

# QUANTUM STATISTICAL MECHANICS IN ARITHMETIC TOPOLOGY

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ABSTRACT. This paper provides a construction of a quantum statistical mechanical system associated to knots in the 3-sphere and cyclic branched coverings of the 3-sphere, which is an analog, in the sense of arithmetic topology, of the Bost–Connes system, with knots replacing primes, and cyclic branched coverings of the 3-sphere replacing abelian extensions of the field of rational numbers. The operator algebraic properties of this system differ significantly from the Bost–Connes case, due to the properties of the action of the semigroup of knots on a direct limit of knot groups. The resulting algebra of observables is a noncommutative Bernoulli product. We describe the main properties of the associated quantum statistical mechanical system and of the relevant partition functions, which are obtained from simple knot invariants like genus and crossing number.

## 1. INTRODUCTION

This paper addresses a question asked to the first author by Masanori Morishita, on the possibility of adapting to 3-manifolds the Bost–Connes construction [5] of a quantum statistical mechanical system associated to the abelian extensions of  $\mathbb{Q}$ , and its generalizations to number fields [13], [14], [24], [33], along the lines of the general “arithmetic topology” program. The latter can be seen as a broad dictionary of analogies between the geometry of knots and 3-manifolds and the arithmetic of number fields, with knots as analogs of primes and 3-manifolds, seen as branched coverings of the 3-sphere, viewed as analogs of number fields. In this paper we answer Morishita’s question by providing explicit constructions of quantum statistical mechanical systems associated to (alternating) knots, to knot groups, and to cyclic branched covers of the 3-sphere, with the latter providing our analog of the abelian extensions of  $\mathbb{Q}$  in the Bost–Connes construction. The structure of the resulting quantum statistical mechanical systems is different from the Bost–Connes case and it leads to an algebra of observables that can be expressed in the form of a Bernoulli crossed product, of the type studied in noncommutative Bernoulli actions in the theory of factors. We relate the geometry and dynamics of our system to known invariants of knots and 3-manifolds.

**1.1. The principle of Arithmetic Topology.** Arithmetic topology originates from insights by John Tate and Michael Artin on topological interpretations of class field theory. The analogy between primes and knots, which is the founding principle of Arithmetic Topology, was first observed by Barry Mazur, David Mumford, and Yuri Manin. The subject developed over the years, with various contributions, such as [19], [30], [32], [42], [43], [49], [51], [54], as a powerful guiding principle outlining parallel results and analogies between the arithmetic of number fields and the topology of 3-manifolds. The basic analogy sees number fields as analogs of compact oriented 3-manifolds, with  $\mathbb{Q}$  playing the role of the 3-sphere  $S^3$ . Here the main idea is that, while number fields are finite extensions of  $\mathbb{Q}$ , ramified at a finite set of primes, all compact oriented 3-manifolds can be described as branched coverings of the 3-sphere, branched along a link. A major point where this analogy does not

carry over is the fact that, while the description of a number field as ramified covering of  $\mathbb{Q}$  is unique, there are many inequivalent ways of describing 3-manifolds as branched covers of the 3-sphere, branched along knots or links (or more generally embedded graph). While this lack of uniqueness for 3-manifolds can be used to make the construction dynamical, see [40], the same dynamics does not apply to number fields. However, the corresponding analogy between knots and primes, that results from this first analogy between number fields and 3-manifolds, has been very fruitful, leading to many new results, ranging from arithmetic analogs for higher linking numbers [42], [43], to arithmetic Chern–Simons theory, [32].

Over the past two decades, the connection between number theory and quantum statistical mechanics was also widely explored, starting with early constructions of statistical systems associated to the primes, [29], [57], the more refined Bost–Connes system [5] which also involves the Galois theory of abelian extensions of  $\mathbb{Q}$ , and subsequent generalizations of this construction to arbitrary number fields, obtained in [24] and further studied in [14], [15], [33], [45]. The purpose of the present paper is to recast the Bost–Connes construction in the setting of arithmetic topology, with the semigroup of knots with the connecting sum operation replacing the multiplicative semigroup of positive integers, and the cyclic branched coverings of the 3-spheres replacing the abelian coverings of  $\mathbb{Q}$ .

**1.2. Bost–Connes system.** We recall briefly the construction of the Bost–Connes algebra and quantum statistical mechanical system from [5] (see also [11] and §3 of [12]). Consider the group ring  $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$  with generators  $e(r)$  with  $r \in \mathbb{Q}/\mathbb{Z}$ . The maps  $\{\sigma_n\}_{n \in \mathbb{N}_p}$  given by

$$(1.1) \quad \sigma_n(e(r)) := e(nr)$$

determine an action of the semigroup  $\mathbb{N}$  by endomorphisms of the group ring  $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ . These endomorphisms have partial inverses  $\alpha_n : \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rightarrow \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ ,

$$(1.2) \quad \alpha_n(e(r)) = \frac{1}{n} \sum_{s: ns=r} e(s)$$

with  $\sigma_n \circ \alpha_n(e(r)) = e(r)$  and  $\alpha_n \circ \sigma_n(e(r)) = e_n \cdot e(r)$ , with  $e_n = n^{-1} \sum_{s: ns=0} e(s)$  an idempotent in  $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ . Thus, one can define the semigroup crossed product. This is the (rational) Bost–Connes algebra  $\mathcal{A}_{BC, \mathbb{Q}} = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$  with generators  $\mu_n$  and  $e(r)$  and relations

$$(1.3) \quad \mu_n^* \mu_n = 1, \quad \mu_n \mu_n^* = e_n, \quad \mu_n \mu_m = \mu_{nm}, \quad \mu_n \mu_m^* = \mu_m^* \mu_n \text{ for } (n, m) = 1,$$

$$(1.4) \quad \mu_n e(r) \mu_n^* = \alpha_n(e(r)), \quad \mu_n^* e(r) \mu_n = \sigma_n(e(r)).$$

The complexification  $\mathcal{A}_{BC, \mathbb{C}} = \mathcal{A}_{BC, \mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$  has a  $C^*$ -algebra completion given by the semigroup crossed product  $\mathcal{A}_{BC} = C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}$ , with the same generators and relations. The time evolution of the Bost–Connes system is defined by  $\sigma_t(\mu_n) = n^{it} \mu_n$  and  $\sigma_t(e(r)) = e(r)$ . The algebra  $\mathcal{A}_{BC}$  has representations on the Hilbert space  $\ell^2(\mathbb{N})$ , parameterized by the choice of an element  $u \in \hat{\mathbb{Z}}^*$ , of the form

$$(1.5) \quad \pi_u(e(r)) \epsilon_m = u(r)^m \epsilon_m, \quad \pi_u(\mu_n) \epsilon_m = \epsilon_{nm},$$

where  $u(r)$  is a root of unity in  $\mathbb{C}$  determined by the embedding of  $\mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}$  specified by the choice of  $u \in \hat{\mathbb{Z}}^*$ , where we identify  $\hat{\mathbb{Z}} = \text{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ .

Given a pair  $(\mathcal{A}, \sigma)$  of a  $C^*$ -algebra and a time evolution  $\sigma : \mathbb{R} \rightarrow \text{Aut}(\mathcal{A})$ , a  $\text{KMS}_\beta$  state for  $(\mathcal{A}, \sigma)$  is a continuous linear functional  $\varphi_\beta : \mathcal{A} \rightarrow \mathbb{C}$  satisfying normalization  $\varphi_\beta(1) = 1$  and positivity  $\varphi_\beta(a^*a) \geq 0$  (that is, a *state* on  $\mathcal{A}$ ) such that, for all  $a, b \in \mathcal{A}$  there is a

function  $F_{a,b}(z)$  that is holomorphic on the strip  $\mathcal{I}_\beta = \{z \in \mathbb{C} : 0 < \Im(z) < \beta\}$  and continuous on the boundary  $\partial\mathcal{I}_\beta$  of the strip, such that

$$(1.6) \quad F_{a,b}(t) = \varphi_\beta(a\sigma_t(b)), \quad F_{a,b}(t + i\beta) = \varphi(\sigma_t(b)a).$$

In other words, the failure of a  $\text{KMS}_\beta$  to be a trace is measured by interpolation by a holomorphic function.

The KMS states of the Bost–Connes system  $(\mathcal{A}_{BC}, \sigma)$  are completely classified and given by the following list of cases (see [5]):

- for every  $0 < \beta \leq 1$  there is a unique  $\text{KMS}_\beta$  state  $\varphi_\beta$  determined by

$$\varphi_\beta(e(\frac{a}{b})) = \frac{f_{-\beta+1}(b)}{f_1(b)}$$

where  $f_k(b) = \sum_{d|b} \mu(d)(b/d)^k$ , with  $\mu$  the Möbius function;

- for every  $\beta > 1$ , the extremal  $\text{KMS}_\beta$  states are given by Gibbs states determined by

$$(1.7) \quad \varphi_{\beta,u}(e(r)) = \frac{\text{Li}_\beta(u(r))}{\zeta(\beta)},$$

where  $\text{Li}_\beta$  is the polylogarithm function,  $u(r)$  is a root of unity, for a given  $u \in \hat{\mathbb{Z}}^*$ , and  $\zeta(\beta)$  is the Riemann zeta function;

- for  $\beta = \infty$  the extremal  $\text{KMS}_\beta$  states are determined by  $\varphi_{\infty,u}(e(r)) = u(r)$ .

The Bost–Connes system is related to the arithmetic of  $\mathbb{Q}$  and the Galois theory of its abelian extensions. Generalizations of this quantum statistical mechanical system were constructed for arbitrary number fields in [24], [33], [45], and further studied in [14], [15].

**1.3. Structure of the paper.** In §2 we develop a quantum statistical mechanics of knots. There is a natural semigroup structure on knots. It is given by the operation of connected sum defined on equivalence classes of *oriented knots*. This operation gives rise to an abelian semigroup  $(\mathcal{K}, \#)$ , which is infinitely generated, with generators the prime knots. Each knot has a unique prime decomposition  $K = K_1 \# \cdots \# K_m$  for some  $m$ , with  $K_j$  prime knots. We focus on knot invariants that behave well with respect to the connected sum operation. In particular, we focus on simple invariants such as the genus and the crossing number. In the latter case, it is at present an open conjecture whether the invariant is additive over connected sums for all knots, but the result is known to hold for alternating knots. Therefore, in the paper we often restrict our attention to alternating knots, purely for the purposes of using these results about the crossing number. Conditionally to the above mentioned conjecture, one can reformulate them in terms of the larger semigroup of knots. We construct a Hamiltonian based on genus and crossing number and we estimate in Theorem 2.3 the range of convergence of the partition function using results of [61], [62] on the rate of growth of multiplicities. We show the uniqueness of KMS states for this system of knots without interaction in Proposition 2.6 and we discuss the type III nature of the high temperature state in Proposition 2.6 and Theorem 2.9.

In §2.3 we return to the original system without interaction of [29] and [5], with prime numbers contributing independent oscillators (in the form of Toeplitz operators) and we discuss how one can try to extend it from the multiplicative semigroup  $\mathbb{N}$  of positive integer to the group  $\mathbb{Q}_+^*$  of positive rational numbers. We show that the Hamiltonian can be extended so that the corresponding partition function is again expressible in terms of the

Riemann zeta function. We show in §2.4 that the same construction extends to the case of the Grothendieck group of the semigroup of (alternating) knots with the connected sum. Again, this result relies on estimates of [61], [62] on the number of alternating knots with fixed genus and crossing number. However, at the level of the algebra of observables of the system, this generalization of the Hamiltonian requires an extension of the algebra by the spectral projections of the Hamiltonian, in order to remain invariant under the time evolution. This extension has the effect of making the time evolution inner, which is not desirable from the operator algebra perspective. We bypass this problem by considering more general systems with interaction involving both knots and 3-manifolds.

In §3, we introduce cyclic branched coverings of  $S^3$ , branched along a knot. We discuss the behavior of knot groups under connected sums of knots, and we construct a directed system of knot groups over the semigroup of knots ordered by “divisibility” with respect to the connected sum operation. We interpret the resulting direct limit as the knot group of a wild knot. We also consider a projective limit, related to changing the order of the cyclic branched cover.

In §4 we construct a more refined system, which is more similar in nature to the Bost–Connes system and which involves not only knots but also the cyclic branched covers of  $S^3$ . We begin by investigating the action of the semigroup of knots with connected sum on the group algebra of the direct limit of the system of knot groups considered in the previous section. We show that, unlike the Bost–Connes case, the endomorphisms  $\sigma_K$  are injective and not surjective. The resulting crossed product system is then more similar to the generalization of the Bost–Connes considered in [38], in relation to the Habiro ring. In particular, we show that the resulting crossed product algebra is in fact a noncommutative Bernoulli shift

$$\bigotimes_{g \in \mathcal{G}_{\mathcal{K}}} C_r^*(\pi) \rtimes \mathcal{G}_{\mathcal{K}},$$

where  $\mathcal{G}_{\mathcal{K}}$  is the Grothendieck group of the semigroup of knots  $(\mathcal{K}, \#)$  and  $\pi = \varinjlim_K \pi_K$  is the direct limit of the system of knot groups. The action of  $\mathcal{G}_{\mathcal{K}}$  is the Bernoulli action that permutes the terms in the crossed product  $\otimes_g C_r^*(\pi)$ . We then include the datum of the branched covers, in the form of a group homomorphism  $\rho : \pi \rightarrow \mathbb{Q}/\mathbb{Z}$ . We construct a projective limit of groups  $\hat{\pi}_{K,n}$  and  $\hat{\pi}_n$ , which correspond to adding  $n$ -th roots of the generators of the knot group. This construction is modelled on the construction of roots of Tate motives in [36]. This construction allows us to replace the algebra  $C_r^*(\pi)$ , which encodes the information about the knot groups, but not about the coverings, with the more refined  $C_r^*(\hat{\pi}_\rho) \rtimes_\alpha \mathbb{N}_\rho$ , where  $\hat{\pi}_\rho$  is the projective limit of the system of the  $\pi_n$  and  $\mathbb{N}_\rho$  is a subsemigroup of  $\mathbb{N}$ , given by those integers that are relatively prime to  $n_\rho$ , which is the order of the root of unity that is the image under the morphism  $\rho$  of the generators of the group  $\pi$ . The semigroup action of  $\mathbb{N}_\rho$  on  $C_r^*(\hat{\pi}_\rho)$  is modeled on the Bost–Connes action, by viewing  $\hat{\pi}_\rho$  as a fibered product inside  $\pi \times \mathbb{Q}/\mathbb{Z}$ .

We then construct time evolutions, first on the algebra  $C_r^*(\hat{\pi}_\rho) \rtimes_\alpha \mathbb{N}_\rho$ , induced by the Bost–Connes time evolution on  $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}$ , and then on the tensor product  $\otimes_g \mathcal{B}_g$ , with  $g \in \mathcal{G}_{\mathcal{K}}$  and  $\mathcal{B}_g = C_r^*(\hat{\pi}_\rho) \rtimes_\alpha \mathbb{N}_\rho$ . In this tensor product case, we take on each factor a version  $\sigma_{t,g}$  of the Bost–Connes time evolution, with the Hamiltonian  $H_{BC}$  scaled by a factor  $f(g)$ , for a function  $f : \mathcal{G}_{\mathcal{K}} \rightarrow \mathbb{N}$ . In Proposition 4.26, we identify a summability condition on the function  $f(g)$  that guarantees that the Hamiltonian has a well defined partition function, which is convergent for  $\beta > 1$ . Here the trace of the operator  $e^{-\beta H}$  in the partition function

is a combination of the operator trace on  $\ell^2(\mathbb{N}_\rho)$  and the von Neumann trace on the group algebra of  $\hat{\pi}_\rho$ . We also show how the KMS states of the Bost–Connes determine KMS states for the system  $(\otimes_g \mathcal{B}_g, \otimes_g \sigma_{t,g})$ . In particular the low temperature states give rise to KMS states  $\Psi_{\beta,f}$  for this system that are Gibbs states with respect to the partition function and the trace described in Proposition 4.26. We then consider the crossed product  $(\otimes_g \mathcal{B}_g) \rtimes \mathcal{G}_\mathcal{K}$  and we show that the KMS states  $\Psi_{\beta,f}$  transform, under the action  $\alpha_h$  of  $h \in \mathcal{G}_\mathcal{K}$  as  $\Psi_{\beta,f} \circ \alpha_h = \Psi_{\beta, \alpha_{h^{-1}}(f)}$ .

Restricting to the subsemigroup  $\mathcal{K}_a$  of alternating knots, we show that the same estimates of [61], [62] on the number of alternating knots with fixed genus and crossing number that we used in §2.4, and the result of Theorem 2.3, imply that a function satisfying the desired convergence properties can be constructed using the crossing number and the genus of knots.

## 2. QUANTUM STATISTICAL MECHANICS OF KNOTS

Consider the semigroup  $(\mathcal{K}, \#)$  or ambient isotopy classes of oriented knots with the connected sum operation. The primary decomposition of knots states that every  $K \in \mathcal{K}$  can be decomposed into a direct sum of prime knots. There are infinitely many prime knots, hence the semigroup  $\mathcal{K}$  is a countably generated free abelian semigroup. A choice of an enumeration of the prime knots gives a (non-canonical) semigroup isomorphism of  $(\mathcal{K}, \#)$  with  $(\mathbb{N}, \cdot)$  by mapping prime knots to the prime numbers. The identification is non-canonical as prime knots, unlike prime numbers, have no natural ordering. However, this identification suggests that the quantum statistical mechanics of creation-annihilation operators constructed out of the primary decomposition in  $\mathbb{N}$  (see [29], [57], and §2 of [5]) can be directly adapted to the semigroup of knots.

Let  $\mathcal{P}_\mathcal{K}$  denote the set of prime knots. As in the case of the semigroup  $\mathbb{N}$ , we can identify  $\ell^2(\mathcal{K})$  with the bosonic Fock space  $\ell^2(\mathcal{K}) = \bigoplus_{n=1}^\infty S^n \ell^2(\mathcal{P}_\mathcal{K})$ , where  $S^n \mathcal{H}$  is the  $n$ -th symmetric power of a Hilbert space  $\mathcal{H}$ , see §2 of [5]. The  $C^*$ -algebra  $C^*(\mathcal{K})$  is generated by isometries  $\mu_K$ , for  $K \in \mathcal{P}_\mathcal{K}$ , with  $\mu_K^* \mu_K = 1$ , and such that, for  $K = K_1 \# \cdots \# K_n$ ,  $\mu_K = \mu_{K_1} \cdots \mu_{K_n}$ . The  $C^*$ -algebra  $C^*(\mathcal{K})$  is an infinite tensor product of Toeplitz algebras  $C^*(\mathcal{K}) = \otimes_{K \in \mathcal{P}_\mathcal{K}} \mathcal{T}K$ .

Let  $\lambda : \mathcal{K} \rightarrow \mathbb{N}$  be a knot invariant that behaves multiplicatively under connected sums,  $\lambda(K_1 \# K_2) = \lambda(K_1) \lambda(K_2)$ . Any such invariant determines a semigroup homomorphism  $\lambda : (\mathcal{K}, \#) \rightarrow (\mathbb{N}, \cdot)$ .

**Example 2.1.** The Alexander polynomial  $\Delta_K(t) \in \mathbb{Z}[t, t^{-1}]$  of a knot  $K$  is multiplicative under connected sums. Thus, for instance, setting  $\lambda(K)$  to be the absolute value of the coefficient of the top degree term of  $\Delta_K(t)$  provides an example of such a semigroup homomorphism  $\lambda : \mathcal{K} \rightarrow \mathbb{N}$ .

Simpler examples can be obtained by considering additive invariants. Let  $\kappa : \mathcal{K} \rightarrow \mathbb{Z}_+$  be a non-negative integer invariant of knots satisfying  $\kappa(K_1 \# K_2) = \kappa(K_1) + \kappa(K_2)$ . For a choice of a positive integer  $q \in \mathbb{N}$  (for example,  $q = 2$ ), the invariant  $\lambda(K) = q^{\kappa(K)}$  satisfies the multiplicative property as above.

**Example 2.2.** There are several examples of knot invariants with values in non-negative integers that behave additively under connected sums: for example, the knot genus  $g(K)$  satisfy additivity  $g(K_1 \# K_2) = g(K_1) + g(K_2)$ .

**2.1. Alternating knots, crossing number, and genus.** A more interesting example is the crossing number  $Cr(K)$ , the minimum number of crossings over all planar diagrams  $D(K)$ . While it is clear that  $Cr(K_1\#K_2) \leq Cr(K_1) + Cr(K_2)$ , it is an open conjecture that the crossing number is in fact additive,  $Cr(K_1\#K_2) = Cr(K_1) + Cr(K_2)$ . It is known that additivity is satisfied for alternating knots [52], and for certain classes of knots, like connected sums of torus knots. A larger class of knots on which additivity is satisfied is identified in [20]. Thus, we can either use  $Cr(K)$  on the entire semigroup  $\mathcal{K}$ , conditionally, or restrict to a subsemigroup  $\mathcal{K}_a$  of alternating knots, or  $\mathcal{K}_t$  generated by those prime knots that are torus knots, or one corresponding to the subclass of [20].

**Theorem 2.3.** *Let  $\mathcal{P}_{\mathcal{K},a} \subset \mathcal{P}_{\mathcal{K}}$  be the set of prime knots that are alternating, and consider the bosonic Fock space  $\ell^2(\mathcal{K}_a) = \bigoplus_n S^n \ell^2(\mathcal{P}_{\mathcal{K},a})$ . The  $C^*$ -algebra  $C^*(\mathcal{K}_a) = \bigotimes_{K \in \mathcal{P}_{\mathcal{K},a}} \tau_K$  acts by bounded operators on the Hilbert space  $\ell^2(\mathcal{K}_a)$ , with  $\mu_K \epsilon_{K'} = \epsilon_{K\#K'}$ . For a fixed  $q \in \mathbb{N}$ , with  $q \geq 2$ , and for all  $t \in \mathbb{R}$ , setting  $\sigma_t(\mu_K) = q^{it(Cr(K)+g(K))} \mu_K$  defines a time evolution  $\sigma : \mathbb{R} \rightarrow \text{Aut}(C^*(\mathcal{K}_a))$ , with Hamiltonian  $H\epsilon_K = (Cr(K) + g(K)) \log(q) \epsilon_K$ . The partition function is given by the series*

$$(2.1) \quad Z_a(\beta) = \text{Tr}(e^{-\beta H}) = \sum_{K \in \mathcal{K}_a} q^{-\beta(Cr(K)+g(K))}$$

converges in the range  $\beta \geq \beta_+ = \log \frac{2^{20}}{3^6} - 6 \log \log 2$  and diverges for  $\beta < \beta_-$ , where  $\beta = \beta_-$  is the unique solution of

$$\beta - 6 \log \left( \frac{q^{-\beta}}{1 - q^{-\beta}} \right) = 2 \log(20) - 6 \log \log 2,$$

with  $\beta_- = \beta_-(q) \leq 1.9391 \dots$  for all  $q \in \mathbb{N}$  with  $q \geq 2$ .

