

THE EXPLICIT LOCAL LANGLANDS CORRESPONDENCE FOR G_2

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ABSTRACT. We develop a general strategy for constructing explicit Local Langlands Correspondences for p -adic reductive groups via reduction to LLC for supercuspidal representations of proper Levi subgroups, using Hecke algebra techniques.

As an example of our general strategy, we construct explicit Local Langlands Correspondence for the exceptional group G_2 over a nonarchimedean local field, with explicit L -packets and explicit matching between the group and Galois sides. We also give a list of characterizing properties for our LLC. For intermediate series, we build on our previous results on Hecke algebras. For principal series, we improve previous works of Muic etc. and obtain more explicit descriptions on both group and Galois sides.

Moreover, we show the existence of non-unipotent *singular* supercuspidal representations of G_2 , and exhibit them in *mixed* L -packets mixing supercuspidal representations with non-supercuspidal ones. Furthermore, our LLC satisfies a list of expected properties, including the compatibility with cuspidal support.

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1. INTRODUCTION

Let F be a non-archimedean local field and \mathbf{G} a connected reductive algebraic group over F . Let G^\vee be the group of \mathbb{C} -points of the reductive group whose root

datum is the coroot datum of \mathbf{G} . The Local Langlands Conjecture predicts a surjective map¹

$$\left\{ \begin{array}{l} \text{irred. smooth} \\ \text{repres. } \pi \text{ of } \mathbf{G}(F) \end{array} \right\} / \text{iso.} \longrightarrow \left\{ \begin{array}{l} L\text{-parameters} \\ \text{i.e. cont. homomorphisms} \\ \varphi_\pi: W_F \times \text{SL}_2(\mathbb{C}) \rightarrow G^\vee \rtimes W_F \end{array} \right\} / G^\vee\text{-conj.},$$

where W_F is the Weil group of F . The fibers of this map, called *L-packets*, are expected to be finite. In order to obtain a bijection between the group side and the Galois side, the above Conjecture was later *enhanced*. On the Galois side, one considers *enhanced L-parameters*.

Many cases of the Local Langlands Conjecture have been established, notably:

- for $\text{GL}_n(F)$: [HT01, Hen00, Sch13];
- for $\text{SL}_n(F)$: [HS12] for $\text{char}(F) = 0$ and [ABPS16b] for $\text{char}(F) > 0$;
- for $\text{GSp}_4(F)$: [GT11] for $\text{char}(F) = 0$ and [Gan15] for $\text{char}(F) > 0$;
- quasi-split classical groups for F of characteristic zero: [Art13, Moe11] etc.

For classical groups, the main methods in literature are either (1) to classify representations of these groups in terms of representations of the general linear groups via twisted endoscopy, and to compare the stabilized twisted trace formula on the general linear group side and the stabilized (twisted) trace formula on the classical group side, or (2) to use the theta correspondence.

In this article, we take a completely different approach—from the above existing literature—to the construction of explicit Local Langlands Correspondences for p -adic reductive groups via reduction to LLC for supercuspidal representations of proper Levi subgroups. We apply this general strategy and construct explicit Local Langlands Correspondence for the exceptional group G_2 over an arbitrary non-archimedean local field of residual characteristic $\neq 2, 3$, with explicit *L-packets* and explicit matching between the group and Galois sides. Our methods are inspired by previous works such as [Mou17, AMS18, AMS21, FOS21, Sol18].

More precisely, we use a combination of the Langlands-Shahidi method, (extended affine) Hecke algebra techniques, Kazhdan-Lusztig theory and generalized Springer correspondence—in particular, the AMS Conjecture on cuspidal support [AMS18, Conjecture 7.8]. For *intermediate series*, i.e. Bernstein series with supercuspidal support “in between” a torus and G itself (in our case, the supercuspidal support has to lie in a Levi subgroup isomorphic to GL_2), we use our previous result on Hecke algebra isomorphisms and local Langlands correspondence for Bernstein series obtained in [AX22], which builds on the work of many others such as [Ber84, Sol22, Sol20, Hei11]. For principal series (i.e. Bernstein series with supercuspidal support in a torus), we improve on previous work of Muic’s [Mui97] and obtain more explicit description on the group side; and we use [Roc98, Ree02, ABPS16a, ABP11, Ram03] to match the group and Galois sides.

¹To avoid overrunning the margins, we use abbreviations “irred.” for “irreducible”, “repres.” for “representations”, “iso.” for “isomorphism”, “cont.” for “continuous” and “conj.” for “conjugacy”.

For supercuspidal representations, we make explicit the theory of [Kal19, Kal21] for the non-singular supercuspidal representations and their L -packets. For *singular*² supercuspidal representations, which are not covered in *loc.cit.*, we use [AMS18, Conjecture 7.8] (see Property 10.1.9) to exhibit them in *mixed* L -packets with non-supercuspidal representations. These mixed L -packets are drastically different from the supercuspidal L -packets of [Kal19, Kal21].

Furthermore, our LLC satisfies several expected properties, including the expectation that $\text{Irr}(S_\varphi)$ parametrizes the internal structure of the L -packet $\Pi_\varphi(G)$, where S_φ is the component group of the centralizer of the (image of the) L -parameter φ .

1.1. Main results. We now state our main results. Let $\text{Irr}^\mathfrak{s}(G)$ be the Bernstein series attached to the inertial class $\mathfrak{s} = [L, \sigma]$ (for more details, see (3.3.2)). Let $\Phi_e(G)$ denote the set of G^\vee -conjugacy classes of enhanced L -parameters for G . Let $\Phi_e^{\mathfrak{s}^\vee}(G) \subset \Phi_e(G)$ be the Bernstein series on the Galois side, whose cuspidal support lies in $\mathfrak{s}^\vee = [L^\vee, (\varphi_\sigma, \rho_\sigma)]$, i.e. the image under LLC for L of \mathfrak{s} (for more details, see §2.4). For any $\mathfrak{s} = [L, \sigma]_G \in \mathfrak{B}(G)$, the LLC for L given by $\sigma \mapsto (\varphi_\sigma, \rho_\sigma)$ is expected to induce a bijection (see [AMS18, Conjecture 2] and Conjecture 10.1.12):

$$(1.1.1) \quad \text{Irr}^\mathfrak{s}(G) \xrightarrow{\sim} \Phi_e^{\mathfrak{s}^\vee}(G).$$

For the group G_2 , by [AX22, Main Theorem], we have such a bijection (1.1.1) for each Bernstein series $\text{Irr}^\mathfrak{s}(G)$ of *intermediate series*. On the other hand, the analogous bijection to (1.1.1) holds for *principal series* Bernstein blocks thanks to [Roc98, Ree02, ABPS16a, AMS18].

Let $G = G_2(F)$ and $p \neq 2, 3$. Combined with the detailed analysis in all of §5 through §9, we construct an explicit Local Langlands Correspondence

$$(1.1.2) \quad \text{LLC}: \text{Irr}(G) \xrightarrow{1-1} \Phi_e(G) \\ \pi \mapsto (\varphi_\pi, \rho_\pi),$$

and obtain the following result (see Theorem 10.2.1).

Theorem 1.1.1. *The explicit Local Langlands Correspondence (1.1.2) verifies $\Pi_{\varphi_\pi}(G) \xrightarrow{\sim} \text{Irr}(S_{\varphi_\pi})$ for any $\pi \in \text{Irr}(G)$, and satisfies (1.1.1) for any $\mathfrak{s} \in \mathfrak{B}(G)$, where $\mathfrak{s}^\vee = [L^\vee, (\varphi_\sigma, \rho_\sigma)]_{G^\vee}$, as well as a list of properties (see §10.1) that uniquely determine it (up to very few minor choices³).*

In other words,

- (1) to each explicitly described $\pi \in \text{Irr}(G)$, we attach an explicit L -parameter φ_π and determine its enhancement ρ_π explicitly;
- (2) to each $\varphi \in \Phi(G)$, we describe (the shape of) its L -packet $\Pi_\varphi(G)$, and give an internal parametrization in terms of $\rho \in \text{Irr}(S_\varphi)$;
- (3) Moreover, for non-supercuspidal representations, we specify the precise parabolic induction that it occurs in.

²which we define to be simply the ones that are *not non-singular* in the sense of [Kal21]

³specified in Theorem 10.2.1

We now comment on several works related to our paper.

A few weeks after our paper first appeared, Gan and Savin had also claimed a construction of a local Langlands correspondence using completely different methods. For the convenience of the reader, we briefly comment on the differences between our work and theirs:

- (1) Our LLC is completely *explicit*, with explicit tables, explicit Kazhdan-Lusztig triples, explicit information on the L -packets and their packet members, etc. In particular, *explicit* computations of unipotent classes (in Kazhdan-Lusztig triples) such as the ones in our paper have found many important applications in number theory (see for example [Tho22]).
- (2) Our construction and proofs of LLC are *purely local*, whereas Gan-Savin's construction is a consequence of the global construction of Kret-Shin [KS16] (and the earlier works of Chenevier etc.). As such, our construction and proofs work for cases even in the lack of global constructions; yet on the other hand, due to the explicit nature of our construction, it is not hard to check local-global compatibility when, indeed, global constructions are made.
- (3) It is still unclear whether Gan-Savin's construction gives *the* LLC nor the right unique characterization. Contrary to what is claimed, Gan-Savin's construction involves many choices. For example, Gan-Savin's construction relies on Bin Xu's construction of L -packets for PGSp_6 , which are not known to be canonical. Moreover, Gan-Savin's characterization does not establish stability.
- (4) On the other hand, our construction is compatible with that of [Kal19, Kal16], for which stability is known [FKS19].
- (5) Our strategy for constructing the LLC works for arbitrary reductive groups and does not rely on the existences of theta correspondences.
- (6) Furthermore, our methods provide explicit local Langlands correspondence in a uniform way, independent of the characteristic of F .

We also note the construction of an LLC for generic supercuspidal representations for G_2 in [HKT19] using global methods and theta correspondences. It would be very interesting to compare our construction with that of *loc. cit.*

Moreover, as mentioned earlier, our work complements that of [Kal19, Kal21] as our constructions cover even the non-supercuspidal representations, and specifically the L -packets that *mix* supercuspidal representations with non-supercuspidal ones, i.e. the L -packets that do not appear in *loc. cit.*. Note that such "mixed" L -packets necessarily consist of *singular* supercuspidal representations not addressed in the constructions of *loc. cit.*

Lastly, it would also be interesting to compare our construction with those of [Zhu20], [FS21] and [GL17].

NOTATION. Let F be a local non-archimedean field. Let \mathfrak{o}_F denote the ring of integers of F , \mathfrak{p}_F the maximal ideal in \mathfrak{o}_F and $k_F := \mathfrak{o}_F/\mathfrak{p}_F$ the residue field of F . The group of units in \mathfrak{o}_F will be denoted \mathfrak{o}_F^\times . We assume that k_F is finite and denote by $q = q_F$ its cardinality. Let $\nu_F := \|\cdot\|_F$ denote the normalized absolute value of F .

We fix a separable closure F_{sep} of F and denote by $W_F \subset \text{Gal}(F_{\text{sep}}/F)$ the absolute Weil group of F . Let $I_F = \text{Gal}(F_{\text{sep}}/F)$ be the inertia group of F , and P_F the wild inertia group (i.e., the maximal pro- p open normal subgroup of I_F). We denote by F_{nr} the maximal unramified extension of F inside F_{sep} and by Fr_F the element of $\text{Gal}(F_{\text{nr}}/F)$ that induces the automorphism $\text{Fr}: a \mapsto a^q$ on the residue field \bar{k}_F of F_{nr} . Then $W_F = I_F \rtimes \langle \text{Fr}_F \rangle$. Let W_F^{der} denote the closure of the commutator subgroup of W_F , and write $W_F^{\text{ab}} = W_F/W_F^{\text{der}}$. We take $W'_F := W_F \times \text{SL}_2(\mathbb{C})$ for the Weil-Deligne group of F .

For n a positive integer, let μ_n denote the group on n th roots of unity and let ζ_n be a primitive n th root of the unity.

For $b \in F^\times/F^{\times 2}$, let $\text{U}_b(1,1)$ be the quasi-split unitary group, and $\text{U}_b(2)$ the compact unitary group in two variables in $F(\sqrt{b})$.

For \mathbf{H} any reductive algebraic group, we denote by \mathbf{H}° the identity component of \mathbf{H} and by $Z_{\mathbf{H}}$ the center of \mathbf{H} . Let h be an element of \mathbf{H} . We denote by $Z_{\mathbf{H}}(h)$ the centralizer of h in \mathbf{H} and by $A_{\mathbf{H}}(h) := Z_{\mathbf{H}}(h)/Z_{\mathbf{H}}(h)^\circ$ the component group of $Z_{\mathbf{H}}(h)$.

Let \mathbf{G} be a connected reductive algebraic group defined over F , let $\mathbf{G}_{\text{ad}} = \mathbf{G}/Z_{\mathbf{G}}$ be the adjoint group of \mathbf{G} and let $\mathbf{A}_{\mathbf{G}}$ denote the maximal split central subtorus of \mathbf{G} . Let $W_{\mathbf{G}} = W_{\mathbf{G}}(\mathbf{T})$ denote the Weyl group of \mathbf{G} with respect to a maximal torus \mathbf{T} . By a Levi subgroup of we mean an F -subgroup \mathbf{L} of \mathbf{G} which is a Levi factor of a parabolic F -subgroup of \mathbf{G} . We denote by G, A, T, L the groups of F -rational points of $\mathbf{G}, \mathbf{A}, \mathbf{T}, \mathbf{L}$, respectively. Let $X^*(L) := X^*(\mathbf{L})_F$ be the group of all rational characters $\chi: \mathbf{L} \rightarrow \text{GL}_1$.

Let $\mathcal{B}(\mathbf{G}, F)$ denote the (enlarged) building of G . For x a point in $\mathcal{B}(\mathbf{G}, F)$, let G_x denote the subgroup of G fixing x , and $G_{x,0} \subset G_x$ the associated parahoric subgroup. Let $G_{x,0^+}$ denote the pro- p unipotent radical of $G_{x,0}$, and $\mathbb{G}_{x,0}$ the reductive quotient of $G_{x,0}$. If τ is a representation of $\mathbb{G}_{x,0}$ we denote by $\tau = \tau_{\mathbb{T},\theta}$ its inflation to $G_{x,0}$.

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2. REVIEW ON ENHANCED L -PARAMETERS

Let \mathbf{G} be a connected reductive algebraic group defined over F , and $G := \mathbf{G}(F)$ the group of F -rational points of \mathbf{G} . The main purpose of this section is to recall in §10.1 a list of properties that are expected to be satisfied by the local Langlands correspondence for G . In later sections, we will use these properties to build an explicit local Langlands correspondence for the exceptional group G_2 .

2.1. Definitions and Examples. Let G^\vee be the complex reductive group with root datum dual to that of G . We suppose that the group \mathbf{G} is F -split.

A Langlands parameter (or L -parameter) for G is a continuous morphism $\varphi: W_F \times \mathrm{SL}_2(\mathbb{C}) \rightarrow G^\vee$ such that $\varphi(w)$ is semisimple for each $w \in W_F$, and that the restriction $\varphi|_{\mathrm{SL}_2(\mathbb{C})}$ of φ to $\mathrm{SL}_2(\mathbb{C})$ is a morphism of complex algebraic groups. Since G is F -split, the Weil group W_F acts trivially on G^\vee . Hence the L -group of G is the direct product $G^\vee \times W_F$ and L -parameters may be taken with values in G^\vee . The group G^\vee acts on the set of L -parameters, and we denote by $\Phi(G)$ the set of G^\vee -conjugacy classes of L -parameters for G .

Definition 2.1.1. [Vog93] The *infinitesimal parameter* of an L -parameter φ for G is a morphism $\lambda_\varphi: W_F \rightarrow G^\vee$ defined by

$$(2.1.1) \quad \lambda_\varphi(w) := \varphi \left(w, \begin{pmatrix} \|w\|^{1/2} & 0 \\ 0 & \|w\|^{-1/2} \end{pmatrix} \right) \quad \text{for any } w \in W_F.$$

Obviously, if φ has trivial restriction to $\mathrm{SL}_2(\mathbb{C})$, then it coincides with its infinitesimal parameter.

Definition 2.1.2. An L -parameter φ (resp. its infinitesimal parameter λ_φ) is

- (1) *bounded* if $\varphi(W_F)$ is bounded;
- (2) *discrete* if $\varphi(W_F)$ (resp. $\lambda_\varphi(W_F)$) is not contained in any proper Levi subgroup of G^\vee ;
- (3) *supercuspidal* if it is discrete and has trivial restriction to $\mathrm{SL}_2(\mathbb{C})$.

Remark 2.1.3. An L -parameter is supercuspidal if and only if its infinitesimal parameter is supercuspidal (since they coincide). On the other hand, if λ_φ is discrete, then φ is also discrete. Indeed, if φ is a non-discrete L -parameter, then $\varphi(W_F) \subset L^\vee$ for some proper Levi subgroup L^\vee of G^\vee . Then, (2.1.1) implies that $\lambda_\varphi(W_F)$ is contained in L^\vee , thus λ_φ is also non-discrete. However, in general there exist discrete L -parameters with non-discrete infinitesimal parameters.

Let $\mathrm{Irr}(G)$ be the set of isomorphism classes of irreducible smooth representations of G . A supercuspidal irreducible representation π of G is compact modulo center, and so a fortiori, it is square integrable modulo center. Let $\mathrm{Irr}_{\mathrm{scusp}}(G)$ denote the subset of $\mathrm{Irr}(G)$ formed by the isomorphism classes of supercuspidal irreducible representations of G . The Langlands correspondence for G is expected to partition $\mathrm{Irr}(G)$ into finite sets, called L -packets, indexed by $\Phi(G)$. If $\pi \in \mathrm{Irr}(G)$, and φ_π denotes the L -parameter of π , we denote by $\Pi_\varphi(G)$ the L -packet attached to φ :

$$(2.1.2) \quad \Pi_\varphi(G) := \{\pi \in \mathrm{Irr}(G) : \varphi_\pi = \varphi\}.$$

In order to both characterize the images of supercuspidal representations under the LLC and parametrize the elements of every L -packet, we need to *enhance the L -parameter* by a representation of a certain finite group (as explained in Definition 2.1.10). In order to do this, we start by recalling in §2.1.1 the notion of *cuspidal unipotent pair* introduced by Lusztig in [Lus84b, Definition 2.4].

2.1.1. *Cuspidal unipotent pairs.* Let \mathcal{G} be a complex Lie group, u a unipotent element in \mathcal{G} and ρ an irreducible representation of $A_{\mathcal{G}}(u)$, which is the group of component of the centralizer of u in \mathcal{G} . The pair (u, ρ) is called *cuspidal* if it determines a \mathcal{G} -equivariant cuspidal local system on the \mathcal{G} -conjugacy class of u in the sense of Lusztig [Lus84b]. We denote by $\mathfrak{B}(U_e(\mathcal{G}))$ the set of \mathcal{G} -conjugacy classes of cuspidal unipotent pairs. The pairs (u, ρ) such that ρ is cuspidal are called *cuspidal unipotent pairs*. If (u, ρ) is cuspidal, then u is a distinguished unipotent element in \mathcal{G} (that is, u does not meet the unipotent variety of any proper Levi subgroup of \mathcal{G}), see [Lus84b, Proposition 2.8]. However, in general, there exist distinguished unipotent elements whose conjugacy classes do not support cuspidal local systems.

Example 2.1.4. The case $\mathcal{G} = T^{\vee}$: The pair (T^{\vee}, triv) is the unique cuspidal unipotent pair in T^{\vee} .

For $\text{GL}_n(\mathbb{C})$ and $\text{SL}_n(\mathbb{C})$, the unipotent conjugacy classes are completely described by the sizes of the Jordan blocks of the elements in these classes. Therefore, the unipotent conjugacy classes correspond to partitions of n . We denote by \mathcal{C}_{ν} (or just ν for abbreviation) the unipotent class corresponding to the partition ν : it consists of unipotent matrices with Jordan blocks of sizes equal to the parts of the partition ν . For example, the trivial class is parametrized by the partition $(1^n) := (1, 1, \dots, 1)$, and the regular unipotent class corresponds to the partition (n) . Note that the centralizer of a unipotent element of $\text{GL}_n(\mathbb{C})$ is always connected.

Example 2.1.5. The pair $((1), \text{triv})$ is a cuspidal unipotent pair in $\text{GL}_1(\mathbb{C})$, and there is no cuspidal unipotent pair in $\text{GL}_n(\mathbb{C})$ for $n \geq 2$. Any Levi subgroup of $\text{GL}_n(\mathbb{C})$ is isomorphic to a product of the form $\text{GL}_{n_1}(\mathbb{C}) \times \text{GL}_{n_2}(\mathbb{C}) \times \dots \times \text{GL}_{n_r}(\mathbb{C})$ with $n_1 + n_2 + \dots + n_r = n$, thus admits a cuspidal pair only if $n_1 = n_2 = \dots = n_r = 1$, i.e. is a maximal torus \mathcal{T} . The pair $((1^n), \text{triv})$ is a cuspidal unipotent pair in \mathcal{T} .

