}$ to the map $\widetilde{\gamma}_{V,V,X}$ in (A.3), we see that $\delta_{V,g,X}$ is the restriction of $\widetilde{\gamma}_{V,V,X}$ to $\rho(g) \otimes \omega((V \otimes V^\vee)\star X)$, hence $d_g = b_g$. This shows $a_g = b_g$ and finishes the proof of Theorem A.4.1. □

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