*Proof.* The adjoint  $\mu_K^*$  acts as  $\mu_K^* \epsilon_{K'} = 0$  if  $K$  does not divide  $K'$  in the semigroup  $(\mathcal{K}_a, \#)$  and  $\mu_K^* \epsilon_{K'} = \epsilon_{K''}$  if  $K' = K\#K''$  in  $\mathcal{K}_a$ . These satisfy the relation  $\mu_K^* \mu_K = 1$ , while  $\mu_K \mu_K^*$  is the orthogonal projection on the subspace of  $\ell^2(\mathcal{K}_a)$  generated by all  $K'$  that are divisible by  $K$  in  $(\mathcal{K}_a, \#)$ . Thus, setting  $\mu_K \epsilon_{K'} = \epsilon_{K\#K'}$  determines a representation of  $C^*(\mathcal{K}_a)$  on  $\ell^2(\mathcal{K}_a)$ . For  $K = K_1\#K_2$ , we have  $\mu_K = \mu_{K_1} \mu_{K_2}$  in  $C^*(\mathcal{K}_a)$  and the time evolution satisfies  $\sigma_t(\mu_K) = q^{it(Cr(K)+g(K))} \mu_K = q^{it(Cr(K_1)+g(K_1))} q^{it(Cr(K_2)+g(K_2))} \mu_{K_1} \mu_{K_2} = \sigma_t(\mu_{K_1}) \sigma_t(\mu_{K_2})$ , since both  $Cr$  and the genus are additive on connected sums of alternating knots. It also clearly satisfies  $\sigma_{t+s}(X) = \sigma_t(\sigma_s(X))$  for  $X \in C^*(\mathcal{K}_a)$  and for all  $t, s \in \mathbb{R}$ . Thus, the time evolution is indeed a 1-parameter family of automorphisms of the algebra, that is, a group homomorphism  $\sigma : \mathbb{R} \rightarrow \text{Aut}(C^*(\mathcal{K}_a))$ . The Hamiltonian  $H$  is determined (up to an arbitrary additive constant) by the covariance relation  $R(\sigma_t(X)) = e^{itH} R(X) e^{-itH}$ , for all  $X \in C^*(\mathcal{K}_a)$  and all  $t \in \mathbb{R}$ , where  $R : C^*(\mathcal{K}_a) \rightarrow \mathcal{B}(\ell^2(\mathcal{K}_a))$  is the representation described above. The densely defined self-adjoint unbounded operator defined by  $H\epsilon_K = (Cr(K) + g(K)) \log(q) \epsilon_K$  satisfies

$$\begin{aligned} e^{itH} R(\mu_K) e^{-itH} \epsilon_{K'} &= q^{-it(Cr(K')+g(K'))} e^{itH} \epsilon_{K\#K'} \\ &= q^{-it(Cr(K')+g(K'))} q^{-it(Cr(K\#K')+g(K\#K'))} \epsilon_{K\#K'} \\ &= q^{it(Cr(K)+g(K))} R(\mu_K) \epsilon_{K'} \\ &= R(\sigma_t(\mu_K)) \epsilon_{K'}. \end{aligned}$$

We have

$$\begin{aligned} \mathrm{Tr}(e^{-\beta H}) &= \sum_{K \in \mathcal{K}_a} \langle \epsilon_K, e^{-\beta H} \epsilon_K \rangle = \sum_{K \in \mathcal{K}_a} q^{-\beta(Cr(K)+g(K))} \\ &= \sum_{n=0}^{\infty} \sum_{g=0}^{\infty} N_{n,g} q^{-\beta(n+g)}, \end{aligned}$$

where  $N_{n,g}$  is the number of alternating knots  $K$  with  $Cr(K) = n$  and  $g(K) = g$ . It was shown in Corollary 3.1 of [60] that

$$N_{n,g} = O(n^{p_g}), \quad \text{for } n \rightarrow \infty,$$

for some  $p_g \in \mathbb{N}$ . A more precise estimate is given in Theorem 1.2 of [61] and in Theorem 1.1 of [62], which show that

$$N_{n,g} \sim C_g n^{6g-4} \quad \text{for } n \rightarrow \infty,$$

where the  $a_n \sim b_n$  means that  $a_n/b_n \rightarrow 1$  for  $n \rightarrow \infty$ . The behavior of  $C_g$  when  $g \rightarrow \infty$  can be estimated from below and above by expressions of the form  $\frac{C^g}{(6g)!}$ , for constants  $C > 0$ , see Theorem 1.1 of [62] for a more precise statement. We first consider the summation in the crossing number  $Cr(K) = n$ , for a fixed genus  $g(K) = g$ , that is, the series

$$1 + \sum_{n=1}^{\infty} N_{n,g} q^{-\beta n}.$$

Using the estimate above, the behavior of this series is controlled by that of the polylogarithm series

$$\mathrm{Li}_{4-6g}(q^{-\beta}) = \sum_{n=1}^{\infty} n^{6g-4} q^{-\beta n},$$

which converges for all  $\beta > 0$ . We then consider the summation in the genus  $g(K) = g$ . The polylogarithm function satisfies

$$\begin{aligned} \mathrm{Li}_{-m}(z) &= \left(z \frac{\partial}{\partial z}\right)^m \frac{z}{1-z} = \sum_{k=0}^m k! S(m+1, k+1) \left(\frac{z}{1-z}\right)^{k+1} \\ &= \frac{1}{(1-z)^{m+1}} \sum_{k=0}^{m-1} \langle m \rangle_k z^{m-k}, \end{aligned}$$

where  $S(a, b)$  are the Stirling numbers of the second kind

$$S(a, b) = \frac{1}{b!} \sum_{j=0}^b (-1)^{b-j} \binom{b}{j} j^a,$$

while

$$\langle m \rangle_k = \sum_{j=0}^{k+1} (-1)^j \binom{m+1}{j} (k-j+1)^m$$

are the Eulerian numbers. The Stirling numbers of the second kind have upper and lower bounds of the form [50]

$$\frac{1}{2}(b^2 + b + 2)b^{a-b-1} - 1 \leq S(a, b) \leq \frac{1}{2} \binom{a}{b} b^{a-b},$$

and, for fixed  $b$ , the asymptotic behavior of  $S(a, b)$  for  $a \rightarrow \infty$  is of the form  $S(a, b) \sim b^a/b!$ . Moreover, the ordered Bell numbers  $\mathbf{b}_a = \sum_{b=0}^a b!S(a, b)$  behave for  $a \rightarrow \infty$  like [58]

$$\mathbf{b}_a \sim \frac{a!}{2(\log(2))^{a+1}}.$$

When  $q^{-\beta} \leq 1/2$ , that is, when  $\beta > \frac{\log 2}{\log q}$ , we have  $q^{-\beta} \leq (1 - q^{-\beta})$ , hence

$$\left(\frac{q^{-\beta}}{1 - q^{-\beta}}\right)^{6g-3} \leq \left(\frac{q^{-\beta}}{1 - q^{-\beta}}\right)^{k+1} \leq \frac{q^{-\beta}}{1 - q^{-\beta}},$$

for all  $k = 0, \dots, 6g - 4$ . Thus, the result of the first summation in  $n = Cr(K)$  can be approximated, for large  $g = g(K)$ , by upper and lower bounds of the form

$$\begin{aligned} \text{Li}_{4-6g}(q^{-\beta}) &\leq \mathbf{b}_{6g-4} \frac{q^{-\beta}}{1 - q^{-\beta}} \sim \frac{(6g-4)!}{2(\log 2)^{6g-4}} \frac{q^{-\beta}}{1 - q^{-\beta}} \\ &\frac{(6g-4)!}{2(\log 2)^{6g-4}} \left(\frac{q^{-\beta}}{1 - q^{-\beta}}\right)^{6g-3} \sim \mathbf{b}_{6g-4} \left(\frac{q^{-\beta}}{1 - q^{-\beta}}\right)^{6g-3} \leq \text{Li}_{4-6g}(q^{-\beta}). \end{aligned}$$

Then, in this range of values of  $\beta$ , the series defining the partition function  $Z(\beta) = \sum_K e^{-\beta H_K}$ , with  $H_K = \langle \epsilon_K, H \epsilon_K \rangle$ , is controlled from above by the behavior of

$$\sum_{g=1}^{\infty} C_g \frac{(6g-4)!}{2(\log 2)^{6g-4}} q^{-\beta g}.$$

Using  $C_g \sim \frac{C^g}{(6g)!}$  we obtain

$$(2.2) \quad C_g \frac{(6g-4)!}{2(\log 2)^{6g-4}} q^{-\beta g} \sim \frac{(\log 2)^4}{2} \frac{e^{g(\log C - \beta - 6 \log \log 2)}}{(6g-3)(6g-2)(6g-1)6g}.$$

When  $\beta \geq \log C - 6 \log \log 2$  the above series converges, with convergence in the case  $\beta = \log C - 6 \log \log 2$  ensured by the polynomial in the denominator. Thus, in the range  $\beta \geq \frac{\log 2}{\log q}$ , the partition function  $Z(\beta) = \text{Tr}(e^{-\beta H})$  converges for all

$$\beta \geq \max\left\{\frac{\log 2}{\log q}, \log C - 6 \log \log 2\right\}.$$

On the other hand, in this same range, the series defining the partition function is controlled from below by a series of the form

$$\sum_{g=1}^{\infty} C_g \frac{(6g-4)!}{2(\log 2)^{6g-4}} \lambda_{\beta}^{6g-3} q^{-\beta g},$$

where  $\lambda_{\beta} = q^{-\beta}/(1 - q^{-\beta})$ . In this case we have

$$(2.3) \quad C_g \frac{(6g-4)!}{2(\log 2)^{6g-4}} \lambda_{\beta}^{6g-3} q^{-\beta g} \sim \frac{(\log 2)^4}{2\lambda_{\beta}^3} \frac{e^{g(\log C - \beta - 6 \log \log 2 + 6 \log \lambda_{\beta})}}{(6g-3)(6g-2)(6g-1)6g}.$$

The corresponding series converges for  $\beta - 6 \log \lambda_{\beta} \geq \log C - 6 \log \log 2$  and diverges for  $\beta - 6 \log \lambda_{\beta} < \log C - 6 \log \log 2$ . Notice that, since in this range we have  $\lambda_{\beta} \leq 1$ , the convergence condition  $\beta \geq \log C - 6 \log \log 2$  for the upper bound also implies this convergence, as it

should, while the divergence condition  $\beta - 6 \log \lambda_\beta < \log C - 6 \log \log 2$  gives a range of divergence for the series defining the partition function  $Z(\beta)$ : we have divergence for

$$\frac{\log 2}{\log q} \leq \beta < 6 \log \lambda_\beta + \log C - 6 \log \log 2.$$

Consider then the case where  $\beta < \frac{\log 2}{\log q}$ . In this case we have  $q^{-\beta} > (1 - q^{-\beta})$ , that is,  $\lambda_\beta > 1$ , and, for all  $k = 0, \dots, 6g - 4$ ,

$$\frac{q^{-\beta}}{1 - q^{-\beta}} \leq \left( \frac{q^{-\beta}}{1 - q^{-\beta}} \right)^{k+1} \leq \left( \frac{q^{-\beta}}{1 - q^{-\beta}} \right)^{6g-3}.$$

In this case, the result of the first summation can be approximated from above, for large  $g = g(K)$ , with

$$\text{Li}_{4-6g}(q^{-\beta}) \leq \mathfrak{b}_{6g-4} \lambda_\beta^{6g-3} \sim \frac{(6g-4)!}{2(\log 2)^{6g-4}} \lambda_\beta^{6g-3},$$

and from below with

$$\frac{(6g-4)!}{2(\log 2)^{6g-4}} \lambda_\beta \sim \mathfrak{b}_{6g-4} \lambda_\beta \leq \text{Li}_{4-6g}(q^{-\beta}).$$

Thus, in this case, the series that determines the partition function is controlled from above by the behavior of the series

$$\sum_{g=1}^{\infty} C_g \frac{(6g-4)!}{2(\log 2)^{6g-4}} \lambda_\beta^{6g-3}.$$

As above, we can estimate this with (2.3). Again, the resulting series converges for  $\beta - 6 \log \lambda_\beta \geq \log C - 6 \log \log 2$ . Here  $\lambda_\beta > 1$ , so this inequality also implies the inequality  $\beta \geq \log C - 6 \log \log 2$ , which in this case gives the convergence of the lower bound, here of the form (2.2). The divergence of the lower bound happens for  $\beta < \log C - 6 \log \log 2$ . Thus, in the range  $\beta < \frac{\log 2}{\log q}$  we have convergence when

$$6 \log \lambda_\beta + \log C - 6 \log \log 2 \leq \beta < \frac{\log 2}{\log q},$$

and divergence for

$$\beta < \min\left\{\frac{\log 2}{\log q}, \log C - 6 \log \log 2\right\}.$$

As in Theorem 1.1 of [62], we can take the constant  $C$  to be  $C = 400$  for the lower bound on  $C_g$  and  $C = 2^{20}/3^6 \sim 1438.38$  for the upper bound on  $C_g$ . Using these values we can estimate that the series  $Z(\beta) = \sum_K \langle \epsilon_K, e^{-\beta H} \epsilon_K \rangle$  defining the partition function converges for

$$\beta \geq \max\left\{\frac{\log 2}{\log q}, \log \frac{2^{20}}{3^6} - 6 \log \log 2\right\}$$

and for

$$\beta - 6 \log \lambda_\beta \geq \log \frac{2^{20}}{3^6} - 6 \log \log 2 \quad \text{and} \quad \beta < \frac{\log 2}{\log q}$$

while it diverges for

$$\beta < \min\left\{\frac{\log 2}{\log q}, 2 \log(20) - 6 \log \log 2\right\}$$

and for

$$\beta - 6 \log \lambda_\beta < 2 \log(20) - 6 \log \log 2 \quad \text{and} \quad \beta \geq \frac{\log 2}{\log q}.$$

Consider the condition that the integer  $q \in \mathbb{N}$  satisfies

$$\frac{\log 2}{\log q} < 2 \log 20 - 6 \log \log 2.$$

We have  $\log 2 = (2 \log 20 - 6 \log \log 2) \log(x)$  for  $x \sim 1.0883$ , hence for all  $q \in \mathbb{N}$  with  $q \geq 2$  the condition above is satisfied. Then the convergence range above reduces to just the first condition  $\beta \geq \log \frac{2^{20}}{3^6} - 6 \log \log 2$ , since in the second case the conditions  $\beta < \frac{\log 2}{\log q}$  and  $\beta \geq \beta - 6 \log \lambda_\beta \geq \log \frac{2^{20}}{3^6} - 6 \log \log 2$  cannot be simultaneously realized since  $\frac{\log 2}{\log q} < \log \frac{2^{20}}{3^6} - 6 \log \log 2$ . Let  $\beta_+ := \log \frac{2^{20}}{3^6} - 6 \log \log 2$ . Similarly, the estimate of the range of divergence gives  $\beta < \frac{\log 2}{\log q}$  or  $\frac{\log 2}{\log q} \leq \beta < 6 \log \lambda_\beta + 2 \log 20 - 6 \log \log 2$ . Note that, in the range  $\beta \geq \frac{\log 2}{\log q}$ , the function  $\beta - 6 \log \lambda_\beta$  is non-negative and monotonically increasing, with a zero at  $\beta = \frac{\log 2}{\log q}$ . Let  $\beta_-$  be the unique value of  $\beta$  where  $\beta - 6 \log \lambda_\beta = 2 \log 20 - 6 \log \log 2 \sim 8.1905$ . The dependence on  $q$  of  $\beta_- = \beta_-(q)$  is monotonically decreasing, with  $\beta_-(q=2) \sim 1.9391$ , and with for example  $\lambda_-(q=10^2) \sim 0.3362$  and  $\lambda_-(q=10^3) \sim 0.2262$ . Then we obtain that the series defining the partition function is divergent in the range  $\beta < \beta_-$ . Note that the function  $F(q) = \beta_+ - 6 \log \lambda_{\beta_+}(q) - (2 \log 20 - 6 \log \log 2)$  is monotonically increasing in the variable  $q$ , with  $F(2) \sim 40.6574$ , hence  $\beta_- < \beta_+$ . Summarizing, we conclude that, for any choice of  $q \in \mathbb{N}$  with  $q \geq 2$ , the series defining the partition function  $Z(\beta)$  is convergent for  $\beta \geq \beta_+$  and divergent for  $\beta < \beta_-$ .  $\square$

**Remark 2.4.** The approximation method we used here, based on the estimates of [62], does not give information on the behavior of the series defining the partition function in the range  $\beta_- \leq \beta < \beta_+$ , but it is reasonable to expect that there will be a point  $\beta_c \in [\beta_-, \beta_+]$  where a phase transition occurs, so that the series defining the partition function converges for all  $\beta > \beta_c$  and diverges for all  $\beta < \beta_c$ .

**Lemma 2.5.** *In the range  $\beta \geq \beta_+$ , where the series (2.1) is convergent, the partition function  $Z(\beta)$  has an Euler product expansion*

$$(2.4) \quad Z_a(\beta) = \prod_{K \in \mathcal{P}_{\mathcal{K},a}} (1 - q^{-\beta(Cr(K)+g(K))})^{-1}.$$

*Proof.* This is a general fact about bosonic Fock spaces and the trace and determinant of operators. As in §2 of [5], we identify  $\ell^2(\mathcal{K}_a) = S\ell^2(\mathcal{P}_{\mathcal{K},a}) := \bigoplus_{n=0}^{\infty} S^n \ell^2(\mathcal{P}_{\mathcal{K},a})$ , the bosonic Fock space given by the sum of the symmetric powers of  $\ell^2(\mathcal{P}_{\mathcal{K},a})$ . Let  $T$  be the densely defined operator on  $\ell^2(\mathcal{P}_{\mathcal{K},a})$  with  $T\epsilon_K = q^{-\beta(Cr(K)+g(K))}\epsilon_K$ , and let  $ST$  be the induced densely defined operator on the Fock space  $\ell^2(\mathcal{K}_a)$ . On a basis element  $\epsilon_{K_1 \# \dots \# K_m} = \epsilon_{K_1} \cdots \epsilon_{K_m}$ , this satisfies  $ST\epsilon_{K_1 \# \dots \# K_m} = q^{-\beta(Cr(K_1)+g(K_1))} \dots q^{-\beta(Cr(K_m)+g(K_m))} \epsilon_{K_1 \# \dots \# K_m}$ . Thus, when the trace of  $ST$  is finite it satisfies

$$\text{Tr}(ST) = \frac{1}{\det(1 - T)}.$$

By direct inspection, we see that  $ST = e^{-\beta H}$  and that  $1/\det(1 - T)$  is the Euler product of (2.4).  $\square$

**2.2. Statistical mechanics of knots without interaction.** We then have, for the  $C^*$ -dynamical system  $(C^*(\mathcal{K}_a), \sigma_t)$  described above, the analog of Proposition 8 of [5].

**Proposition 2.6.** *For every  $\beta > 0$  there is a unique  $KMS_\beta$  state for  $(C^*(\mathcal{K}_a), \sigma_t)$ , which is the infinite tensor product of unique  $KMS_\beta$  states  $\phi_{\beta,K}$  for  $K \in \mathcal{P}_{\mathcal{K},a}$ , on the Toeplitz algebra  $\tau_K$  with the induced time evolution, with eigenvalue list*

$$(2.5) \quad \Sigma(\phi_{\beta,K}) = \{(1 - q^{-\beta(Cr(K)+g(K))})q^{-\beta n(Cr(K)+g(K))}\}_{n \in \mathbb{N}}.$$

For  $\beta \geq \beta_+$  the  $KMS$  state is a Gibbs state of the form

$$\phi_\beta(X) = \frac{1}{Z(\beta)} \text{Tr}(X e^{-\beta H}), \quad \forall X \in C^*(\mathcal{K}_a),$$

while for  $\beta < \beta_-$  the  $KMS$  state is of type III.

*Proof.* On the Toeplitz algebra  $\tau_K$ , for some  $K \in \mathcal{P}_{\mathcal{K},a}$ , the induced time evolution is determined by  $\sigma_t(\mu_K) = q^{it(Cr(K)+g(K))} \mu_K$ . A  $KMS_\beta$  state on  $(\tau_K, \sigma_t)$  will necessarily vanish on all eigenvectors of the time evolution with  $\sigma_t(X) = \lambda^{it} X$  where  $\lambda \neq 1$ , while by the  $KMS$  condition it will satisfy

$$\begin{aligned} \varphi_{\beta,K}(\mu_K \mu_K^*) &= \varphi_{\beta,K}(\mu_K^* \sigma_{i\beta}(\mu_K)) = q^{-\beta(Cr(K)+g(K))} \varphi_{\beta,K}(\mu_K^* \mu_K) \\ &= q^{-\beta(Cr(K)+g(K))} \varphi_{\beta,K}(1) = q^{-\beta(Cr(K)+g(K))}. \end{aligned}$$

The complementary projection  $1 - \mu_K \mu_K^*$  then has  $\varphi_{\beta,K}(1 - \mu_K \mu_K^*) = 1 - q^{-\beta(Cr(K)+g(K))}$ . On powers  $\mu_K^n (\mu_K^*)^m$  the  $KMS$  state vanishes unless  $n = m$ , in which case  $\varphi_{\beta,K}(\mu_K^n (\mu_K^*)^n) = q^{-\beta n(Cr(K)+g(K))}$ , by the same argument. Note that, since we are working with alternating knots  $n(Cr(K) + g(K)) = Cr(K \# \cdots \# K) + g(K \# \cdots \# K)$ , with the connected sum taken  $n$  times. The same argument used in Proposition 8 of [5] then shows that this determines uniquely the  $KMS_\beta$  state  $\phi_{\beta,K}$ , and the fact that this implies the uniqueness of the  $KMS_\beta$  state on the tensor product  $C^*$ -algebra  $C^*(\mathcal{K}_a) = \otimes_{K \in \mathcal{P}_{\mathcal{K},a}} \tau_K$ . As in case (b) of Proposition 8 of [5] the finiteness of  $Z(\beta) = \text{Tr}(e^{-\beta H})$  for  $\beta \geq \beta_+$  shows that the  $KMS_\beta$  state is of the Gibbs form (by uniqueness, since the Gibbs state is clearly a  $KMS_\beta$  state). In the range  $\beta < \beta_-$  where the series defining the partition function is divergent, one uses the same argument used in [5], based on the result of [2]. Namely, as in Lemma 2.14 of [2], if  $\{\lambda_{\nu,i}\}$  is the eigenvalue list of an infinite tensor product  $M = \otimes_\nu M_\nu$  of type I factors, then  $M$  is of type I if and only if  $\sum_\nu |1 - \lambda_{\nu,1}| < \infty$ ; of type II if and only if  $n_\nu < \infty$  for all  $\nu$  and  $\sum_{\nu,i} |n_\nu^{-1/2} - \lambda_{\nu,i}^{1/2}|^2 < \infty$ ; and, when  $\lambda_{\nu,1} \geq \delta$  for some  $\delta$  for all  $\nu$ ,  $M$  is of type III if and only if

$$\sum_{\nu,i} \lambda_{\nu,i} \inf\{|\frac{\lambda_{\nu,1}}{\lambda_{\nu,i}} - 1|^2, C\} = \infty$$

for some (hence all)  $C > 0$ . In our case, with the eigenvalue list (2.5), we have  $\lambda_{\nu,1} = 1 - q^{-\beta(Cr(K)+g(K))}$  hence  $|1 - \lambda_{\nu,1}| = q^{-\beta(Cr(K)+g(K))}$ . In the range  $\beta < \beta_-$  the series  $\sum_K q^{-\beta(Cr(K)+g(K))}$  is divergent, hence type I is excluded. Similarly, type II is excluded because  $n_\nu = \infty$ . For a fixed  $\beta$ , the condition  $\lambda_{\nu,1} \geq \delta$  is satisfied with  $\delta = 1 - q^{-\beta}$ , and we have

$$\sum_{\nu,j} \lambda_{\nu,j} \inf\{|\frac{\lambda_{\nu,1}}{\lambda_{\nu,j}} - 1|^2, C\} \sim \sum_{K,j} (1 - q^{-\beta(Cr(K)+g(K))}) q^{-\beta j(Cr(K)+g(K))} = \infty$$

hence the factor is type III. □

**Remark 2.7.** Notice that, since we do not have in this case a complete analysis of the behavior of the partition function in the intermediate range  $\beta_- \leq \beta < \beta_+$ , we do not have in this case the direct analog of case (c) of Proposition 8 of [5].

**Lemma 2.8.** *For a fixed  $q \in \mathbb{N}$ ,  $q \geq 2$ , there is a unique solution  $\tilde{\beta}_- = \tilde{\beta}_-(q)$ , with  $\tilde{\beta}_- > \frac{\log 2}{\log q}$ , to the equation*

$$(2.6) \quad \beta - 6 \log \lambda_\beta + 6 \log \beta = \log C - 6 \log \log q,$$

where  $C = 400$  and

$$\lambda_\beta = \frac{q^{-\beta}}{1 - q^{-\beta}}.$$

The value  $\tilde{\beta}_-(q)$  satisfies  $\tilde{\beta}_-(q) < \beta_-(q)$ , where  $\beta_-(q)$  is, as in Theorem 2.3, the unique solution of  $\beta - 6 \log \lambda_\beta = \log C - 6 \log \log 2$ .