Example 2.1.6. The cuspidal unipotent pairs in $\text{SL}_n(\mathbb{C})$ are the pairs $(\mathcal{C}_{(n)}, \rho)$, where ρ is an order- n character of μ_n (see [Lus84b, (10.3.2)]).

Example 2.1.7. A Levi subgroup of $\text{SL}_n(\mathbb{C})$ is isomorphic to a product of the form $S(\text{GL}_{n_1}(\mathbb{C}) \times \text{GL}_{n_2}(\mathbb{C}) \times \dots \times \text{GL}_{n_r}(\mathbb{C}))$, i.e. the subgroup of $\text{GL}_{n_1}(\mathbb{C}) \times \text{GL}_{n_2}(\mathbb{C}) \times \dots \times \text{GL}_{n_r}(\mathbb{C})$ consisting of elements with determinant equal to 1. We will denote it as $\mathcal{M}_{n_1, \dots, n_r}$.

The case where $n = 3$ will be of special interest to us, as we have SL_3 as a pseudo-Levi subgroup of $G_{2, \mathbb{C}}$. Denote by $\rho[\zeta_3]$ and $\rho[\zeta_3^2]$ the two order-3 characters of μ_3 . Thus the cuspidal unipotent pairs in $\mathcal{M}_{(3)} = \text{SL}_3(\mathbb{C})$ are $(\mathcal{C}_{(3)}, \rho[\zeta_3])$ and $(\mathcal{C}_{(3)}, \rho[\zeta_3^2])$. The group $\mathcal{M}_{2,1}$ has no cuspidal unipotent pair (see the proof of [Lus84b, (10.3.2)]), and the unique cuspidal unipotent pair in the maximal torus $\mathcal{M}_{(1^3)}$ of \mathcal{G} is $((1^3), \text{triv})$.

2.1.2. *L-packets.* We shall use the following notations:

$$(2.1.3) \quad Z_{G^\vee}(\varphi) := Z_{G^\vee}(\varphi(W'_F)) \quad \text{and} \quad \mathfrak{G}_\varphi := Z_{G^\vee}(\varphi(W_F)).$$

We also consider the following component groups

$$(2.1.4) \quad A_\varphi := Z_{G^\vee}(\varphi)/Z_{G^\vee}(\varphi)^\circ \quad \text{and} \quad S_\varphi := Z_{G^\vee}(\varphi)/Z_{G^\vee} \cdot Z_{G^\vee}(\varphi)^\circ.$$

We recall that $A_{\mathfrak{G}_\varphi}(u_\varphi)$ denotes the component group of $Z_{\mathfrak{G}_\varphi}(u_\varphi)$. By [Mou17, § 3.1],

$$(2.1.5) \quad A_\varphi \simeq A_{\mathfrak{G}_\varphi}(u_\varphi), \text{ where } u_\varphi := \varphi\left(1, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\right).$$

Remark 2.1.8. In the case where $\mathbf{G} = \mathbf{G}_2$, we have $Z_{G^\vee} = \{1\}$. Thus $S_\varphi = Z_{G^\vee}(\varphi)/Z_{G^\vee}(\varphi)^\circ = A_\varphi$.

Definition 2.1.9. An *enhancement* of φ is an irreducible representation ρ of S_φ . The pairs (φ, ρ) are called *enhanced L-parameters* for G .

The group G^\vee acts on the set of enhanced L -parameters in the following way

$$(2.1.6) \quad g \cdot (\varphi, \rho) = (g\varphi g^{-1}, g \cdot \rho).$$

We denote by $\Phi_e(G)$ the set of G^\vee -conjugacy classes of enhanced L -parameters.

Definition 2.1.10. An enhanced L -parameter $(\varphi, \rho) \in \Phi_e(G)$ is called *cuspidal* if φ is discrete and (u_φ, ρ) is a cuspidal unipotent pair in \mathfrak{G}_φ .

We denote by $\Phi_{e,\text{cusp}}(G)$ the subset of $\Phi_e(G)$ consisting of G^\vee -conjugacy classes of cuspidal enhanced L -parameters.

Example 2.1.11. The proper Levi subgroups of $\text{SO}_4(\mathbb{C})$ are isomorphic to one of the following groups: $\text{GL}_2(\mathbb{C})$, $\text{GL}_1(\mathbb{C}) \times \text{SO}_2(\mathbb{C})$, $\text{GL}_1(\mathbb{C}) \times \text{GL}_1(\mathbb{C})$.

The group $\text{SO}_4(\mathbb{C})$ has a unique cuspidal unipotent pair: $((3, 1), \rho)$ (see [Lus84b, Corollary 13.4]). The group $\text{GL}_1(\mathbb{C}) \times \text{SO}_2(\mathbb{C})$ has no cuspidal unipotent pair, and $((1^2), \text{triv})$ is the unique cuspidal unipotent pair in $\text{GL}_1(\mathbb{C}) \times \text{GL}_1(\mathbb{C})$.

The construction of enhanced L -parameters (φ, ρ) for G is based on *the generalized Springer correspondence* for the group \mathfrak{G}_φ , which we now recall in §2.2.1.

2.2. Complex groups. In this subsection, we recall several results involving complex reductive groups. Let \mathfrak{G} be a connected reductive group over \mathbb{C} , and let $U(\mathfrak{G})$ denote the unipotent variety of \mathfrak{G} .

2.2.1. *Generalized Springer Correspondence.* Let $D_{\mathfrak{G}}^b(U(\mathfrak{G}))$ be the constructible \mathfrak{G} -equivariant derived category on $U(\mathfrak{G})$, and $\text{Perv}_{\mathfrak{G}}(U(\mathfrak{G}))$ its subcategory of \mathfrak{G} -equivariant perverse sheaves. We denote by $U_e(\mathfrak{G})$ the set of \mathfrak{G} -conjugacy classes of pairs (u, ρ) , with $u \in \mathfrak{G}$ unipotent and $\rho \in \text{Irr}(A_{\mathfrak{G}}(u))$, where $A_{\mathfrak{G}}(u) := Z_{\mathfrak{G}}(u)/Z_{\mathfrak{G}}(u)^\circ$. The elements of $U_e(\mathfrak{G})$ are called *enhanced unipotent classes*.

Let $\text{IC}(\mathcal{C}, \mathcal{E})$ be the Deligne-Goresky-MacPherson intersection cohomology complex (see [BBD82], [GM83] or [Lus84b, (0.1)]) of the closure of \mathcal{C} with coefficients in \mathcal{E} . The simple objects in $\text{Perv}_{\mathfrak{G}}(U(\mathfrak{G}))$ are the $\text{IC}(\mathcal{C}, \mathcal{E})$, where \mathcal{C} is a unipotent class in \mathfrak{G} and \mathcal{E} is an irreducible \mathfrak{G} -equivariant $\overline{\mathbb{Q}}_\ell$ -local system on \mathcal{C} . We recall that

there is a canonical bijection $\rho \mapsto \mathcal{E}_\rho$ between $\text{Irr}(A_{\mathfrak{G}}(u))$, where $u \in \mathcal{C}$, and the set of isomorphism classes of irreducible \mathfrak{G} -equivariant local systems on \mathcal{C} .

Let $\mathcal{P} = \mathcal{L}\mathcal{U}$ be a parabolic subgroup of \mathfrak{G} , with Levi factor \mathcal{L} and unipotent radical \mathcal{U} . By a \mathcal{P} -resolution of an algebraic variety X , we mean a variety Y endowed with a free \mathcal{P} -action and a smooth \mathcal{P} -equivariant morphism $Y \rightarrow X$. From [BL94, §3.7], we recall the integration functor

$$(2.2.1) \quad \gamma_{\mathcal{P}}^{\mathfrak{G}}: D_{\mathcal{P}}^b(\mathbf{U}(\mathfrak{G})) \rightarrow D_{\mathfrak{G}}^b(\mathbf{U}(\mathfrak{G})),$$

given by: for any object A of $D_{\mathcal{P}}^b(\mathbf{U}(\mathfrak{G}))$ and Y a \mathfrak{G} -resolution of $\mathbf{U}(\mathfrak{G})$,

$$(2.2.2) \quad (\gamma_{\mathcal{P}}^{\mathfrak{G}}A)(Y) := (q_Y)_!A(Y)[2 \dim \mathfrak{G}/\mathcal{P}],$$

where $q_Y: \mathcal{P} \backslash Y \rightarrow \mathfrak{G} \backslash Y$ is the quotient functor and $A(Y)$ is defined by regarding Y as a \mathcal{P} -resolution of $\mathbf{U}(\mathfrak{G})$. Let

$$(2.2.3) \quad m: \mathbf{U}(\mathcal{P}) \hookrightarrow \mathbf{U}(\mathfrak{G}) \quad \text{and} \quad p: \mathbf{U}(\mathcal{P}) \rightarrow \mathbf{U}(\mathcal{L})$$

denote the inclusion and projection, respectively. Then the parabolic induction functor is the functor

$$(2.2.4) \quad \mathbf{I}_{\mathcal{L} \subset \mathcal{P}}^{\mathfrak{G}} := \gamma_{\mathcal{P}}^{\mathfrak{G}} \circ m_! \circ p^* : \text{Perv}_{\mathcal{L}}(\mathbf{U}(\mathcal{L})) \rightarrow \text{Perv}_{\mathfrak{G}}(\mathbf{U}(\mathfrak{G})).$$

If $\mathcal{F}_{\mathcal{L}}$ is a simple object in $\text{Perv}_{\mathcal{L}}(\mathbf{U}(\mathcal{L}))$, then $\mathbf{i}_{\mathcal{L} \subset \mathcal{P}}^{\mathfrak{G}}(\mathcal{F}_{\mathcal{L}})$ is semisimple. A simple object \mathcal{F} in $\text{Perv}_{\mathfrak{G}}(\mathbf{U}(\mathfrak{G}))$ is called *cuspidal* if for any simple object $\mathcal{F}_{\mathcal{L}}$ in $\text{Perv}_{\mathcal{L}}(\mathbf{U}(\mathcal{L}))$, \mathcal{F} does not occur in $\mathbf{i}_{\mathcal{L} \subset \mathcal{P}}^{\mathfrak{G}}(\mathcal{F}_{\mathcal{L}})$ (equivalently, if $\mathbf{r}_{\mathcal{L} \subset \mathcal{P}}^{\mathfrak{G}}(\mathcal{F}) = 0$) for any proper parabolic subgroup \mathcal{P} of \mathfrak{G} with Levi factor \mathcal{L} . The functor $\mathbf{i}_{\mathcal{L} \subset \mathcal{P}}^{\mathfrak{G}}$ is left adjoint to $\mathbf{r}_{\mathcal{L} \subset \mathcal{P}}^{\mathfrak{G}} := p_! \circ m^*$.

Let $\mathcal{F}_\rho := \text{IC}(\mathcal{C}, \mathcal{E}_\rho)$, where $(\mathcal{C}, \rho) \in \mathbf{U}_e(\mathfrak{G})$. Then \mathcal{F}_ρ occurs as a summand of $\mathbf{i}_{\mathcal{L} \subset \mathcal{P}}^{\mathfrak{G}}(\text{IC}(\mathcal{C}_{\text{cusp}}, \mathcal{E}_{\text{cusp}}))$, for some quadruple $(\mathcal{P}, \mathcal{L}, \mathcal{C}_{\text{cusp}}, \mathcal{E}_{\text{cusp}})$, where \mathcal{P} is a parabolic subgroup of \mathfrak{G} with Levi subgroup \mathcal{L} and $(\mathcal{C}_{\text{cusp}}, \mathcal{E}_{\text{cusp}})$ is a cuspidal enhanced unipotent class in \mathcal{L} (see [Lus84b, § 6.2]). Moreover, the triple $(\mathcal{P}, \mathcal{L}, \mathcal{C}_{\text{cusp}}, \mathcal{E}_{\text{cusp}})$ is unique up to \mathfrak{G} -conjugation (see [Lus84b, Proposition 6.3]). We denote by $\epsilon := \rho_{\mathcal{E}_{\text{cusp}}}$ the (equivalence class of) irreducible representation of $A_{\mathfrak{G}}(u)$ which corresponds to $\mathcal{E}_{\text{cusp}}$, and by $\mathfrak{t} := (\mathcal{L}, (\mathcal{C}_{\text{cusp}}, \epsilon))_{\mathfrak{G}}$ the \mathfrak{G} -conjugacy class of $(\mathcal{L}, (\mathcal{C}_{\text{cusp}}, \epsilon))$. We call \mathfrak{t} the *cuspidal support* of (\mathcal{C}, ρ) and denote by $\mathfrak{B}(\mathbf{U}_e(\mathfrak{G}))$ the set of cuspidal supports for the group \mathfrak{G} . Then the *cuspidal support map* for $\mathbf{U}_e(\mathfrak{G})$ is defined to be the map

$$(2.2.5) \quad \text{Sc}_{\mathfrak{G}}: \mathbf{U}_e(\mathfrak{G}) \rightarrow \mathfrak{B}(\mathbf{U}_e(\mathfrak{G})),$$

which sends the \mathfrak{G} -conjugacy class of (\mathcal{C}, ρ) to its cuspidal support $\mathfrak{t} = (\mathcal{L}, (\mathcal{C}_{\text{cusp}}, \epsilon))_{\mathfrak{G}}$. For simplicity, we often refer to the unipotent class $\mathcal{C}_{\text{cusp}}$ in \mathcal{L} by a unipotent element v in it.

By [Lus84b, Theorem 9.2], the fiber of (2.2.5) is in bijection with $\text{Irr}(W_{\mathfrak{t}})$, where $W_{\mathfrak{t}} := N_{\mathfrak{G}}(\mathcal{L})/\mathcal{L}$ is a finite Weyl group, i.e.

$$(2.2.6) \quad \text{Sc}_{\mathfrak{G}}^{-1}(\mathfrak{t}) \simeq \text{Irr}(W_{\mathfrak{t}}) \quad \text{for any cuspidal support } \mathfrak{t} = (\mathcal{L}, (\mathcal{C}_{\text{cusp}}, \epsilon))_{\mathfrak{G}}.$$

2.3. Cuspidal support map of enhanced L -parameters. Let (φ, ρ) be an enhanced L -parameter for G . Recall that $u_\varphi := \varphi(1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})$. Then u_φ is a unipotent element of the (possibly disconnected) complex reductive group \mathfrak{G}_φ defined in (2.1.3), and $\rho \in \text{Irr}(A_{\mathfrak{G}_\varphi}(u_\varphi))$ by (2.1.5). Let $\mathfrak{t}_\varphi := (\mathcal{L}^\varphi, (v^\varphi, \epsilon^\varphi))$ denote the cuspidal support of (u_φ, ρ) , i.e.

$$(2.3.1) \quad (\mathcal{L}^\varphi, (v^\varphi, \epsilon^\varphi)) := \text{Sc}_{\mathfrak{G}_\varphi}(u_\varphi, \rho).$$

In particular, $(v^\varphi, \epsilon^\varphi)$ is a cuspidal unipotent pair in \mathcal{L}^φ .

Upon conjugating φ with a suitable element of $Z_{\mathfrak{G}_\varphi}(u_\varphi)$, we may assume that the identity component of \mathcal{L}^φ contains $\varphi\left(\left(1, \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}\right)\right)$ for all $z \in \mathbb{C}^\times$. Recall that by the Jacobson–Morozov theorem (see for example [Car93, § 5.3]), any unipotent element v of \mathcal{L}^φ can be extended to a homomorphism of algebraic groups

$$(2.3.2) \quad j_v: \text{SL}_2(\mathbb{C}) \rightarrow \mathcal{L}^\varphi \text{ satisfying } j_v\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = v.$$

Moreover, by [Kos59, Theorem 3.6], this extension is unique up to conjugation in $Z_{\mathcal{L}^\varphi}(v)^\circ$. We shall call a homomorphism j_v satisfying these conditions to be *adapted to φ* .

By [AMS18, Lemma 7.6], up to G^\vee -conjugacy, there exists a unique homomorphism $j_v: \text{SL}_2(\mathbb{C}) \rightarrow \mathcal{L}^\varphi$ which is adapted to φ , and moreover, the cocharacter

$$(2.3.3) \quad \chi_{\varphi, v}: z \mapsto \varphi\left(1, \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}\right) \cdot j_v\left(\begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}\right)$$

has image in $Z_{\mathcal{L}^\varphi}^\circ$. We define an L -parameter $\varphi_v: W_F \times \text{SL}_2(\mathbb{C}) \rightarrow Z_{G^\vee}(Z_{\mathcal{L}^\varphi}^\circ)$ by

$$(2.3.4) \quad \varphi_v(w, x) := \varphi(w, 1) \cdot \chi_{\varphi, v}(\|w\|^{1/2}) \cdot j_v(x) \quad \text{for any } w \in W_F \text{ and any } x \in \text{SL}_2(\mathbb{C}).$$

Remark 2.3.1. Let $w \in W_F$ and $x_w := \begin{pmatrix} \|w\|^{1/2} & 0 \\ 0 & \|w\|^{-1/2} \end{pmatrix}$. By (2.1.1), we have

$$(2.3.5) \quad \begin{aligned} \lambda_{\varphi_v}(w) &= \varphi_v(w, x_w) = \varphi(w, 1) \cdot \chi_{\varphi, v}(\|w\|^{1/2}) \cdot j_v(x_w) \\ &= \varphi(w, 1) \cdot \varphi(1, x_w) \cdot j_v(x_w^{-1}) \cdot j_v(x_w) = \varphi(w, x_w) = \lambda_\varphi(w). \end{aligned}$$

Definition 2.3.2. [AMS18, Definition 7.7] The *cuspidal support* of (φ, ρ) is

$$(2.3.6) \quad \text{Sc}(\varphi, \rho) := (Z_{G^\vee}(Z_{\mathcal{L}^\varphi}^\circ), (\varphi_{v^\varphi}, \epsilon^\varphi)).$$

2.4. Bernstein series of L -enhanced paramters. Let L^\vee be the Langlands dual group of L and $\iota_{L^\vee}: L^\vee \hookrightarrow G^\vee$ the canonical embedding. Define

$$(2.4.1) \quad \mathfrak{X}_{\text{nr}}(L^\vee) := \{\zeta: W_F/I_F \rightarrow Z_{L^\vee}^\circ\}.$$

There is a canonical bijection between $\mathfrak{X}_{\text{nr}}(L^\vee)$ and $\mathfrak{X}_{\text{nr}}(L)$ (see [Hai14, §3.3.1]). The group $\mathfrak{X}_{\text{nr}}(L^\vee)$ acts on the set of cuspidal enhanced L -parameters for L in the following way: Given $(\varphi, \rho) \in \Phi_e(L)$ and $\xi \in \mathfrak{X}_{\text{nr}}(L^\vee)$, we define $(\xi\varphi, \varrho) \in \Phi_e(L)$ by $\xi\varphi := \varphi$ on $I_F \times \text{SL}_2(\mathbb{C})$ and $(\xi\varphi)(\text{Fr}_F) := \tilde{\xi}\varphi(\text{Fr}_F)$. Here $\tilde{\xi} \in Z_{L^\vee \times I_F}^\circ$ represents ζ . We denote by $\mathfrak{X}_{\text{nr}}(L^\vee).(\varphi_\sigma, \rho_\sigma)$ the orbit of $(\varphi_\sigma, \rho_\sigma) \in \Phi_e(L)$.

Definition 2.4.1. Let $\mathfrak{s}^\vee := [L^\vee, (\varphi_\sigma, \rho_\sigma)]_{G^\vee}$ be the G^\vee -conjugacy class of

$$(L^\vee, \mathfrak{X}_{\text{nr}}(L^\vee) \cdot (\varphi_\sigma, \rho_\sigma)).$$

Let $\mathfrak{B}(G^\vee)$ be the set of such \mathfrak{s}^\vee .

We define

(2.4.2)

$$N_{G^\vee}(\mathfrak{s}_{L^\vee}^\vee) := \{n \in N_{G^\vee}(L^\vee) : {}^n(\varphi_c, \varrho_c) \simeq (\varphi_c, \varrho_c) \otimes \chi^\vee \text{ for some } \chi^\vee \in \mathfrak{X}_{\text{nr}}(L^\vee)\},$$

where $\mathfrak{s}_{L^\vee}^\vee = [L^\vee, (\varphi_\sigma, \rho_\sigma)]_{G^\vee} \in \mathfrak{B}(L^\vee)$ and denote by $W_{G^\vee}^{\mathfrak{s}^\vee}$ the extended finite Weyl group $W(L^\vee) := N_{G^\vee}(\mathfrak{s}_{L^\vee}^\vee)/L^\vee$.

By [AMS18, (115)], the set $\Phi_e(G)$ is partitioned into *series à la Bernstein* as

$$(2.4.3) \quad \Phi_e(G) = \prod_{\mathfrak{s}^\vee \in \mathfrak{B}(G^\vee)} \Phi(G)^{\mathfrak{s}^\vee},$$

where for each \mathfrak{s}^\vee , the subset $\Phi_e(G)^{\mathfrak{s}^\vee}$ is defined to be the fiber of \mathfrak{s}^\vee under the supercuspidal map Sc defined in (2.3.6).