*Proof.* For  $\beta = \log 2 / \log q$  we have  $(\beta - 6 \log \lambda_\beta)|_{\beta=\frac{\log 2}{\log q}} = \frac{\log 2}{\log q}$  hence

$$(\beta - 6 \log \lambda_\beta + 6 \log \beta)|_{\beta=\frac{\log 2}{\log q}} = \frac{\log 2}{\log q} + 6 \log \log 2 - 6 \log \log q < \log C - 6 \log \log q,$$

since we have seen in Theorem 2.3 that, for all  $q \in \mathbb{N}$  with  $q \geq 2$ ,

$$(2.7) \quad \frac{\log 2}{\log q} < \log C - 6 \log \log 2.$$

For  $\beta \geq \log 2 / \log q$  the function  $f(\beta, q) := \beta - 6 \log \lambda_\beta + 6 \log \beta$  is monotonically increasing and unbounded for  $\beta \rightarrow \infty$  (see the plot in Figure 1, hence there will be a unique  $\tilde{\beta}_- = \tilde{\beta}_-(q)$  where (2.6) holds. Finally, we see that at  $\beta = \beta_-(q)$  we have

$$(\beta - 6 \log \lambda_\beta + 6 \log \beta)|_{\beta=\beta_-(q)} = \log C - 6 \log \log 2 + 6 \log \beta_-(q).$$

Notice that we have  $\beta_-(q) > \log 2 / \log q$  because of (2.7), hence we find  $\beta_- - 6 \log \lambda_{\beta_-} + 6 \log \beta_- > \log C - 6 \log \log q$ , hence  $\beta_-(q) > \tilde{\beta}_-(q)$ .  $\square$

Thus, the range  $\beta < \tilde{\beta}_-(q)$  is contained in the range of divergence of the series defining the partition function, as we have seen in Theorem 2.3.

**Theorem 2.9.** *Let  $\tilde{\beta}_- = \tilde{\beta}_-(q)$  be as in Lemma 2.8. For  $\beta < \tilde{\beta}_-(q)$ , the unique  $KMS_\beta$  state is of type  $III_{q^{-\beta}}$ .*

*Proof.* The argument is similar to Lemma 4.5.1 of [27] and Lemma 2.4 of [45]. We need to show that  $q^{-\beta}$  belongs to the asymptotic ratio set, see Definition 3.2 and Lemma 3.6 of [2]. As in Lemma 4.5.1 of [27], for given  $\beta$ , let  $N \in \mathbb{N}$  be chosen so that  $\beta N > \beta_+$  and, for a chosen  $K \in \mathcal{P}_{\mathcal{K},a}$ , consider the projector  $e_K = 1 - \mu_K^N(\mu_K^*)^N$  in the Toeplitz algebra  $\tau_K$ , and the projection  $e = \prod_{K \in \mathcal{P}_{\mathcal{K},a}} e_K$ , as weak limit of projections in the tensor product von Neumann algebra. Since  $\beta N > \beta_+$  we have, using the Euler product of Lemma 2.5,

$$\phi_\beta(e) = \prod_{K \in \mathcal{P}_{\mathcal{K},a}} (1 - q^{-\beta N(Cr(K)+g(K))}) = Z(\beta N)^{-1} \neq 0,$$

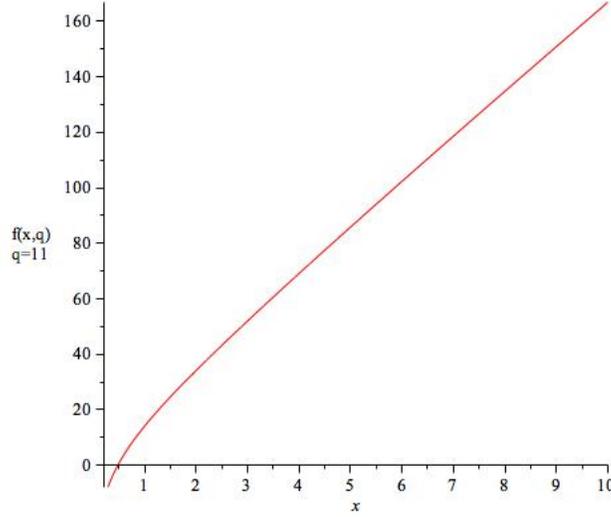


FIGURE 1. The function  $f(\beta, q)$  for  $\beta > \frac{\log 2}{\log q}$  and  $q = 11$ .

hence  $e \neq 0$ . Setting  $\tilde{\phi}_{\beta,e}(X) = \phi_{\beta}(X)/\phi_{\beta}(e)$  determines a KMS state on the compression of the algebra with the projection  $e$ . For each prime knot  $K \in \mathcal{P}_{\mathcal{K},a}$  we similarly have  $\tilde{\phi}_{\beta,e,K}(X) = \phi_{\beta,K}(X)(1 - q^{-\beta N(Cr(K)+g(K))})^{-1}$ . The eigenvalue list of  $\tilde{\phi}_{\beta,e,K}$  is then

$$\Sigma(\tilde{\phi}_{\beta,e,K}) = \left\{ \frac{(1 - q^{-\beta(Cr(K)+g(K))})q^{-\beta n(Cr(K)+g(K))}}{(1 - q^{-\beta N(Cr(K)+g(K))})} \right\}_{n \in \mathbb{N}}.$$

By the results of [62], the number of knots  $K$  in  $\mathcal{K}_a$  with a given value  $Cr(K) + g(K) = n$  is given by

$$(2.8) \quad N(n) = \#\{K \in \mathcal{K}_a \mid Cr(K) + g(K) = n\} \sim \sum_{g=1}^n \frac{C^g}{(6g)!} (n - g + 1)^{6g-4}.$$

Thus, we have  $N(n)$  knots  $K_1, \dots, K_{N(n)}$  for which  $q^{\beta(Cr(K_i)+g(K_i))} = q^{-\beta n}$ . Consider two disjoint sets  $\mathcal{X}_1(n) = \{K_1, \dots, K_{N(2n)}\}$ , the set of knots with  $q^{\beta(Cr(K_i)+g(K_i))} = q^{-\beta 2n}$  and a subset  $\mathcal{X}_2(n) = \{K'_1, \dots, K'_{N(2n)}\}$  of the same cardinality of the set of knots with  $q^{\beta(Cr(K_i)+g(K_i))} = q^{-\beta(2n+1)}$ . Consider the set  $\mathcal{F}_n$  of functions from the set  $\mathcal{X}(n) = \mathcal{X}_1(n) \cup \mathcal{X}_2(n)$  to the set  $\underline{N} = \{0, \dots, N - 1\}$ , namely  $\mathcal{F}_n = \mathcal{F}(\mathcal{X}(n), \underline{N})$ . In this set, consider the delta functions  $\delta_{K_i}$  and  $\delta_{K'_i}$  for  $i = 1, \dots, N(2n)$ . Setting

$$\mu(f) = \prod_{i=1}^{N(2n)} \lambda_{K_i, f(K_i)} \lambda_{K'_i, f(K'_i)},$$

where

$$\lambda_{K_i, j} = \frac{(1 - q^{2n\beta})}{(1 - q^{-2nN\beta})} q^{-\beta 2nj},$$

defines a measure on the set  $\mathcal{F}_n$ . This satisfies

$$\mu(\delta_{K_i}) = \mu(\delta_{K_1}) = \left( \frac{(1 - q^{2n\beta})}{(1 - q^{-2nN\beta})} \right)^{N(2n)} \cdot \left( \frac{(1 - q^{(2n+1)\beta})}{(1 - q^{-(2n+1)N\beta})} \right)^{N(2n)} \cdot q^{-\beta 2n} =: \mu(n).$$

The measure of the set  $\{\delta_{K_i}\}$  is equal to  $\mu(\{\delta_{K_i}\}) = N(2n)\mu(n)$ .

By (2.8), the behavior of the series  $\sum_n N(2n)\mu(n)$  can be estimated in terms of the behavior of

$$\sum_n N(2n)q^{-\beta 2n} = \sum_n \sum_{k+g=2n} N_{k,g}q^{-\beta(k+g)},$$

where  $N_{k,g}$  is the number of alternating knots with  $Cr(K) = k$  and  $g(K) = g$ . Note that this is a subseries of the series  $\sum_g \sum_k N_{k,g}q^{-\beta(k+g)}$ , whose behavior we analyzed in Theorem 2.3. In particular, we know that for  $\beta < \beta_-$  the series  $\sum_g \sum_k N_{k,g}q^{-\beta(k+g)}$  diverges. We now need to check whether the subseries corresponding to the terms with  $k + g$  even also diverges. We first show that we can express this series in terms of the Lerch transcendent, replacing the polylogarithms used in the case of the full series in Theorem 2.3.

Let  $\Phi(z, s, \alpha)$  be the Lerch transcendent

$$(2.9) \quad \Phi(z, s, \alpha) = \sum_{\ell \geq 0} \frac{z^\ell}{(\alpha + \ell)^s}.$$

We can then write the series above as

$$(2.10) \quad \begin{aligned} \sum_{k,g \geq 0: k+g \text{ even}} N_{k,g}q^{-\beta(k+g)} &\sim \sum_{g \geq 0} q^{-2\beta g} \frac{C^g}{(6g)!} \sum_{\ell \geq 0} q^{-2\beta \ell} (g + 2\ell)^{6g-4} \\ &= \sum_{g \geq 0} \frac{C^g 2^{6g-4} q^{-2\beta g}}{(6g)!} \Phi(q^{-2\beta}, 4 - 6g, \frac{g}{2}). \end{aligned}$$

The Lerch transcendent  $\Phi(z, s, \alpha)$  has a Taylor expansion

$$(2.11) \quad \Phi(z, s, \alpha) = z^{-\alpha} \left( \Gamma(1-s)(-\log(z))^{s-1} + \sum_{j \geq 0} \zeta(s-j, \alpha) \frac{\log^j(z)}{j!} \right),$$

which is valid for  $|\log(z)| < 2\pi$ ,  $s \notin \mathbb{N}$  and  $\alpha \notin \mathbb{Z}_{\leq 0}$ . In our setting we have  $z = q^{-2\beta}$ , hence  $|\log(z)| = 2\beta \log(q)$ . One can check that the function

$$H(q) := (\beta - 6 \log \lambda_\beta + 6 \log \beta)|_{\beta = \frac{\pi}{\log q}} - (\log C - 6 \log \log q)$$

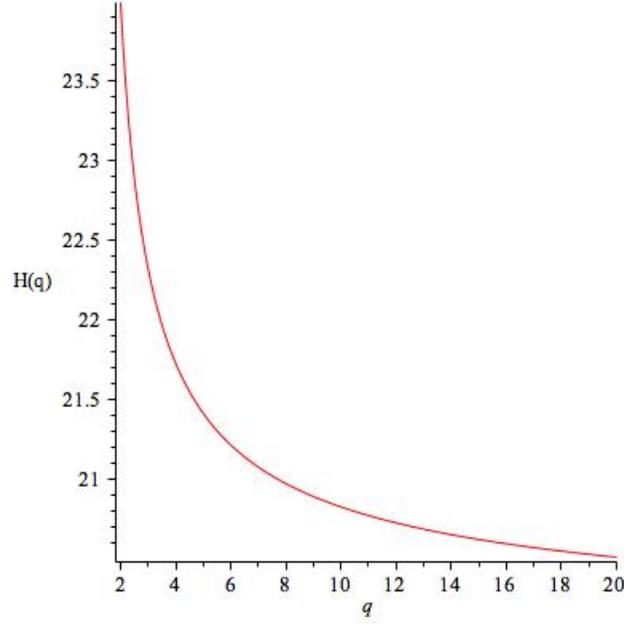
is positive for  $q \geq 2$  (see the plot in Figure 2), hence  $\frac{\pi}{\log q} > \tilde{\beta}_-(q)$ . Thus, in the range  $\beta < \tilde{\beta}_-(q)$  the Taylor expansion above applies.

Then we have

$$\Phi(q^{-2\beta}, 4 - 6g, \frac{g}{2}) = q^{\beta g} \left( \frac{(6g-4)!}{(2\beta \log(q))^{6g-3}} + \sum_{j \geq 0} \zeta(4 - 6g - j, \frac{g}{2}) \frac{(-2\beta \log(q))^j}{j!} \right).$$

Thus, the general term of series above has a leading contribution of the form

$$(2.12) \quad \frac{C^g q^{-\beta g}}{(6g)(6g-1)(6g-2)(6g-3)} \cdot \frac{2^{6g-4}}{(2\beta \log(q))^{6g-3}}.$$


 FIGURE 2. The function  $H(q)$  for  $q \geq 2$ .

Notice that this is the analog of the leading term of the form

$$\frac{C^g q^{-\beta g}}{(6g)(6g-1)(6g-2)(6g-3)} \cdot \frac{1}{2(\log 2)^{6g-4}}$$

for the full series, as in (2.2) of Theorem 2.3. Arguing in a similar way, we then see that the series

$$\sum_g \frac{C^g q^{-\beta g}}{(6g)(6g-1)(6g-2)(6g-3)2(\beta \log(q))^{6g-3}}$$

is divergent in the range  $\beta < \tilde{\beta}_-(q)$ , hence so is the series (2.10). Thus, we obtain the divergence result

$$\sum_{n \geq n_0} N(2n)\mu(n) = \infty.$$

We then proceed in the same way as in Lemma 4.5.1 of [27]. The bijection  $\Psi_n : \mathcal{X}_1(n) \rightarrow \mathcal{X}_2(n)$  determines a bijection of the set of delta functions, and we have

$$\frac{\mu(\Psi_n(\delta_{K_i}))}{\mu(\delta_{K_i})} = \frac{\lambda_{K_i,0}\lambda_{K'_i,1}}{\lambda_{K_i,1}\lambda_{K'_i,0}} = \frac{q^{-\beta(2n+1)}}{q^{-\beta 2n}} = q^{-\beta},$$

which shows that  $q^{-\beta}$  is in the asymptotic ratio set.  $\square$

**2.3. Intermezzo: statistical physics of  $\mathbb{Q}_+^*$ .** In preparation for the construction we will illustrate in the following section, we discuss here a toy model, based on the multiplicative group  $\mathbb{Q}_+^*$  and its reduced group algebra  $C_r^*(\mathbb{Q}_+^*)$  acting on the Hilbert space  $\ell^2(\mathbb{Q}_+^*)$ . Here we regard  $\mathbb{Q}_+^*$  as the discrete infinitely generated abelian group, generated by the primes,  $\mathbb{Q}_+^* = \prod_{p \in \mathcal{P}} p^{\mathbb{Z}}$ . We want to construct a quantum statistical mechanical system whose algebra of observables contains  $C_r^*(\mathbb{Q}_+^*)$ , with Hilbert space of states  $\ell^2(\mathbb{Q}_+^*)$  and with a Hamiltonian generator densely defined on  $\ell^2(\mathbb{Q}_+^*)$ , so that the partition function  $Z(\beta) = \text{Tr}(e^{-\beta H})$  is finite for sufficiently large  $\beta > 0$ .

It is easy to construct such a system for the multiplicative semigroup  $\mathbb{N}$  and the  $C^*$ -algebra  $C_r^*(\mathbb{N})$  acting on  $\ell^2(\mathbb{N})$ . For instance by considering the ‘‘system without interaction’’ of [5] with time evolution  $\sigma_t(\mu_n) = n^{it}\mu_n$  and with  $H\epsilon_n = \log(n)\epsilon_n$ . However, the natural extension of this system from the semigroup  $\mathbb{N}$  to the group  $\mathbb{Q}_+^*$  by  $\sigma_t(\mu_{a/b}) = (a/b)^{it}\mu_{a/b}$  will no longer satisfy the condition  $\text{Tr}(e^{-\beta H}) < \infty$  for large  $\beta$ .

We proceed in a slightly different way, motivated by the analogies between quantum statistical mechanical systems and spectral triples discussed in [23]. We consider first the case of a single prime  $p$  and the group  $p^{\mathbb{Z}} \simeq \mathbb{Z}$ , and then the case of the group  $\mathbb{Q}_+^*$ .

Recall that a spectral triple  $(\mathcal{A}, \mathcal{H}, D)$  is the datum of an involutive algebra  $\mathcal{A}$ , a representation of  $\mathcal{A}$  by bounded operators on  $\mathcal{H}$  and a self-adjoint operator  $D$ , densely defined on  $\mathcal{H}$ , with compact resolvent  $(D^2 + 1)^{-1/2} \in \mathcal{K}$  and such that the commutators  $[D, a]$  with all  $a \in \mathcal{A}$  are bounded operators on  $\mathcal{H}$ , see [10].

**Lemma 2.10.** *Consider the algebra  $\mathbb{C}[p^{\mathbb{Z}}] \subset C^*(p^{\mathbb{Z}}) \simeq C(S^1)$ , acting on  $\ell^2(p^{\mathbb{Z}})$ , and the operator defined by  $D_p\epsilon_{p^n} = n \log(p)\epsilon_{p^n}$ . The datum  $(\mathbb{C}[p^{\mathbb{Z}}], \ell^2(p^{\mathbb{Z}}), D_p)$  is a spectral triple.*

*Proof.* It is easy to check that all properties are satisfied. We check explicitly the bounded commutator condition. Let  $\delta_{p^m}$  be the operator on  $\ell^2(p^{\mathbb{Z}})$  corresponding to the delta function  $\delta_{p^m} \in \mathbb{C}[p^{\mathbb{Z}}]$ . It acts by  $\delta_{p^m}\epsilon_{p^n} = \epsilon_{p^{n+m}}$ . Thus, we have

$$(D_p\delta_{p^m} - \delta_{p^m}D_p)\epsilon_{p^n} = ((m+n)\log(p) - n\log(p))\epsilon_{p^{n+m}} = m\log(p)\epsilon_{p^{n+m}},$$

hence the bounded commutator condition is satisfied for all element of the dense subalgebra  $\mathbb{C}[p^{\mathbb{Z}}]$  of  $C_r^*(p^{\mathbb{Z}})$ .  $\square$

**Remark 2.11.** Notice that, in the case of the group  $\mathbb{Q}_+^*$ , the operator  $D$  acting on a basis element  $\epsilon_r$  of  $\ell^2(\mathbb{Q}_+^*)$  as  $D\epsilon_r = (n_1 \log(p_1) + \dots + n_k \log(p_k))\epsilon_r$ , for  $r = p_1^{n_1} \dots p_k^{n_k} \in \mathbb{Q}_+^*$ , with  $n_i \in \mathbb{Z}$  has bounded commutators with elements of the dense subalgebra  $\mathbb{C}[\mathbb{Q}_+^*]$  of  $C_r^*(\mathbb{Q}_+^*)$ , by the same argument of Lemma 2.10. However,  $D$  does not have compact resolvent, since the set  $\{\sum_{i=1}^k n_i \log(p_i) \mid n_i \in \mathbb{Z}, p_i \in \mathcal{P}, k \in \mathbb{N}\}$  is dense in  $\mathbb{R}$ , hence  $D$  does not determine a spectral triple for  $C_r^*(\mathbb{Q}_+^*)$ .

We now modify the operator above, using the polar decomposition of the Dirac operator  $D_p = |D_p|F$ , with  $F$  the sign operator.

**Lemma 2.12.** *Consider the operator  $H_p = |D_p|$  acting on a basis  $\epsilon_{p^n}$  of  $\ell^2(p^{\mathbb{Z}})$  as*

$$(2.13) \quad H_p \epsilon_{p^n} = |n| \log(p) \epsilon_{p^n}.$$

*The operator  $e^{-\beta H_p}$  is trace class for all  $\beta > 0$  with*

$$(2.14) \quad \text{Tr}(e^{-\beta H_p}) = 1 + 2 \sum_{n \in \mathbb{N}} p^{-\beta n} = \frac{(1 - p^{-2\beta})}{(1 - p^{-\beta})^2}.$$

*Proof.* For all  $\beta > 0$ , we have

$$\begin{aligned} \mathrm{Tr}(e^{-\beta H_p}) &= \sum_{n \in \mathbb{Z}} \langle \epsilon_{p^n}, e^{-\beta H_p} \epsilon_{p^n} \rangle = 1 + 2 \sum_{n \in \mathbb{N}} p^{-\beta n} = 1 + \frac{2p^{-\beta}}{(1 - p^{-\beta})} \\ &= \frac{1}{1 - p^{-\beta}} + \frac{p^{-\beta}}{1 - p^{-\beta}} = \frac{1 + p^{-\beta}}{1 - p^{-\beta}} = \frac{(1 - p^{-2\beta})}{(1 - p^{-\beta})^2} \end{aligned}$$

where the first term corresponds to  $\mathrm{Ker}(H_p) = \mathbb{C}\epsilon_1$ .  $\square$

This in turn determines an operator  $H$  on  $\ell^2(\mathbb{Q}_+^*)$  with the following properties.

**Lemma 2.13.** *Consider the operator  $H$  acting on the basis elements  $\epsilon_r$  of  $\ell^2(\mathbb{Q}_+^*)$ , for  $r = p_1^{n_1} \cdots p_k^{n_k} \in \mathbb{Q}_+^*$ , with  $n_i \in \mathbb{Z}$ , as*

$$(2.15) \quad H \epsilon_r = (|n_1| \log(p_1) + \cdots + |n_k| \log(p_k)) \epsilon_r.$$

Then for  $\beta > 1$  the operator  $e^{-\beta H}$  is trace class with

$$(2.16) \quad \mathrm{Tr}(e^{-\beta H}) = \frac{\zeta^2(\beta)}{\zeta(2\beta)},$$

where  $\zeta(\beta)$  is the Riemann zeta function.

*Proof.* We have  $\mathrm{Ker}(H) = \mathbb{C}\epsilon_1$ , with  $\epsilon_1$  the basis vector of  $\ell^2(\mathbb{Q}_+^*)$  corresponding to the unit in  $\mathbb{Q}_+^*$ . The spectrum of  $H$  is given by  $\mathrm{Spec}(H) = \{\log(n) \mid n \in \mathbb{N}\}$ . The trace is then computed by

$$\mathrm{Tr}(e^{-\beta H}) = \sum_{r \in \mathbb{Q}_+^*} \langle \epsilon_r, e^{-\beta H} \epsilon_r \rangle = \sum_{\lambda \in \mathrm{Spec}(H)} m_\lambda e^{-\beta \lambda},$$

where  $m_\lambda$  is the multiplicity. For  $\lambda = n_1 \log(p_1) + \cdots + n_k \log(p_k) = \log(n)$  with  $n_i \in \mathbb{N}$  and  $p_i \in \mathcal{P}$ , and  $n = p_1^{n_1} \cdots p_k^{n_k}$ , the multiplicity is  $m_\lambda = 2^k$ , with  $k$  the number of distinct prime factors in  $r = p_1^{\pm n_1} \cdots p_k^{\pm n_k}$ , since for each  $n_i$  we have two choices of  $\pm n_i \in \mathbb{Z}$ . Thus, we can rewrite the series computing the partition function as

$$\mathrm{Tr}(e^{-\beta H}) = \sum_{n \in \mathbb{N}} \frac{2^{\omega(n)}}{n^{-\beta}},$$

where  $\omega(n)$  is the number of distinct prime factors of  $n \in \mathbb{N}$ . It is known by Theorem 301, p.335 of [25] that this series converges for  $\beta > 1$  with sum

$$\sum_{n \in \mathbb{N}} \frac{2^{\omega(n)}}{n^{-\beta}} = \frac{\zeta^2(\beta)}{\zeta(2\beta)},$$

with  $\zeta(\beta) = \sum_{n \in \mathbb{N}} n^{-\beta}$  the Riemann zeta function. This can be seen easily by the form (2.14) of the Euler factors, since we have

$$\frac{\zeta^2(\beta)}{\zeta(2\beta)} = \prod_p \frac{(1 - p^{-2\beta})}{(1 - p^{-\beta})^2} = \prod_p (1 + 2 \sum_{k \in \mathbb{N}} p^{-\beta k}) = \sum_{n \geq 1} 2^{\omega(n)} n^{-\beta}.$$

$\square$

Consider the algebra of bounded operators  $\mathcal{B}(\ell^2(\mathbb{Q}_+^*))$ . The operator  $H$  described above determines a time evolution of the form

$$(2.17) \quad \sigma_t(T) = e^{itH} T e^{-itH}, \quad \forall t \in \mathbb{R}, \quad \forall T \in \mathcal{B}(\ell^2(\mathbb{Q}_+^*)),$$

with partition function as in (2.16)

$$Z(\beta) = \text{Tr}(e^{-\beta H}) = \frac{\zeta^2(\beta)}{\zeta(2\beta)}.$$

The subalgebra  $C_r^*(\mathbb{Q}_+^*) \subset \mathcal{B}(\ell^2(\mathbb{Q}_+^*))$  is not preserved by the time evolution (2.17). We have the following result, which is analogous to Lemma 5.7 of [23].

**Proposition 2.14.** *The smallest  $C^*$ -subalgebra  $\mathcal{A} \subset \mathcal{B}(\ell^2(\mathbb{Q}_+^*))$  that contains  $C_r^*(\mathbb{Q}_+^*)$  is invariant under the time evolution (2.17) is generated by  $C_r^*(\mathbb{Q}_+^*)$  and by projections*

$$\Pi_{(k,\ell)} \epsilon_{a/b} = \begin{cases} \epsilon_{a/b} & k|a \text{ and } \ell|b \\ 0 & \text{otherwise} \end{cases}$$

The time evolution (2.17) acts on the algebra  $\mathcal{A}$  by inner automorphisms.

*Proof.* Consider a generator  $\delta_r$ , for  $r \in \mathbb{Q}_+^*$ , of the algebra  $C_r^*(\mathbb{Q}_+^*)$ . The operator  $e^{itH} \delta_r e^{-itH}$  acts on a basis element  $\epsilon_{r'}$  as

$$e^{itH} \delta_r e^{-itH} \epsilon_{r'} = n(rr')^{it} n(r')^{-it} \delta_r \epsilon_{r'},$$

where for  $r = p_1^{n_1} \cdots p_k^{n_k}$  in  $\mathbb{Q}_+^*$ , with  $n_i \in \mathbb{Z}$ , we have  $n(r) = p_1^{|n_1|} \cdots p_k^{|n_k|}$  in  $\mathbb{N}$ . If  $r' = a/b$ , with  $a, b \in \mathbb{N}$  with  $(a, b) = 1$ , and  $r = u/v$  with  $u, v \in \mathbb{N}$  with  $(u, v) = 1$ , then

$$\frac{n(r'r)}{n(r')} = n(r) \cdot (b, u) \cdot (a, v).$$

Thus, for  $r = u/v$ , we can rewrite the operator above as

$$e^{itH} \delta_r e^{-itH} = \sum_{k|u} \sum_{\ell|v} n(r)^{it} k^{it} \ell^{it} \delta_r \Pi_{(k,\ell)},$$

where  $\Pi_{(k,\ell)} \epsilon_{r'} = \epsilon_{r'}$  if  $k|a$  and  $\ell|b$  and zero otherwise, where  $r' = a/b$ . The term of the sum with  $k = 1$  and  $\ell = 1$  corresponds to the operator  $n(r)^{it} \delta_r$ . Thus, this shows that the smallest  $C^*$ -subalgebra  $\mathcal{A} \subset \mathcal{B}(\ell^2(\mathbb{Q}_+^*))$  that contains  $C_r^*(\mathbb{Q}_+^*)$  and that is invariant under the time evolution (2.17) is the  $C^*$ -subalgebra  $\mathcal{A} \subset \mathcal{B}(\ell^2(\mathbb{Q}_+^*))$  generated by the  $\delta_r$  and by the projections  $\Pi_{(k,\ell)}$ . Let  $\Pi_n$  be the spectral projections of the operator  $H$  corresponding to the eigenvalues  $\log(n)$  with  $n \in \mathbb{N}$ . We see that these are in the algebra  $\mathcal{A}$  generated by the  $\delta_r$  and the  $\Pi_{(k,\ell)}$ . Indeed we have  $\Pi_n \epsilon_r = \epsilon_r$  when  $n = n(r)$  and zero otherwise, so that we have  $\Pi_n = \sum_{k,\ell: k\ell=n} \Pi_{(k,\ell)}$ . The unitary operator  $e^{itH}$  is a bounded operator that is in the  $C^*$ -algebra generated by the spectral projections of  $H$ . Thus, the time evolution (2.17) acts on the algebra  $\mathcal{A}$  by inner automorphisms.  $\square$

**2.4. Semigroup and Grothendieck group.** We will see later in the paper that, in addition to the abelian semigroup  $(\mathcal{K}, \#)$  of oriented knots with the connected sum operation, we also need to consider the associated Grothendieck group.

Let  $\mathcal{G}_{\mathcal{K}}$  denote the universal enveloping abelian group (Grothendieck group) of the semigroup  $(\mathcal{K}, \#)$ . The decomposition into prime knots shows that  $(\mathcal{K}, \#)$  is a free abelian semigroup on a countable set of generators given by the prime knots. Thus,  $(\mathcal{K}, \#)$  is non-canonically isomorphic to the semigroup  $(\mathbb{N}, \cdot)$ , and its enveloping group  $\mathcal{G}_{\mathcal{K}}$  is non-canonically isomorphic to the multiplicative group  $\mathbb{Q}_+^*$ . The universal enveloping abelian group  $\mathcal{G}_{\mathcal{K}}$  of  $(\mathcal{K}, \#)$  can be identified with pairs  $(K, K')$ , up to the equivalence relation  $(K, K') \sim (K \# K'', K' \# K'')$  for all  $K'' \in \mathcal{K}$ . We write the equivalence classes of pairs as formal differences, denoted by  $K \ominus K'$ .

In the case of the semigroup  $\mathcal{K}_a$  of alternating knots with the connected sum operation, freely generated by the set  $\mathcal{P}_{\mathcal{K},a}$  of alternating prime knots, we similarly construct the enveloping abelian group  $\mathcal{G}_{\mathcal{K},a}$ . It is also non-canonically isomorphic to  $\mathbb{Q}_+^*$ .

**2.5. Statistical physics of the group  $\mathcal{G}_{\mathcal{K},a}$ .** We show that the construction presented above of a quantum statistical mechanical system for  $\mathbb{Q}_+^*$  with partition function  $\zeta^2(\beta)/\zeta(2\beta)$ , with  $\zeta(\beta)$  the Riemann zeta function, can be generalized to the case of the group  $\mathcal{G}_{\mathcal{K},a}$ .

Let  $K \ominus K' = (a_1 K_1 \# \cdots \# a_j K_j) \ominus (b_1 K'_1 \# \cdots \# b_\ell K'_\ell)$  be an element of  $\mathcal{G}_{\mathcal{K},a}$  with primary decompositions  $K = a_1 K_1 \# \cdots \# a_m K_m$  and  $K' = b_1 K'_1 \# \cdots \# b_\ell K'_\ell$ , where the  $K_i$  and  $K'_j$  are all distinct prime knots, with multiplicities  $a_i$  and  $b_j$ . Let  $\epsilon_{K \ominus K'}$  be the corresponding basis element of  $\ell^2(\mathcal{G}_{\mathcal{K},a})$ . For a knot  $K$ , let  $\omega(K)$  denote the number of distinct prime knots in its primary decomposition, namely  $\omega(K) = m$  for  $K = a_1 K_1 \# \cdots \# a_m K_m$  with the  $K_i$  prime.

**Proposition 2.15.** *Consider the operator  $H$  acting on  $\ell^2(\mathcal{G}_{\mathcal{K},a})$ , which acts on basis elements as*

$$(2.18) \quad H \epsilon_{K \ominus K'} = \left( \sum_{i=1}^m (a_i (Cr(K_i) + g(K_i))) + \sum_{j=1}^{\ell} b_j (Cr(K'_j) + g(K'_j)) \right) \log(q) \epsilon_{K \ominus K'}.$$

*This is an unbounded densely defined operator, such that  $e^{-\beta H}$  is trace class for all  $\beta \geq \beta_+$ , satisfying*

$$(2.19) \quad Z_{\mathcal{G}_{\mathcal{K},a}}(\beta) := \text{Tr}(e^{-\beta H}) = \frac{Z_a^2(\beta)}{Z_a(2\beta)},$$

where  $Z_a(\beta)$  is the partition function of Theorem 2.3.

*Proof.* The argument is very similar to the case of  $\mathbb{Q}_+^*$  discussed above. By Lemma 2.5, we can write

$$\frac{Z_a^2(\beta)}{Z_a(2\beta)} = \prod_{K \in \mathcal{P}_{\mathcal{K},a}} \frac{1 - q^{-2\beta(Cr(K)+g(K))}}{(1 - q^{-\beta(Cr(K)+g(K))})^2}.$$

We then write this as

$$\prod_{K \in \mathcal{P}_{\mathcal{K},a}} \frac{1 + q^{-\beta(Cr(K)+g(K))}}{1 - q^{-\beta(Cr(K)+g(K))}} = \prod_{K \in \mathcal{P}_{\mathcal{K},a}} \left( \frac{1}{1 - q^{-\beta(Cr(K)+g(K))}} + \frac{q^{-\beta(Cr(K)+g(K))}}{1 - q^{-\beta(Cr(K)+g(K))}} \right)$$

$$= \prod_{K \in \mathcal{P}_{\mathcal{K},a}} \left( 1 + 2 \sum_{n \geq 1} q^{-\beta n (Cr(K) + g(K))} \right).$$

On the other hand, we have

$$\mathrm{Tr}(e^{-\beta H}) = \sum_{K \ominus K' \in \mathcal{G}_{\mathcal{K},a}} \langle \epsilon_{K \ominus K'}, e^{-\beta H} \epsilon_{K \ominus K'} \rangle = \sum_{\lambda \in \mathrm{Spec}(H)} m_\lambda e^{-\beta \lambda},$$

where  $m_\lambda$  are the multiplicities. By (2.18) the operator is diagonal on the basis  $\epsilon_{K \ominus K'}$  with eigenvalues  $q^{-\beta((Cr(K)+g(K))+(Cr(K')+g(K'))}$ . The multiplicities are  $2^{m+\ell}$  for  $K = a_1 K_1 \# \cdots \# a_m K_m$  and  $K' = b_1 K'_1 \# \cdots \# b_\ell K'_\ell$ , since all the other basis vectors in the same eigenspace are obtained by moving some of the  $K_i$  and  $K'_j$  factors to the other side of  $\ominus$ , hence for each primary term in the decomposition there are two choices. Thus, we obtain

$$\mathrm{Tr}(e^{-\beta H}) = \sum_{K \in \mathcal{K}_a} 2^{\omega(K)} q^{-\beta(Cr(K)+g(K))}.$$

We then see by rewriting this in the Euler product form that the identity (2.19) holds.  $\square$

We have the analog of Proposition 2.14, which is proved by the same argument.

**Proposition 2.16.** *The smallest  $C^*$ -subalgebra  $\mathcal{A} \subset \mathcal{B}(\ell^2(\mathcal{G}_{\mathcal{K},a}))$  that contains  $C_r^*(\mathcal{G}_{\mathcal{K},a})$  and is invariant under the time evolution*

$$(2.20) \quad \sigma_t(T) = e^{itH} T e^{-itH}, \quad T \in \mathcal{B}(\ell^2(\mathcal{G}_{\mathcal{K},a})),$$

with  $H$  as in (2.18), is generated by  $C_r^*(\mathcal{G}_{\mathcal{K},a})$  and by projections

$$\Pi_{(K_1, K_2)} \epsilon_{K \ominus K'} = \begin{cases} \epsilon_{K \ominus K'} & K_1 | K \text{ and } K_2 | K' \\ 0 & \text{otherwise} \end{cases}$$

The time evolution (2.20) acts on the algebra  $\mathcal{A}$  by inner automorphisms.

The fact that this time evolution is inner is undesirable from the operator-algebraic point of view. We will return to discuss this problem in the following sections. A first step towards improving the system described in this section is to introduce interaction terms in the quantum statistical mechanical system, as one does in the case of the Bost–Connes system by passing from the algebra  $C_r^*(\mathbb{N})$  to the crossed product algebra  $C_r^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}$ . In our setting the analogous step will consist in passing from a quantum statistical mechanical system associated to knots to one associated to 3-manifolds.

### 3. KNOT GROUPS, 3-MANIFOLDS, AND CYCLIC BRANCHED COVERS

**3.1. Cyclic branched coverings of the 3-sphere.** Let  $\mathcal{K}$  denote the set of ambient isotopy classes of (oriented) knots in  $S^3$ . For simplicity of notation, in the following we will write  $K$  for a knot and also for its equivalence class up to ambient isotopy.

It is well known, [1], [26], [44], that every smooth oriented closed 3-manifold can be realized (non-uniquely) as a branched cover of the 3-sphere, branched along a knot. Moreover, it is also well known that an  $n$ -fold branched covering of the 3-sphere, branched along a knot  $K$ , is entirely determined by the datum of a representation

$$(3.1) \quad \rho : \pi_1(S^3 \setminus K) \rightarrow S_n,$$

where  $\pi_1(S^3 \setminus K)$  is the fundamental group of the knot complement, and  $S_n$  is the symmetric group of permutations of  $n$  elements, [1].

An  $n$ -fold branched covering of the 3-sphere  $S^3$ , branched along a knot  $K$ , is said to be *cyclic* or *abelian* if the corresponding homomorphism (3.1) factors through the abelianization  $\pi_1(S^3 \setminus K)^{ab} = H_1(S^3 \setminus K, \mathbb{Z}) = \mathbb{Z}$ , as a homomorphism  $\rho : H_1(S^3 \setminus K, \mathbb{Z}) \rightarrow \mathbb{Z}/n\mathbb{Z}$  with values in the subgroup of cyclic permutations  $\mathbb{Z}/n\mathbb{Z} \subset S_n$ . In particular, for a given knot  $K$ , there is a unique connected cyclic branched covering  $Y_n(K)$ . We write  $\pi_{K,n} : Y_n(K) \rightarrow S^3$  for the corresponding projection map. The remaining elements in  $\text{Hom}(H_1(S^3 \setminus K, \mathbb{Z}), \mathbb{Z}/n\mathbb{Z})$  correspond to coverings that have multiple components.

It is known, [4], [65], that, if one knows the cyclic coverings  $Y_p(K)$  for three distinct primes  $p$ , this uniquely identifies the knot  $K$ . In other words, given a knot  $K$ , there are at most two distinct primes  $p \neq p'$  for which there exist some inequivalent knot  $K'$  with homeomorphic branched cyclic coverings,  $Y_p(K) \simeq Y_p(K')$  and  $Y_{p'}(K) \simeq Y_{p'}(K')$ .

To the purpose of building an analog of the Bost–Connes system in the setting of arithmetic topology, we think of cyclic branched coverings of the 3-sphere  $S^3$  as an analog of abelian extensions of  $\mathbb{Q}$ .

**3.2. Knots semigroup.** For the purpose of our construction, we will consider a semigroup

$$(3.2) \quad \mathcal{S} = \mathcal{K} \times \mathbb{N},$$

where  $\mathcal{K} = (\mathcal{K}, \#)$  is the semigroup of oriented knots with the direct sum operation, as in the previous section, and  $\mathbb{N}$  is the multiplicative semigroup of positive integers. The idea behind this choice is that an element  $(K, n) \in \mathcal{S}$  specifies the branch locus and of the order of a cyclic branched covering of  $S^3$ . The semigroup  $\mathcal{S}$  is generated by the pairs  $(K, p)$  where  $K$  is a prime knot and  $p$  is a prime number.

The quantum statistical mechanical model we discussed in the previous section for the semigroup  $\mathcal{K}_a$  and its group completion  $\mathcal{G}_{\mathcal{K},a}$  extend to the product  $\mathcal{K}_a \times \mathbb{N}$  as follows.

**Lemma 3.1.** *Let  $N : \mathcal{K}_a \times \mathbb{N} \rightarrow \mathbb{R}_+^*$  be a semigroup homomorphism. Then setting  $\sigma_t(\mu_{K,n}) = N(n, K)^{it} \mu_{K,n}$  defines a time evolution of  $C_r^*(\mathcal{K}_a \times \mathbb{N})$ . In particular, taking*

$$N(K, n) = n q^{(Cr(K)+g(K))}$$

*determines a time evolution with partition function  $\zeta(\beta)Z_a(\beta)$ , where  $\zeta(\beta)$  is the Riemann zeta function and  $Z_a(\beta)$  is as in Theorem 2.3, for  $\beta > \max\{\beta_+(q), 1\}$ .*

*Proof.* The argument is analogous to Theorem 2.3. The time evolution  $\sigma_t(\mu_{K,n}) = N(n, K)^{it} \mu_{K,n}$  with  $N(K, n) = n q^{(Cr(K)+g(K))}$  is implemented by a Hamiltonian of the form

$$H \epsilon_{K,n} = ((Cr(K) + g(K)) \log(q) + \log(n)) \epsilon_{K,n}$$

on the canonical basis of  $\ell^2(\mathcal{K}_a \times \mathbb{N})$ , with partition function

$$Z_{\mathcal{K}_a \times \mathbb{N}}(\beta) = \text{Tr}(e^{-\beta H}) = \sum_{K,n} \langle \epsilon_{K,n}, e^{-\beta H} \epsilon_{K,n} \rangle = \sum_{K,n} q^{-\beta(Cr(K)+g(K))} n^{-\beta} = Z_a(\beta) \cdot \zeta(\beta),$$

where  $Z_a(\beta)$  is the partition function of Theorem 2.3 and  $\zeta(\beta)$  is the Riemann zeta function. The operator  $e^{-\beta H}$  is trace class in the range  $\beta > \max\{\beta_+(q), 1\}$ .  $\square$

**3.3. Wirtinger presentations and connected sums.** The fundamental group  $\pi_1(S^3 \setminus K)$  of a knot complement has an explicit presentation, associated to the choice of a planar diagram  $D(K)$  representing the knot  $K$ . It is given by the Wirtinger presentation  $\mathcal{W}(D(K))$ . Let  $N_D$  be the number of crossings in the planar diagram  $D = D(K)$ . Then in the presentation  $\mathcal{W}(D(K))$  there are  $N$  generators  $a_i$ , identified with loops circling around the oriented arcs given by the two parts of the lower branch at each crossing (drawn as two arcs in the planar diagram). At each crossing one imposes a relation, which is either of the form  $a_i a_j^{-1} a_{i+1}^{-1} a_j = 1$  or  $a_i a_j a_{i+1}^{-1} a_j^{-1} = 1$ , depending on the orientations at the crossing, see §4.2.3 of [59] for more details. The following fact is well known. We reproduce it here for the reader's convenience.

**Lemma 3.2.** *Let  $K = K_1 \# K_2$  be a connected sum. Then the fundamental groups satisfy*

$$(3.3) \quad \pi_1(S^3 \setminus (K_1 \# K_2)) = \pi_1(S^3 \setminus K_1) *_{\mathbb{Z}} \pi_1(S^3 \setminus K_2).$$

*Proof.* Choose planar diagrams  $D_1 = D(K_1)$  and  $D_2 = D(K_2)$ . Let  $N_i$  be the number of crossings in  $D_i$ . In these diagrams, let us number the arcs so that  $a_1$  and  $b_1$  are, respectively, the arcs where the connected sum operation is performed. Let  $D = D(K_1 \# K_2)$  be the resulting planar diagram for the connected sum knot. Let  $\mathcal{W}(D_1) = \langle a_1, \dots, a_{N_1} \mid r_1, \dots, r_{N_1} \rangle$  and  $\mathcal{W}(D_2) = \langle b_1, \dots, b_{N_2} \mid s_1, \dots, s_{N_2} \rangle$  be the Wirtinger presentations of  $\pi_1(S^3 \setminus K_i)$  associated to these planar diagrams. Let  $f_i : \mathbb{Z} \rightarrow \pi_1(S^3 \setminus K_i)$  denote the homomorphisms that map the generators of  $\mathbb{Z}$  to the generators, in the respective Wirtinger presentations as above, given by the arcs chosen for the connected sum:  $f_1(1) = a_1$  and  $f_2(1) = b_1$ . The amalgamated product in (3.3) is the resulting pushout diagram of groups

$$(3.4) \quad \begin{array}{ccc} \mathbb{Z} & \xrightarrow{f_1} & \pi_1(S^3 \setminus K_1) \\ \downarrow f_2 & & \downarrow \\ \pi_1(S^3 \setminus K_2) & \longrightarrow & \pi_1(S^3 \setminus K_1) *_{\mathbb{Z}} \pi_1(S^3 \setminus K_2) \end{array}$$

where  $\pi_1(S^3 \setminus K_1) *_{\mathbb{Z}} \pi_1(S^3 \setminus K_2)$  has a presentation of the form

$$\langle a_1, \dots, a_{N_1}, b_1, \dots, b_{N_2} \mid r_1, \dots, r_{N_1}, s_1, \dots, s_{N_2}, a_1 b_1^{-1} \rangle,$$

which agrees with the Wirtinger presentation  $\mathcal{W}(D)$  of  $D = D(K_1 \# K_2)$ , hence the pushout group is isomorphic to  $\pi_1(S^3 \setminus (K_1 \# K_2))$ .  $\square$

**Corollary 3.3.** *The groups  $\pi_1(S^3 \setminus K)$  form a direct system, with respect to the directed set  $\mathcal{K}$ , partially ordered by divisibility with respect to the direct sum operation, with maps*

$$(3.5) \quad \varphi_{K',K} : \pi_1(S^3 \setminus K') \rightarrow \pi_1(S^3 \setminus K), \quad \text{for } K' | K,$$

*Proof.* A knot  $K'$  divides a knot  $K$  in the semigroup  $(\mathcal{K}, \#)$  if there is some other knot  $K''$  such that  $K = K' \# K''$ . Defining a partial order by setting  $K' \leq K$  if  $K'$  divides  $K$  makes  $\mathcal{K}$  into a directed set. As in the previous lemma, we then have a group homomorphism  $\varphi_{K',K} : \pi_1(S^3 \setminus K') \rightarrow \pi_1(S^3 \setminus K)$  given by the corresponding map in the pushout diagram (3.4). These morphisms satisfy  $\varphi_{K,K} = 1$  and  $\varphi_{K_2,K_3} \circ \varphi_{K_1,K_2} = \varphi_{K_1,K_3}$  when  $K_1 | K_2$  and  $K_2 | K_3$ , hence the groups  $\pi_1(S^3 \setminus K)$  form a direct system.  $\square$

We can then consider the direct limit of this direct system,

$$(3.6) \quad \pi := \varinjlim_{K \in \mathcal{K}} \pi_1(S^3 \setminus K) = \varinjlim_{K \in \mathcal{K}} \pi_K,$$

where we use the shorthand notation  $\pi_K := \pi_1(S^3 \setminus K)$ . The direct limit  $\pi$  is given by equivalence classes of elements  $\gamma_K \in \pi_K$ , under the relation  $\gamma_K \sim \gamma_{K'}$  if there is some  $K''$  in  $\mathcal{K}$  such that  $K|K''$  and  $K'|K''$  with  $\varphi_{K,K''}(\gamma_K) = \varphi_{K',K''}(\gamma_{K'})$ . Setting  $[\gamma_K] \cdot [\gamma_{K'}] := \varphi_{K,K\#K'}(\gamma_K) \cdot \varphi_{K',K\#K'}(\gamma_{K'})$ , with the product in  $\pi_{K\#K'}$  determines a product on  $\pi$  that is independent of representatives.