2.5. Explicit Construction of enhanced L -parameters. We give a construction of enhanced L -parameters for the irreducible constituents of any parabolically induced representations from any supercuspidal pair $(L, \sigma)_G$ for which the “crude local Langlands correspondence” is established. We emphasize that it does not require knowledge of the internal structure of the corresponding L -packets for L , but only the definition of φ_σ . Our construction extends—and is inspired by—[KL87], [Ree02, §4.2], [ABPS16a, §4.1] and [ABPS17b, §5], which all treat principal series cases (i.e. when \mathbf{L} is a torus).

Let L be a Levi subgroup of G and let σ be an irreducible supercuspidal representation of L . Suppose that an L -parameter $\varphi_\sigma : W_F \times \text{SL}_2(\mathbb{C}) \rightarrow L^\vee$ for L has been constructed. Then we define

$$(2.5.1) \quad t := \varphi_\sigma(\text{Fr}_F, 1) \text{ and } H_\sigma^\vee := Z_{G^\vee}(\varphi_\sigma|_{I_F}).$$

Let u be a unipotent element of H_σ^\vee such that $tut^{-1} = u^q$. Similar to (2.3.2), the unipotent element u can be extended to a morphism

$$(2.5.2) \quad j_u : \text{SL}_2(\mathbb{C}) \rightarrow H_\sigma^\vee \text{ satisfying } j_u \left(\begin{smallmatrix} 1 & \\ 0 & 1 \end{smallmatrix} \right) = u.$$

Definition 2.5.1. We attach to the pair (σ, u) an L -parameter $\varphi_{\sigma, u} : W_F \times \text{SL}_2(\mathbb{C}) \rightarrow G^\vee$ defined by

$$(2.5.3) \quad \begin{aligned} \varphi_{\sigma, u}(w, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) &:= \varphi_\sigma(w) \cdot j_u \left(\begin{smallmatrix} \|w\|^{1/2} & 0 \\ 0 & \|w\|^{-1/2} \end{smallmatrix} \right)^{-1}, \text{ for any } w \in W_F. \\ \varphi_{\sigma, u}(1, \begin{pmatrix} 1 & \\ 0 & 1 \end{pmatrix}) &:= u. \end{aligned}$$

We define the group

$$(2.5.4) \quad \mathcal{G}_{\sigma, u} := \mathcal{G}_{\varphi_{\sigma, u}} = Z_{G^\vee}(\varphi_{\sigma, u}(W_F)),$$

and we observe that $H_\sigma^\vee = \mathcal{G}_{\varphi_{\sigma, u}|_{I_F}}$.

2.5.1. Loop groups.

Proposition 2.5.2. [Ree10, Proposition 3.8] *Suppose \mathcal{G} is a connected reductive group over \mathbb{C} and σ is a semisimple automorphism of \mathcal{G} . Then $(\mathcal{G}^\sigma)^\circ$ is a reductive quotient of the twisted loop group $L^\theta \mathcal{G}$ where θ is a pinned automorphism of \mathcal{G} in the same outer class as σ . Moreover $\mathcal{G}^\sigma = (\mathcal{G}^\sigma)^\circ$ is connected if \mathcal{G} is simply connected.*

Proof. By a classical result of Steinberg's [Ste68, p.51], σ stabilizes a Borel subgroup and a maximal torus $\mathcal{T} \subset \mathcal{B} \subset \mathcal{G}$. Thus we may write by [Ree10, Lemma 3.2] $\sigma = \theta \circ c_t$, where θ preserves some pinning for $(\mathcal{B}, \mathcal{T})$, $t \in \mathcal{T}^\theta$ and c_t is the conjugation by t . Write $t = s \cdot t_{\text{uni}}$, where $s \in X_*(\mathcal{T}^\theta) \otimes \mathbb{R}_+$ and $t_{\text{uni}} \in X_*(\mathcal{T}^\theta) \otimes S^1$. By taking $\sigma_{\text{uni}} := \theta \circ c_{t_{\text{uni}}}$, we have $(\mathcal{G}^\sigma)^\circ = ((\mathcal{G}^{\sigma_{\text{uni}}})^\circ)^{c_s}$.

Consider $\mathbb{R}/\mathbb{Z} \cong S^1$ with which we take a lift $\tilde{t}_{\text{uni}} \in X_*(\mathcal{T}^\theta) \otimes \mathbb{R}$ of t_{uni} . Then \tilde{t}_{uni} corresponds to a point on some facet \mathcal{F} on the apartment in the building of $L^\theta \mathcal{G}$ associated to the maximal untwisted loop (i.e. $\mathbb{C}((\varpi))$ -split) torus $L^\theta((\mathcal{T}^\theta)^\circ) = L((\mathcal{T}^\theta)^\circ)$. We have that $(\mathcal{G}^{\sigma_{\text{uni}}})^\circ$ is the reductive quotient at this facet \mathcal{F} . Consider $\mathbb{R} \xrightarrow[\sim]{\text{exp}} \mathbb{R}_+$, and let $\tilde{s} \in X_*(\mathcal{T}^\theta) \otimes \mathbb{R}$ be the pullback of s . Let \mathcal{F}' be the facet containing $\mathcal{F} + \tilde{\epsilon}\tilde{s}$ for all $0 < \epsilon \ll 1$. Since $(\mathcal{G}^\sigma)^\circ = ((\mathcal{G}^{\sigma_{\text{uni}}})^\circ)^{c_s}$ is a Levi subgroup of $(\mathcal{G}^{\sigma_{\text{uni}}})^\circ$, we have that $(\mathcal{G}^\sigma)^\circ$ is the reductive quotient at \mathcal{F}' . This proves the proposition except for the last statement, which is also a result of Steinberg's. \square

3. REVIEW ON REPRESENTATIONS OF p -ADIC GROUPS

Let $\mathfrak{R}(G)$ denote the category of all smooth complex representations of G . This is an abelian category admitting arbitrary coproducts.

3.1. Supercuspidal support. Let $\pi \in \text{Irr}(G)$. There exists a parabolic subgroup $P = LU$ of G and a supercuspidal irreducible representation σ of L such that π embeds in $i_P^G \sigma$. If $P' = L'U'$ is a parabolic subgroup of G and σ' a supercuspidal irreducible representation of L' , then π is isomorphic to a subquotient of $i_{P'}^G \sigma'$ if and only if there exists an element of G conjugating (L, σ) and (L', σ') . The G -conjugacy class $(L, \sigma)_G$ of (L, σ) is called the *supercuspidal support* of π . We denote by Sc the map defined by $\text{Sc}(\pi) := (L, \sigma)_G$.

Two supercuspidal pairs (L, σ_1) and (L, σ_2) are G -conjugate if and only if σ_1 and σ_2 are in the same orbit under $W_G(L) := N_G(L)/L$.

3.2. Langlands classification. Let \mathbf{B} be a Borel subgroup of \mathbf{G} defined over F , and let $\mathbf{T} \subset \mathbf{B}$ be a maximal F -torus in \mathbf{G} . A parabolic subgroup P of G , with Levi subgroup L is said to be *standard* if $P \supset \mathbf{B}(F)$ and $L \supset \mathbf{T}(F)$. Let $\mathfrak{a}_L^* := \mathbb{R} \otimes X^*(L)$. We denote by $\nu \mapsto \chi_\nu$ the isomorphism from \mathfrak{a}_L^* to the group of positive real valued unramified characters of L as defined in [SZ18, (2)].

Definition 3.2.1. A *standard triple* (P, π, ν) for G consists of:

- P a standard parabolic subgroup of G ;
- π an irreducible tempered representation of the standard Levi subgroup L of P ;

- $\nu \in \mathfrak{a}_L^*$ a real parameter which is *regular* and *positive*, i.e. properly contained in the positive chamber determined by P (for more details, see for instance [SZ18, §1.3]).

Let (P, π, ν) be a standard triple for G . Let $i_P^G: \mathfrak{R}(L) \rightarrow \mathfrak{R}(G)$ be the normalized parabolic induction functor. Then $i_P^G(\pi \otimes \chi_\nu)$ has a unique irreducible quotient

$$(3.2.1) \quad J(P, \pi, \nu) := J(i_P^G(\pi \otimes \chi_\nu)),$$

called the *Langlands quotient* of $i_P^G(\pi \otimes \chi_\nu)$, and the map J defines a bijection between the set of all standard triples for G and $\text{Irr}(G)$ (see [Kon03, Theorem 3.5]).

Let \mathbf{U} denote the unipotent radical of the Borel subgroup \mathbf{B} and $U := \mathbf{U}(F)$. A character $\chi: U \rightarrow \mathbb{C}^\times$ is called *generic* if the stabilizer of χ in $\mathbf{T}(F)$ is exactly $\mathbf{Z}_G(F)$. Such a pair $\mathfrak{w} := (U, \chi)$ is called a *Whittaker datum* for G . There are only finitely many G -conjugacy classes of Whittaker data for G , since the group $\mathbf{G}_{\text{ad}}(F)$ acts transitively on the set of these pairs.

Definition 3.2.2. A smooth irreducible representation π of G is called *\mathfrak{w} -generic* if $\text{Hom}_G(\pi, \text{Ind}_U^G \chi) \neq 0$.

3.3. Bernstein series. Let L be a Levi subgroup of a parabolic subgroup P of G , and let $\mathfrak{X}_{\text{nr}}(L)$ denote the group of unramified characters of L . Let σ be an irreducible supercuspidal smooth representation of L .

Notation 3.3.1. We write:

- $(L, \sigma)_G$ for the G -conjugacy class of the pair (L, σ) ;
- $\mathfrak{s} := [L, \sigma]_G$ for the G -conjugacy class of the pair $(L, \mathfrak{X}_{\text{nr}}(L) \cdot \sigma)$.

We denote by $\mathfrak{B}(G)$ the set of such classes \mathfrak{s} . We set $\mathfrak{s}_L := [L, \sigma]_L$.

Let $\mathfrak{R}^{\mathfrak{s}}(G)$ be the full subcategory of $\mathfrak{R}(G)$ whose objects are the representations (π, V) such that every G -subquotient of π is equivalent to a subquotient of a parabolically induced representation $i_P^G(\sigma')$, where i_P^G is the functor of normalized parabolic induction and $\sigma' \in \mathfrak{X}_{\text{nr}}(M) \cdot \sigma$. The categories $\mathfrak{R}^{\mathfrak{s}}(G)$ are indecomposable and split the full smooth category $\mathfrak{R}(G)$ into a direct product (see [Ber84]):

$$(3.3.1) \quad \mathfrak{R}(G) = \prod_{\mathfrak{s} \in \mathfrak{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G).$$

We denote by $\text{Irr}^{\mathfrak{s}}(G)$ the classes of irreducible objects in $\mathfrak{R}^{\mathfrak{s}}(G)$ –i.e. irreducible representations whose supercuspidal support lies in \mathfrak{s} – and call $\text{Irr}^{\mathfrak{s}}(G)$ the *Bernstein series* attached to \mathfrak{s} . By (3.3.1), Bernstein series give a partition of the set $\text{Irr}(G)$ of isomorphism classes of irreducible smooth irreducible representations of G :

$$(3.3.2) \quad \text{Irr}(G) = \bigsqcup_{\mathfrak{s} \in \mathfrak{B}(G)} \text{Irr}^{\mathfrak{s}}(G).$$

For $\mathfrak{s} = [L, \sigma]_G \in \mathfrak{B}(G)$, we define

$$(3.3.3) \quad N_G(\mathfrak{s}_L) := \{g \in G : {}^g L = L \text{ and } {}^g \sigma \simeq \chi \otimes \sigma, \text{ for some } \chi \in \mathfrak{X}_{\text{nr}}(L)\},$$

where $\mathfrak{s}_L = [L, \sigma]_L \in \mathfrak{B}(L)$ and denote by $W_G^{\mathfrak{s}}$ the extended finite Weyl group $N_G(\mathfrak{s}_L)/L$.

3.4. Formal degrees. Let \mathbf{A} denote the maximal F -split torus of the center of \mathbf{G} , and $A := \mathbf{A}(F)$. Let π be an irreducible square-integrable representation of G on a Hilbert space V . Let $\text{fdeg}(\pi) \in \mathbb{R}_{>0}$ denote the formal degree of π . By definition, we have

$$(3.4.1) \quad \int_{G/A} (\pi(g)v_1, v'_1) \overline{(\pi(g)(v_2, v'_2))} dg = \text{fdeg}(\pi)^{-1} \cdot (v_1, v'_1) \overline{(v_2, v'_2)}$$

holds for all $v_1, v'_1, v_2, v'_2 \in V$, where (\cdot, \cdot) is a $\mathbf{G}(F)$ -invariant inner product on V . We remark that $\text{fdeg}(\pi)$ depends on the choice of the Haar measure dg on G . We normalize the measure dg as in [GG99]. When \mathbf{G} splits over an unramified extension of F , by [FOS21, (6)], the volume of the parahoric subgroup $G_{x,0}$ is then

$$(3.4.2) \quad \text{Vol}(G_{x,0}) = q^{-\dim(\mathbb{G}_{x,0})/2} |\mathbb{G}_{x,0}|_{p'}.$$

3.5. Unipotent representations. Let $x \in \mathcal{B}(G, F)$ and let τ be an irreducible cuspidal unipotent representation of $\mathbb{G}_{x,0}$. Let τ^\vee denote the contragredient of the inflation τ of τ to $G_{x,0}$. Consider the algebra $\mathcal{H}(G, \tau)$ of locally constant locally supported $\text{End}(\tau^\vee)$ -valued functions f satisfying $f(p_1 g p_2) = \tau^\vee(p_1) f(g) \tau^\vee(p_2)$ for any $p_1, p_2 \in G_{x,0}$ and $g \in G$.

We have an anti-involution on $\mathcal{H}(G, \tau)$ given on the Iwahori-Matsumoto basis (T_w) by $\overline{T_w} := T_{w^{-1}}$. It induces a Hermitian form h on $\mathcal{H}(G, \tau)$ defined by $h(f_1, f_2) := (f_1 \overline{f_2})(1)$. Let $\mathcal{H}^2(G, \tau)$ denote the corresponding completion. A finite dimensional simple left $\mathcal{H}(G, \tau)$ -module $\pi^{\mathcal{H}}$ is called square-integrable if it may be realized as an $\mathcal{H}(G, \tau)$ -submodule $M_{\pi^{\mathcal{H}}}$ of $\mathcal{H}^2(G, \tau)$. There is a positive real number $d(\pi^{\mathcal{H}})$, depending only on the isomorphism class of $\pi^{\mathcal{H}}$, such that

$$(3.5.1) \quad h(f_1, f_2) \cdot \overline{h(g_1, g_2)} = d(\pi^{\mathcal{H}}) \cdot h(f_1 \overline{g_1}, f_2 \overline{g_2}) \quad \text{for any } f_1, f_2, g_1, g_2 \in M_{\pi^{\mathcal{H}}}.$$

Proposition 3.5.1. *Let π be a unipotent square-integrable irreducible representation of G such that $\pi^{\mathcal{H}} := \text{Hom}_{G_{x,0}}(\tau, \pi) \neq 0$. We have*

$$(3.5.2) \quad \text{fdeg}(\pi) = \frac{\dim \tau}{\text{Vol}(G_{x,0})} \cdot d(\pi^{\mathcal{H}}).$$

Proof. See [Ree97, Proposition 9.1]. □

Generalizing his work with Kazhdan [KL87], Lusztig proved in [Lus95] that, when \mathbf{G} is simple of adjoint type, the unipotent representations of G are in bijective correspondence with G^\vee -conjugacy classes of triples (t, u, ρ) , where $s \in G^\vee$ is semisimple, u is unipotent such that $tut^{-1} = u^q$, and ρ is the isomorphism class of an irreducible representation of the component group of the mutual centralizer in G^\vee of t and u , such that ρ is trivial on the center of G^\vee . This result was extended to an arbitrary group G in [FOS21].

One can replace u by $n = \ln(u)$, a nilpotent element in the Lie algebra \mathfrak{g}^\vee of G^\vee so that an indexing triple is (t, n, ρ) , where t is a semisimple element in G^\vee , n a nilpotent element in \mathfrak{g}^\vee such that $\text{Ad}(t)n = qn$, and an irreducible representation ρ of the component group $A(t, n) := Z_{G^\vee}(t, n)/Z_{G^\vee}(t, n)^\circ$, where $Z_{G^\vee}(s, n) := Z_{G^\vee}(t) \cap$

$Z_{G^\vee}(n)$ and $Z_{G^\vee}(n)$ is taken with respect to the adjoint action of G^\vee on \mathfrak{g}^\vee . We will use this form of the indexing triples in the examples in later sections.

3.6. Supercuspidal representations. In this section, we briefly recall the general construction of supercuspidal representations of $G = \mathbf{G}(F)$, where \mathbf{G} is an arbitrary tamely ramified reductive group and the residual characteristic p of F does not divide the order of the Weyl group of \mathbf{G} . We also describe different kinds of supercuspidal representations that will occur: regular (resp. non-singular) supercuspidal representations, depth-zero supercuspidal representations and unipotent supercuspidal representations.

3.6.1. Deligne-Lusztig theory. Let $\underline{\mathbb{G}}$ be a connected reductive algebraic group defined over a finite field \mathbb{F}_q . We denote by $\underline{\mathbb{G}}^\vee$ a connected reductive algebraic group defined over \mathbb{F}_q with root datum dual to that of $\underline{\mathbb{G}}$. Recall from classical Deligne-Lusztig theory [DL76, §10] and [Lus84a, (8.4.4)] that the set of equivalence classes of irreducible representations of $\underline{\mathbb{G}}(\mathbb{F}_q)$ decomposes into a disjoint union

$$(3.6.1) \quad \text{Irr}(\underline{\mathbb{G}}(\mathbb{F}_q)) \xrightarrow{\sim} \coprod_{(s)} \mathcal{E}(\underline{\mathbb{G}}(\mathbb{F}_q), s),$$

where (s) is the $\underline{\mathbb{G}}^\vee(\mathbb{F}_q)$ -conjugacy class of a semisimple element s of $\underline{\mathbb{G}}^\vee(\mathbb{F}_q)$, and $\mathcal{E}(\underline{\mathbb{G}}, s)$ is the Lusztig series defined as below in (3.6.3). More precisely, the decomposition is obtained as follows: as proved in [DL76, Corollary 7.7], for any irreducible representation τ of $\underline{\mathbb{G}}(\mathbb{F}_q)$, there exists an \mathbb{F}_q -rational maximal torus $\underline{\mathbb{T}}$ of $\underline{\mathbb{G}}$ and a character θ of $\underline{\mathbb{T}}(\mathbb{F}_q)$ such that τ occurs in the Deligne-Lusztig (virtual) character $R_{\underline{\mathbb{T}}}^{\underline{\mathbb{G}}}(\theta)$, i.e. such that $\langle \tau, R_{\underline{\mathbb{T}}}^{\underline{\mathbb{G}}}(\theta) \rangle_{\underline{\mathbb{G}}(\mathbb{F}_q)} \neq 0$, where the character of τ is also denoted by τ , and $\langle \cdot, \cdot \rangle_{\underline{\mathbb{G}}(\mathbb{F}_q)}$ is the usual scalar product on the space of class functions on $\underline{\mathbb{G}}(\mathbb{F}_q)$:

$$(3.6.2) \quad \langle f_1, f_2 \rangle_{\underline{\mathbb{G}}(\mathbb{F}_q)} = |\underline{\mathbb{G}}(\mathbb{F}_q)|^{-1} \sum_{g \in \underline{\mathbb{G}}(\mathbb{F}_q)} f_1(g) \overline{f_2(g)}.$$

If $\theta = 1$ (i.e. the trivial character of $\underline{\mathbb{T}}(\mathbb{F}_q)$), then the representation τ is called *unipotent*.

If the pairs $(\underline{\mathbb{T}}, \theta)$ and $(\underline{\mathbb{T}}', \theta')$ are not $\underline{\mathbb{G}}(\mathbb{F}_q)$ -conjugate, then $R_{\underline{\mathbb{T}}}^{\underline{\mathbb{G}}}(\theta)$ and $R_{\underline{\mathbb{T}}'}^{\underline{\mathbb{G}}}(\theta')$ are orthogonal to each other with respect to $\langle \cdot, \cdot \rangle_{\underline{\mathbb{G}}(\mathbb{F}_q)}$, but they may have a common constituent as they are virtual characters. This has motivated the introduction of the following weaker notion of conjugacy: the pairs $(\underline{\mathbb{T}}, \theta)$ and $(\underline{\mathbb{T}}', \theta')$ are called *geometrically conjugate* if there exists a $g \in \underline{\mathbb{G}}$, such that $\underline{\mathbb{T}}' = {}^g \underline{\mathbb{T}}$ and such that for any non-negative integer m , we have $\theta' \circ N_{\text{Fr}^m | \text{Fr}} = \theta \circ N_{\text{Fr}^m | \text{Fr}} \circ \text{ad}(g)$, where $\text{Fr}: \underline{\mathbb{G}} \rightarrow \underline{\mathbb{G}}$ is the geometric Frobenius endomorphism associated to the \mathbb{F}_q -structure of $\underline{\mathbb{G}}$ (hence we have $\underline{\mathbb{G}}^{\text{Fr}} = \underline{\mathbb{G}}(\mathbb{F}_q)$), and $N_{\text{Fr}^m | \text{Fr}}: \underline{\mathbb{T}}^{\text{Fr}^m} \rightarrow \underline{\mathbb{T}}^{\text{Fr}}$ is the norm map given by $N_{\text{Fr}^m | \text{Fr}}(t) := t \cdot \text{Fr}(t) \cdot \text{Fr}^2(t) \cdots \text{Fr}^{m-1}(t)$.

The $\underline{\mathbb{G}}(\mathbb{F}_q)$ -conjugacy classes of pairs $(\underline{\mathbb{T}}, \theta)$ as above are in one-to-one correspondence with the $\underline{\mathbb{G}}^\vee(\mathbb{F}_q)$ -conjugacy classes of pairs $(\underline{\mathbb{T}}^\vee, s)$ where s is a semisimple element of $\underline{\mathbb{G}}^\vee(\mathbb{F}_q)$, and $\underline{\mathbb{T}}^\vee$ is an \mathbb{F}_q -rational maximal torus of $\underline{\mathbb{G}}^\vee$ containing s .

Then the *Lusztig series* $\mathcal{E}(\underline{\mathbb{G}}, s)$ is defined as

$$(3.6.3) \quad \{\tau \in \text{Irr}(\underline{\mathbb{G}}) : \tau \text{ occurs in } R_{\underline{\mathbb{T}}}^{\underline{\mathbb{G}}}(\theta) \text{ where } (\underline{\mathbb{T}}, \theta)_{\underline{\mathbb{G}}} \text{ corresponds to } (\underline{\mathbb{T}}^{\vee}, s)_{\underline{\mathbb{G}}^{\vee}}\}.$$

By definition, $\mathcal{E}(\underline{\mathbb{G}}, 1)$ consists only of unipotent representations.

By the various works of Lusztig, there is a bijection

$$(3.6.4) \quad \begin{aligned} \mathcal{E}(\underline{\mathbb{G}}(\mathbb{F}_q), s) &\xrightarrow{\sim} \mathcal{E}(\underline{\mathbb{G}}_s^{\vee}(\mathbb{F}_q), 1) \\ \tau &\mapsto \tau_{\text{unip}}, \end{aligned}$$

where $\underline{\mathbb{G}}_s^{\vee} := Z_{\underline{\mathbb{G}}^{\vee}}(s)$ denotes the centralizer of s in $\underline{\mathbb{G}}^{\vee}$, and hence plays the role of an ‘‘endoscopic’’ group of $\underline{\mathbb{G}}$:

- In the case where $\underline{\mathbb{G}}$ has connected center, the existence of the bijection (3.6.4) was established in [Lus84a, Theorem 4.23].
- For an arbitrary $\underline{\mathbb{G}}$, the group $\underline{\mathbb{G}}_s^{\vee}$ may be disconnected, and one needs to first extend the notion of Deligne-Lusztig character to this case in the following way. Let $\underline{\mathbb{G}}_s^{\vee, \circ}$ be the identity component of $\underline{\mathbb{G}}_s^{\vee}$. For $\underline{\mathbb{T}}^{\vee}$ an \mathbb{F}_q -rational maximal torus of $\underline{\mathbb{G}}_s^{\vee, \circ}$ and θ^{\vee} a character of $\underline{\mathbb{T}}^{\vee}(\mathbb{F}_q)$, we define

$$(3.6.5) \quad R_{\underline{\mathbb{T}}^{\vee}}^{\underline{\mathbb{G}}_s^{\vee}}(\theta^{\vee}) := \text{Ind}_{\underline{\mathbb{G}}_s^{\vee, \circ}(\mathbb{F}_q)}^{\underline{\mathbb{G}}_s^{\vee}(\mathbb{F}_q)}(R_{\underline{\mathbb{T}}^{\vee}}^{\underline{\mathbb{G}}_s^{\vee, \circ}}(\theta^{\vee})).$$

Then $\mathcal{E}(\underline{\mathbb{G}}_s^{\vee}(\mathbb{F}_q), 1)$ is defined as the set of irreducible constituents of $R_{\underline{\mathbb{T}}^{\vee}}^{\underline{\mathbb{G}}_s^{\vee}}(1)$ and the equivalence (3.6.4) was established in [Lus88, §12].

Proposition 3.6.1. *The bijection (3.6.4) satisfies the following properties:*

- (1) *It sends a Deligne-Lusztig character $R_{\underline{\mathbb{T}}}^{\underline{\mathbb{G}}}(\theta)$ in $\underline{\mathbb{G}}$ (up to a sign) to a Deligne-Lusztig character $R_{\underline{\mathbb{T}}^{\vee}}^{\underline{\mathbb{G}}_s^{\vee}}(1)$, where 1 denotes the trivial character of $\underline{\mathbb{T}}^{\vee}$ (see [Lus88, §12]).*
- (2) *It preserves cuspidality in the following sense:*
if $\tau \in \text{Irr}(\underline{\mathbb{G}}(\mathbb{F}_q), s)$ is cuspidal, then
 - if $s \in \underline{\mathbb{G}}^{\vee}(\mathbb{F}_q)$, then the largest \mathbb{F}_q -split torus in the center of $\underline{\mathbb{G}}_s^{\vee}$ coincides with the largest \mathbb{F}_q -split torus in the center of $\underline{\mathbb{G}}^{\vee}$ (see [Lus84a, (8.4.5)]),*
 - the unipotent representation τ_{unip} is cuspidal.*
- (3) *The dimension of τ is given by*

$$(3.6.6) \quad \dim(\tau) = \frac{|\underline{\mathbb{G}}_{x_0}|_{p'}}{|(Z_{\underline{\mathbb{G}}_{x_0}^{\vee}}(s))(\mathbb{F}_q)|_{p'}} \dim(\tau_{\text{unip}}),$$

where $|\cdot|_{p'}$ is the largest prime-to- p factor of $|\cdot|$ (see [DM91, Remark 13.24]).

Let $W_{\underline{\mathbb{G}}}(\underline{\mathbb{T}}) := N_{\underline{\mathbb{G}}}(\underline{\mathbb{T}})/\underline{\mathbb{T}}$ and $\theta \in \text{Irr}(\underline{\mathbb{T}}(\mathbb{F}_q))$. We define

$$(3.6.7) \quad W_{\underline{\mathbb{G}}}(\underline{\mathbb{T}}, \theta)^{\text{Fr}} := \{w \in W_{\underline{\mathbb{G}}}(\underline{\mathbb{T}})^{\text{Fr}} : w\theta = \theta\}.$$

By [DL76, Theorem 6.8], we have

$$(3.6.8) \quad \langle R_{\underline{\mathbb{T}}}^{\underline{\mathbb{G}}}(\theta), R_{\underline{\mathbb{T}}}^{\underline{\mathbb{G}}}(\theta) \rangle_{\underline{\mathbb{G}}(\mathbb{F}_q)} = |W_{\underline{\mathbb{G}}}(\underline{\mathbb{T}}, \theta)^{\text{Fr}}|,$$

where $\langle \cdot, \cdot \rangle_{\mathbb{G}(\mathbb{F}_q)}$ is as in (3.6.2). We recall that θ is said to be *in general position* if $W_{\mathbb{G}}(\mathbb{T}, \theta)^{\text{Fr}} = \{1\}$ (see [DL76, Definition 5.15]).

Lemma 3.6.9. Let $\tau \in \mathcal{E}(\mathbb{G}(\mathbb{F}_q), s)$ be such that $\mathbb{G}_s^\vee := Z_{\mathbb{G}^\vee}(s)$ is a torus. Then $\tau = \pm R_{\mathbb{T}}^{\mathbb{G}}(\theta)$, where θ is in general position.

Proof. When $\mathbb{G}_s^\vee := Z_{\mathbb{G}^\vee}(s)$ is a torus, we have $\tau_{\text{unip}} = 1 = R_{\mathbb{T}^\vee}^{Z_{\mathbb{G}^\vee}(s)}(1)$. By Proposition 3.6.1(1), we obtain $\tau = \pm R_{\mathbb{T}}^{\mathbb{G}}(\theta)$. Therefore

$$(3.6.10) \quad \langle R_{\mathbb{T}}^{\mathbb{G}}(\theta), R_{\mathbb{T}}^{\mathbb{G}}(\theta) \rangle_{\mathbb{G}} = \langle \tau, \tau \rangle_{\mathbb{G}} = 1.$$

Then it follows from (3.6.8) that $|W_{\mathbb{G}}(\mathbb{T}, \theta)^{\text{Fr}}| = 1$, i.e. θ is in general position. \square

3.6.2. Depth-zero supercuspidal representations. Let \mathbf{G} be a connected reductive algebraic group defined over F and \mathbf{S} an F -torus of \mathbf{G} . Then \mathbf{S} possesses an ft-Néron model (a smooth group scheme over \mathfrak{o}_F of finite type), the neutral connected component of it is called the *connected Néron model* of \mathbf{S} and denoted by \mathfrak{S}° (see [BLR90, §10]). Let $S := \mathbf{S}(F)$ and $S_0 := \mathfrak{S}^\circ(\mathfrak{o}_F)$.

We suppose that the torus \mathbf{S} is a *maximally unramified elliptic maximal torus* (as defined in [Kal19, Definition 3.4.2]), i.e. such that $\mathbf{S} = Z_{\mathbf{G}}(\mathbf{T})$, where \mathbf{T} is the maximal unramified subtorus of \mathbf{S} . Since \mathbf{T} is a maximal split torus over F_{nr} , we have the apartment $\mathcal{A}_{\text{red}}(\mathbf{S}, F_{\text{nr}}) \subset \mathcal{B}_{\text{red}}(\mathbf{G}, F_{\text{nr}})$. Since \mathbf{S} is defined over F and elliptic, this apartment is Frobenius stable, and contains a unique Frobenius-fixed point, which we denote by x . Then x is a vertex of $\mathcal{B}_{\text{red}}(\mathbf{G}, F_{\text{nr}})$ by [Kal19, Lemma 3.4.3]. Therefore $G_{x,0}$ is a maximal parahoric subgroup of G . We denote by $\mathbb{G}_{x,0}$ its reductive quotient, which is the group of \mathbb{F}_q -points of a connected reductive algebraic group $\mathbb{G}_{x,0}$ over $\overline{\mathbb{F}_q}$.

The special fiber of the (automatically connected) ft-Néron model of \mathbf{T} embeds canonically as an elliptic (i.e. anisotropic modulo center) maximal torus \mathbb{T} of the reductive group \mathbb{G}_x . More explicitly, $\mathbb{T}(k_{F'}) \subset \mathbb{G}_x(k_{F'})$ is the image in $\mathbf{G}(F')_{x,0+}$ of $\mathbf{S}(F') \cap \mathbf{G}(F')_x$, or equivalently of $\mathbf{T}(F') \cap \mathbf{G}(F')_x$, for every unramified extension F' of F [DeB06, Lemma 2.2.1(3)]. Every elliptic maximal torus of $\mathbb{G}_{x,0}$ arises in this way by [DeB06, Lemma 2.3.1].

In [DL76, Definition 5.15], Deligne and Lusztig defined two regularity conditions for a character θ of $\mathbb{T}(\mathbb{F}_q)$, which we now recall:

- θ is said to be *in general position* if its stabilizer in $(N_{\mathbb{G}}(\mathbb{T})/\mathbb{T})(\mathbb{F}_q)$ is trivial.
- θ is said to be *non-singular* if it not orthogonal to any coroot.

If the centre of \mathbb{G} is connected, θ is non-singular if and only if it is in general position by [DL76, Proposition 5.16]. The character θ is said to be *regular* if its stabilizer in $(N_{\mathbf{G}}(\mathbf{S})/\mathbf{S})(F)$ is trivial (see [Kal19, Definition 3.4.16]). If θ is regular, then it is in general position (see [Kal19, Fact 3.4.18]).

Definition 3.6.2. [Kal19, Definition 3.4.16] Let θ_F be a depth-zero character of S . Then θ_F is said to be *regular* if its restriction to S_0 equals the inflation of a regular character of $\mathbb{T}(\mathbb{F}_q)$.

Let π be an irreducible depth-zero supercuspidal representation of G . Then there exists a vertex $x \in \mathcal{B}_{\text{red}}(G, F)$ and an irreducible cuspidal representation τ of $\mathbb{G}_x(\mathbb{F}_q)$, such that the restriction of π to $G_{x,0}$ contains the inflation of τ (see [Mor96, §1-2] or [MP96a, Proposition 6.6]). The normalizer $N_G(G_{x,0})$ of $G_{x,0}$ in G is a totally disconnected group that is compact mod center, which, by [BT84, proof of (5.2.8)], coincides with the fixator $G_{[x]}$ of $[x]$ under the action of G on the reduced building of \mathbf{G} . The π is compactly induced from a representation of $N_G(G_{x,0})$:

$$(3.6.11) \quad \pi = \text{c-Ind}_{G_{[x]}}^G(\tau).$$

Definition 3.6.3. [Kal19, Definition 3.4.19] The representation π is said to be *regular* if $\tau = \pm R_{\mathbb{T}}^{\mathbb{G}_x,0}(\theta)$ for some pair (\mathbb{T}, θ) , where \mathbb{T} is an elliptic maximal torus of $\mathbb{G}_{x,0}$ and θ is a regular character of $\mathbb{T}(\mathbb{F}_q)$.

Let π be an irreducible depth-zero supercuspidal representation of G . Let $x \in \mathcal{B}_{\text{red}}(G, F)$ and $\tau \in \text{Irr}_{\text{cusp}}(\mathbb{G}_x(\mathbb{F}_q))$ such that the restriction of π to $G_{x,0}$ contains τ .

Proposition 3.6.4. *If \mathbf{G} is simply connected, then π is regular if and only if $\tau = \pm R_{\mathbb{T}}^{\mathbb{G}_x,0}(\theta)$, where \mathbb{T} is an elliptic maximal torus of $\mathbb{G}_{x,0}$ and θ is a character of $\mathbb{T}(\mathbb{F}_q)$ that is in general position.*

In particular, if $\tau \in \mathcal{E}(\mathbb{G}_{x,0}(\mathbb{F}_q), s)$ and $Z_{\mathbb{G}_{x,0}^\vee}(s)$ is a torus, then π is regular.

Proof. By [Kal19, Lemma 3.4.10], we have an exact sequence

$$(3.6.12) \quad 1 \rightarrow N_{G_{x,0}}(\mathbf{S})/S_0 \rightarrow N_{\mathbf{G}}(\mathbf{S})(F)/S \rightarrow G_x/[G_{x,0} : S] \rightarrow 1,$$

and the natural map

$$(3.6.13) \quad N_{G_{x,0}}(\mathbf{S})/S_0 \longrightarrow N_{\mathbb{G}_x}(\mathbb{T})(\mathbb{F}_q)/\mathbb{T}(\mathbb{F}_q)$$

is bijective. Since \mathbf{G} is simply connected, the parahoric subgroup $G_{x,0}$ coincides with the fixator G_x in G of x (see [BT84, §5.2.9]). Combining (3.6.12) and (3.6.13), we have

$$(3.6.14) \quad N_{G_x}(\mathbf{S})/S_0 \simeq N_{\mathbf{G}}(\mathbf{S})(F)/S \simeq N_{\mathbb{G}_x}(\mathbb{T})(\mathbb{F}_q)/\mathbb{T}(\mathbb{F}_q).$$

Thus by (3.6.14), θ is regular if and only if it is in general position. The last assertion follows from Lemma 3.6.9. \square

Suppose \mathbf{G} is simply connected, then $G_{[x]} = G_{x,0}$ for any $x \in \mathcal{B}(\mathbf{G}, F)$. Let $\pi \in \text{Irr}(G)$ be a depth-zero supercuspidal representation of G . By (3.6.11), there is $\tau \in \text{Irr}(\mathbb{G}_{x,0})$ such that

$$(3.6.15) \quad \pi = \text{c-Ind}_{G_{x,0}}^G \tau =: i_{\mathbb{G}_{x,0}}^G \tau.$$

Proposition 3.6.5. *The formal degree of π is given by*

$$(3.6.16) \quad \text{fdeg}(\pi) = \frac{q^{\dim(\mathbb{G}_{x,0})/2} \dim(\tau_{\text{unip}})}{|(Z_{\mathbb{G}_{x,0}^\vee}(s))^\vee(\mathbb{F}_q)|_{p'}}.$$

Proof. We have (see for instance [Sch21, Lemma 18] or [DR09, § 5.3])

$$(3.6.17) \quad \text{fdeg}(\pi) = \frac{\dim \tau}{\text{Vol}(G_{x,0})}.$$

Combining (3.6.6) and (3.4.2), we have

$$(3.6.18) \quad \text{fdeg}(\pi) = \frac{|G_{x,0}|_{p'} \dim(\tau_{\text{unip}})}{|(Z_{\mathbb{G}_{x,0}^\vee}(s))(\mathbb{F}_q)|_{p'} \cdot q^{-\dim(\mathbb{G}_{x,0})/2} |G_{x,0}|_{p'}} = \frac{q^{\dim(\mathbb{G}_{x,0})/2} \dim(\tau_{\text{unip}})}{|(Z_{\mathbb{G}_{x,0}^\vee}(s))^\vee(\mathbb{F}_q)|_{p'}}.$$

□

3.6.3. Positive-depth supercuspidal representations. In this section, we assume that \mathbf{G} splits over a tamely ramified extension of F , and that p does not divide the order of the Weyl group $W_{\mathbf{G}}$ of \mathbf{G} . Let E/F be a finite extension. By a twisted E -Levi subgroup of \mathbf{G} , we mean an F -subgroup \mathbf{G}' of \mathbf{G} such that $\mathbf{G}' \otimes_F E$ is a Levi subgroup of $\mathbf{G} \otimes_F E$. If E/F is tamely ramified, then \mathbf{G}' is called a tamely ramified twisted Levi subgroup of \mathbf{G} . A tamely ramified twisted Levi sequence in \mathbf{G} is a finite sequence $\vec{\mathbf{G}} = (\mathbf{G}^0, \mathbf{G}^1, \dots, \mathbf{G}^d)$ of twisted E -Levi subgroups of \mathbf{G} , with E/F tamely ramified (see [Yu01, p 586]). We recall that an anisotropic algebraic group over F is a linear algebraic group that does not contain non-trivial F -split tori.

Definition 3.6.6. A *cuspidal \mathbf{G} -datum* is a tuple $\mathcal{D} := (\vec{\mathbf{G}}, y, \vec{r}, \pi^0, \vec{\phi})$ consisting of

- (1) a tamely ramified Levi sequence $\vec{\mathbf{G}} = (\mathbf{G}^0 \subset \mathbf{G}^1 \subset \dots \subset \mathbf{G}^d = \mathbf{G})$ of twisted E -Levi subgroups of \mathbf{G} , such that $Z_{\mathbf{G}^0}/Z_{\mathbf{G}}$ is anisotropic;
- (2) a point y in $\mathcal{B}(\mathbf{G}^0, F) \cap \mathcal{A}(\mathbf{T}, E)$, whose projection to the reduced building of G^0 is a vertex, where \mathbf{T} is a maximal torus of \mathbf{G}^0 (hence of \mathbf{G}^i) that splits over E ;
- (3) a sequence $\vec{r} = (r_0, r_1, \dots, r_d)$ of real numbers such that $0 < r_0 < r_1 < \dots < r_{d-1} \leq r_d$ if $d > 0$, and $0 \leq r_0$ if $d = 0$;
- (4) an irreducible depth-zero supercuspidal representation π^0 of G^0 ;
- (5) a sequence $\vec{\phi} = (\phi_0, \phi_1, \dots, \phi_d)$ of characters, where ϕ_i is a character of G^i which is trivial on G_{y, r_i+}^i and nontrivial on G_{y, r_i}^i for $0 \leq i \leq d-1$, such that
 - ϕ_d is trivial on G_{y, r_d+}^d and nontrivial on G_{y, r_d}^d if $r_{d-1} < r_d$, and $\phi_d = 1$ if $r_{d-1} = r_d$ (with r_{-1} defined to be 0).
 - Moreover, ϕ_i is \mathbf{G}^{i+1} -generic of depth r_i relative to y in the sense of [Yu01, §9] for $0 \leq i \leq d-1$.

All supercuspidal representations arise from cuspidal \mathbf{G} -data if p does not divide the order of the Weyl group of G (see [Fin21]). When $\mathbf{G} = G_2$, this condition says that $p \neq 2, 3$.

Let \mathbf{S} be a maximal torus in \mathbf{G} and be a character of depth zero. As before, let \mathbf{T} denote the maximal unramified subtorus of \mathbf{S} . Let F'/F be an unramified extension splitting \mathbf{T} and $R_{\text{res}}(\mathbf{G}, \mathbf{T})$ denote the set of restrictions to \mathbf{T} of the absolute roots in $R(\mathbf{G}, \mathbf{S})$.