In addition to considering the direct limit  $\pi$  of the directed system of the groups  $\pi_K$  with the homomorphisms  $\varphi_{K,K\#K'}$ , it will be convenient for our purposes to also consider the direct product

$$(3.7) \quad \tilde{\pi} := \prod_{K \in \mathcal{K}} \pi_1(S^3 \setminus K) = \prod_{K \in \mathcal{K}} \pi_K,$$

without imposing the equivalence relations of the direct limit. There are then induced morphisms

$$(3.8) \quad \varphi_{K'} : \tilde{\pi} \rightarrow \tilde{\pi}, \quad \varphi_{K'} = (\varphi_{K,K\#K'})_{K \in \mathcal{K}}$$

given coordinatewise by the morphisms  $\gamma_K \mapsto \varphi_{K,K\#K'}(\gamma_K)$  of the direct system.

**3.4. Wild knots and fundamental groups.** Wilder knots are a class of wild knots with a single wild point, obtained as infinite connected sums of a sequence  $K_n$  of tame knots. It is well known (see [16], [37]) that such Wilder knots have knot group isomorphic to the infinite amalgamated product  $\pi_\infty := \pi_1(S^3 \setminus K_\infty) = \pi_{K_1} *_{\mathbb{Z}} \pi_{K_2} *_{\mathbb{Z}} \pi_{K_3} \cdots *_{\mathbb{Z}} \pi_{K_N} *_{\mathbb{Z}} \cdots$ . More generally wild knots have knot groups that are obtained as direct limits of knot groups of tame knots, [16]. The direct limit  $\pi = \varinjlim_K \pi_K$  described above has a similar interpretation.

**Lemma 3.4.** *The direct limit  $\pi = \varinjlim_K \pi_K$  is the knot group  $\pi = \pi_1(S^3 \setminus K_\infty)$  of a wild knot  $K_\infty$  with a Cantor set of wild points.*

*Proof.* The construction of the wild knot  $K_\infty$  is modeled on the direct system of groups  $\pi_K$  under the order relation in the semigroup  $\mathcal{K}$  given by divisibility. Choose an enumeration of the prime knots. Any such choice determines a bijection between the set of prime knots and the set of prime numbers, and a corresponding isomorphisms of semigroups  $(\mathcal{K}, \#) \simeq (\mathbb{N}, \cdot)$ . We write the chosen enumeration of the prime knots as  $\{K_p\}$  where  $p$  ranges over prime numbers. Starting with the unknot in  $S^3$ , construct a Wilder knot given by the infinite connected sum of all the prime knots  $K_p$ . This has a single wild point lying on the initial unknot. At the successive step repeat the procedure in each of the prime knots of the previous level, namely insert in each the full sequence of prime knots  $K_p$ , with a single tame point for each knot of the previous level, which we locate at the intersection of those knots with the original unknot. In the limit the resulting wild knot  $K_\infty$  has set of wild points that is compact, totally disconnected, with each wild point an accumulation point of other wild points. The fundamental group of the knot complement of  $K_\infty$  is then obtained as in [16] as the direct limit  $\pi$ .  $\square$

**Remark 3.5.** In a rooted tree, we say that a vertex has level  $N$  if it is connected to the root by a path of  $N$  edges. Let  $\mathcal{T}$  be the non-locally-finite labelled rooted tree with root vertex labelled by the unknot. The root vertex (level zero) is connected to a countable infinity of vertices labelled by the prime knots (level one). In turn each of these vertices is connected to another countable set of vertices labelled by the prime knots (level two), and so on, with each vertex at level  $N$  connected to a countable set of vertices at level  $N + 1$ ,

labelled by the prime knots. The number of vertices at a given level  $N$  is a countable union of countable sets, hence countable. Let  $E\mathcal{T}$  be the resulting infinite set of edges. The wild knot  $K_\infty$  of Lemma 3.4 can be described as obtained by performing a connected sum along each of the edges of the tree  $\mathcal{T}$ .

**3.5. Projective limits and cyclic coverings.** For an arbitrary knot  $K$ , the abelianization of the fundamental group  $\pi_K = \pi_1(S^3 \setminus K)$  is always just the infinite cyclic group generated by the meridian

$$(3.9) \quad \pi_1(S^3 \setminus K)^{ab} = H_1(S^3 \setminus K) = \mathbb{Z}.$$

In particular, by the form of the relations in the Wirtinger presentation, one sees that any representation of the group  $\pi_K$  into an abelian group  $H$  will necessarily map all the generators of  $\pi_K$  to a same element of  $H$ .

Let us consider again the representation  $\rho = \rho_{K,n} : \pi_1(S^3 \setminus K) \rightarrow \mathbb{Z}/n\mathbb{Z}$  that corresponds to the unique connected cyclic branched cover  $Y_n(K)$  of  $S^3$ , branched along  $K$ . This representation sends all the generators of  $\pi_K$  to a primitive  $n$ -th root of unity, and it corresponds to the quotient homomorphism  $\rho_{K,n} : \mathbb{Z} = \pi_1(S^3 \setminus K)^{ab} \rightarrow \mathbb{Z}/n\mathbb{Z}$ .

Consider again the maps  $\sigma_m : \mathbb{Z}/nm\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$  that raise to the  $m$ -th power and determine the projective system of the  $\mathbb{Z}/n\mathbb{Z}$ , with the indices  $n \in \mathbb{N}$  ordered by divisibility, with limit the profinite completion of  $\mathbb{Z}$ ,

$$\varprojlim_{n \in \mathbb{N}} \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}.$$

We have the simple compatibility condition of the maps  $\rho_{K,n}$ ,

$$(3.10) \quad \begin{array}{ccc} \mathbb{Z} = H_1(S^3 \setminus K, \mathbb{Z}) & \xrightarrow{\rho_{nm}} & \mathbb{Z}/nm\mathbb{Z} \\ & \searrow \rho_n & \downarrow \sigma_m \\ & & \mathbb{Z}/n\mathbb{Z}. \end{array}$$

The induced map to the projective limit is just the canonical map  $\rho : \mathbb{Z} \rightarrow \hat{\mathbb{Z}}$  of the integers to their pro-finite completion.

Using the identification  $\hat{\mathbb{Z}} = \text{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ , we can think of the resulting map  $\rho : \mathbb{Z} = H_1(S^3 \setminus K, \mathbb{Z}) \rightarrow \hat{\mathbb{Z}}$  as describing a locally trivial fibration over  $S^3 \setminus K$  with fiber the set of all roots of unities, identified with  $\mathbb{Q}/\mathbb{Z}$ , extended to  $S^3$  with branch locus  $K$ . This space can be regarded as a limit, in the category of topological spaces, of the cyclic branched coverings  $Y_n(K)$ . We denote it by  $Y_{\hat{\mathbb{Z}}}(K)$ .

When we consider simultaneously the inverse limit of the fibers, and direct limit of the branch loci knots, we obtain a space  $Y_{\hat{\mathbb{Z}}}(K_\infty)$ , which is a branched cover of  $S^3$  branched along the wild knot  $K_\infty$  with fiber the set of roots of unity. The covering is specified by a representation of the direct limit group  $\pi$  to  $\hat{\mathbb{Z}}$ . The space  $Y_{\hat{\mathbb{Z}}}(K_\infty)$  is the geometric object underlying the construction of the quantum statistical mechanical system that we describe in the coming section.

## 4. QUANTUM STATISTICAL MECHANICS OF 3-MANIFOLDS

In this section we combine the constructions of the previous section with the quantum statistical mechanics of knots, to construct an analog of the Bost–Connes system associated to cyclic branched coverings of the 3-sphere. We show that the properties of the resulting quantum statistical mechanical system are significantly different from the Bost–Connes case and are related to noncommutative Bernoulli crossed products.

**4.1. Group rings.** Thus, in order to construct a replacement for the algebra  $C^*(\mathbb{Q}/\mathbb{Z}) = C(\hat{\mathbb{Z}})$  of the Bost–Connes system, which will account for all the possible choices of a knot  $K$  and a cyclic branched cover of some order  $n$ , we need to introduce appropriate group rings. We first deal with the part of the information that concerns the knot complements and the knot groups  $\pi_K = \pi_1(S^3 \setminus K)$  and then, in §4.6 below, we combine this part of the construction with the information on the choice of the cyclic branched coverings coming from the  $\hat{\mathbb{Z}}$  datum.

We consider group rings  $\mathbb{Q}[\pi_K]$ , for each knot group  $\pi_K = \pi_1(S^3 \setminus K)$ , and also the group ring  $\mathbb{Q}[\tilde{\pi}]$ , with  $\tilde{\pi}$  the direct product of the  $\pi_K$  as in (3.7), and the group ring  $\mathbb{Q}[\pi]$ , with  $\pi$  the direct limit of the  $\pi_K$ .

Note that, unlike the group ring  $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$  of the Bost–Connes system, the group rings  $\mathbb{Q}[\pi_K]$ ,  $\mathbb{Q}[\tilde{\pi}]$  and  $\mathbb{Q}[\pi]$  are noncommutative, hence the corresponding  $C^*$ -algebra completions, which we will discuss later, can no longer be written as algebras of continuous function on a dual group. If one considers the abelianization  $\pi^{ab}$ , all the maps of the direct system of the groups  $\pi_K$  induce the identity on the homology groups, hence  $\pi^{ab} = \mathbb{Z}$ , and one would simply obtain the commutative group ring  $\mathbb{Q}[\pi^{ab}] = \mathbb{Q}[\mathbb{Z}] = \mathbb{Q}[t, t^{-1}]$ .

**4.2. Semigroup action and crossed product.** In the following, we consider the semigroup  $\mathcal{K}$  acting as endomorphisms of  $\mathbb{Q}[\tilde{\pi}]$  via the morphisms

$$(4.1) \quad \sigma_K : \gamma_{K'} \mapsto \varphi_{K', K \# K'}(\gamma_{K'}),$$

for  $\gamma_{K'}$  in  $\pi_{K'} \subset \tilde{\pi}$ .

As we have seen in Corollary 3.3, the maps  $\varphi_{K, K \# K'}$  satisfy  $\varphi_{K \# K', K \# K' \# K''} \circ \varphi_{K, K \# K'} = \varphi_{K, K \# K' \# K''}$ . Thus, the homomorphism  $\sigma_K : \tilde{\pi} \rightarrow \tilde{\pi}$  defined by (4.1) is indeed a semigroup action, since we have

$$\begin{aligned} \sigma_{K_1 \# K_2}(\gamma_{K'}) &= \varphi_{K', K_1 \# K_2 \# K'}(\gamma_{K'}) \\ &= \varphi_{K_1 \# K', K_1 \# K_2 \# K'}(\varphi_{K', K_1 \# K'}(\gamma_{K'})) = \sigma_{K_2}(\sigma_{K_1}(\gamma_{K'})). \end{aligned}$$

We use the same notation for the induced morphism of the group ring  $\sigma_K : \mathbb{Q}[\tilde{\pi}] \rightarrow \mathbb{Q}[\tilde{\pi}]$ .

By direct inspection of the respective Wirtinger presentations, as in Lemma 3.2, we see that the generators  $\{a, a_2, \dots, a_{N_1}, b_2, \dots, b_{N_2}\}$  of  $\pi_{K_1 \# K_2}$  satisfy  $a = \varphi_{K_1, K_1 \# K_2}(a_1) = \varphi_{K_2, K_1 \# K_2}(b_1)$ . The remaining generators  $a_i = \varphi_{K_1, K_1 \# K_2}(a_i)$  have a preimage in  $\pi_{K_1}$  but no preimage in  $\pi_{K_2}$ , and vice versa for the  $b_i$ . Thus, an element  $\gamma_{K_1 \# K_2} \in \pi_{K_1 \# K_2}$  has either one preimage or none in  $\pi_{K_1}$  and  $\pi_{K_2}$ . Let  $\mathcal{R}_K$  denote the range of the homomorphism  $\sigma_K$  acting on  $\mathbb{Q}[\tilde{\pi}]$ . As a subring of  $\mathbb{Q}[\tilde{\pi}]$ ,  $\mathcal{R}_K$  is generated by all the elements  $\gamma_{K \# K'}$  in some  $\pi_{K \# K'} \subset \tilde{\pi}$  that are in the range of  $\varphi_{K', K \# K'}$ . Then, by the observation above, there is a ring homomorphism  $\eta_K : \mathcal{R}_K \rightarrow \mathbb{Q}[\tilde{\pi}]$  given by  $\eta_K(\gamma_{K \# K'}) = \gamma_{K'}$  for  $\gamma_{K \# K'} = \varphi_{K', K \# K'}(\gamma_{K'})$ , satisfying  $\sigma_K \circ \eta_K = id|_{\mathcal{R}_K}$  and  $\eta_K \circ \sigma_K = id|_{\mathbb{Q}[\tilde{\pi}]}$ .

**Remark 4.1.** The behavior of the endomorphisms  $\sigma_K$  here is significantly different from the case of the endomorphisms  $\sigma_n$  of the Bost–Connes system. Indeed, the  $\sigma_K$  are injective, while the  $\sigma_n$  are surjective. The case we are looking at here resembles closely the adaptation of the Bost–Connes system to the Habiro ring considered in [38], where a similar injectivity condition is satisfied by the  $\sigma_n$ . Our construction here follows closely the setting of [38] and of §4.7 of [39].

Let  $\mathcal{G}_{\mathcal{K}}$  denote the universal enveloping abelian group (Grothendieck group) of the semi-group  $(\mathcal{K}, \#)$ , as in §2.4.

**Lemma 4.2.** *The direct limit of the ring homomorphisms  $\sigma_K : \mathbb{Q}[\tilde{\pi}] \rightarrow \mathbb{Q}[\tilde{\pi}]$  satisfies*

$$(4.2) \quad \varinjlim_{K \in \mathcal{K}} (\sigma_K : \mathbb{Q}[\tilde{\pi}] \rightarrow \mathbb{Q}[\tilde{\pi}]) \cong \bigotimes_{h \in \mathcal{G}_{\mathcal{K}}} \mathbb{Q}[\pi].$$

*Proof.* First note that, since  $\pi$  is the direct limit of the groups  $\pi_K$ , the group ring  $\mathbb{Q}[\pi]$  is the direct limit of the group rings  $\mathbb{Q}[\pi_K]$ . Moreover, because  $\tilde{\pi}$  is the direct product of the groups  $\pi_K$ , the group ring is a tensor product  $\mathbb{Q}[\tilde{\pi}] = \bigotimes_{K \in \mathcal{K}} \mathbb{Q}[\pi_K]$ . Let  $\psi_K : \pi_K \rightarrow \pi$  be the maps to the direct limit determined by the direct system. They satisfy  $\psi_{K\#K'} \circ \varphi_{K,K\#K'} = \psi_K$ , for all  $K, K' \in \mathcal{K}$ . We denote by the same symbol the resulting morphisms on the group rings. We have commutative diagrams

$$(4.3) \quad \begin{array}{ccc} \bigotimes_K \mathbb{Q}[\pi_K] & \xrightarrow{\psi=(\psi_K)} & \bigotimes_K \mathbb{Q}[\pi] \\ \downarrow \sigma_{K'} & & \downarrow \hat{\sigma}_{K'} \\ \bigotimes_K \mathbb{Q}[\pi_K] & \xrightarrow{\psi=(\psi_K)} & \bigotimes_K \mathbb{Q}[\pi], \end{array}$$

where on the right hand side the morphism  $\hat{\sigma}_{K'} : \bigotimes_K \mathbb{Q}[\pi] \rightarrow \bigotimes_K \mathbb{Q}[\pi]$  shifts the indices, mapping the copy of  $\mathbb{Q}[\pi]$  in the  $K$ -th position to the copy in the  $K\#K'$ -th position. Since the maps  $\psi_K$  are the maps to the direct limit of the system of the  $\pi_K$ , the direct limit of the system on the left column reduces to that of the right column, or equivalently, the induced morphism between the direct limits is an isomorphism

$$\psi : \varinjlim_{K' \in \mathcal{K}} (\sigma_{K'} : \mathbb{Q}[\tilde{\pi}] \rightarrow \mathbb{Q}[\tilde{\pi}]) \xrightarrow{\cong} \varinjlim_{K' \in \mathcal{K}} (\hat{\sigma}_{K'} : \bigotimes_K \mathbb{Q}[\pi] \rightarrow \bigotimes_K \mathbb{Q}[\pi]).$$

Elements in the limit on the right hand side are rational combinations of equivalence classes of elements  $g_{K,K'} \in \pi$  with  $K, K' \in \mathcal{K}$  under the equivalence relation induced by the maps  $\hat{\sigma}_{K''}$ , given by the shifting of indices  $g_{K,K'} \sim g_{K\#K'', K'\#K''}$ , where  $(K, K') \sim (K\#K'', K'\#K'')$  is the relation that defines the elements  $h = K \ominus K'$  in the Grothendieck group  $\mathcal{G}_{\mathcal{K}}$  of the abelian semigroup  $\mathcal{K}$ . Thus, we can identify

$$\varinjlim_{K' \in \mathcal{K}} (\hat{\sigma}_{K'} : \bigotimes_K \mathbb{Q}[\pi] \rightarrow \bigotimes_K \mathbb{Q}[\pi]) = \bigotimes_{h \in \mathcal{G}_{\mathcal{K}}} \mathbb{Q}[\pi].$$

□

We now consider a crossed product construction analogous to the version of the Bost–Connes construction given in [38].

**Definition 4.3.** *Let  $\mathcal{A}_{\tilde{\pi}, \mathcal{K}}$  be the  $\mathbb{Q}$ -algebra generated by  $\mathbb{Q}[\tilde{\pi}]$  and generators  $\mu_K, \mu_K^*$  for  $K \in \mathcal{K}$  with the relations  $\mu_K^* \mu_K = 1$  and*

$$(4.4) \quad \mu_K \sigma_K(\gamma_{K'}) = \gamma_{K'} \mu_K, \quad \mu_K^* \gamma_{K'} = \sigma_K(\gamma_{K'}) \mu_K^*.$$

**Remark 4.4.** Unlike what happens with the Bost–Connes algebra, the elements  $e_K = \mu_K \mu_K^*$  does not belong to the algebra  $\mathbb{Q}[\tilde{\pi}]$ . However, as we see below, these elements belong to the direct limit  $\varinjlim_{K \in \mathcal{K}} (\sigma_K : \mathbb{Q}[\tilde{\pi}] \rightarrow \mathbb{Q}[\tilde{\pi}])$  described above.

Let  $\mathcal{A}_K$  be the rings generated by all the elements of the form  $\mu_K \gamma_{K'} \mu_K^*$  with  $\gamma_{K'} \in \tilde{\pi}$ . When  $K$  is the unknot we just have  $\mathbb{Q}[\tilde{\pi}]$ . A direct analog of Lemma 2.2 of [38] shows that the endomorphisms  $\sigma_K : \mathbb{Q}[\tilde{\pi}] \rightarrow \mathbb{Q}[\tilde{\pi}]$  extend to morphisms  $\sigma_K : \mathcal{A}_{K'} \rightarrow \mathcal{A}_{K'}$  when  $K \nmid K'$  and to morphism  $\sigma_K : \mathcal{A}_{K'} \rightarrow \mathcal{A}_{K''}$  when  $K|K'$  with  $K' = K \# K''$ . Setting  $\alpha_K(a) = \mu_K a \mu_K^*$  gives homomorphisms (since  $\mu_K^* \mu_K = 1$ ) mapping  $\alpha_K : \mathcal{A}_{K'} \rightarrow \mathcal{A}_{K \# K'}$  satisfying

$$\alpha_K(\sigma_K(a)) = e_K a e_K, \quad \sigma_K(\alpha_K(a)) = a,$$

where the idempotents  $e_K = \mu_K \mu_K^*$  map  $\mathbb{Q}[\tilde{\pi}]$  by  $\gamma_{K'} \mapsto e_K \gamma_{K'} e_K$  to the subring  $e_K \mathcal{R}_K e_K \subset \mathcal{A}_K$ , with  $\mathcal{R}_K$  the range of  $\sigma_K$  as above. All this can be seen easily by essentially the same argument as in Lemma 2.2 of [38]. Moreover, as in Lemma 2.3 of [38] we then have the following identification.

**Lemma 4.5.** *The algebra  $\mathcal{A}_{\tilde{\pi}, \mathcal{K}}$  described above is the direct limit  $\varinjlim_{K \in \mathcal{K}} (\sigma_K : \mathbb{Q}[\tilde{\pi}] \rightarrow \mathbb{Q}[\tilde{\pi}])$ .*

*Proof.* There are homomorphisms  $\mathcal{A}_K \hookrightarrow \mathcal{A}_{K'}$  whenever  $K' = K \# K''$  in  $\mathcal{K}$ , determined by identifying  $\mu_K a \mu_K^* = \mu_{K'} \sigma_{K''}(a) \mu_{K'}^*$ . Thus, we can identify the algebra  $\mathcal{A}_{\tilde{\pi}, \mathcal{K}}$  with the direct limit

$$\mathcal{A}_{\tilde{\pi}, \mathcal{K}} = \varinjlim_{K \in \mathcal{K}} \mathcal{A}_K = \cup_{K \in \mathcal{K}} \mathcal{A}_K.$$

The morphisms  $\sigma_K$  and  $\alpha_K$  described above are compatible with the direct system of the  $\mathcal{A}_K$  and determine an invertible morphism between the direct limits, hence giving the identification

$$\varinjlim_{K \in \mathcal{K}} \mathcal{A}_K \cong \varinjlim_{K \in \mathcal{K}} (\sigma_K : \mathbb{Q}[\tilde{\pi}] \rightarrow \mathbb{Q}[\tilde{\pi}]).$$

□

In particular, as in Lemma 2.3 of [38] we see that the morphisms induced by the  $\sigma_K$  on the direct limit become invertible.

**Lemma 4.6.** *The maps induced on the direct limit  $\otimes_{\mathcal{G}_{\mathcal{K}}} \mathbb{Q}[\pi]$  by the  $\sigma_K : \mathbb{Q}[\tilde{\pi}] \rightarrow \mathbb{Q}[\tilde{\pi}]$  are isomorphisms.*

*Proof.* In terms of the algebra  $\mathcal{A}_{\tilde{\pi}, \mathcal{K}}$ , the elements  $e_K = \mu_K \mu_K^*$  are idempotents, hence we can write them as  $e_K = 1 - p_K$  for some projection  $p_K$ . Using the relations (4.4) we see that these satisfy  $\sigma_K(p_K) = \mu_K^* (1 - \mu_K \mu_K^*) \mu_K = 0$ . By the injectivity of the  $\sigma_K$  this gives  $e_K = 1$ . Thus, the  $\mu_K$  satisfy both  $\mu_K^* \mu_K = 1$  and  $\mu_K \mu_K^* = 1$  are therefore unitaries, not just isometries. The  $\sigma_K$  are then automorphisms with inverses  $\alpha_K$ . Equivalently, in terms of the direct system  $\sigma_K : \mathbb{Q}[\tilde{\pi}] \rightarrow \mathbb{Q}[\tilde{\pi}]$ , elements in the direct limit are sequences  $g_{K \ominus K'}$ , with formal differences  $K \ominus K' \in \mathcal{G}_{\mathcal{K}}$ , where  $g_{K \ominus K' \# K''} = \hat{\sigma}_{K''}(g_{K \ominus K'})$ , hence in the direct limit the maps induced by the  $\sigma_K$  are surjective as well as injective. □

Thus, the resulting crossed product algebra is a group crossed product, which is just given by the Bernoulli action that shifts the tensor factors indices,

$$(4.5) \quad \bigotimes_{h \in \mathcal{G}_{\mathcal{K}}} \mathbb{Q}[\pi] \rtimes \mathcal{G}_{\mathcal{K}}.$$

**4.3. Operator algebras: von Neumann algebra.** Given a discrete group  $\Gamma$ , one can consider the action by bounded operators on the Hilbert space  $\ell^2(\Gamma)$  given by the left (or right) action of the group on itself. This determines a representation of the algebra  $\mathbb{C}[\Gamma] = \mathbb{Q}[\Gamma] \otimes_{\mathbb{Q}} \mathbb{C}$  on  $\ell^2(\Gamma)$ . We drop the explicit labeling of the left/right regular representation, and simply write  $R : \mathbb{C}[\Gamma] \rightarrow \mathcal{B}(\ell^2(\Gamma))$ . The reduced group  $C^*$ -algebra  $C_r^*(\Gamma)$  is the norm completion of  $R(\mathbb{C}[\Gamma])$  in  $\mathcal{B}(\ell^2(\Gamma))$  and the von Neumann algebra  $\mathcal{N}(\Gamma)$  is the double commutant  $R(\mathbb{C}[\Gamma])''$ . The group von Neumann algebra  $\mathcal{N}(\Gamma)$  has a finite trace given by  $\tau(R(\gamma)) = 1$  if  $\gamma = 1$  and  $\tau(R(\gamma)) = 0$  otherwise. Every von Neumann algebra can be decomposed as a direct integral of factors. A group von Neumann algebra  $\mathcal{N}(\Gamma)$  is a factor if and only if  $\Gamma$  has the infinite conjugacy classes (ICC) property, namely the conjugacy classes of all nontrivial elements  $\gamma \neq 1$  in  $\Gamma$  are infinite.