Definition 3.6.7. [Kal21, Definition 3.1.1] A depth-zero character $\theta_F: \mathbf{S}(F) \rightarrow \mathbb{C}^\times$ is called *F-non-singular* if for every $\alpha \in R_{\text{res}}(\mathbf{G}, \mathbf{T})$ the character $\theta_F \circ \mathbf{N}_{F'|F} \circ \alpha^\vee: F' \rightarrow \mathbb{C}^\times$ has non-trivial restriction to $\mathfrak{o}_{F'}$.

Definition 3.6.8. [Kal21, Definition 3.4.1] Let $\theta_F: S \rightarrow \mathbb{C}^\times$ be a character. The pair (\mathbf{S}, θ_F) is said to be *tame F-non-singular elliptic* in \mathbf{G} if

- \mathbf{S} is elliptic and its splitting extension E/F is tame;
- Inside the connected reductive subgroup \mathbf{G}^0 of \mathbf{G} with maximal torus \mathbf{S} and root system

$$R := \{\alpha \in R(\mathbf{G}, \mathbf{S}) \mid \theta_F(N(\alpha^\vee(E_{0+}^\times))) = 1\},$$

the torus \mathbf{S} is maximally unramified.

- The character θ_F is *F-non-singular* with respect to \mathbf{G}^0 in the sense of Definition 3.6.7.

In [Kal19, Kal21], Kaletha describes how to construct supercuspidal representations $\pi_{\mathbf{S}, \theta_F}$ of G from tame *F-non-singular elliptic* pairs (\mathbf{S}, θ_F) in \mathbf{G} . The representation $\pi_{\mathbf{S}, \theta_F}$ is obtained in two steps. One starts by unfolding the pair (\mathbf{S}, θ_F) into a cuspidal \mathbf{G} -datum $\mathcal{D}_{\mathbf{S}, \theta_F} := (\vec{\mathbf{G}}, y, \vec{r}, \pi^0, \vec{\phi})$ as in Definition 3.6.6. The properties of \mathbf{S} and θ_F provided by Definition 3.6.8 allow us to go to the reductive quotient and use the theory of Deligne-Lusztig cuspidal representations in order to construct π^0 , the so-called depth-zero part of the datum $\mathcal{D}_{\mathbf{S}, \theta_F}$. The second step involves applying Yu's construction [Yu01] to the obtained \mathbf{G} -datum.

Since p does not divide the order of the Weyl group of \mathbf{G} , it does not divide the order of the fundamental group of \mathbf{G}_{der} .

Definition 3.6.9. [Kal19, Definition 3.7.9] A supercuspidal representation of G is said to be *regular* if it arises via Yu's construction from a cuspidal G -datum \mathcal{D} such that the representation π^0 of G^0 is regular in the sense of Definition 3.6.3.

Let $\mathbf{S}_0 \subset \mathbf{S}$ be the connected component of the intersection of kernels of all elements of R_{0+} , and $R(\mathbf{G}, \mathbf{S}_0)$ be the image of $R(\mathbf{G}, \mathbf{S}) \setminus R_{0+}$ under the restriction map $X^*(\mathbf{S}) \rightarrow X^*(\mathbf{S}_0)$.

Definition 3.6.10. [Kal21, Definition 4.1.4] A *torally wild supercuspidal L-packet datum* of \mathbf{G} is a tuple $(\mathbf{S}, j^\vee, \chi_0, \theta_F)$, where

- \mathbf{S} is a torus of dimension equal to the absolute rank of \mathbf{G} , defined over F and split over a tame extension of F ;
- $j^\vee: \mathbf{S}^\vee \hookrightarrow \mathbf{G}^\vee$ is an embedding of complex reductive groups whose \mathbf{G}^\vee -conjugacy class is Γ_F -stable;
- $\chi_0 = (\chi_{\alpha_0})_{\alpha_0}$ are tamely ramified χ -data for $R(\mathbf{G}, \mathbf{S}_0)$;
- $\theta_F: \mathbf{S}(F) \rightarrow \mathbb{C}^\times$ is a character; subject to the condition that (\mathbf{S}, θ_F) is a tame *F-non-singular elliptic* pair.

Formal degrees of arbitrary-depth tame supercuspidal representations in the sense of [Yu01] can be computed as in [Sch21, Theorem A]. Let \mathbf{G} be a semisimple F -group, and let \mathcal{D} be a cuspidal \mathbf{G} -datum with associated supercuspidal representation π . Let

R_i denote the absolute root system of \mathbf{G}^i , for the twisted Levi sequence $(\mathbf{G}^i)_{0 \leq i \leq d}$. Let $\exp_q(t) := q^t$.

Proposition 3.6.11. *The formal degree of π is given by*

$$(3.6.19) \quad \text{fdeg}(\pi) = \frac{\dim \rho}{[G_{[y]}^0 : G_{y,0+}^0]} \exp_q \left(\frac{1}{2} \dim G + \frac{1}{2} \dim \mathbb{G}_{y,0}^0 + \frac{1}{2} \sum_{i=0}^{d-1} r_i (|R_{i+1}| - |R_i|) \right).$$

Remark 3.6.12. The Formal Degree Conjecture of [HHI08], which describes the formal degree $\text{fdeg}(\pi)$ of any irreducible smooth representation π of G in terms of adjoint gamma factor, has been proved for regular supercuspidals in [Sch21, Theorem B], for non-singular supercuspidals in [Oha21, Theorem 9.2], and for unipotent supercuspidals in [FOS20, Theorem 3].

4. PRELIMINARIES ON G_2

Let $\mathbf{G} = G_2$ be the exceptional group of type G_2 over F , and let $G = G_2(F)$ denote its group of F -rational points. Let \mathbf{T} be a maximal F -split torus in \mathbf{G} and $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ be the canonical basis of \mathbb{R}^3 , equipped with the scalar product $(\ |)$ for which this basis is orthonormal. Then $\alpha := \varepsilon_1 - \varepsilon_2$ and $\beta := -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$ define a basis of the set $R(\mathbf{G}, \mathbf{T})$ of roots of \mathbf{G} with respect to \mathbf{T} , and

$$(4.0.1) \quad R(\mathbf{G}, \mathbf{T})^+ := \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}$$

is a subset of positive roots in $R(\mathbf{G}, \mathbf{T})$ (see [Bou68, Planche IX]). We have

$$(4.0.2) \quad (\alpha|\alpha) = 2, \quad (\beta|\beta) = 6, \quad \text{and} \quad (\alpha|\beta) = -3.$$

The root α is a short root, and β is a long root. Let $\mathbf{B} = \mathbf{T}\mathbf{U}$ be the corresponding Borel subgroup in \mathbf{G} and $\overline{\mathbf{B}} = \mathbf{T}\overline{\mathbf{U}}$ for the opposite Borel subgroup. For any root $\gamma \in R(\mathbf{G}, \mathbf{T})$, we denote by γ^\vee the corresponding coroot and by s_γ the fundamental reflection in W defined by γ :

$$(4.0.3) \quad s_\gamma(\gamma') := \gamma' - \langle \gamma', \gamma^\vee \rangle \gamma = \gamma' - \frac{2(\gamma'|\gamma)}{(\gamma|\gamma)} \gamma, \quad \text{for any } \gamma' \in R(\mathbf{G}, \mathbf{T}).$$

We set $a := s_\alpha$, $b := s_\beta$ and $r := ba$.

The group $X^*(T)$ of rational characters of T has the following description

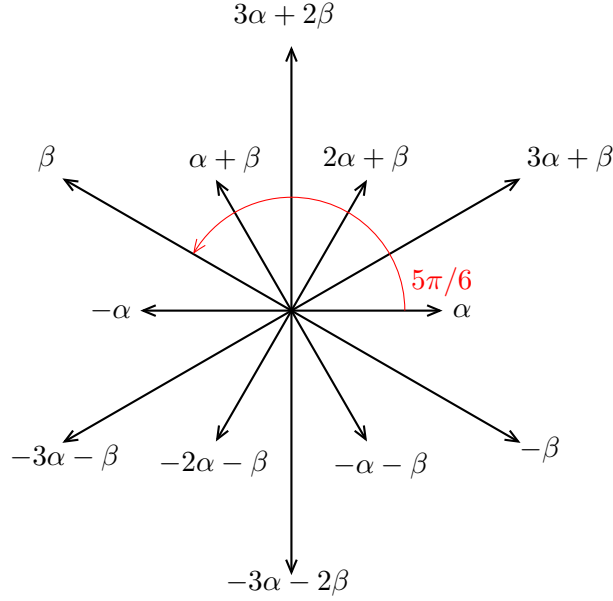
$$(4.0.4) \quad X^*(T) = \mathbb{Z}(2\alpha + \beta) + \mathbb{Z}(\alpha + \beta).$$

We identify T with $F^\times \times F^\times$ by

$$(4.0.5) \quad \begin{aligned} \eta_\alpha: T &\longrightarrow F^\times \times F^\times \\ \mathbf{t} &\mapsto ((2\alpha + \beta)(\mathbf{t}), (\alpha + \beta)(\mathbf{t})). \end{aligned}$$

Let $\eta'_\alpha: F^\times \times F^\times \rightarrow T$ be the inverse morphism of η_α . We have

$$(4.0.6) \quad \begin{cases} \alpha(\eta'_\alpha(t_1, t_2)) = t_1 t_2^{-1}, & \beta(\eta'_\alpha(t_1, t_2)) = t_1^{-1} t_2^2 \\ a(\eta'_\alpha(t_1, t_2)) = (t_2, t_1), & b(\eta'_\alpha(t_1, t_2)) = (t_1, t_1 t_2^{-1}) \end{cases}.$$

FIGURE 1. Diagram for $G_2(F)$

Any pair (ξ_1, ξ_2) of characters of F^\times define a character of T that we denote by $\xi_1 \otimes \xi_2$. For each root $\gamma \in R(\mathbf{G}, \mathbf{T})$, we fix root group homomorphisms $x_\gamma: F \rightarrow G$ and \mathbb{Z} -homomorphisms $\zeta_\gamma: \mathrm{SL}_2(F) \rightarrow G$ as in [BT72, (6.1.3) (b)]. We have

$$(4.0.7) \quad x_\gamma(u) = \zeta_\gamma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad x_{-\gamma}(u) = \zeta_{-\gamma} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \quad \text{and} \quad \gamma^\vee(t) = \zeta_\gamma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

For $\gamma \in \{\alpha, \beta\}$, let \mathbf{P}_γ be the maximal standard parabolic subgroup of \mathbf{G} generated by γ , and \mathbf{M}_γ be the centralizer of the image of $(\gamma')^\vee$ in \mathbf{G} , where γ' is the unique positive root orthogonal to γ , i.e.

$$(4.0.8) \quad \gamma' := \begin{cases} 3\alpha + \beta & \text{if } \gamma = \alpha, \\ 3\alpha + 2\beta & \text{if } \gamma = \beta. \end{cases}$$

Then \mathbf{M}_γ is a Levi factor for \mathbf{P}_γ , and $\mathbf{M}_\alpha, \mathbf{M}_\beta$ are representatives for the two conjugacy classes of maximal Levi subgroups of \mathbf{G} .

We extend $\zeta_\gamma: \mathrm{SL}_2(F) \rightarrow M_\gamma := \mathbf{M}_\gamma(F)$ to an isomorphism $\zeta_\gamma: \mathrm{GL}_2(F) \rightarrow M_\gamma$ via

$$(4.0.9) \quad \zeta_\gamma \left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \right) := \zeta_{\gamma'} \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right), \quad \text{for } t \in F^\times.$$

Then the restriction to T of the inverse map of ζ_γ coincides with the isomorphism $\eta_\gamma: T \xrightarrow{\sim} F^\times \times F^\times$, where η_α is as in (4.0.5), while

$$(4.0.10) \quad \begin{aligned} \eta_\beta: T &\longrightarrow F^\times \times F^\times \\ \mathbf{t} &\longmapsto ((\alpha + \beta)(\mathbf{t}), \alpha(\mathbf{t})). \end{aligned}$$

It follows from (4.0.6) that

$$(4.0.11) \quad (\eta_\beta \circ \eta'_\alpha)(t_1, t_2) = (t_2, t_1 t_2^{-1}), \quad \text{for } t_1, t_2 \in F^\times.$$

The long roots $\{\pm\beta, \pm(3\alpha + \beta), \pm(3\alpha + 2\beta)\}$ form the root system of $\mathrm{SL}_3(F)$. Hence $\mathrm{SL}_3(F)$ embeds in $G_2(F)$ as the subgroup generated by the groups U_γ with γ long.

The Weyl group W of \mathbf{G} is generated by the reflections s_α and s_β satisfying relations

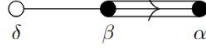
$$(4.0.12) \quad s_\alpha^2 = s_\beta^2 = 1 \quad \text{and} \quad (s_\alpha s_\beta)^6 = 1.$$

Thus W is the dihedral group of order 12. Let $r(\vartheta) \in W$ denote the rotation through ϑ with center at the origin. The elements of W are described in Table 1, along with their actions on the character $\xi_1 \otimes \xi_2$ of $T \cong F^\times \times F^\times$.

w	w	$w(\xi_1 \otimes \xi_2)$
1	1	$\xi_1 \otimes \xi_2$
s_α	a	$\xi_2 \otimes \xi_1$
s_β	b	$\xi_1 \xi_2 \otimes \xi_2^{-1}$
$r(\pi/3)$	ab	$\xi_2^{-1} \otimes \xi_1 \xi_2$
$r(5\pi/3)$	ba	$\xi_1 \xi_2 \otimes \xi_1^{-1}$
$s_{3\alpha+\beta}$	aba	$\xi_1^{-1} \otimes \xi_1 \xi_2$
$s_{\alpha+\beta}$	bab	$\xi_1 \otimes \xi_1^{-1} \xi_2^{-1}$
$r(2\pi/3)$	$abab$	$\xi_1^{-1} \xi_2^{-1} \otimes \xi_1$
$r(4\pi/3)$	$baba$	$\xi_2 \otimes \xi_1^{-1} \xi_2^{-1}$
$s_{2\alpha+\beta}$	$ababa$	$\xi_1^{-1} \xi_2^{-1} \otimes \xi_2$
$s_{3\alpha+2\beta}$	$babab$	$\xi_2^{-1} \otimes \xi_1^{-1}$
$r(\pi)$	$bababa$	$\xi_1^{-1} \otimes \xi_2^{-1}$

TABLE 1.

The group \mathbf{G} is simply connected. The nodes of its extended Dynkin diagram are $\delta := -3\alpha - 2\beta$ (the opposite of the highest root), α and β (in particular δ is the extended node, and α, β are respectively short and long roots):



The affine Weyl group of G is generated by s_δ , s_α and s_β . There are three G -conjugacy classes of maximal parahoric subgroups of G , obtained by deleting one node from the extended Dynkin diagram. We denote by G_{x_0} (deleting node δ), G_{x_1} (deleting node α) and G_{x_2} (deleting node β) a set of representatives of these classes. Their reductive quotients are $\mathbb{G}_{x_0} \simeq \mathbf{G}_2(\mathbb{F}_q)$, $\mathbb{G}_{x_1} \simeq \mathbf{SL}_3(\mathbb{F}_q)$ and $\mathbb{G}_{x_2} \simeq \mathbf{SO}_4(\mathbb{F}_q) \simeq \mathbf{SL}_2(\mathbb{F}_q) \times \mathbf{SL}_2(\mathbb{F}_q)/\{\pm 1\} \simeq \mathbf{SO}_4(\mathbb{F}_q)$, see [GY03, p.338].

Remark 4.0.1. By (3.4.2) the volume of the Iwahori subgroup $\mathcal{J}_{\mathbf{G}_2}$ of \mathbf{G}_2 is

$$(4.0.13) \quad \text{Vol}(\mathcal{J}_{\mathbf{G}_2}) = \frac{(q-1)^2}{q},$$

since the reductive quotient of \mathcal{J} is a torus isomorphic to $\mathbb{F}_q^\times \times \mathbb{F}_q^\times$.

Let G^\vee denote the Langlands dual group of G , i.e. the complex Lie group with root datum dual to that of G . We identify T^\vee with $\mathbb{C}^\times \times \mathbb{C}^\times$ by

$$(4.0.14) \quad \begin{aligned} \eta_{\beta^\vee}: T^\vee &\longrightarrow \mathbb{C}^\times \times \mathbb{C}^\times \\ \mathbf{t} &\longmapsto ((2\beta^\vee + \alpha^\vee)(\mathbf{t}), (\beta^\vee + \alpha^\vee)(\mathbf{t})). \end{aligned}$$

The group G^\vee is an exceptional group of type \mathbf{G}_2 , with positive roots the set

$$(4.0.15) \quad R(G^\vee, T^\vee)^+ = \{\alpha^\vee, \beta^\vee, \alpha^\vee + \beta^\vee, \alpha^\vee + 2\beta^\vee, \alpha^\vee + 3\beta^\vee, 2\alpha^\vee + 3\beta^\vee\},$$

in which α^\vee is a long root and β^\vee a short root. Let \mathfrak{g}^\vee and \mathfrak{t}^\vee denote the Lie algebras of G^\vee and T^\vee respectively. The adjoint action of G^\vee on \mathfrak{g}^\vee defines a Cartan decomposition

$$(4.0.16) \quad \mathfrak{g}^\vee = \mathfrak{t}^\vee \oplus \bigoplus_{\gamma^\vee \in R(G^\vee, T^\vee)} \mathfrak{g}_{\gamma^\vee}^\vee, \quad \text{with } \mathfrak{g}_{\gamma^\vee}^\vee = \mathbb{C}e_{\gamma^\vee}.$$

For each $\gamma^\vee \in R(G^\vee, T^\vee)$, let $U_{\gamma^\vee}^\vee$ be the associated root subgroup in G^\vee . We fix root group homomorphisms $x_{\gamma^\vee}: \mathbb{C} \rightarrow U_{\gamma^\vee}^\vee$.

4.0.17. Let $U(G^\vee)$ denote the unipotent variety of G^\vee . For two unipotent classes \mathcal{C} , \mathcal{C}' in G^\vee , we write $\mathcal{C}' \leq \mathcal{C}$ if \mathcal{C}' is contained in the Zariski closure of \mathcal{C} . The relation \leq defines a partial ordering on $U(G^\vee)$. In the Bala-Carter classification, the unipotent classes in G^\vee are

$$(4.0.18) \quad 1 \leq A_1 \leq \tilde{A}_1 \leq \mathbf{G}_2(a_1) \leq \mathbf{G}_2.$$

They are described in the following Table 2 (see [CM84, §8.4]). Amongst these five unipotent classes, three are special: 1, $\mathbf{G}_2(a_1)$ and \mathbf{G}_2 . In [Ram03], 1, A_1 , \tilde{A}_1 , $\mathbf{G}_2(a_1)$ and \mathbf{G}_2 are referred to as the *trivial*, *minimal*, *subminimal*, *subregular* and *regular* (also called *principal*) unipotent orbit, respectively.

By (4.0.16), for any $\gamma \in R(G^\vee, T^\vee)^+$, we can choose $e_\gamma \in \mathfrak{g}_\gamma^\vee$, $f_\gamma \in \mathfrak{g}_{-\gamma}^\vee$ and $h_\gamma \in \mathfrak{t}^\vee$ such that

$$(4.0.19) \quad [h_{\gamma'}, e_\gamma] = \gamma(h_{\gamma'})e_\gamma, \quad [h_{\gamma'}, f_\gamma] = -\gamma(h_{\gamma'})f_\gamma \quad \text{and} \quad [e_\gamma, f_\gamma] = h_\gamma.$$

Label	Diagram	Dimension
1	0 0	0
A_1	1 0	6
\tilde{A}_1	0 1	8
$G_2(a_1)$	2 0	10
G_2	2 2	12

TABLE 2. : Bala-Carter classification for $G_2(\mathbb{C})$.

The weighted Dynkin diagram of the minimal orbit A_1 shows that $\beta^\vee(h) = 1$ and $\alpha^\vee(h) = 0$, if (e, f, h) is an SL_2 -triple where e is a nilpotent element which corresponds to A_1 . From the table above, we know that $h = h_{2\beta^\vee + \alpha^\vee}$ works. It follows that $e_{2\beta^\vee + \alpha^\vee}$ is a representative of the nilpotent orbit corresponding to A_1 . The root $\beta^\vee + \alpha^\vee$ is short. Since all the short roots are conjugate under the Weyl group $W^\vee = N_{G^\vee}(T^\vee)/T^\vee$ of G^\vee , the corresponding spaces $\mathfrak{g}_{\gamma^\vee}^\vee$ are also conjugate under W^\vee . Since the orbits are closed under multiplication by a non-zero scalar and $\dim \mathfrak{g}_{\gamma^\vee}^\vee = 1$, any non-zero vector in a space $\mathfrak{g}_{\gamma^\vee}^\vee$ with γ^\vee a short root spans the minimal nilpotent orbit of G^\vee .

We recall the Springer correspondence for $G_2(\mathbb{C})$ from [Car93, p.427] in Table 3: to each irreducible character, it attaches a pair (u, ρ) consisting of (the conjugacy class of) a unipotent element u in $G^\vee = G_2(\mathbb{C})$ and an irreducible representation ρ of the group $A_u := \pi_0(Z_{G^\vee}(u))$. There is one ρ missing, which is the case where the component group A_u is the symmetric group S_3 and ρ is the sign character of S_3 .

characters of W	unipotent	A_u	enhancement
$\phi_{1,6}$	triv	triv	triv
$\phi''_{1,3}$	A_1	triv	triv
$\phi_{2,2}$	\tilde{A}_1	triv	triv
$\phi_{2,1}$	$G_2(a_1)$	S_3	triv
$\phi'_{1,3}$	$G_2(a_1)$	S_3	$\rho_{2,1}$
$\phi_{1,0}$	G_2	triv	triv

TABLE 3. : Springer correspondence for $G_2(\mathbb{C})$.

4.0.20. The group $\mathrm{SO}_4(\mathbb{C})$: We realize the group $\mathrm{SO}_4(\mathbb{C}) \simeq \mathrm{SL}_2^{\mathrm{lr}} \times \mathrm{SL}_2^{\mathrm{sr}} / \{\pm 1\}$ as the subgroup of $\mathrm{G}_2(\mathbb{C})$ generated by T^\vee and the images of x_{α^\vee} and $x_{2\beta^\vee + \alpha^\vee}$. Then the Weyl group of $\mathrm{SO}_4(\mathbb{C})$ with respect to T^\vee is $\{1, s_{\alpha^\vee}, s_{2\beta^\vee + \alpha^\vee}, s_{\alpha^\vee} \cdots s_{2\beta^\vee + \alpha^\vee}\}$. The group $\mathrm{SO}_4(\mathbb{C})$ admits 4 unipotent classes that are labelled as $(3, 1)$, $(2, 2)'$, $(2, 2)''$, (1) (see for instance [Lus84b, § 11.3]). A representative of the nilpotent class corresponding to $(2, 2)'$ is e_{α^\vee} , and a representative of the nilpotent class corresponding to $(2, 2)''$ is $e_{2\beta^\vee + \alpha^\vee}$. The closure order on the unipotent classes of $\mathrm{SO}_4(\mathbb{C})$ is given by the following:

$$(4.0.21) \quad \begin{array}{ccc} & (3, 1) & \\ & / \quad \backslash & \\ (2, 2)' & & (2, 2)'' \\ & \backslash \quad / & \\ & (1) & \end{array} .$$

The classes $(3, 1)$ and (1) are said to be non-degenerate, and the classes $(2, 2)'$, $(2, 2)''$ degenerate.

unipotent class in $\mathrm{SO}_4(\mathbb{C})$	unipotent class in $\mathrm{G}_2(\mathbb{C})$	$A_{\mathrm{SO}_4}(u)$
$(3, 1)$	$\mathrm{G}_2(a_1)$	μ_2
$(2, 2)'$	A_1	$\{1\}$
$(2, 2)''$	\tilde{A}_1	$\{1\}$
(1)	1	$\{1\}$

TABLE 4. Unipotent classes of $\mathrm{SO}_4(\mathbb{C})$.

4.0.22. The group $\mathrm{SL}_3(\mathbb{C})$: The long roots $\{\pm\alpha^\vee, \pm(3\beta^\vee + \alpha^\vee), \pm(3\beta^\vee + 2\alpha^\vee)\}$ form the root system of $\mathrm{SL}_3(\mathbb{C})$. Hence $\mathrm{SL}_3(\mathbb{C})$ embeds in $\mathrm{G}_2(\mathbb{C})$ as the subgroup generated by T^\vee and the images of x_{γ^\vee} with γ^\vee long. We can define an explicit such embedding $j_3: \mathrm{SL}_3(\mathbb{C}) \hookrightarrow \mathrm{G}_2(\mathbb{C})$ as

$$(4.0.23) \quad j_3 \begin{pmatrix} 1 & u \\ & 1 \\ & & 1 \end{pmatrix} = x_{\alpha^\vee}(u), \quad j_3 \begin{pmatrix} 1 & & \\ & 1 & \\ & u & 1 \end{pmatrix} = x_{3\beta^\vee + \alpha^\vee}(u)$$

$$(4.0.24) \quad j_3 \begin{pmatrix} 1 & & \\ u & 1 & \\ & & 1 \end{pmatrix} = x_{-\alpha^\vee}(u) \quad \text{and} \quad j_3 \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_1^{-1}t_2^{-1} \end{pmatrix} = \eta'_{\beta^\vee}(t_1, t_2),$$

where

$$(4.0.25) \quad \eta'_{\beta^\vee}: \mathbb{C}^\times \times \mathbb{C}^\times \rightarrow T^\vee$$

is the inverse morphism of η_{β^\vee} .

In general, the center of $\mathcal{G} := \mathrm{SL}_n(\mathbb{C})$ is isomorphic to the group μ_n of n -th roots of unity of n . If $u \in \mathcal{C}_{(n)}$ (the regular unipotent class), then the component group A_u of the centralizer of u in \mathcal{G} is isomorphic to μ_n . In particular, the unipotent classes of $\mathrm{SL}_3(\mathbb{C})$ can be summarized as in Table 5.

unipotent class in $\mathrm{SL}_3(\mathbb{C})$	unipotent class in $G_2(\mathbb{C})$	$A_{\mathrm{SL}_3}(u)$
3	$G_2(a_1)$	μ_3
(2, 1)	A_1	{1}
(1, 1, 1)	1	{1}

TABLE 5. : Unipotent classes of $\mathrm{SL}_3(\mathbb{C})$.

4.0.26. The group $\mathrm{GL}_2(\mathbb{C})$: There are two different cases depending on whether $\mathrm{GL}_2(\mathbb{C})$ corresponds to a Levi subgroup of $G_2(\mathbb{C})$ attached to a short or a long root γ^\vee . We have $W_G^s = \{1, s_{\gamma^\vee}\}$. The unipotent classes in this case are summarized in Table 6.

unipotent class in $\mathrm{GL}_2(\mathbb{C})$	representative of nilpotent orbit	unipotent class in $G_2(\mathbb{C})$	$A_{\mathrm{GL}_2}(u)$
regular	e_{γ^\vee}	$\begin{cases} \tilde{A}_1 & \text{if } \gamma^\vee \text{ is long} \\ A_1 & \text{if } \gamma^\vee \text{ is short} \end{cases}$	1
trivial	0	1	1

TABLE 6. : Unipotent classes of $\mathrm{GL}_2(\mathbb{C})$.

5. THE GALOIS SIDE FOR G_2

In this section, we study Langlands parameters into $G^\vee = G_2(\mathbb{C})$, and determine all non-supercuspidal members in their L -packets assuming the properties in §10.1. We assume $p \neq 2, 3$.

Proposition 5.0.1. *The following are equivalent for $\varphi : W_F \times \mathrm{SL}_2(\mathbb{C}) \rightarrow G^\vee$:*

- (1) $\varphi(W_F)$ is abelian.
- (2) $Z_{G^\vee}(S)$ contains a maximal torus of G^\vee .

Proof. We claim that any abelian subgroup of $G^\vee = G_2(\mathbb{C})$ is contained in a torus except for a single exception that is a unique conjugacy class of abelian subgroups isomorphic to μ_2^3 . The claim is equivalent to the statement that $Z_{G^\vee}(S)$ contains a

maximal torus of G^\vee for any $S \subset G^\vee$ abelian. Since closed subgroup of G^\vee satisfies the descending chain condition, it suffices to prove this for finitely generated abelian group. Using Proposition 2.5.2, for $s \in S$ non-trivial we have $Z_{G^\vee}(s)$ is isomorphic to one of $\mathrm{SL}_3(\mathbb{C})$, $\mathrm{SO}_4(\mathbb{C})$, $\mathrm{GL}_2(\mathbb{C})$ or $\mathrm{GL}_1(\mathbb{C})^2$. The whole centralizer $Z_{G^\vee}(S) = Z_{Z_{G^\vee}(s)}(S)$ contains a maximal torus of $\mathrm{SL}_3(\mathbb{C})$, $\mathrm{SO}_4(\mathbb{C})$, $\mathrm{GL}_2(\mathbb{C})$ or $\mathrm{GL}_1(\mathbb{C})^2$, except for the case of $\mathrm{SO}_4(\mathbb{C})$ for which an exception happens when S maps to the unique subgroup of $\mathrm{SO}_4(\mathbb{C})/\mu_2 \cong (\mathrm{PGL}_2(\mathbb{C}))^2$ that is isomorphic to μ_2^2 . Together with s , this implies that S is an order 8 subgroup of G^\vee isomorphic to μ_2^3 . \square

Recall that we have chosen in (4.0.15) a set of positive roots for G^\vee and its maximal torus T^\vee . We denote by $\lambda : (\mathbb{C}^\times)^2 \cong T^\vee$ the isomorphism satisfying $\alpha(\lambda(z_1, z_2)) = z_1$ and $\beta(\lambda(z_1, z_2)) = z_2$. The Weyl group $W_{G^\vee}(T^\vee)$ is generated by two reflections, one sending $\lambda(z_1, z_2) \mapsto \lambda(z_1^{-1}, z_1 z_2)$ and the other sending $\lambda(z_1, z_2) \mapsto \lambda(z_1 z_2^3, z_2^{-1})$.

Example 5.0.2. $\lambda(1, \zeta_3)$ has centralizer isomorphic to $\mathrm{SL}_3(\mathbb{C})$, and is conjugate to $\lambda(1, \zeta_3^{-1})$, making its orbit the unique one with this centralizer.

5.0.1. There are two obvious strategies to start with:

- (1) Classify φ in terms of $\varphi|_{\mathrm{SL}_2(\mathbb{C})}$;
- (2) Classify φ in terms of $\varphi|_{W_F}$.

The two strategies are somewhat orthogonal. As in line with standard literature, we also use the second strategy in the cases where $\varphi(\mathrm{SL}_2(\mathbb{C})) = 1$. One of the upshots in our approach is that we use a combination of both strategies.

5.1. Classification of φ in terms of $\varphi|_{\mathrm{SL}_2(\mathbb{C})}$. This is the first strategy. For a general Langlands parameter $\varphi : W_F \times \mathrm{SL}_2(\mathbb{C}) \rightarrow G^\vee$, the possible shapes for $\varphi|_{\mathrm{SL}_2(\mathbb{C})}$ are in bijection with unipotent orbits of G^\vee (see §4.0.17 for the notations for unipotent orbits). We have the following cases.

- (1) $\varphi\left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right)$ is regular, i.e. it lies in the G_2 unipotent class. In this case, we have $Z_{G^\vee}(\varphi(\mathrm{SL}_2(\mathbb{C}))) = \{1\}$ and the L -parameter φ is discrete, by Property 10.1.2, we obtain a singleton L -packet consisting of the Steinberg representation St_{G_2} of G .
- (2) $\varphi\left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right)$ is subregular, i.e. it lies in the subregular unipotent class $G_2(a_1)$. In this case, $Z_{G^\vee}(\varphi(\mathrm{SL}_2(\mathbb{C}))) \cong S_3$. To see this: $\varphi|_{\mathrm{SL}_2(\mathbb{C})}$ factors through $\mathrm{SL}_2(\mathbb{C}) \hookrightarrow \mathrm{SL}_3(\mathbb{C}) \hookrightarrow G^\vee$. Up to G^\vee -conjugation, we may assume that $\varphi\left(\begin{smallmatrix} t & 0 \\ & t^{-1} \end{smallmatrix}\right) = \lambda(t, 1)$, such that $Z_{G^\vee}(\{\lambda(t, 1) \mid t \in \mathbb{C}^\times\}) \cong \mathrm{GL}_2(\mathbb{C})$ is given by the short root β of G^\vee . This $\mathrm{GL}_2(\mathbb{C})$ acts on the sum of root spaces corresponding to $\{\alpha^\vee, \alpha^\vee + \beta^\vee, \alpha^\vee + 2\beta^\vee, \alpha^\vee + 3\beta^\vee\}$, which is a 4-dimensional representation of $\mathrm{GL}_2(\mathbb{C})$ that is isomorphic to $\mathrm{Sym}^3(\mathrm{std})$. One can check that $\varphi\left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right)$ is an element in general position in $\mathrm{Sym}^3(\mathrm{std})$ and thus has centralizer S_3 , i.e. the permutation group of the 3 roots of a separable polynomial of degree 3.

In this case, $\varphi(W_F)$ is any subgroup of S_3 , and φ is always discrete. We remark that the order 2 element in $S_3 \cong Z_{G^\vee}(\varphi(\mathrm{SL}_2(\mathbb{C})))$ has centralizer isomorphic to $\mathrm{SO}_4(\mathbb{C})$, while the order 3 element has centralizer isomorphic to $\mathrm{SL}_3(\mathbb{C})$. The subregular unipotent $\varphi\left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right)$ lies in the regular unipotent

orbit in both $SO_4(\mathbb{C})$ and $SL_3(\mathbb{C})$ by Tables 4 and 5. Since the group $\varphi(W_F)$ can have order 1, 2, 3 or 6, we have an L -packet of size 2+1, 1+1, 1+2 or 1+0 respectively (here we write the number of non-supercuspidal representations plus the number of supercuspidal representations).

- The case where $|\varphi(W_F)| = 1$ is the unipotent case, and will be addressed in Theorem 5.1.2(3f) and Table 15. More precisely, this L -packet has size 3: it contains two unipotent representations in the principal series, the representations $\pi(1)$ and $\pi(1)'$ from Table 15, and one unipotent supercuspidal representation from §10.3.1(1).
 - The case where $|\varphi(W_F)| = 2$ will be discussed in §5.2.1(3a).
 - The case where $|\varphi(W_F)| = 3$ will be discussed in §5.2.1(2a).
 - Lastly the case where $|\varphi(W_F)| = 6$ will be discussed in §5.2.1(2c).
- (3) When $\varphi\left(\begin{pmatrix} 1 & \\ & 0 & 1 \end{pmatrix}\right)$ is in the A_1 (resp. A_1) unipotent class, i.e. $\varphi(SL_2(\mathbb{C}))$ is a short (resp. long) root $SL_2(\mathbb{C})$ in G^\vee , its centralizer $Z_{G^\vee}(\varphi(SL_2(\mathbb{C}))) \cong SL_2(\mathbb{C})$ is a long (resp. short) root $SL_2(\mathbb{C})$. We have two sub-cases:
- (a) $\varphi(W_F) \subset Z_{G^\vee}(\varphi(SL_2(\mathbb{C})))$ is not contained in a torus of $Z_{G^\vee}(\varphi(SL_2(\mathbb{C})))$, in which case φ is discrete. This case is treated in §5.2.3.
 - (b) $\varphi(W_F)$ is contained in a torus $T^\vee \subset Z_{G^\vee}(\varphi(SL_2(\mathbb{C})))$. In this case, either $Z_{G^\vee}(\varphi) = T^\vee$, or $Z_{G^\vee}(\varphi) = Z_{G^\vee}(\varphi(SL_2(\mathbb{C})))$. We always have $S_\varphi = 1$, and the L -packet consists of a single principal series by Proposition 5.0.1.
- (4) $\varphi(SL_2(\mathbb{C}))$ is trivial. We leave it to later case-by-case discussions in §5.2.1 and §5.2.2.

As a consequence of the above discussions, we immediately obtain the following.

Corollary 5.1.1. *When φ is non-discrete, either $\varphi(SL_2(\mathbb{C}))$ is trivial, or $\varphi(SL_2(\mathbb{C}))$ is a short or long $SL_2(\mathbb{C})$ in G^\vee . In the latter case(s), $S_\varphi = \{1\}$ and the L -packet consists of a single principal series.*

Let $F_{2,2}$ denote the bi-quadratic extension over F with $\text{Gal}(F_{2,2}/F) \cong \mu_2^2$. We now study our LLC using Properties 10.1.1, 10.1.2, 10.1.6 and 10.1.9.

Theorem 5.1.2. *A parabolically induced representation of $G = G_2(F)$ has two or more non-isomorphic constituents exactly in the following cases:*

- (1) *The parabolically induced representation $i_P^G \sigma$ from a Levi subgroup $M \cong GL_2(F)$ of a supercuspidal representation σ of M s.t.*
 - (a) *The Levi $M = M_\beta \cong GL_2(F)$ is associated to the long root and $\sigma \cong \sigma_{S_3} \otimes \nu_F^{\pm 1}$, where σ_{S_3} is the unique supercuspidal representation of GL_2 whose corresponding L -parameter under the LLC for GL_2 has image S_3 , and $\otimes \nu_F^{\pm 1}$ denotes tensoring with $\nu_F^{\pm 1} \circ \det : GL_2(F) \rightarrow \mathbb{C}^\times$. In this case, the parabolic induction $i_P^G(\sigma)$ has a discrete constituent $\pi(\sigma)$ and another non-tempered constituent $J(\sigma)$, where $\pi(\sigma)$ corresponds to the L -parameter in 5.2.1(2c) such that $\varphi|_{SL_2(\mathbb{C})}$ meets the $G_2(a_1)$ unipotent class, while $J(\sigma)$ has trivial $\varphi|_{SL_2(\mathbb{C})}$.*
 - (b) *The central character ω_σ of σ has order 2 (and is non-trivial); moreover, if $\sigma \cong \sigma_{S_3}$, then $GL_2(F)$ cannot be the one attached to the long root. In this case, $i_P^G \sigma$ has two non-isomorphic constituents π and π' that are*

both non-discrete and tempered, one of which is generic. Both π and π' correspond to the L -parameters φ in 5.2.1(4c) for the depth-zero case, or 5.2.2(2e) for the positive-depth case. In particular, we always have $\varphi|_{\mathrm{SL}_2(\mathbb{C})} \equiv 1$ in this case.

- (c) The central character $\omega_\sigma \equiv \nu_F^{\pm 1}$. In this case, the parabolic induction has a discrete constituent $\pi(\sigma)$ which corresponds to the L -parameters in §5.2.3, and another non-tempered constituent $J(\sigma)$ which has trivial $\varphi|_{\mathrm{SL}_2(\mathbb{C})}$.
- (2) The parabolically induced representation from the order 2 character χ of $(F^\times)^2$ such that $\varphi_\chi: W_F \rightarrow (\mathbb{C}^\times)^2$ satisfies $\ker(\phi) = W_{F_{2,2}}$ for the bi-quadratic extension $F_{2,2}$ over F .

In this case, the parabolically induced representation has two non-isomorphic constituents π and π' that are both non-discrete and tempered, one of which is generic. Both π and π' correspond to the L -parameter φ given in 5.2.1(3c). In particular, we have $\varphi|_{\mathrm{SL}_2(\mathbb{C})} \equiv 1$.

Representations in the above two cases (1) and (2) are the only non-discrete representations of G whose L -packets are not singleton packets. In both cases, the L -packet has two members that are exactly the two irreducible constituents in the same parabolic induction.

- (3) The parabolically induced representation of $\theta: T \rightarrow \mathbb{C}^\times$ with $\theta \circ \lambda^\vee = \nu_F$ for some coroot $\lambda: F^\times \rightarrow T$. We have the following cases:
- (a) There is a unique such $\lambda \in R^\vee(\mathbf{G}, \mathbf{T})$. In this case, the parabolically induced representation has two non-isomorphic constituents. They are both non-discrete.
- (b) There are exactly two such $\lambda \in R^\vee(\mathbf{G}, \mathbf{T})$ that are both short with 120° angle, or both long with 60° angle. In this case, the parabolic induction again has two non-isomorphic constituents, both non-discrete.
- (c) There are two such $\lambda \in R^\vee(\mathbf{G}, \mathbf{T})$ and they are perpendicular to each other. In this case, the parabolic induction has four constituents. One of them is discrete and corresponds to an L -parameter with $Z_{G^\vee}(\varphi(W_F)) = \mathrm{SO}_4(\mathbb{C})$ and $\varphi|_{\mathrm{SL}_2(\mathbb{C})}$ subregular.
- (d) There are exactly two such $\lambda \in R^\vee(\mathbf{G}, \mathbf{T})$ that are both long with 120° angle. In this case, the parabolic induction has four constituents. One of them is discrete and corresponds to an L -parameter with $Z_{G^\vee}(\varphi(W_F)) = \mathrm{SL}_3(\mathbb{C})$ and $\varphi|_{\mathrm{SL}_2(\mathbb{C})}$ subregular.
- (e) There are two such $\lambda \in R^\vee(\mathbf{G}, \mathbf{T})$ with 150° angle in between. In this case, the parabolic induction has the Steinberg representation as a constituent, and has three other non-discrete constituents.
- (f) There are four such $\lambda \in R^\vee(\mathbf{G}, \mathbf{T})$. In this case, the parabolic induction has 5 constituents. Two of them are discrete and correspond to the two non-supercuspidal members in the L -packet (which is a unipotent L -packet of size 3 containing also the supercuspidal representation $\pi[-1]$) for the L -parameter with $\varphi(W_F) = 1$ and $\varphi|_{\mathrm{SL}_2(\mathbb{C})}$ subregular.

In cases (3b), (3e) and (3f), the character $\theta: T \rightarrow \mathbb{C}^\times$ is unique up to the action of the Weyl group W_G (one character for each of the two configurations in (3b)). In case (3c), we have that θ (up to W_G -action) is determined by an order-2 character $F^\times \rightarrow \mathbb{C}^\times$, i.e. by a quadratic extension. In case (3d), we have that θ (up to W_G -action) is determined by an order-3 character up to inversion, i.e. it is determined up to a Galois cubic extension.