The question of whether the knot groups  $\pi_K$  (in the non-torus case) satisfy the ICC property was stated as an open problem (Problem 3) in [17]. It was then proved in Corollary 11.1 of [18] that indeed the knot groups  $\pi_K$  are ICC if and only if the knot  $K$  is not a torus knot. A direct product of groups is ICC if and only if each of its factors is, hence the group  $\tilde{\pi}$  is not ICC because the factors  $\pi_K$  corresponding to torus knots are not ICC.

**Lemma 4.7.** *The countably generated group  $\pi = \varinjlim_K \pi_K$  has the ICC property.*

*Proof.* First observe that the groups  $\pi_K$ , for any non-prime knot  $K$ , have the ICC property. This follows immediately from the topological property that all torus knots are prime knots, hence by the characterization of Corollary 11.1 of [18] the knot group of every non-prime knot is ICC. Moreover, by Proposition 5.1 of [18], if  $\Gamma = \Gamma_1 *_{\Gamma_0} \Gamma_2$  is an amalgamated product of discrete groups, with respect to a common subgroup  $\Gamma_0$  that is not of index 2 (non-degenerate case), then  $\Gamma$  has the ICC property if at least one of the two groups  $\Gamma_1, \Gamma_2$  is ICC. (For a more general characterization of the ICC property for amalgamated products see §5.6 of [48].) As in Lemma 3.4, we identify the direct limit  $\pi$  with the knot group  $\pi_1(S^3 \setminus K_\infty)$  of a wild knot  $K_\infty$  obtained from a tree of connected sums. As we have seen in Lemma 3.4, after choosing an enumeration  $K_p$  of the prime knots, we can describe  $K_\infty$  as the result of constructing a necklace given by the infinite connected sum  $K_{p_1} \# K_{p_2} \# \cdots \# K_{p_n} \# \cdots$  of the prime knots in the chosen order, followed iteratively by repeatedly inserting by connected sum similar necklaces into each of the knots at the previous stage, see Remark 3.5. Consider a finite subset  $K_{p_i}, i = 1, \dots, N$  of prime knots and the tame knot obtained as their direct sum  $K = K_{p_1} \# \cdots \# K_{p_N}$ . Let  $\hat{K}_i, i = 1, \dots, N$  be the wild knots consisting of all the successive iterative level to be inserted by connected sum into each of the  $K_{p_i}$ , and let  $\hat{K}$  be the remaining wild knot given by the infinite connected sum of the remaining prime knots  $\hat{K} = K_{p_{N+1}} \# K_{p_{N+2}} \# \cdots$  and all the successive iterative levels inserted into these. Then we can describe the resulting wild knot as  $K_\infty = K \# \hat{K} \# \hat{K}_1 \# \cdots \# \hat{K}_N$ . By the previous observations  $\pi_K$  has the ICC property hence the amalgamated product  $\pi_{K \# \hat{K}} = \pi_K *_{\mathbb{Z}} \pi_{\hat{K}}$  also does, and the same applies to the remaining connected sums with the  $\hat{K}_i$ .  $\square$

**Remark 4.8.** As mentioned above, the ICC property for a group  $\Gamma$  corresponds to  $\mathcal{N}(\Gamma)$  being a  $II_1$  factor. Moreover, it is known [9] that if the group  $\Gamma$  is amenable then  $\mathcal{N}(\Gamma)$  is isomorphic to the hyperfinite type  $II_1$  factor  $R$ . However, knot groups are non-amenable (see [22]), even though they are  $K$ -amenable (Theorem 5.18 of [41]).

After changing to  $\mathbb{C}$ -coefficients, the crossed product (4.5) also has a von Neumann algebra completion

$$(4.6) \quad \bigotimes_{h \in \mathcal{G}_K} \mathcal{N}(\pi) \rtimes \mathcal{G}_K,$$

which is a special case of the class of Bernoulli crossed products first studied in [9], and more recently in [47], [63]. For simplicity of notation, here we just write  $\otimes$  instead of the commonly used  $\bar{\otimes}$ , for tensor products in the von Neumann algebra context. The algebra (4.9) is represented on the Hilbert space  $L^2(\mathcal{N}(\pi), \tau) \otimes \ell^2(\mathcal{G}_K)$  or equivalently  $\ell^2(\pi) \otimes \ell^2(\mathcal{G}_K)$ . Other representations can be constructed using a unitary representation  $\mathcal{V}$  of  $\pi$  and replacing  $\ell^2(\pi)$  with  $\ell^2(\pi) \otimes \mathcal{V}$ . An explicit example of how this twisting by a representation  $\mathcal{V}$  can be obtained is discussed briefly in the following §4.4.

**4.4. Twisting by de Rham representations.** Because the abelianizations of the knot groups are all equal to  $\pi_K^{ab} = \mathbb{Z}$ , one-dimensional representations of  $\pi_K$  by unitary operators correspond to character homomorphisms

$$\text{Hom}(\pi_K, U(1)) = \text{Hom}(\mathbb{Z}, U(1)) = U(1).$$

For each  $K \in \mathcal{K}$ , consider then a choice of a phase  $\theta_K \in \mathbb{R}/\mathbb{Z}$ . With the above identification, this determines a homomorphism, which we still denote by  $\theta_K : \pi_K \rightarrow U(1)$ , which sends all the generators of  $\pi_K$  to the same element  $\lambda_K = \exp(2\pi i \theta_K) \in U(1)$ .

While the 1-dimensional representations of  $\pi_K$  are only of this trivial nature, with all generators acting as the same phase factor  $\lambda_K$ , it is well known that the knot groups  $\pi_K$  have interesting higher dimensional representations. In particular, already in the 2-dimensional case, one has an interesting family of representations, the so called de Rham representations. In general, representations of  $\pi_K$  are related to roots of the Alexander polynomial.

The de Rham representations of knot groups are homomorphisms  $\pi_K \rightarrow \text{GL}_2(\mathbb{C})$ . For each root  $r_K$  of the Alexander polynomial  $\Delta_K(t)$  of the knot  $K$ , there are, up to conjugation,  $2k_r$  de Rham representations of  $\pi_K$ , where  $k_r$  is the largest  $k$  such that the  $k$ -th order Alexander polynomial (that is, the greatest common divisor of the determinants of the  $(n - k + 1) \times (n - k + 1)$  minors of the Alexander matrix) satisfies  $\Delta_k(r) \neq 0$ , see [21]. In a de Rham representation associated to a root  $r$  of the Alexander polynomial  $\Delta_K(t)$  the generators of  $\pi_K$  are represented as  $2 \times 2$ -matrices of the form

$$\begin{pmatrix} \sqrt{r} & x \\ 0 & \frac{1}{\sqrt{r}} \end{pmatrix}.$$

In order to avoid the abelian representations where  $x$  is the same for all generators, one only considers *based* representations, where one of the  $x$ , say for the first generator in a given Wirtinger presentation of  $K$ , is equal to zero, while all the others are nonzero. the list of the elements  $x$  associated to a set of the remaining generators of  $\pi_K$  gives a vector in the kernel of the Alexander matrix  $A_K(t)$  at  $t = r$ , see [7], [21].

For a knot  $K$  let  $\mathcal{V}_K$  be the representation of  $\pi_K$  given by the complex vector space  $\mathcal{V}_K = \bigoplus_r \mathcal{V}_{K,r}$ , where  $r$  ranges over roots of the Alexander polynomial  $\Delta_K(t)$  and  $\mathcal{V}_{K,r}$  is a 2-dimensional de Rham representation of  $\pi_K$ , constructed as above. We denote by  $R_K = \bigoplus_r R_{K,r}$  the resulting representation of  $\pi_K$  on the vector space  $\mathcal{V}_K$ .

**Lemma 4.9.** *For a connected sum  $K = K_1 \# K_2$ , the representation satisfies  $\mathcal{V}_{K_1 \# K_2} = \mathcal{V}_{K_1} \oplus \mathcal{V}_{K_2}$  with  $R_{K_1 \# K_2} = R_{K_1} \oplus R_{K_2}$ . Let  $\Phi_{K_i, K_1 \# K_2}$ ,  $i = 1, 2$ , denote the inclusions of*

the direct factors  $\mathcal{V}_{K_i}$  in  $\mathcal{V}_{K_1\#K_2}$ . Under the direct system of homomorphisms  $\varphi_{K_i, K_1\#K_2} : \pi_{K_i} \rightarrow \pi_{K_1\#K_2}$ , the representations satisfy the compatibility condition

$$(4.7) \quad \Phi_{K_i, K_1\#K_2} \circ R_{K_i}(\gamma_{K_i}) = R_{K_1\#K_2}(\varphi_{K_i, K_1\#K_2}(\gamma_{K_i})) \circ \Phi_{K_i, K_1\#K_2}$$

*Proof.* The Alexander polynomial is given by the determinant  $\Delta_K(t) = \det(V_K - tV_K^T)$ , where  $V_K$  is a Seifert matrix of the knot. For a connected sum  $K = K_1\#K_2$ , given Seifert matrices  $V_{K_1}$  and  $V_{K_2}$ , the direct sum  $V_K = V_{K_1} \oplus V_{K_2}$  is a Seifert matrix for  $K$ . Correspondingly, the Alexander matrix  $A_K(t) = V_K - tV_K^T$  also satisfies  $A_K(t) = A_{K_1}(t) \oplus A_{K_2}(t)$  for  $K = K_1\#K_2$ . Thus, the set of roots of  $\Delta_{K_1\#K_2}(t)$  is the union of the sets of roots of  $\Delta_{K_i}$ , which implies that the vector space is a direct sum  $\mathcal{V}_K = \mathcal{V}_{K_1} \oplus \mathcal{V}_{K_2}$ . The vectors in the kernel of the Alexander matrix at a given root also correspond to those of  $A_{K_1}(t)$  and  $A_{K_2}(t)$ , depending on the root, hence the representations also split as  $R_K = R_{K_1} \oplus R_{K_2}$ . To check the compatibility conditions, notice that we are working with based representations. We can always assume that the generator in the given Wirtinger presentation of  $\pi_{K_1}$  with  $x = 0$  in the de Rham representation corresponds to the arc where the connected sum is performed, which matches then with the action of the corresponding generator of  $\pi_{K_2}$ , so that the resulting representation given by  $R_{K_1} \oplus R_{K_2}$  on  $\mathcal{V}_K$  is indeed a representation of the amalgamated product  $\pi_K = \pi_{K_1} *_Z \pi_{K_2}$ .  $\square$

**Remark 4.10.** If we want the elements of  $\pi_K$  to be represented as unitary, rather than just as invertible operators, then we should consider  $SU(2)$  representations of the knot group  $\pi_K$ , as in [31], rather than  $GL_2(\mathbb{C})$  representations as above. For our purposes two-dimensional representations will suffice, but the construction we obtain can be generalized to the higher dimensional representations obtained as in [3], [28], [55].

**Corollary 4.11.** *The compatibility condition (4.7) satisfied by the de Rham representations  $\mathcal{V}_K$  implies that they induce a representation of the direct limit  $\pi = \varinjlim_K \pi_K$  on the space  $\mathcal{V} = \varinjlim_K \mathcal{V}_K$ , obtained as the direct limit under the direct system of morphisms  $\Phi_{K, K\#K'}$ .*

*Proof.* An element in  $\mathcal{V}$  is an equivalence class of elements  $v_K \in \mathcal{V}_K$  under the relation  $v_K \sim \Phi_{K, K\#K'}(v_K)$ . Defining the action  $[\gamma_K] \in \pi$  on  $[v_K] \in \mathcal{V}$  as  $R([\gamma_K])[v_K] := [R_K(\gamma_K)v_K]$  is well defined, since the compatibility condition (4.7) implies that

$$\Phi_{K, K\#K'}(R_K(\gamma_K)v_K) = R_{K\#K'}(\varphi_{K, K\#K'}(\gamma_K))\Phi_{K, K\#K'}v_K.$$

$\square$

**4.5. Operator algebras:  $C^*$ -algebra.** We now consider the reduced  $C^*$ -algebras of the knot groups  $\pi_K$  and of the direct limit  $\pi$ .

**Lemma 4.12.** *The reduced group  $C^*$  algebra of the direct limit  $\pi = \varinjlim_K \pi_K$  satisfies*

$$(4.8) \quad C_r^*(\pi) = \varinjlim_{K \in \mathcal{K}} C_r^*(\pi_K) = \star_{C_r^*(\mathbb{Z}), \mathcal{T}} C_r^*(\pi_K),$$

where  $\star_{C_r^*(\mathbb{Z}), \mathcal{T}} C_r^*(\pi_K)$  denotes the infinite amalgamated product of the reduced  $C^*$ -algebras  $C_r^*(\pi_K)$  along the common subalgebra  $C_r^*(\mathbb{Z}) = C(S^1)$ , performed as in the amalgamated products of groups, along the tree  $\mathcal{T}$ , in Remark 3.5.

*Proof.* By Proposition 2.5 of [53], the reduced  $C^*$ -algebra of the direct limit  $\pi$  is an amalgamated product of  $C^*$ -algebras. More precisely, by Lemma 3.4, we identify the direct limit  $\pi$  with the knot group  $\pi_1(S^3 \setminus K_\infty)$  of the wild knot  $K_\infty$  obtained from a tree of connected sums obtained by successively inserting with connected sums in each of the knots of a necklace given by the infinite connected sum of the prime knots additional necklaces of the same kind, and so on iteratively, see Remark 3.5. Thus, the direct limit group can be identified as an infinite sequence of amalgamated products  $\pi_K *_{\mathbb{Z}} \pi_{K'}$ , over a common subgroup  $\mathbb{Z}$ , corresponding to each successive connected sum  $K \# K'$ . As shown in Proposition 2.5 of [53], the reduced  $C^*$ -algebra of a countably infinite amalgamated product of discrete countable groups, all performed along a same common subgroup, is an amalgamated product of  $C^*$ -algebras,  $C_r^*(\pi) = \star_{C_r^*(\mathbb{Z}), \mathcal{T}} C_r^*(\pi_K)$ , where the amalgamated products are performed in the same way as for the groups, using the notation  $\pi = *_{\mathbb{Z}, \mathcal{T}} \pi_K$  to indicate the infinite amalgamated product as in Lemma 3.4 and Remark 3.5, with the connected sums performed along the edges of the tree  $\mathcal{T}$  as in Remark 3.5. The reduced amalgamated free product of reduced group  $C^*$ -algebras is taken with respect to the conditional expectations. Namely, by Theorem 2.2 of [53], given a family of unital  $C^*$ -algebras  $A_j$  all containing a sub- $C^*$ -algebra  $B$  with  $1 \in B$ . If there are conditional expectations  $E_j : A_j \rightarrow B$  with faithful GNS representations, then there is a unique  $C^*$ -algebra  $A$ , the amalgamated product of the  $A_j$  along  $B$ , with the properties that  $B \subset A$  with  $1_A \in B$ , with a conditional expectation  $E : A \rightarrow B$  with a faithful GNS representation; with inclusions  $A_j \subset A$  extending the inclusion  $B \subset A$ , so that  $A$  is generated as a  $C^*$ -algebra by the  $A_j$ , which form a free family of subalgebras, with the expectations given by restrictions  $E|_{A_j} = E_j$ . The freeness condition means that  $E(a_1 \cdots a_n) = 0$  whenever  $a_i \in A_{j_i}$  with  $j_i \neq j_{i+1}$  and all  $a_i \in \text{Ker}(E)$ . It is shown in Theorem 2.3 and Proposition 2.5 of [53] that these conditions hold in the case of amalgamated products of reduced group  $C^*$ -algebras as above. Lemma 2.6 of [53], together with Lemma 3.4 above, also shows that  $C_r^*(\pi) = \varinjlim_{K \in \mathcal{K}} C_r^*(\pi_K)$ .  $\square$

At the level of  $C^*$ -algebras, one can similarly consider the crossed product

$$(4.9) \quad \bigotimes_{h \in \mathcal{G}_{\mathcal{K}}} C_r^*(\pi) \rtimes \mathcal{G}_{\mathcal{K}},$$

acting on the same Hilbert space  $\ell^2(\pi) \otimes \ell^2(\mathcal{G}_{\mathcal{K}})$ . As in the case of von Neumann algebras above, we simply write  $\otimes$  for the completed tensor products in the operator algebra context.

**4.6. The combined system.** We now combine the previous construction, based on the direct system of the knot groups  $\pi_K$  and the action of the semigroup  $\mathcal{K}$ , with the information on the choice of the cyclic branched cover, by combining the algebra constructed above with the Bost–Connes algebra, via the representations  $\rho_{K,n} : \pi_K \rightarrow \mathbb{Z}/n\mathbb{Z}$  that specify the unique connected cyclic branched cover  $Y_n(K)$  of  $S^3$  of order  $n$ , branched along  $K$ .

Let  $\mathfrak{r}_\infty$  be the group of all roots of unity of arbitrary order, which we identify with  $\mathfrak{r}_\infty \simeq \mathbb{Q}/\mathbb{Z}$ . For any  $n$ , let  $\mathfrak{r}_n \simeq \mathbb{Z}/n\mathbb{Z}$  be the group of roots of unity of order  $n$ , with  $\mathfrak{r}_n \subset \mathfrak{r}_\infty$ .

**Remark 4.13.** Notational warning: we avoid the more standard notation  $\mu_n$  and  $\mu_\infty$  for the groups of roots of unity, to avoid a conflict with the Bost–Connes notation, that we follow below, where  $\mu_n$  is used for the isometries in the crossed product algebra.

Any group homomorphism  $\rho_K : \pi_K \rightarrow \mathfrak{r}_\infty$  or  $\rho : \pi \rightarrow \mathfrak{r}_\infty$  factors through the abelianizations  $\pi_K^{ab} = \mathbb{Z}$  and  $\pi^{ab} = \mathbb{Z}$ , hence it maps all the generators to an element  $\zeta \in \mathfrak{r}_\infty$ , of some

order  $n$ . Thus, the homomorphisms  $\rho_K$  and  $\rho$  determine representations  $\rho_{K,n} : \pi_K \rightarrow \mathbb{Z}/n\mathbb{Z}$  and  $\rho_n : \pi \rightarrow \mathbb{Z}/n\mathbb{Z}$ . Let  $\mathcal{R} \subset \text{Hom}(\pi, \mathfrak{t}_\infty)$  and  $\mathcal{R}_K \subset \text{Hom}(\pi_K, \mathfrak{t}_\infty)$  be the subsets of homomorphisms such that the corresponding  $\rho_{K,n}$  and  $\rho_n$  determine the unique connected cyclic branched cover.

Consider then the pullback diagrams of groups

$$(4.10) \quad \begin{array}{ccc} \hat{\pi}_{K,n} & \longrightarrow & \mathfrak{t}_\infty \\ \downarrow & & \downarrow \sigma_n \\ \pi_K & \xrightarrow{\rho_K} & \mathfrak{t}_\infty \end{array}$$

and

$$(4.11) \quad \begin{array}{ccc} \hat{\pi}_n & \longrightarrow & \mathfrak{t}_\infty \\ \downarrow & & \downarrow \sigma_n \\ \pi & \xrightarrow{\rho} & \mathfrak{t}_\infty \end{array}$$

where  $\sigma_n : \mathfrak{t}_\infty \rightarrow \mathfrak{t}_\infty$  is the endomorphism  $\sigma_n : \zeta \mapsto \zeta^n$ , that is, the homomorphism  $\sigma_n : \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$  mapping  $\sigma_n : r \mapsto nr$ . The pullback groups are given by  $\hat{\pi}_{K,n} = \{(\gamma, \zeta) \in \pi_K \times \mathfrak{t}_\infty \mid \rho(\gamma) = \zeta^n\}$  and  $\hat{\pi}_n = \{(\gamma, \zeta) \in \pi \times \mathfrak{t}_\infty \mid \rho(\gamma) = \zeta^n\}$ .

**Lemma 4.14.** *Let  $\mathcal{S} \subset \mathbb{N}$  be a subsemigroup, with the partial ordering defined by the divisibility relation. The groups  $\hat{\pi}_{K,n}$  and  $\hat{\pi}_n$  form projective systems with respect to  $n \in \mathcal{S}$ , with epimorphisms  $\hat{\sigma}_{n/m} : \hat{\pi}_{K,n} \rightarrow \hat{\pi}_{K,m}$  for  $m|n$  in  $\mathcal{S}$ , and similarly for the  $\hat{\pi}_n$ , with respective projective limits  $\hat{\pi}_{K,\rho_K,\mathcal{S}}$  and  $\hat{\pi}_{\rho,\mathcal{S}}$ , which depend both on the initial choice of the morphism  $\rho_K \in \mathcal{R}_K$  (respectively,  $\rho \in \mathcal{R}$ ) and on the semigroup  $\mathcal{S}$ .*

*Proof.* We illustrate the argument for  $\pi_K$ ; the case of the direct limit  $\pi$  is analogous. When  $m|n$  in  $\mathcal{S}$  we have a commutative diagram

$$(4.12) \quad \begin{array}{ccc} \hat{\pi}_{K,n} & \longrightarrow & \mathfrak{t}_\infty \\ \searrow \hat{\sigma}_{n/m} & & \downarrow \sigma_{n/m} \\ & \hat{\pi}_{K,m} & \longrightarrow \mathfrak{t}_\infty \\ \searrow & \downarrow & \downarrow \sigma_m \\ & \pi_K & \xrightarrow{\rho_K} \mathfrak{t}_\infty \end{array}$$

where the arrow  $\hat{\sigma}_{n/m} : \hat{\pi}_{K,n} \rightarrow \hat{\pi}_{K,m}$  is determined by the universal property. We have  $\hat{\sigma}_{n/m}(\gamma, \zeta) = (\gamma, \sigma_{n/m}(\zeta))$ , with  $\rho(\gamma) = \zeta^n = (\sigma_{n/m}(\zeta))^m$ , hence we obtain a projective system of epimorphisms  $\hat{\sigma}_{n/m} : \hat{\pi}_{K,n} \rightarrow \hat{\pi}_{K,m}$  for  $m|n$ . The construction of these pullback diagrams and the groups  $\hat{\pi}_{K,n}$  of the projective system depend on the initial choice of the homomorphism  $\rho_K \in \mathcal{R}_K \subset \text{Hom}(\pi_K, \mathfrak{t}_\infty)$  and on the semigroup  $\mathcal{S}$ , hence the resulting projective limit  $\hat{\pi}_{K,\rho_K,\mathcal{S}} = \varprojlim_{n \in \mathcal{S}} \hat{\pi}_{K,n}$  also depends on  $\rho_K$  and  $\mathcal{S}$ .  $\square$

As mentioned above, the representation and  $\rho_K \in \mathcal{R}_K$  maps the generators of  $\pi_K$  to a single element  $\zeta$  in the set  $\mathcal{P}(n_{\rho_K})$  of primitive roots of unity of some order  $n_{\rho_K}$ . An arbitrary element  $\gamma \in \pi_K$  maps to some  $\rho(\gamma) = \zeta_{n_{\rho_K}}^{n\gamma} \in \mathfrak{t}_{n_{\rho_K}} \subset \mathfrak{t}_\infty$ . Similarly for  $\rho \in \mathcal{R}$ .