Proof. For non-discrete L -parameters, a case-by-case study conducted in §5.2.1 and §5.2.2 shows that the only cases where $S_\varphi \neq 1$ are: §5.2.1(3c), §5.2.1(4c) and §5.2.2(2e). These cases belong to case (2) in Theorem 5.1.2. For discrete Langlands parameters, Lemma 5.2.1 below implies that $\varphi|_{\mathrm{SL}_2(\mathbb{C})} \not\equiv 1$ is necessary to obtain non-supercuspidal representations. Hence one can refer to the classification of φ in §5.1, and match them with the case-by-case study in sections §5.2.1, §5.2.2 and §5.2.3 to obtain a complete list of discrete non-supercuspidal representations of G . These results are listed in cases (1a), (1c), (3c), (3d), (3e) and (3f).

Outside the above discussed cases, we have $S_\varphi = \{1\}$. Therefore, in order to have two different constituents, we need two non-supercuspidal L -parameters with the same supercuspidal support. We start with a Langlands parameter $\varphi: W_F \times \mathrm{SL}_2(\mathbb{C}) \rightarrow G^\vee$, and consider the *trivially* enhanced Langlands parameter $(\varphi, 1)$. Then Property 10.1.9 specifies the supercuspidal support as follows: We take a maximal torus $T^\vee \subset \mathcal{G}_\varphi := Z_{G^\vee}(\varphi(W_F))$. Let $L^\vee := Z_{G^\vee}(T^\vee)$. The cuspidal support $\mathrm{Sc}(\varphi, 1)$ of $(\varphi, 1)$ is the trivially enhanced Langlands parameter $(\bar{\varphi}, 1)$, where $\bar{\varphi}: W_F \rightarrow L^\vee$ is

$$(5.1.1) \quad \bar{\varphi}(w) := \varphi(w, \begin{pmatrix} \|w\|^{1/2} & 0 \\ 0 & \|w\|^{-1/2} \end{pmatrix}).$$

Suppose there exists more than one φ with the same $\bar{\varphi}$. Denote by ϕ their common $\bar{\varphi}$. We have $\varphi|_{\mathrm{SL}_2(\mathbb{C})} \not\equiv 1$ for at least one φ . The case where φ is discrete is considered in the case-by-case study in sections §5.2.1, §5.2.2 and §5.2.3. We devote the rest of this proof to the case where φ is non-discrete.

By Corollary 5.1.1, when φ is non-discrete, we have that $H^\vee = T^\vee$ is a maximal torus. The group $L^\vee := T^\vee \cdot \varphi(\mathrm{SL}_2(\mathbb{C}))$ is a (short root or long root) Levi subgroup isomorphic to $\mathrm{GL}_2(\mathbb{C})$, for which we denote by $\lambda^\vee \in R^\vee(L^\vee, T^\vee)$ given by $\lambda^\vee(t) = \varphi\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right)$ with λ as the corresponding root. The fact that $\varphi|_{W_F}: W_F \rightarrow T^\vee$ commutes with $\varphi(\mathrm{SL}_2(\mathbb{C}))$ is equivalent to the fact that $\lambda \circ \varphi|_{W_F} \equiv 1$, which is more-over equivalent to saying that

$$(5.1.2) \quad \lambda \circ \phi \equiv \|\cdot\| \text{ as characters on } W_F.$$

By Local Langlands for tori, $\phi = \bar{\varphi}: W_F \rightarrow T^\vee$ corresponds to some $\theta: T \rightarrow \mathbb{C}^\times$, and (5.1.2) is equivalent to saying that $\theta \circ \lambda \equiv \|\cdot\|$. Every step of the argument above is reversible, therefore, if we start with a $\phi: W_F \rightarrow T^\vee$ satisfying (5.1.2) for some root $\lambda \in R(G^\vee, T^\vee)$, we obtain a corresponding non-discrete φ with $\varphi|_{\mathrm{SL}_2(\mathbb{C})} \not\equiv 1$ and $\bar{\varphi} = \phi$. It then remains to resolve the question of whether there can be more than one choice for such λ .

If there is only one choice for such λ , then we obtain two φ 's satisfying $\bar{\varphi} = \phi$; one of them satisfies $\varphi|_{\mathrm{SL}_2(\mathbb{C})} \equiv 1$ and $\varphi|_{W_F} = \phi$. This is precisely case (3a).

If, however, there are more than one such λ , then they generate a lattice

$$\Lambda := \langle \lambda \in R(G^\vee, T^\vee) \mid \lambda \circ \phi \equiv \|\cdot\| \rangle.$$

We have $\Lambda \subset X^*(T^\vee)$ with index at most 3. When $\Lambda = X^*(T^\vee)$, we have that ϕ is uniquely determined by λ 's. There can be either two or four such λ 's of the configuration in case (3b), (3e) or (3f). In case (3b), the two choices of λ are conjugate under the Weyl group action, and thus there is only one possible φ satisfying $\bar{\varphi} = \phi$ and $\varphi|_{\mathrm{SL}_2(\mathbb{C})} \not\equiv 1$.

In case (3e), we have two different choices for λ , giving rise to two non-discrete φ 's satisfying $\bar{\varphi} = \phi$. Nevertheless, there is also a discrete φ with $\bar{\varphi} = \phi$, which is the one satisfying $\varphi|_{W_F} \equiv 1$ and $\varphi\left(\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}\right)$ is the regular unipotent class.

In case (3f), we have two different choices for λ , giving rise to two non-discrete φ 's satisfying $\bar{\varphi} = \phi$. Nevertheless, there is also a discrete φ with $\bar{\varphi} = \phi$, which is the one satisfying $\varphi|_{W_F} \equiv 1$ and $\varphi\left(\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}\right)$ is the subregular unipotent class. Moreover, there is a non-trivial local system on the subregular unipotent orbit (which is the unique one of rank 2) that lives in the principal series. This gives one another non-trivial enhancement of the same φ giving our θ as the supercuspidal support.

In case (3c), we have two different choices for λ , giving rise to two non-discrete φ 's satisfying $\bar{\varphi} = \phi$. We also have a non-discrete L -parameter satisfying $\varphi|_{W_F} = \phi$ and $\varphi|_{\mathrm{SL}_2(\mathbb{C})} \equiv 1$. Suppose that the two choices for λ are denoted by λ_1 and λ_2 , where λ_1 is short. Write $\lambda'_1 := \frac{3}{2}\lambda_1 - \frac{1}{2}\lambda_2$ and $\lambda'_2 := \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2$. Thus we also have $\lambda'_i \in R(G^\vee, T^\vee)$, but $\lambda'_i \circ \phi$ is different from $\|\cdot\|$ by an order-two character. This order-two character corresponds to $\eta : W_F \rightarrow T^\vee$ where $Z_{G^\vee}(\eta) \cong \mathrm{SO}_4(\mathbb{C})$ such that $\lambda_1, \lambda_2 \in R(Z_{G^\vee}(\eta), T^\vee)$. We then consider the discrete Langlands parameter $\varphi : W_F \times \mathrm{SL}_2(\mathbb{C}) \rightarrow G^\vee$ such that $\varphi|_{W_F} = \eta$ and $\varphi(\mathrm{SL}_2(\mathbb{C})) \subset Z_{G^\vee}(\eta) \cong \mathrm{SO}_4(\mathbb{C})$ meets the regular unipotent orbit in $\mathrm{SO}_4(\mathbb{C})$. The trivial enhancement of this L -parameter also has supercuspidal support ϕ (while the other enhancement is cuspidal).

Case (3d) is completely analogous to (3c), except that η now has order three and $Z_{G^\vee}(\eta) \cong \mathrm{SL}_3(\mathbb{C})$. We note that there exists a Weyl group element that stabilizes everything else but sends η to η^{-1} . This finishes the proof. \square

5.1.3. Before we proceed, let us summarize the results of Theorem 5.1.2 in the following Table 7 and Table 8.

For a character $\theta : T \rightarrow \mathbb{C}^\times$, we write $\theta = \xi_1 \otimes \xi_2$, where $\xi_1 = \theta \circ \beta^\vee$ and $\xi_2 = \theta \circ (\alpha^\vee + \beta^\vee)$ with α^\vee, β^\vee as in (4.0.15). We remark that this identification is compatible with (4.0.5). Thanks to Theorem 5.1.2, we have a table for parabolic inductions $I(\xi_1 \otimes \xi_2)$ that contain at least two non-isomorphic irreducible constituents that are principal series. In the second to last column of Table 7, each item begins with $\varphi\left(\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}\right)$ as its first tag, followed by tags among {d., n.d., t., n.t., g., n.g.} for the adjectives ‘‘discrete’’, ‘‘non-discrete’’, ‘‘tempered’’, ‘‘non-tempered’’, ‘‘generic’’, and ‘‘non-generic’’, respectively. On the other hand, all the principal series not appearing in Table 7 should have only one irreducible constituent up to isomorphism.

Label	Input	$\xi_1 \otimes \xi_2$	Constituents of $i_P^G(\xi_1 \otimes \xi_2)$	Table
(2) §5.2.1(3c)	$\eta_2^2 \equiv (\eta'_2)^2 \equiv 1,$ $1 \neq \eta_2 \neq \eta'_2 \neq 1$	$\eta_2 \otimes \eta'_2$	(1, n.d., t., g.) (1, n.d., t., n.g.)	Table 23
(3a) (3b)	$\eta : F^\times \rightarrow \mathbb{C}^\times$ s.t. $\eta^2 \neq 1 \neq (\eta\nu_F)^2,$ $\eta \neq \nu_F^2, \eta \neq \nu_F^{-3}$	$\eta \otimes \nu_F$	$(\tilde{A}_1, \text{n.d.}), (1, \text{n.t.}),$ the first tempered if and only if $\ \eta\nu_F^{1/2}\ \equiv 1$	Table 13
(3a) (3b)	$\eta : F^\times \rightarrow \mathbb{C}^\times$ s.t. $\eta^3 \neq 1, \nu_F^{-3}$	$\eta \otimes \eta\nu_F$	$(A_1, \text{n.d.}), (1, \text{n.t.}),$ 1.51.5 the first tempered if and only if $\ \eta\nu_F^{1/2}\ \equiv 1$	Table 21 Table 22 Table 24
(3c) §5.2.1(3a)	E/F quadratic	$\nu_F \otimes \eta_{E/F}$	$(G_2(a_1), \text{d.}, \text{g.}),$ $(\tilde{A}_1, \text{n.t.}), (A_1, \text{n.t.}), (1, \text{n.t.})$	Table 16 Table 17
(3d) §5.2.1(2a)	E/F Galois cubic	$\eta_{E/F} \otimes \eta_{E/F}\nu_F$	$(G_2(a_1), \text{d.}, \text{g.}),$ $(A_1, \text{n.t.}), (A_1, \text{n.t.}), (1, \text{n.t.})$	Table 19 Table 20
(3e)	None	$\nu_F \otimes \nu_F^2$	$(G_2, \text{d.}, \text{g.}),$ $(\tilde{A}_1, \text{n.t.}), (A_1, \text{n.t.}), (1, \text{n.t.})$	Table 18
(3f)	None	$1 \otimes \nu_F$	$(G_2(a_1), \text{d.}, \text{g.}), (G_2(a_1), \text{d.}, \text{n.g.}),$ $(\tilde{A}_1, \text{n.t.}), (A_1, \text{n.t.}), (1, \text{n.t.})$	Table 15

TABLE 7. Principal series with more than one non-isomorphic constituents.

5.1.4. On the other hand, for a supercuspidal representation σ of $\text{GL}_2(F)$, we also consider the parabolic induction $i_P^G(\sigma)$, whose irreducible constituents are referred to as *intermediate series*. We denote by $\omega_\sigma : \text{GL}_1(F) \rightarrow \mathbb{C}^\times$ the central character of the supercuspidal representation σ . We also denote by $\sigma \otimes \nu_F^s$ the representation σ tensored with $\text{GL}_2(F) \xrightarrow{\det} \text{GL}_1(F) \xrightarrow{\nu_F^s} \mathbb{C}$, such that $\omega_{\sigma \otimes \nu_F^s} = \omega_\sigma \cdot \nu_F^{2s}$.

We shall also consider a particular supercuspidal representation of $\text{GL}_2(F)$ that is denoted as σ_{S_3} and is characterized by the property that: its corresponding Langlands parameter $W_F \rightarrow \text{GL}_2(\mathbb{C})$ has image isomorphic to S_3 ; such a supercuspidal representation exists if and only if $q \equiv -1(3)$, in which case it is unique, at depth-zero and has central character ω_σ unramified of order two. We have the following Table 8 for parabolic inductions that contain at least two non-isomorphic irreducible

constituents that are intermediate series.

Cases	Choice of $\mathrm{GL}_2(F)$	Condition on σ	Constituents of $i_P^G(\sigma)$
3(1c) 5.2.3	short	$\omega_\sigma \equiv \nu_F^{\pm 1}$	$(\tilde{A}_1, \mathrm{d.}, \mathrm{g.}), (1, \mathrm{n.t.})$
3(1c) 5.2.3	long	$\omega_\sigma \equiv \nu_F^{\pm 1}$	$(A_1, \mathrm{d.}, \mathrm{g.}), (1, \mathrm{n.t.})$
3(1b) 5.2.1(4c) 5.2.2(2e)	short	$\omega_\sigma^2 \equiv 1,$ $\omega_\sigma \not\equiv 1$	$(1, \mathrm{n.d.}, \mathrm{t.}, \mathrm{g.}), (1, \mathrm{n.d.}, \mathrm{t.}, \mathrm{n.g.})$
3(1b) 5.2.1(4c) 5.2.2(2e)	long	$\omega_\sigma^2 \equiv 1,$ $\omega_\sigma \not\equiv 1, \sigma \not\cong \sigma_{S_3}$	$(1, \mathrm{n.d.}, \mathrm{t.}, \mathrm{g.}), (1, \mathrm{n.d.}, \mathrm{t.}, \mathrm{n.g.})$
3(1a) 5.2.1(2c)	long	$\sigma \cong \sigma_{S_3} \otimes \nu_F^{\pm 1}$	$(G_2(a_1), \mathrm{d.}, \mathrm{g.}), (1, \mathrm{n.t.})$

TABLE 8. Intermediate series with more than 1 non-isomorphic constituents.

5.2. L -parameters for G_2 .

5.2.1. *Depth-zero L -parameters.* In the rest of this chapter, we follow the second strategy outlined in §5.0.1 to finish the classification of φ , i.e. we classify φ in terms of $\varphi|_{W_F}$. Let us denote by $G_0^\vee := Z_{G^\vee}(\varphi(P_F))$ and $S^\vee := Z_{G^\vee}(\varphi(I_F)) = Z_{G_0^\vee}(\varphi(I_F))$. Recall that $\mathfrak{G}_\varphi := Z_{G^\vee}(\varphi(W_F)) = Z_{S^\vee}(\varphi(\mathrm{Fr}))$, and we have $S_\varphi = \pi_0(Z_{\mathfrak{G}_\varphi}(\varphi(\mathrm{SL}_2(\mathbb{C}))))$.

For each configuration of $\varphi|_{W_F}$, we have that φ is discrete if and only if $(\mathfrak{G}_\varphi)^\circ$ is semisimple and $\varphi(1, \begin{pmatrix} 1 & \\ & 0 \end{pmatrix})$ is a distinguished unipotent element $u = u_\varphi$ in $(\mathfrak{G}_\varphi)^\circ$. In this case, $\rho \in \mathrm{Irr}(S_\varphi)$ is cuspidal in the sense of §2.1.1. In fact, the following holds (thanks to the small rank of G_2).

Lemma 5.2.1. *For discrete L -parameters, except in the case where $\varphi|_{W_F}$ is trivial and $S_\varphi \cong S_3$, we have that u is regular unipotent in \mathfrak{G}_φ , and $\rho \in \mathrm{Irr}(S_\varphi)$ is cuspidal if and only if either ρ is non-trivial or \mathfrak{G}_φ is finite.*

Proof. The assertion is obvious when \mathfrak{G}_φ is finite. We now assume that $\dim \mathfrak{G}_\varphi > 0$. Each connected Dynkin component for \mathfrak{G}_φ is of type A_1 , A_2 or G_2 . The only non-regular but distinguished unipotent appears in the G_2 case for which $\varphi|_{W_F}$ is trivial

and $S_\varphi \cong S_3$. For the remaining cases, u is regular. Pulling back to the simply-connected cover of \mathcal{G}_φ , any local system on the regular unipotent orbit of $\mathrm{SL}_2(\mathbb{C})$, $\mathrm{SL}_3(\mathbb{C})$ or $G_2(\mathbb{C})$ is cuspidal if and only if it is non-trivial. \square

We remark that when u is regular unipotent, we have $S_\varphi = Z_{\mathcal{G}_\varphi}$.

We are now ready to proceed with a detailed case-by-case discussion. The centralizer $S^\vee := Z_{G^\vee}(\varphi(I_F))$ is the fixed subgroup of a finite-order automorphism of G^\vee . Since G^\vee is simply-connected, S^\vee is connected. It can be one of the following cases:

- (1) $S^\vee \cong G^\vee$, i.e. $\varphi(I_F)$ is trivial. In this case, the L -parameter is unipotent and an explicit LLC has been constructed by [Lus95] (see also [Ree94, p.480] and [Mor96]).
- (2) $S^\vee \cong \mathrm{SL}_3(\mathbb{C})$. In this case, $\varphi(I_F)$ is generated by a conjugate of $\lambda(1, \zeta_3)$. Since $\lambda(1, \zeta_3)$ is also conjugate to $\lambda(1, \zeta_3^{-1})$, this makes $\varphi|_{I_F}$ unique. By Proposition 2.5.2, $\mathcal{G}_\varphi := Z_{S^\vee}(\varphi(\mathrm{Fr}))$ can be SL_3 , GL_2 , GL_1^2 , Sp_2 , SO_3 or GL_1 . In the case $\mathcal{G}_\varphi = \mathrm{GL}_2$, GL_1^2 or GL_1 , we have that \mathcal{G}_φ is not semisimple and thus φ is not discrete. In these cases $Z_{G^\vee}(\varphi)$ is always a torus making $S_\varphi = \{1\}$. In the cases $\mathcal{G}_\varphi = \mathrm{GL}_2$ or GL_1^2 , Fr acts on $\mathrm{SL}_3(\mathbb{C})$ by inner automorphism, necessarily fixing $Z_{\mathrm{SL}_3(\mathbb{C})} = \varphi(I_F)$. Hence $\varphi(W_F)$ is abelian. By Proposition 5.0.1 the L -packet consists of only a principal series. When $\mathcal{G}_\varphi \cong \mathrm{GL}_1$, $\varphi(W_F)$ is non-abelian and by Proposition 5.0.1 we get an L -packet consists of a single non-discrete constituent of a parabolic induction from $\mathrm{GL}_2(F)$. We discuss the rest of the cases:
 - (a) $\mathcal{G}_\varphi = \mathrm{SL}_3$. This is to say that $\varphi(W_F) \subset Z_{G^\vee}(\mathrm{SL}_3(\mathbb{C})) = \varphi(I_F)$, i.e. $\varphi|_{W_F}$ factors through $\mathrm{Gal}(E/F)$ for some degree 3 ramified Galois extension, which necessarily forces $q \equiv 1(3)$. In such case there are 3 choices of E . For each choice we have a unique discrete Langlands parameter with $S_\varphi = \mu_3$ by Lemma 5.2.1. Any such L -packet consists of one non-supercuspidal representation and two supercuspidal ones. When φ is not discrete, the centralizer $Z_{\mathcal{G}_\varphi}(\varphi(\mathrm{SL}_2(\mathbb{C})))$ is either $\mathrm{SL}_3(\mathbb{C})$ or $\mathrm{GL}_1(\mathbb{C})$ so that $S_\varphi = \{1\}$. By Proposition 5.0.1 we obtain an L -packet consisting of a single principal series.
 - (b) $\mathcal{G}_\varphi = \mathrm{Sp}_2(\mathbb{C})$. In this case $\varphi(\mathrm{Fr})$ acts on $S^\vee = \mathrm{SL}_3(\mathbb{C})$ by an outer automorphism. The unique outer class for SL_3 is represented by $g \mapsto (g^t)^{-1}$ which acts as inversion on $Z_{\mathrm{SL}_3(\mathbb{C})} = \varphi(I_F)$. This implies $q \equiv -1(3)$. Moreover, such an outer automorphism with fixed subgroup $\mathrm{Sp}_2(\mathbb{C})$ is unique up to $\mathrm{SL}_3(\mathbb{C})$ -conjugacy because in Proposition 2.5.2 there is a unique Sp_2 vertex of the loop group $\mathrm{SU}_3/\mathbb{C}((t))$. Hence there exists a unique such φ that is discrete, with $S_\varphi = Z_{\mathrm{Sp}_2(\mathbb{C})} = \mu_2$. The L -packet consists of one non-supercuspidal representation and one supercuspidal representation. By Property 10.1.9, the non-supercuspidal is the generic constituent of the parabolic induction from M_β –the long root $\mathrm{GL}_2(F)$ – of the depth-zero supercuspidal representation of $\mathrm{GL}_2(F)$ corresponding to $W_F \xrightarrow{\varphi^\tau} (\mathbb{Z}/3) \rtimes (\mathbb{Z}/4) \xrightarrow{\tau} \mathrm{GL}_2(\mathbb{C})$ with some unramified twist specified by Property 10.1.9. Here τ is the unique irreducible yet faithful

representation of $(\mathbb{Z}/3) \rtimes (\mathbb{Z}/4)$, and φ^τ maps I_F onto $\mathbb{Z}/3$ and some lift Fr to $(0, 1)$. More precisely, this depth-zero supercuspidal representation of $\text{GL}_2(F)$ corresponds to the tame L -parameter $W_F \rightarrow \text{GL}_2(\mathbb{C})$ that sends a generator of tame inertia to $\begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^{-1} \end{pmatrix}$ and a lift of Fr to $\begin{pmatrix} 0 & iq^{1/2} \\ iq^{1/2} & 0 \end{pmatrix}$. In fact, this L -parameter is also of long SL_2 -type as in §5.2.3, and the above description is generalized there.