**Definition 4.15.** *Given  $\rho \in \mathcal{R}$  (respectively,  $\rho_K \in \mathcal{R}_K$ ), Let  $\mathbb{N}_\rho \subset \mathbb{N}$  (respectively,  $\mathbb{N}_{\rho_K} \subset \mathbb{N}$ ) be the subsemigroup of  $n \in \mathbb{N}$  with  $(n, n_\rho) = 1$  (respectively,  $(n, n_{\rho_K}) = 1$ ), that is,*

the multiplicative semigroup generated by those primes  $p \in \mathcal{P}$  that do not occur in the primary decomposition of  $n_\rho$  (respectively,  $n_{\rho_K}$ ). We use the notation  $\hat{\pi}_\rho := \hat{\pi}_{\rho, \mathbb{N}_\rho}$  and  $\hat{\pi}_{K, \rho_K} := \hat{\pi}_{K, \rho_K, \mathbb{N}_{\rho_K}}$  for the corresponding projective limits.

**Remark 4.16.** The effect of passing to the pullbacks  $\hat{\pi}_{K, n}$  and  $\hat{\pi}_n$  is to introduce  $n$ -th roots for the elements of the knot groups  $\pi_K$  and of their limit  $\pi$ . Indeed, for each element  $\gamma$  of  $\pi_K$ , with  $\rho(\gamma) = \zeta_{n_{\rho_K}}^{n_\gamma}$ , there are  $n$  corresponding elements in  $\hat{\pi}_{K, n}$  of the form  $(\gamma, \zeta)$  with  $\zeta^n = \zeta_{n_{\rho_K}}^{n_\gamma}$ . The projective limits  $\hat{\pi}_{K, \rho_K}$  and  $\hat{\pi}_\rho$  contain roots of the elements of  $\pi_K$  (or of  $\pi$ ) for arbitrary order in  $\mathbb{N}_{\rho_K}$  (respectively,  $\mathbb{N}_\rho$ ).

**Remark 4.17.** The construction of the pullbacks  $\hat{\pi}_{K, n}$  and  $\hat{\pi}_n$  and projective limits  $\hat{\pi}_{K, \rho_K}$  and  $\hat{\pi}_\rho$  is analogous to the construction of formal roots of Tate motives in §4.2 of [36].

**Proposition 4.18.** For all  $n \in \mathbb{N}_\rho$ , there are homomorphisms  $\sigma_n : \hat{\pi}_\rho \rightarrow \hat{\pi}_\rho$  given by

$$(4.13) \quad \sigma_n(\gamma, \zeta) := (\gamma, \zeta^n).$$

The maps  $\{\sigma_n\}_{n \in \mathbb{N}_\rho}$  of (4.13) determine an action of the semigroup  $\mathbb{N}_\rho$  by endomorphisms of the group ring  $\mathbb{Q}[\hat{\pi}_\rho]$ . The endomorphisms  $\sigma_n$  have partial inverses  $\alpha_n : \mathbb{Q}[\hat{\pi}_\rho] \rightarrow \mathbb{Q}[\hat{\pi}_\rho]$ ,

$$(4.14) \quad \alpha_n(\delta_{(\gamma, \zeta)}) = \frac{1}{n} \sum_{\eta: \eta^n = \zeta} \delta_{(\gamma, \eta)}$$

satisfying  $\sigma_n \circ \alpha_n(\delta_{(\gamma, \zeta)}) = \delta_{(\gamma, \zeta)}$  and  $\alpha_n \circ \sigma_n(\delta_{(\gamma, \zeta)}) = e_n \cdot \delta_{(\gamma, \zeta)}$ , where  $e_n = n^{-1} \sum_{\xi: \xi^n = 1} \delta_{(1, \xi)}$  is an idempotent in  $\mathbb{Q}[\hat{\pi}_\rho]$ . The case of  $\hat{\pi}_{K, \rho_K}$  is analogous.

*Proof.* An element  $(\gamma, \zeta)$  belongs to  $\hat{\pi}_n$  when  $\rho(\gamma) = \zeta^n$ , that is,  $\zeta^n = \zeta_{n_\rho}^{n_\gamma}$ . An element  $(\gamma, \zeta)$  with  $\gamma \in \pi$  and  $\zeta \in \mathfrak{r}_\infty$  is in  $\hat{\pi}_\rho$  when there is some  $m \in \mathbb{N}_\rho$  such that  $\zeta^m = \zeta_{n_\rho}^{n_\gamma}$ . That is,  $\zeta \in \cup_{m \in \mathbb{N}_\rho} \sigma_m^{-1}(\mathfrak{r}_{n_\rho})$ . Suppose given  $(\gamma, \zeta) \in \hat{\pi}_\rho$  and  $n \in \mathbb{N}_\rho$ . We need to check that the element  $\sigma_n(\gamma, \zeta) := (\gamma, \zeta^n)$  is also in  $\hat{\pi}_\rho$ . Let  $m \in \mathbb{N}_\rho$  be such that  $\zeta^m = \rho(\gamma) = \zeta_{n_\rho}^{n_\gamma}$ . We need to check whether there exists an  $N \in \mathbb{N}_\rho$  such that  $\zeta^{nN} = \zeta_{n_\rho}^{n_\gamma}$ . Observe that, since  $(n, n_\rho) = 1$ , there is a unique solution  $k$  to the congruence equation  $nk = 1 \pmod{n_\rho}$ . This is obtained by reducing modulo  $n_\rho$  the relation  $nk + n_\rho \ell = 1$ , which is satisfied by a pair of  $k, \ell \in \mathbb{Z}$ , because  $(n, n_\rho) = 1$ . Such  $k$  is unique modulo  $n_\rho$ , since if  $k'$  is another solution,  $n(k - k') = 0 \pmod{n_\rho}$  implies  $n_\rho | (k - k')$  since  $(n, n_\rho) = 1$ . Note that  $(k, n_\rho)$  divides  $nk + n_\rho \ell$ , hence  $(k, n_\rho) = 1$ . Then  $N = mk$  satisfies  $\zeta^{nN} = \zeta^m$ . For  $(\gamma, \zeta) \in \hat{\pi}_\rho$ , let  $\delta_{(\gamma, \zeta)}$  be the corresponding generator of the group ring  $\mathbb{Q}[\hat{\pi}_\rho]$ . The maps (4.13) extend to endomorphisms of  $\mathbb{Q}[\hat{\pi}_\rho]$  by  $\sigma_n(\delta_{(\gamma, \zeta)}) = \delta_{(\gamma, \zeta^n)}$ . Since we clearly have  $\sigma_n \circ \sigma_m = \sigma_{nm}$ , the maps (4.13) determine a semigroup action of  $\mathbb{N}_\rho$  by endomorphisms of  $\mathbb{Q}[\hat{\pi}_\rho]$ . For the endomorphisms  $\alpha_n : \mathbb{Q}[\hat{\pi}_\rho] \rightarrow \mathbb{Q}[\hat{\pi}_\rho]$  of (4.14) we also need to check that, for  $(\gamma, \zeta) \in \hat{\pi}_\rho$  and  $n \in \mathbb{N}_\rho$ , if  $\eta \in \mathfrak{r}_\infty$  is such that  $\eta^n = \zeta$ , then  $(\gamma, \eta)$  is also in  $\hat{\pi}_\rho$ . This can be seen immediately, since we know that there is some  $m \in \mathbb{N}_\rho$ , such that  $\zeta^m = \rho(\gamma)$ , hence we also have  $\eta^{nm} = \zeta^m = \rho(\gamma)$ , hence  $(\gamma, \eta) \in \hat{\pi}_\rho$ . Thus, the  $\alpha_n$  of (4.14) are well defined. It is then also immediate to verify that we have

$$\sigma_n \circ \alpha_n(\delta_{(\gamma, \zeta)}) = \frac{1}{n} \sum_{\eta: \eta^n = \zeta} \sigma_n(\delta_{(\gamma, \eta)}) = \frac{1}{n} \sum_{\eta: \eta^n = \zeta} \delta_{(\gamma, \zeta)} = \delta_{(\gamma, \zeta)},$$

$$\alpha_n \circ \sigma_n(\delta_{(\gamma, \zeta)}) = \frac{1}{n} \sum_{\eta: \eta^n = \zeta^n} \delta_{(\gamma, \eta)} = \frac{1}{n} \sum_{\xi: \xi^n = 1} \delta_{(1, \xi)} \cdot \delta_{(\gamma, \zeta)},$$

since solutions of  $\eta^n = \zeta^n$  are of the form  $\xi\zeta$ . The element  $e_n = n^{-1} \sum_{\xi: \xi^n = 1} \delta_{(1, \xi)}$  is an idempotent since we have

$$e_n \cdot e_n = \frac{1}{n} \sum_{\xi_1: \xi_1^n = 1} \frac{1}{n} \sum_{\xi_2: \xi_2^n = 1} \delta_{(1, \xi_1 \xi_2)} = \frac{1}{n} \sum_{\chi: \chi^n = 1} \delta_{(1, \chi)} = e_n.$$

□

Thus, we can form the semigroup crossed product algebra as in the Bost–Connes case, as a direct consequence of the previous proposition.

**Corollary 4.19.** *The semigroup crossed product algebra  $\mathcal{A}_{\hat{\pi}_\rho, \mathbb{Q}} := \mathbb{Q}[\hat{\pi}_\rho] \rtimes_\alpha \mathbb{N}_\rho$  has generators unitaries  $\delta_{(\gamma, \zeta)}$ , for  $(\gamma, \zeta) \in \hat{\pi}_\rho$ , and isometries  $\mu_n$ , for  $n \in \mathbb{N}_\rho$ , satisfying*

$$\begin{aligned} \mu_n^* \mu_n &= 1, & \mu_n \mu_n^* &= e_n, & \mu_n \mu_m &= \mu_{nm}, & \mu_n \mu_m^* &= \mu_m^* \mu_n \text{ for } (n, m) = 1, \\ \mu_n \delta_{(\gamma, \zeta)} \mu_n^* &= \alpha_n(\delta_{(\gamma, \zeta)}), & \mu_n^* \delta_{(\gamma, \zeta)} \mu_n &= \sigma_n(\delta_{(\gamma, \zeta)}). \end{aligned}$$

The  $C^*$ -algebra  $C_r^*(\hat{\pi}_\rho) \rtimes_\alpha \mathbb{N}_\rho$ , with the same generators and relations, is a  $C^*$ -algebra completion of  $\mathcal{A}_{\hat{\pi}_\rho, \mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ .

We can now combine this construction with the one described in the previous subsections and define the  $C^*$ -algebra of observables of the combined system to be the following.

**Definition 4.20.** *The  $C^*$ -algebra of observable of the quantum statistical mechanical system of cyclic branched coverings of  $S^3$  and knots is given by the Bernoulli crossed product*

$$(4.15) \quad \bigotimes_{g \in \mathcal{G}_K} (C_r^*(\hat{\pi}_\rho) \rtimes_\alpha \mathbb{N}_\rho) \rtimes \mathcal{G}_K.$$

**Remark 4.21.** In the following we will refer to  $C_r^*(\hat{\pi}_\rho) \rtimes_\alpha \mathbb{N}_\rho$  and its associated quantum statistical mechanics as “the inner system”, and to (4.15) as “the combined system” or the “total system”.

**4.7. Quantum statistical mechanics of the inner system.** By Lemma 4.14 and Definition 4.15, we have  $\hat{\pi}_\rho \subset \pi \times \mathbb{Q}/\mathbb{Z}$  and  $\mathbb{N}_\rho \subset \mathbb{N}$ . In order to construct a quantum statistical mechanical system on  $C_r^*(\hat{\pi}_\rho) \rtimes_\alpha \mathbb{N}_\rho$  that incorporates the usual Bost–Connes dynamics, we start by considering the algebra

$$(4.16) \quad C_r^*(\pi) \otimes C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N},$$

where  $\mathbb{N}$  acts on  $C^*(\mathbb{Q}/\mathbb{Z})$  with the Bost–Connes endomorphisms (1.1), with  $\mathcal{A}_{BC} = C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}$  given in terms of generators and relations as in (1.3), (1.4). On the algebra (4.16), we consider the time evolution  $\sigma_t(\gamma \otimes a) = \gamma \otimes \sigma_t(a)$ , with  $\gamma \in \pi$  and  $a \in \mathcal{A}_{BC}$ , where  $\sigma_t(a)$  is the Bost–Connes time evolution. We consider then representations of (4.16) on the Hilbert space  $\mathcal{H} = L^2(\pi, \tau) \otimes \ell^2(\mathbb{N})$ , where  $\tau$  is the von Neumann trace on the group von Neumann algebra, with  $\tau(1) = 1$  and  $\tau(\gamma) = 0$ , for  $\gamma \neq 1$ , given by

$$(4.17) \quad \pi_u(\gamma \otimes a) \xi(\gamma') \otimes \epsilon_m = R(\gamma) \xi(\gamma') \otimes \pi_u(a) \epsilon_m,$$

for  $\xi \in L^2(\pi, \tau)$  and  $\epsilon_m$  the standard basis of  $\ell^2(\mathbb{N})$ , where  $R(\gamma)$  is the right regular representation of  $C_r^*(\pi)$  on  $L^2(\pi, \tau)$  and  $\pi_u(a)$  is the Bost–Connes representation (1.5) of  $\mathcal{A}_{BC}$  on  $\ell^2(\mathbb{N})$ .

**Lemma 4.22.** *In the representation (4.17), the time evolution  $\sigma_t$  is implemented by the Hamiltonian  $H = 1 \otimes H_{BC}$ , where  $H_{BC}$  is the Hamiltonian of the Bost–Connes system.*

*Proof.* For  $H \xi(\gamma') \otimes \epsilon_m = \log(m) \xi(\gamma') \otimes \epsilon_m$ , we have

$$e^{itH} \pi_u(\gamma \otimes a) e^{-itH} = R(\gamma) \otimes e^{itH_{BC}} \pi_u(a) e^{-itH_{BC}} = R(\gamma) \otimes \pi_u(\sigma_t(a)) = \pi_u(\sigma_t(\gamma \otimes a)).$$

□

**Proposition 4.23.** *The functionals  $\psi_\beta := \tau \otimes \varphi_\beta$ , with  $\tau$  the von Neumann trace and  $\varphi_\beta$  a  $\text{KMS}_\beta$  state of the Bost–Connes system are KMS states of  $(C_r^*(\pi) \otimes \mathcal{A}_{BC}, \sigma)$ . Indeed, all KMS states are of this form.*

*Proof.* To see that the functionals  $\psi_\beta = \tau \otimes \varphi_\beta$  satisfy the  $\text{KMS}_\beta$  condition, consider elements  $X, Y \in C_r^*(\pi) \otimes \mathcal{A}_{BC}$  of the form  $X = c \otimes a$  and  $Y = c' \otimes a'$ , with  $c, c' \in C_r^*(\pi)$  and  $a, a' \in \mathcal{A}_{BC}$ . Then set  $\tilde{F}_{X,Y}(z) := \tau(cc')F_{aa'}(z)$ , where  $F_{aa'}(z)$  is the holomorphic function expressing the  $\text{KMS}_\beta$  condition for the state  $\varphi_\beta$  on the algebra  $\mathcal{A}_{BC}$ . The function  $\tilde{F}_{X,Y}$  is clearly holomorphic on  $\mathcal{I}_\beta$  and continuous on  $\partial\mathcal{I}_\beta$  because  $F_{aa'}(z)$  is. Moreover, it satisfies

$$\tilde{F}_{X,Y}(t) = \tau(cc')\varphi_\beta(a\sigma_t(a')) = \psi_\beta(X\sigma_t(Y))$$

$$\tilde{F}_{X,Y}(t + i\beta) = \tau(cc')\varphi_\beta(\sigma_t(a')a) = \tau(c'c)\varphi_\beta(\sigma_t(a')a) = \psi_\beta(\sigma_t(Y)X),$$

hence it expresses the  $\text{KMS}_\beta$  condition for  $\psi_\beta$ . Conversely, suppose given a  $\text{KMS}_\beta$  state  $\psi_\beta$  for  $(C_r^*(\pi) \otimes \mathcal{A}_{BC}, \sigma)$ . It is known (see for instance §5.3.1 of [6]), that the KMS condition expressed as above, in terms of interpolation of  $\psi_\beta(X\sigma_t(Y))$  and  $\psi_\beta(\sigma_t(Y)X)$  by a holomorphic function  $F_{X,Y}(z)$ , is equivalent to the property that, for all  $X, Y$  in a dense involutive subalgebra  $\mathcal{A}_{an}$  of “analytic elements” (also called “entire elements”) the state satisfies  $\psi_\beta(XY) = \psi_\beta(Y\sigma_{i\beta}(X))$ . In particular, for elements in  $\mathcal{A}_{an}$  of the form  $c \otimes 1$  and  $c' \otimes 1$ , we have  $\psi_\beta(cc' \otimes 1) = \psi_\beta(c'c \otimes 1)$ . Indeed, since  $\sigma_t(c \otimes 1) = c \otimes 1$  for  $t \in \mathbb{R}$ , elements of the form  $c \otimes 1$  are always in  $\mathcal{A}_{an}$  with the analytic extension of the time evolution still trivially given by  $\sigma_z(c \otimes 1) = c \otimes 1$ . Thus, the KMS state  $\psi_\beta$  restricted to elements of the form  $c \otimes 1$  has to be a trace, and therefore it has to agree with the unique von Neumann trace  $\tau$ . Consider then elements of  $\mathcal{A}_{an}$  of the form  $X = 1 \otimes a$  and  $Y = 1 \otimes b$ , with  $a, b \in \mathcal{A}_{BC}$ . Then  $\sigma_t(X) = 1 \otimes \sigma_t(a)$  with  $\sigma_t(a)$  the Bost–Connes time evolution. Thus, the analytic continuation is also of the form  $\sigma_z(X) = 1 \otimes \sigma_z(a)$ , that is,  $a \in \mathcal{A}_{an,BC}$  is an analytic element of the Bost–Connes algebra with the corresponding analytic continuation of the time evolution. Thus, we have  $\psi_\beta(1 \otimes ab) = \psi_\beta(1 \otimes b\sigma_{i\beta}(a))$ , which implies that, when restricted to  $1 \otimes \mathcal{A}_{BC}$ , the state satisfies  $\psi_\beta(1 \otimes a) = \varphi_\beta(a)$  for some  $\text{KMS}_\beta$  state  $\varphi_\beta$  of the Bost–Connes system. Thus, for elements of the form  $c \otimes a$ , with  $c \in C_r^*(\pi)$  and  $a \in \mathcal{A}_{BC}$  one obtains  $\psi_\beta(c \otimes a) = \tau(c)\varphi_\beta(a)$ . □

**Remark 4.24.** When restricted to  $C_r^*(\pi_\rho) \rtimes_\alpha \mathbb{N}_\rho$ , the  $\text{KMS}_\beta$  states  $\psi_\beta = \tau \otimes \varphi_\beta$  of  $(C_r^*(\pi) \otimes \mathcal{A}_{BC}, \sigma)$  define KMS states of the system  $(C_r^*(\pi_\rho) \rtimes_\alpha \mathbb{N}_\rho, \sigma)$  with the induced time evolution.

**4.8. Properties of the algebra of observables of the combined system.** We consider here the  $C^*$ -algebra (4.15) and the corresponding von Neumann algebra

$$(4.18) \quad \bigotimes_{g \in \mathcal{G}_K} (\mathcal{N}(\hat{\pi}_\rho) \rtimes_\alpha \mathbb{N}_\rho) \rtimes \mathcal{G}_K,$$

where  $\mathcal{N}(\hat{\pi}_\rho)$  is the group von Neumann algebra of  $\hat{\pi}_\rho$ . This von Neumann algebra belongs to the class of noncommutative Bernoulli crossed products [8].

4.8.1. *Tensor product system.* In order to construct a quantum statistical mechanical system for this algebra, compatible with the construction considered above for the inner system, we first extend the construction of the inner system to the tensor product

$$\otimes_{g \in \mathcal{G}_{\mathcal{K}}} C_r^*(\pi_\rho) \rtimes_\alpha \mathbb{N}_\rho.$$

Let  $\mathcal{B}_g = C_r^*(\pi_\rho) \rtimes_\alpha \mathbb{N}_\rho$  denote the  $g$ -th factor in the above tensor product algebra. An element  $g \in \mathcal{G}_{\mathcal{K}}$  is an equivalence class  $g = K \ominus K'$  of pairs  $(K, K')$  of knots, up to the equivalence defining the Grothendieck group  $\mathcal{G}_{\mathcal{K}}$  of the semigroup  $(\mathcal{K}, \#)$ .

On the algebra  $\mathcal{B}_{K \ominus K'}$  we consider a time evolution similar to the one considered in §4.7, induced on  $C_r^*(\pi_\rho) \rtimes_\alpha \mathbb{N}_\rho$  from a time evolution  $\sigma_t(\gamma \otimes a) = \gamma \otimes \sigma_{t, K \ominus K'}(a)$  on  $C_r^*(\pi) \otimes \mathcal{A}_{BC}$ , where for  $a \in \mathcal{A}_{BC}$  we now take  $\sigma_{t, K \ominus K'}(a)$  to be a scaled version of the Bost–Connes time evolution of the form  $\sigma_{t, K \ominus K'}(e(r)) = e(r)$  and

$$(4.19) \quad \sigma_{t, K \ominus K'}(\mu_n) = n^{it f(K \ominus K')} \mu_n,$$

where  $f : \mathcal{G}_{\mathcal{K}} \rightarrow \mathbb{R}_+^*$  is a function, whose properties we specify below. On the tensor product  $\otimes_{g \in \mathcal{G}_{\mathcal{K}}} \mathcal{B}_g$  we consider the time evolution  $\sigma_t = \otimes_g \sigma_{t, g}$ .

**Definition 4.25.** For bounded linear operators on  $L^2(\pi, \tau) \otimes \ell^2(\mathbb{N}_\rho)$  of the form  $R(\gamma) \otimes T$ , we define  $\text{Tr}_\tau(R(\gamma) \otimes T) := \tau(\gamma) \text{Tr}(T)$ , where  $\text{Tr}$  is the operator trace on  $\mathcal{B}(\ell^2(\mathbb{N}_\rho))$ . The operator  $R(\gamma) \otimes T$  is  $\text{Tr}_\tau$ -class if  $\text{Tr}_\tau(R(\gamma) \otimes T)$  is finite, that is, if  $T$  is trace-class. In particular, for a Hamiltonian of the form  $1 \otimes H$  as in Lemma 4.22, we define the partition function as

$$(4.20) \quad Z_\tau(\beta) = \text{Tr}_\tau(1 \otimes e^{-\beta H}).$$

**Proposition 4.26.** Let  $\mathcal{H} = \otimes_{g \in \mathcal{G}_{\mathcal{K}}} \mathcal{H}_g$  be the Hilbert space with  $\mathcal{H}_g = L^2(\pi, \tau) \otimes \ell^2(\mathbb{N}_\rho)$ , with the algebra  $\otimes_{g \in \mathcal{G}_{\mathcal{K}}} \mathcal{B}_g$  acting on  $\mathcal{H}$  with the action  $\pi_{u, g}$  of  $\mathcal{B}_g$  on  $\mathcal{H}_g$  as in (4.17). Let  $H$  be the Hamiltonian implementing the time evolution  $\sigma_t = \otimes_g \sigma_{t, g}$  in this representation. Consider a function  $f : \mathcal{G}_{\mathcal{K}} \rightarrow \mathbb{N}$  with  $f(g) = 1$  for  $g$  the class of the unknot and  $f(g) \geq 2$  for all other  $g \in \mathcal{G}_{\mathcal{K}}$ . Also assume that  $f$  satisfies

$$(4.21) \quad \sum_{g \in \mathcal{G}_{\mathcal{K}}} f(g)^{-1} < \infty.$$

Then the operator  $e^{-\beta H}$  is  $\text{Tr}_\tau$ -class if and only if  $\beta > 1$  and the partition function of the system is given by

$$(4.22) \quad Z_\tau(\beta) = \prod_{g \in \mathcal{G}_{\mathcal{K}}} \zeta_{n_\rho}(f(g)\beta) < \infty,$$

where  $\zeta_m(s)$  is the Riemann zeta function with the Euler factors of primes  $p$  with  $p|m$  removed.

*Proof.* For  $\otimes_{g \in \mathcal{G}_{\mathcal{K}}} \mathcal{B}_g$  represented on  $\mathcal{H} = \otimes_g \mathcal{H}_g$  by the representation  $\otimes_g \pi_{u, g}$ , the time evolution  $\sigma_t = \otimes_g \sigma_{t, g}$  is implemented on  $\mathcal{H}$  by a Hamiltonian of the form  $H = \otimes_g (1 \otimes H_g)$ , where  $H_g \epsilon_m = f(g) \log m \epsilon_m$ , on the standard orthonormal basis  $\{\epsilon_m\}$  of  $\ell^2(\mathbb{N}_\rho)$ . Definition 4.25 extends to the case of a tensor product  $\mathcal{H} = \otimes_g \mathcal{H}_g$  with each  $\mathcal{H}_g = L^2(\pi, \tau) \otimes \ell^2(\mathbb{N}_\rho)$  and a Hamiltonian of the form  $H = \otimes_g (1 \otimes H_g)$ . For such an operator we write in shorthand notation

$$(4.23) \quad e^{-\beta H} = \otimes_g (1 \otimes e^{-\beta H_g}).$$

The trace is then given by

$$(4.24) \quad \mathrm{Tr}_\tau(e^{-\beta H}) = \prod_g \mathrm{Tr}_\tau(1 \otimes e^{-\beta H_g}) = \prod_g \mathrm{Tr}(e^{-\beta H_g}).$$

On a given  $\mathcal{H}_g$  the Hamiltonian  $H_g$  has

$$\mathrm{Tr}(e^{-\beta H_g}) = \sum_{n \in \mathbb{N}_\rho} n^{-f(g)\beta},$$

which converges for  $\beta > f(g)^{-1}$ , since the sum is less than or equal to  $\sum_{n \geq 1} n^{-f(g)\beta}$ , which converges for  $\beta > f(g)^{-1}$  to  $\zeta(f(g)\beta)$  with  $\zeta(s)$  the Riemann zeta function. Since the summation is only on  $\mathbb{N}_\rho$  instead of  $\mathbb{N}$ , the sum of the series is equal to

$$\sum_{n \in \mathbb{N}_\rho} n^{-f(g)\beta} = \zeta_{n_\rho}(f(g)\beta),$$

where

$$\zeta_m(s) = \prod_{p \nmid m} (1 - p^{-s})^{-1} = \sum_{n \in \mathbb{N}: (n, m) = 1} n^{-s}.$$

Thus, the operator  $1 \otimes e^{-\beta H_g}$  is  $\mathrm{Tr}_\tau$ -class for  $\beta > f(g)^{-1}$  and satisfies  $\mathrm{Tr}_\tau(1 \otimes e^{-\beta H_g}) = \zeta(f(g)\beta)$ . Thus, the partition function of the system is given by the infinite product

$$\mathrm{Tr}_\tau(e^{-\beta H}) = \prod_g \mathrm{Tr}_\tau(1 \otimes e^{-\beta H_g}) = \prod_g \zeta_{n_\rho}(f(g)\beta).$$

The convergence of this depends on each of the factors  $1 \otimes e^{-\beta H_g}$  being  $\mathrm{Tr}_\tau$ -class and on the convergence of the infinite product  $\prod_g \zeta_{n_\rho}(f(g)\beta)$  in the range of  $\beta$  where the  $\mathrm{Tr}_\tau$ -class condition is satisfied. Since  $f(g) \geq 1$  for all  $g \in \mathcal{G}_K$ , and each  $1 \otimes e^{-\beta H_g}$  is  $\mathrm{Tr}_\tau$ -class for  $\beta > f(g)^{-1}$ , then all these operators are simultaneously  $\mathrm{Tr}_\tau$ -class in the range  $\beta > 1$ . In particular, since  $\min_g f(g) = 1$ , then  $\beta > 1$  is exactly the range where  $\mathrm{Tr}_\tau$ -class condition holds. We then use the fact that the Riemann zeta function satisfies

$$\zeta(s) = \sum_{n \geq 1} n^{-s} = 1 + \sum_{n \geq 2} n^{-s} \leq 1 + \int_2^\infty \frac{dx}{x} = 1 - \frac{1}{2(s-1)} \leq 1.$$

This gives, for  $f(g) \geq 1$  and  $\beta > 1$ ,

$$0 < \zeta_{n_\rho}(f(g)\beta) \leq \zeta(f(g)\beta) \leq 1 - \frac{1}{2(f(g)\beta - 1)}.$$

Thus, the convergence of the infinite product  $\prod_g \zeta(f(g)\beta)$  is controlled by the convergence of the infinite product

$$\prod_g \left( 1 - \frac{1}{2(f(g)\beta - 1)} \right).$$

The convergence assumption (4.21) implies the convergence of  $\sum_g (f(g)\beta - 1)^{-1}$ . Recall that, for  $a_\ell$  a sequence of complex numbers with  $\sum_\ell |a_\ell|^2 < \infty$  the convergence of the infinite product  $\prod_\ell (1 + a_\ell)$  is equivalent to the convergence of the series  $\sum_\ell a_\ell$ . Since  $f(g) \geq 2$  for all  $g$  except the unknot, for  $\beta > 1$  we also have  $(f(g)\beta - 1)^{-1} < 1$  for all  $g$  except the unknot. Thus, the convergence of  $\sum_g (f(g)\beta - 1)^{-1}$  also implies the convergence of  $\sum_g (f(g)\beta - 1)^{-2}$ . Thus the convergence of the series  $\sum_g (f(g)\beta - 1)^{-1}$  is in fact equivalent to the convergence of the product  $\prod_g (1 - \frac{1}{2(f(g)\beta - 1)})$ . Thus, under the convergence

assumption (4.21), we obtain that the operator  $e^{-\beta H}$  of (4.23) is  $\text{Tr}_\tau$ -class in the range  $\beta > 1$  and the partition function satisfies (4.22).  $\square$

We have seen in Proposition 4.23 how to obtain  $\text{KMS}_\beta$  states  $\psi_\beta = \tau \otimes \varphi_\beta$  on the algebra  $\mathcal{B}_g = C_r^*(\hat{\pi}_\rho) \rtimes_\alpha \mathbb{N}_\rho$  from  $\text{KMS}$  states  $\varphi_\beta$  of the Bost–Connes system and the von Neumann trace  $\tau$ . We focus now in particular on the extremal low temperature  $\text{KMS}$  states of the Bost–Connes system,  $\varphi_\beta = \varphi_{\beta,u}$  of (1.7), with  $u \in \hat{\mathbb{Z}}^*$ . Let  $\varphi_{\beta,u,g}$  denote an extremal low temperature  $\text{KMS}$  state for the Bost–Connes system with Hamiltonian  $H_g = f(g)H_{BC}$ , where  $H_{BC}$  is the restriction to  $\ell^2(\mathbb{N}_\rho)$  of the usual Bost–Connes Hamiltonian, acting on  $\ell^2(\mathbb{N})$  by  $H_{BC}\epsilon_m = \log(m)\epsilon_m$ . Let  $\psi_{\beta,u,g} = \tau \otimes \varphi_{\beta,u,g}$  be the corresponding  $\text{KMS}_\beta$  state on the system  $(\mathcal{B}_g, \sigma_{t,g})$ .

Given a function  $F : \mathcal{G}_\mathcal{K} \rightarrow \mathcal{B}(\mathcal{H})$ , for a fixed Hilbert space  $\mathcal{H}$ , consider the operator  $\otimes_{g \in \mathcal{G}_\mathcal{K}} F(g)$  acting on the product  $\otimes_{g \in \mathcal{G}_\mathcal{K}} \mathcal{H}_g$  with  $\mathcal{H}_g = \mathcal{H}$ , for all  $g \in \mathcal{G}_\mathcal{K}$ . In particular, we can write elements in the algebra  $\otimes_{g \in \mathcal{G}_\mathcal{K}} \mathcal{B}_g$  in the form of functions

$$F : \mathcal{G}_\mathcal{K} \rightarrow \mathcal{B}(L^2(\pi, \tau) \otimes \ell^2(\mathbb{N}_\rho)).$$

Then Proposition 4.26 implies that we obtain a  $\text{KMS}_\beta$  state on the tensor product system  $(\otimes_{g \in \mathcal{G}_\mathcal{K}} \mathcal{B}_g, \sigma_t = \otimes_g \sigma_{t,g})$  as follows.

**Corollary 4.27.** *Let  $f : \mathcal{G}_\mathcal{K} \rightarrow \mathbb{N}$  be a function satisfying the same hypotheses as in Proposition 4.26. Then  $\Psi_{\beta,u,f} = \otimes_g \psi_{\beta,u,g}$  is a  $\text{KMS}_\beta$  state on the crossed product system  $(\otimes_{g \in \mathcal{G}_\mathcal{K}} \mathcal{B}_g, \sigma_t = \otimes_g \sigma_{t,g})$ . It is explicitly given in the Gibbs form*

$$(4.25) \quad \Psi_{\beta,u,f}(F) = \frac{\text{Tr}_\tau(e^{-\beta f H_{BC}} F)}{Z_\tau(\beta)},$$

with  $Z_\tau(\beta)$  as in (4.22).

*Proof.* As in Proposition 4.26, we have

$$\text{Tr}_\tau(e^{-\beta f H_{BC}} F) = \prod_g \text{Tr}_\tau((1 \otimes e^{-\beta H_g}) F(g)).$$

Moreover, for an element  $F(g) = 1 \otimes a_g$ , with  $a_g \in \mathcal{A}_{BC}$  represented via the representation  $\pi_u$ , the above is equal to  $\prod_g \text{Tr}(e^{-\beta H_g} \pi_u(a_g))$  and one obtains

$$\prod_g \frac{\text{Tr}(e^{-\beta H_g} \pi_u(a_g))}{\zeta(f(g)\beta)} = \prod_g \varphi_{\beta,u,g}(a_g).$$

$\square$

**4.8.2. Bernoulli crossed product.** As above, given a function  $F : \mathcal{G}_\mathcal{K} \rightarrow \mathcal{B}(\mathcal{H})$ , we consider the operator  $\otimes_{g \in \mathcal{G}_\mathcal{K}} F(g)$  on  $\otimes_{g \in \mathcal{G}_\mathcal{K}} \mathcal{H}_g$ , with  $\mathcal{H}_g = \mathcal{H}$ , for all  $g \in \mathcal{G}_\mathcal{K}$ . Consider the action of the group  $\mathcal{G}_\mathcal{K}$  on the set of functions  $F : \mathcal{G}_\mathcal{K} \rightarrow \mathcal{B}(\mathcal{H})$  given by

$$(4.26) \quad \alpha_h(F)(g) := F(h^{-1}g), \quad \text{for } h, g \in \mathcal{G}_\mathcal{K}.$$

As above, we write elements in the algebra  $\otimes_{g \in \mathcal{G}_\mathcal{K}} \mathcal{B}_g$  in this way.

**Proposition 4.28.** *The time evolution determined by (4.19) on the algebra  $\otimes_g \mathcal{B}_g$  extends to the crossed product algebra  $(\otimes_g \mathcal{B}_g) \rtimes \mathcal{G}_\mathcal{K}$  by setting*

$$(4.27) \quad \sigma_t(U_h) = e^{it(f - \alpha_h(f)) H_{BC}} U_h,$$

where  $U_h$ , for  $h \in \mathcal{G}_\mathcal{K}$  are the unitaries implementing the crossed product action  $\alpha_h(F) = U_h F U_h^*$  for  $F = \otimes_g F(g) \in \otimes_g \mathcal{B}_g$ .

*Proof.* Let  $\mathbb{H} : \mathcal{G}_\mathcal{K} \rightarrow \mathcal{B}(\mathcal{H})$  be the function  $\mathbb{H}(g) = 1 \otimes H_g = 1 \otimes f(g) H_{BC}$ , with  $H_{BC} \in \mathcal{B}(\ell^2(\mathbb{N}_\rho))$  the Bost–Connes Hamiltonian. We then write the time evolution on functions  $F : \mathcal{G}_\mathcal{K} \rightarrow \mathcal{B}(\mathcal{H})$  as

$$(4.28) \quad \sigma_t(F)(g) = e^{it\mathbb{H}(g)} F(g) e^{-it\mathbb{H}(g)}.$$

For  $h \in \mathcal{G}_\mathcal{K}$ , let  $U_h$  be the unitary operator on  $\otimes_{g \in \mathcal{G}_\mathcal{K}} \mathcal{H}_g$ , with  $\mathcal{H}_g = L^2(\pi, \tau) \otimes \ell^2(\mathbb{N}_\rho)$ , which acts as  $(U_h \xi)_g = \xi_{hg}$ , where we write elements of  $\otimes_{g \in \mathcal{G}_\mathcal{K}} \mathcal{H}_g$  as  $\xi = \otimes_g \xi_g$ , with  $\xi_g \in L^2(\pi, \tau) \otimes \ell^2(\mathbb{N}_\rho)$ . We then have  $U_h F U_h^* \xi = \alpha_h(F) \xi$ . This action satisfies

$$U_h \sigma_t(F) U_h^* = U_h e^{it\mathbb{H}} F e^{-it\mathbb{H}} U_h^* = \alpha_h(e^{it\mathbb{H}} F e^{-it\mathbb{H}}),$$

where  $(\alpha_h(e^{it\mathbb{H}} F e^{-it\mathbb{H}}) \xi)_g = e^{itf(h^{-1}g) H_{BC}} F(h^{-1}g) e^{-itf(h^{-1}g) H_{BC}} \xi_g$ . On the other hand, we have

$$\sigma_t(U_h F U_h^*) = e^{it\mathbb{H}} U_h F U_h^* e^{-it\mathbb{H}} = e^{it\mathbb{H}} \alpha_h(F) e^{-it\mathbb{H}},$$

where  $(e^{it\mathbb{H}} \alpha_h(F) e^{-it\mathbb{H}} \xi)_g = e^{itf(g) H_{BC}} F(h^{-1}g) e^{-itf(g) H_{BC}} \xi_g$ . This implies that the action of  $\mathcal{G}_\mathcal{K}$  transforms the time evolution as  $\sigma_{h,t} := \alpha_h(\sigma_t)$  with

$$(4.29) \quad \sigma_{h,t}(F)(g) = e^{it\alpha_h(\mathbb{H})(g)} F(g) e^{-it\alpha_h(\mathbb{H})(g)}.$$

Moreover, we obtain (4.27), since

$$\sigma_t(U_h) = e^{it\mathbb{H}} U_h e^{-it\mathbb{H}} = e^{it\mathbb{H}} e^{-it\alpha_h(\mathbb{H})} U_h = e^{itf H_{BC}} e^{-it\alpha_h(f) H_{BC}} U_h.$$

This determines how the time evolution extends to the crossed product  $(\otimes_g \mathcal{B}_g) \rtimes \mathcal{G}_\mathcal{K}$ .  $\square$

Let  $\psi_{\beta,g}$  denote a KMS $_\beta$  state, obtained as in Remark 4.24, for the system  $(\mathcal{B}_g, \sigma_{t,g})$ , where  $\sigma_{t,g}$  is the time evolution (4.19) with Hamiltonian  $H(g) = f(g) H_{BC}$ , and the algebra is  $\mathcal{B}_g = C_r^*(\hat{\pi}_\rho) \rtimes_\alpha \mathbb{N}_\rho$  as above. We denote by  $\Psi_{\beta,u,f}$  the KMS $_\beta$  state on the system  $(\otimes_g \mathcal{B}_g, \otimes_g \sigma_{t,g})$  determined by the  $\psi_{\beta,u,g}$  as in Corollary 4.27.

**Lemma 4.29.** *Under the action  $\alpha_h$  of  $h \in \mathcal{G}_\mathcal{K}$ , the KMS $_\beta$  state  $\Psi_{\beta,u,f}$  of Corollary 4.27 satisfies*

$$(4.30) \quad \Psi_{\beta,u,f} \circ \alpha_h = \Psi_{\beta,u,\alpha_{h^{-1}}(f)}.$$

*Proof.* We have

$$\Psi_{\beta,u,f}(\alpha_h(F)) = \Psi_{\beta,u,f}(U_h F U_h^*) = \Psi_{\beta,u,f}(\sigma_{-i\beta}(U_h^*) U_h F) = \Psi_{\beta,u,f}(e^{-\beta(\alpha_{h^{-1}}(f) - f) H_{BC}} F).$$

On the other hand, we also have

$$\begin{aligned} \Psi_{\beta,u,\alpha_{h^{-1}}(f)}(F) &= \frac{\text{Tr}_\tau(e^{-\beta\alpha_{h^{-1}}(f) H_{BC}} F)}{Z_\tau(\beta)} \\ &= \frac{\text{Tr}_\tau(e^{-\beta f H_{BC}} e^{-\beta(\alpha_{h^{-1}}(f) - f) H_{BC}} F)}{Z_\tau(\beta)} = \Psi_{\beta,u,f}(e^{-\beta(\alpha_{h^{-1}}(f) - f) H_{BC}} F). \end{aligned}$$

$\square$

**4.9. Knot invariants and the function  $f(g)$ .** We now show how to construct a function  $f : \mathcal{G}_{\mathcal{K}} \rightarrow \mathbb{N}$  that satisfies the hypotheses of Proposition 4.26 and Corollary 4.27, using knot invariants. As in §2.5, we write elements of  $\mathcal{G}_{\mathcal{K}}$  in terms of primary decomposition. Let  $K \ominus K' = (a_1 K_1 \# \cdots \# a_j K_j) \ominus (b_1 K'_1 \# \cdots \# b_\ell K'_\ell)$  be an element of  $\mathcal{G}_{\mathcal{K}}$  with primary decompositions  $K = a_1 K_1 \# \cdots \# a_m K_m$  and  $K' = b_1 K'_1 \# \cdots \# b_\ell K'_\ell$ , where the  $K_i$  and  $K'_j$  are all distinct prime knots, with multiplicities  $a_i$  and  $b_j$ . Since we eliminate all possible common factors from the primary decomposition of  $K$  and  $K'$ , this description of elements  $g = K \ominus K' \in \mathcal{G}_{\mathcal{K}}$  is unique. We also use, as in §2.5, the notation  $\omega(K)$  for the number of distinct prime knots in its primary decomposition of a knot  $K$ . It is then convenient to consider knot invariants that are additive under connected sums, and for which there is a good estimate of the rate of growth of the multiplicities.

To this purpose, we proceed as in §2.5, and we restrict from the Grothendieck group  $\mathcal{G}_{\mathcal{K}}$  of the semigroup  $(\mathcal{K}, \#)$  of all knots with the connected sum operation, to the subsemigroup  $(\mathcal{K}_a, \#)$  of alternating knots and its Grothendieck group  $\mathcal{G}_{\mathcal{K},a}$ , so that we can again use the genus and the crossing numbers as invariants. This means that, for the purpose of this section, we will be restricting to the Bernoulli crossed product

$$(4.31) \quad (\otimes_{g \in \mathcal{G}_{\mathcal{K},a}} \mathcal{B}_g) \rtimes \mathcal{G}_{\mathcal{K},a},$$

where, as before,  $\mathcal{B}_g = C_r^*(\hat{\pi}_\rho) \rtimes_\alpha \mathbb{N}_\rho$ .

**Proposition 4.30.** *For  $K \ominus K' \in \mathcal{G}_{\mathcal{K},a}$ , represented through its primary decomposition  $K \ominus K' = (a_1 K_1 \# \cdots \# a_j K_j) \ominus (b_1 K'_1 \# \cdots \# b_\ell K'_\ell)$  with no common prime factors, the function*

$$(4.32) \quad f(K \ominus K') = q^{\lceil \beta_+ \rceil (\sum_{i=1}^m a_i (Cr(K_i) + g(K_i)) + \sum_{j=1}^\ell b_j (Cr(K'_j) + g(K'_j)))},$$

with  $\lceil \beta_+ \rceil$  the smallest integer greater than or equal to the value  $\beta_+$  of Theorem 2.3, satisfies the hypotheses of Proposition 4.26 and Corollary 4.27.

*Proof.* The function  $f(g)$  takes values in  $\mathbb{N}$ , since  $q \geq 2$  is a fixed integer, and it takes value  $f(g) = 1$  only when  $g$  is the unknot, since only in that case the exponent is zero. Thus, we only need to check that the convergence property  $\sum_g f(g)^{-1} < \infty$  is satisfied. By Theorem 2.3 we know that

$$\sum_{K \in \mathcal{K}_a} f(K)^{-1} < \infty,$$

where  $f(K) = q^{\lceil \beta_+ \rceil (Cr(K) + g(K))}$ , while by (2.19) and Proposition 2.15 we see that also

$$\sum_{K \ominus K' \in \mathcal{G}_{\mathcal{K},a}} f(K \ominus K')^{-1} < \infty.$$

□

In particular, we can then see more explicitly the action  $f \mapsto \alpha_{h^{-1}}(f)$  that determines the transformation property of the KMS $_\beta$  state  $\Psi_{\beta,u,f}$  as in Lemma 4.29.

**Corollary 4.31.** *For  $h = \pm K$  in  $\mathcal{P}_a$ , the action  $f(g) \mapsto \alpha_{h^{-1}}(f)(g)$  raises or lowers by one the multiplicity of the prime factor  $K$  in the primary decomposition of  $g = K \ominus K' = (a_1 K_1 \# \cdots \# a_j K_j) \ominus (b_1 K'_1 \# \cdots \# b_\ell K'_\ell)$ .*

*Proof.* It suffices to see the effect of the action of an element  $h \in \mathcal{G}_{\mathcal{K},a}$  given by a single prime knot  $K \in \mathcal{P}_a$  with either a positive or a negative exponent. This gives either

$$\alpha_K(f)(K_1 \ominus K_2) = f(K_1 \# K \ominus K_2),$$

or, respectively,

$$\alpha_{-K}(f)(K_1 \ominus K_2) = f(K_1 \ominus K_2 \# K).$$

Since the definition of the function  $f$  depends on the primary decomposition of  $K_1 \ominus K_2$  without common factors, the result depends on whether  $K$  is a prime factor of either  $K_1$  or  $K_2$ . By analogy to the case of integers, for a knot  $K$ , we denote by  $(K_i, K)$  the connected sum of all the prime factors (with multiplicity) common to  $K_i$  and  $K$  and we denote by  $K_i/(K_i, K)$  the result of removing  $(K_i, K)$  from the primary decomposition of  $K_i$ . Since  $K$  is a single prime knot,  $(K_i, K) = K$  if it is non-trivial, that is, if  $K|K_i$  and it is the trivial knot otherwise. Similarly  $K_i/(K_i, K) = K_i/K$  in the first case and  $K_i/(K_i, K) = K_i$  in the second. Note that, if  $K_1 \ominus K_2$  is represented in a primary decomposition without common factors, then  $K$  can divide either  $K_1$  or  $K_2$  or neither, but it cannot divide both. Thus, the result of  $\alpha_{\pm K}(f)(K_1 \ominus K_2)$  is simply to lower or rise by one the power of  $K$  in the primary decomposition.  $\square$

**Remark 4.32.** It would be interesting to see if the construction presented in this paper can be extended to incorporate other, more sophisticated invariants of knots. For example, the type of (twisted)  $L^2$ -Alexander-Conway invariants of knots considered in [34], [35] are naturally defined in terms of the von Neumann algebra  $\mathcal{N}(\pi_K)$  of the knot group and appear to be suitable for the quantum statistical mechanical setting considered here.

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