If otherwise φ is non-discrete, then $\varphi(\text{SL}_2(\mathbb{C})) = 1$, $S_\varphi = \pi_0(\mathcal{G}_\varphi) = \{1\}$. By Property 10.1.9, we have a singleton L -packet consisting of a tempered representation contained in the parabolic induction from the long root $\text{GL}_2(F)$ given by the finite-order homomorphism $\tau: W_F \rightarrow \text{GL}_2(\mathbb{C})$ described in the previous paragraph.

- (c) $\mathcal{G}_\varphi = \text{SO}_3$. Again in this case $\varphi(\text{Fr})$ acts on $S^\vee = \text{SL}_3(\mathbb{C})$ by an outer automorphism and hence $q \equiv -1(3)$. There is again a unique discrete such φ . In this case $S_\varphi = \{1\}$ and the L -packet consists of a single non-supercuspidal generic discrete series. Under 10.1.9 it is given by the generic constituent of the parabolically induced representation from the long root $\text{GL}_2(F)$ of the depth-zero supercuspidal representation of $\text{GL}_2(F)$ corresponding to $W_F \rightarrow S_3 \hookrightarrow \text{GL}_2(\mathbb{C})$ with some unramified twist specified by Property 10.1.9. Specifically, it is the depth-zero supercuspidal representation corresponding to the tame L -parameter $W_F \rightarrow \text{GL}_2(\mathbb{C})$ that sends a generator of tame inertia group to $\begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^{-1} \end{pmatrix}$ and a lift of Fr to $\begin{pmatrix} 0 & q^{1/2} \\ q^{1/2} & 0 \end{pmatrix}$. We remark that compared to the non-supercuspidal representation in (2b), the two depth-zero supercuspidal representations of $\text{GL}_2(F)$ differ by an order 4 unramified twist.

If otherwise φ is non-discrete, then $\varphi(\text{SL}_2(\mathbb{C})) = 1$, $S_\varphi = \pi_0(\mathcal{G}_\varphi) = \{1\}$. By the 10.1.9, we have a singleton L -packet consisting of a tempered generic representation contained in the parabolic induction from the long root $\text{GL}_2(F)$ given by the finite-order homomorphism from W_F to $\text{GL}_2(\mathbb{C})$ that sends a generator of tame inertia to $\begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^{-1} \end{pmatrix}$ and some lift of Fr to $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ so that $\varphi(W_F) \cong S_3$, somewhat similar to the previous paragraph.

- (3) $S^\vee \cong \text{SO}_4(\mathbb{C})$. In this case, $\varphi(I_F) = Z_{\text{SO}_4(\mathbb{C})}$ is up to conjugate the unique order 2 subgroup of G^\vee . This is because $\varphi(I_F)$ has to be generated by either $\lambda(1, -1)$, $\lambda(-1, 1)$ or $\lambda(-1, -1)$, yet the above three elements are themselves conjugate.

- (a) If φ is discrete, we need the centralizer $\mathcal{G}_\varphi := Z_{S^\vee}(\varphi(\text{Fr}))$ to be semisimple which is only possible when $\mathcal{G}_\varphi = S^\vee$, i.e. $\varphi(\text{Fr}) \in Z_{G^\vee}(S^\vee) = Z_{S^\vee} = \mu_2$. In this case $\varphi|_{W_F}$ factors through $\text{Gal}(E/F)$ for some degree 2 ramified Galois extension. There are two such choices. For each choice we have a unique Langlands parameter with $S_\varphi = Z_{\text{SO}_4(\mathbb{C})} = \mu_2$. Each L -packet consists of one non-supercuspidal representation and one supercuspidal representation.

- (b) Suppose φ is non-discrete. Then $\varphi(\text{Fr})$ commutes with $\varphi(I_F) = Z_{S^\vee} \cong \mu_2$. The centralizer $Z_{G^\vee}(\varphi) \subset S^\vee \cong \text{SO}_4(\mathbb{C})$ is connected except for one case when $\mathfrak{G}_\varphi = Z_{G^\vee}(\varphi) = \text{S}(\text{O}_2(\mathbb{C}) \times \text{O}_2(\mathbb{C}))$, in which case $\varphi(\text{Fr})$ is mapped to a non-central order 2 element in $\text{SO}_4(\mathbb{C})$ (and all of them live in a single orbit). Outside the exception case, by Proposition 5.0.1 we always get a singleton L -packet consisting of a representation in the principal series.
- (c) For the exception case, we have a unique such L -packet. By Property 10.1.9, it has two tempered members. Each of them is contained in a principal series coming from a biquadratic order 2 character.
- (4) $S^\vee \cong \text{GL}_2(\mathbb{C})$. In this case, $\mathfrak{G}_\varphi = Z_{S^\vee}(\varphi(\text{Fr}))$ can be GL_2 , GL_1 , Sp_2 or O_2 .
- (a) The only discrete φ arises in the unique semisimple case $\mathfrak{G}_\varphi = \text{Sp}_2$. In this case φ is of SL_2 -type in §5.2.3.
- (b) Otherwise φ is non-discrete. We note that $\varphi(I_F)$ is contained in $Z_{S^\vee} \cong Z_{\text{GL}_2(\mathbb{C})} = \text{GL}_1(\mathbb{C})$. In the case when $\mathfrak{G}_\varphi \cong \text{GL}_2(\mathbb{C})$, $S_\varphi = \{1\}$ and $\varphi(\text{Fr})$ commutes with Z_{S^\vee} , and hence we fall into the situation of Proposition 5.0.1. When $\mathfrak{G}_\varphi \cong \text{Sp}_2$ or GL_1 , $\varphi(W_F)$ is non-abelian yet $S_\varphi = \{1\}$. By the 10.1.9, we get an L -packet consisting of a single representation contained in the parabolic induction from $\text{GL}_2(F)$ (short root $\text{GL}_2(F)$ in G if $S^\vee \subset G^\vee$ is short root $\text{GL}_2(\mathbb{C})$, and both long otherwise) given by $\varphi|_{W_F}$ which has non-abelian image and thus necessarily supercuspidal.
- (c) Lastly, when $\mathfrak{G}_\varphi = \text{O}_2(\mathbb{C})$, we have $\varphi(\text{SL}_2(\mathbb{C})) = 1$, but $S_\varphi \cong \mu_2$. In this case, $S^\vee \cong \text{GL}_2(\mathbb{C})$ is contained in its normalizer which is isomorphic to $\text{SO}_4(\mathbb{C}) \subset G^\vee$, while $\varphi(\text{Fr})$ normalizes S^\vee in an outer manner. Up to inner conjugation, there is a unique automorphism of $\text{GL}_2(\mathbb{C})$ with fixed subgroup isomorphic to $\text{O}_2(\mathbb{C})$. Hence within $\text{SO}_4(\mathbb{C})$, up to conjugation,

$$\varphi(I_F) \subset \left\{ \begin{pmatrix} s & & & \\ & s^{-1} & & \\ & & s & \\ & & & s^{-1} \end{pmatrix} \mid s \in S^1 \subset \mathbb{C}^\times \right\}, \quad \varphi(\text{Fr}) = \begin{pmatrix} & & & 1 \\ & & & \\ & & 1 & \\ 1 & & & \end{pmatrix}.$$

Also, recall $\varphi(\text{SL}_2(\mathbb{C})) = 1$. We need $\varphi(I_F)$ large enough such that

$$Z_{G^\vee}(\varphi) = \left\{ \begin{pmatrix} z & & & \\ & z^{-1} & & \\ & & z & \\ & & & z^{-1} \end{pmatrix} \mid z \in \mathbb{C}^\times \right\} \sqcup \left\{ \begin{pmatrix} z^{-1} & & & z \\ & & & \\ & & z^{-1} & \\ & & & z \end{pmatrix} \mid z \in \mathbb{C}^\times \right\}.$$

By Property 10.1.9, the L -packet consists of two non-discrete tempered representations, both contained in a parabolic induction from $\text{GL}_2(F)$ (short root $\text{GL}_2(F)$ in G if $S^\vee \subset G^\vee$ is short root $\text{GL}_2(\mathbb{C})$, and both long otherwise) given by $\varphi|_{W_F}$ that has non-abelian image and thus necessarily supercuspidal.

We now make a few remarks. We have $Z_{\text{SO}_4(\mathbb{C})}(\text{O}_2(\mathbb{C})) \cong \text{O}_2(\mathbb{C})$, while $Z_{\text{SO}_4(\mathbb{C})}(\text{SO}_2(\mathbb{C})) \cong \text{GL}_2(\mathbb{C})$. In particular, the supercuspidal representation of $\text{GL}_2(F)$ is characterized by the property that its L -parameter has image in $\text{O}_2(\mathbb{C})$ where Fr is sent to the non-trivial component and I_F is sent to $\text{SO}_2(\mathbb{C})$. In other words, the central character of the supercuspidal representation of $\text{GL}_2(F)$ is the unramified order-two character. Conversely, any supercuspidal representation of $\text{GL}_2(\mathbb{C})$ whose

central character has order-two has its corresponding Langlands parameter mapping into some $O_2(\mathbb{C})$ (but not into $SO_2(\mathbb{C})$) by Lemma 5.2.4. By Property 10.1.9, its parabolic induction to G has a non-discrete constituent living in an L -packet of 2 members. By Property 10.1.1, these 2 members are both tempered. See also §5.2.2(2e) for the case where the supercuspidal representation of $GL_2(F)$ has positive-depth, and its central character is again of order two but possibly ramified.

- (5) $S^\vee \cong GL_1(\mathbb{C})^2$ is a maximal torus of G^\vee . We have the following cases:
- (a) For discrete φ , $\varphi(\text{Fr})$ acts on S^\vee through an elliptic element in $W_{\mathbf{G}}$, such that $S_\varphi \cong (\mu_2)^2$, μ_3 and μ_1 , respectively, when the Weyl group element has order 2, 3 or 6. In this case, the Langlands parameter is regular supercuspidal, and the Local Langlands has been determined in [Kal19]. The size of the L -packet is 4, 3 or 1 respectively.
 - (b) For non-discrete φ , $\varphi(\text{Fr})$ acts on S^\vee through some non-elliptic element w in the Weyl group $W_{\mathbf{G}}$, giving us a non-discrete φ . Thus we have that w is either trivial or a reflection. When w is trivial, $S_\varphi = \pi_0(S^\vee)$ is connected and Proposition 5.0.1 applies. When w is a reflection, we have $S_\varphi = \pi_0(Z_{S^\vee}(\text{Fr})) \cong (X_*(S^\vee)_{\text{Fr}})_{\text{tor}} = \{1\}$. Hence we have a singleton L -packet consisting of an irreducible constituent of the parabolic induction of a supercuspidal representation of $GL_2(F)$.

5.2.2. *Positive-depth L -parameters.* In this case, G_0^\vee is a proper Levi subgroup of G^\vee , which is either $GL_2(\mathbb{C})$ or $GL_1(\mathbb{C})^2$. We have

- (1) When $G_0^\vee \cong GL_1(\mathbb{C})^2$ is a maximal torus of G^\vee , the whole W_F acts on G_0^\vee through a subgroup of the Weyl group, giving $S_\varphi \cong (\mu_2)^2$, μ_3 , μ_2 or trivial, and thus an L -packet with 1, 2, 3 or 4 members. The members are regular supercuspidal representations if $X_*(G_0^\vee)^{W_F}$ is trivial. Otherwise, W_F acts on $X_*(G_0^\vee)$ via at most a single reflection. In this case we have $S_\varphi = \{1\}$ (as in §5.2.1(5a)). When W_F acts trivially on $X_*(G_0^\vee)$, Proposition 5.0.1 applies. Otherwise, when W_F acts by a reflection we have a singleton L -packet consisting of a constituent of the parabolic induction from a supercuspidal representation of $GL_2(F)$.
- (2) When $G_0^\vee \cong GL_2(\mathbb{C})$, there are several cases for which $S^\vee = Z_{G_0^\vee}(\varphi(I_F))$. We first assume φ is discrete:
 - (a) $S^\vee \supset SL_2(\mathbb{C})$. In this case we necessarily have $\mathcal{G}_\varphi = SL_2(\mathbb{C})$ for φ to be discrete. This implies that φ is of SL_2 -type, and we refer to §5.2.3.
 - (b) S^\vee is a maximal torus, in which case $\mathcal{G}_\varphi = S_\varphi = S^\vee[2] \cong (\mu_2)^2$ is a regular supercuspidal L -parameter with 4 members in its L -packet.
 - (c) S^\vee is a torus of rank 1. In this case $\mathcal{G}_\varphi = S_\varphi = S^\vee[2] \cong \mu_2$ is a regular supercuspidal L -parameter with 2 members in its L -packet.
 - (d) $S^\vee \cong O_2(\mathbb{C})$ is disconnected. In this case $\mathcal{G}_\varphi = S_\varphi \cong (\mu_2)^2$ is a non-singular supercuspidal L -parameter with 4 members in its L -packet.

Lastly, we turn to the non-discrete case in which $\dim \mathcal{G}_\varphi > 0$. We claim that we always have \mathcal{G}_φ connected and thus $S_\varphi = \{1\}$ except when $Z_{G^\vee}(\varphi) = \mathcal{G}_\varphi = S^\vee \cong O_2(\mathbb{C})$. This is done by investigating the possibilities for \mathcal{G}_φ in

cases (2a)(2b)(2c) above. In the last case (2d) when $S^\vee \cong O_2(\mathbb{C})$, to have non-discrete L -parameter we have either $\mathcal{G}_\varphi \cong SO_2(\mathbb{C})$ or $\mathcal{G}_\varphi \cong O_2(\mathbb{C})$. In the former case we have again $S_\varphi = \{1\}$. Yet in the latter case:

- (e) When $\mathcal{G}_\varphi \cong O_2(\mathbb{C})$, the L -parameter has the same centralizer and double centralizer (a different $O_2(\mathbb{C})$) as in §5.2.1(4c). Hence the L -packet consists of two non-discrete tempered constituents of parabolic inductions of supercuspidal representations of $GL_2(F)$ with the same properties as described in §5.2.1(4c), except for one difference: the central character of the supercuspidal representation of $GL_2(F)$ may be unramified or ramified. Since $\mathcal{G}_\varphi \cong O_2(\mathbb{C})$, we have $Z_{G^\vee}(\mathcal{G}_\varphi) \cong O_2(\mathbb{C})$, therefore the central character is the order-two character specified by $\varphi: W_F \rightarrow \pi_0(Z_{G^\vee}(\mathcal{G}_\varphi))$.

5.2.3. L -parameters and representations of SL_2 -type.

Definition 5.2.2. We say that an L -parameter $\varphi: W_F \times SL_2(\mathbb{C}) \rightarrow G^\vee$ is of *short SL_2 -type*, if it is discrete and $\varphi(SL_2(\mathbb{C}))$ is a short root SL_2 in $G^\vee = G_2(\mathbb{C})$. We say that an L -parameter $\varphi: W_F \times SL_2(\mathbb{C}) \rightarrow G^\vee$ is of *long SL_2 -type*, if it is discrete and $\varphi(SL_2(\mathbb{C}))$ is a long root SL_2 in $G^\vee = G_2(\mathbb{C})$. We say that it is of *SL_2 -type* if it is of either short or long SL_2 -type.

When $\varphi(SL_2(\mathbb{C}))$ is a short (resp. long) root $SL_2(\mathbb{C})$, we have that $Z_{G^\vee}(\varphi(SL_2(\mathbb{C})))$ is a long (resp. short) root $SL_2(\mathbb{C})$, such that the two $SL_2(\mathbb{C})$'s live in an $SO_4(\mathbb{C}) \subset G^\vee = G_2(\mathbb{C})$. In particular, we have the following.

Lemma 5.2.3. *A discrete L -parameter is of short (resp. long) SL_2 -type if and only if $\varphi(W_F)$ lies in a long (reps. short) $SL_2(\mathbb{C}) \subset G^\vee$. An L -parameter for which $\varphi(W_F)$ is contained in such an $SL_2(\mathbb{C})$ is discrete if and only if $\varphi|_{W_F}$ is a discrete L -parameter into $SL_2(\mathbb{C})$. Moreover, two L -parameters of short (resp. long) SL_2 -type are conjugate if and only if the corresponding discrete L -parameter into $SL_2(\mathbb{C})$ are conjugate.*

Proof. It only remains to show the last statement. Note that the normalizer of a short or long $SL_2(\mathbb{C})$ in G^\vee is the aforementioned $SO_4(\mathbb{C})$. The conjugacy actions of this $SO_4(\mathbb{C})$ on $SL_2(\mathbb{C})$ are all inner (factoring through the corresponding $PGL_2(\mathbb{C})$), hence the claim. \square

Recall from 2.1.2(3) the definition for a discrete L -parameter $\varphi: W_F \rightarrow H^\vee$ (for some H^\vee) to be *supercuspidal*—a terminology that makes sense since it is expected, and indeed implied by Property 10.1.9, that the L -packet $\Pi_\varphi(G)$ consists of only supercuspidal representations. We will study the supercuspidal L -parameters φ such that $\varphi(W_F)$ is contained in $SL_2(\mathbb{C})$. We have the following group-theoretic result on the LLC for GL_2 (or PGL_2):

Lemma 5.2.4. *Suppose $H \subset GL_2(\mathbb{C})$ is a solvable subgroup with $Z_{SL_2(\mathbb{C})}(H)$ finite. Then H is contained in the normalizer of a unique maximal torus $T^\vee \subset GL_2(\mathbb{C})$, and $Z_{SL_2(\mathbb{C})}(H) = Z_{SL_2(\mathbb{C})} = \mu_2$.*

5.2.1. Hence any supercuspidal L -parameter into $\mathrm{SL}_2(\mathbb{C})$ always corresponds to a *compound L -packet* (i.e. a Vogan L -packet) of two members, consisting of an irreducible supercuspidal representation of $\mathrm{PGL}_2(F)$ and another finite-dimensional irreducible representation of the compact inner form of $\mathrm{PGL}_2(F)$. We observe that for our original L -parameter φ , we have $Z_{G^\vee}(\varphi) = Z_{\mathrm{SL}_2(\mathbb{C})}(\varphi(W_F)) = \mu_2$ by Lemma 5.2.4. Hence the L -packet $\Pi_\varphi(G)$ consists of a non-supercuspidal representation and a supercuspidal representation by Lemma 5.2.1. Let P be a parabolic subgroup of G with Levi subgroup isomorphic to $\mathrm{GL}_2(F)$. Here the $\mathrm{GL}_2(F)$ corresponds to the short (resp. long) root if φ is of short (resp. long) SL_2 -type. By Property 10.1.9, the non-supercuspidal member has the following supercuspidal support: $\varphi|_{W_F}: W_F \rightarrow \mathrm{SL}_2(\mathbb{C})$, which gives us a supercuspidal τ of $\mathrm{PGL}_2(F)$. We consider $\tau_s := \tau \otimes (\nu_F \circ \det)^s$ as a representation of $\mathrm{GL}_2(F)$. Then the non-supercuspidal of G is the generic constituent in $i_P^G \tau_{\pm \frac{1}{2}}$ (both $s = \pm \frac{1}{2}$ work).

5.2.4. *Table for discrete L -parameters.* As a summary, we give a table for discrete L -parameters that are neither unipotent nor supercuspidal. In each row, any Galois-theoretic input as indicated gives a (unique) discrete L -parameter.

Label	Galois-theoretic input	$\varphi(W_F)$	\mathcal{G}_φ	S_φ	$\varphi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)$
§5.2.1(2a) §6.0.1(2c)	E/F ramified Galois cubic (only exist when $q \equiv 1(3)$)	$\mathbb{Z}/3$	SL_3	μ_3	subreg.
§5.2.1(2b) §6.0.1(1c)	$q \equiv -1(3)$	$(\mathbb{Z}/3) \rtimes (\mathbb{Z}/4)$	SL_2 (long)	μ_2	long
§5.2.1(2c)	$q \equiv -1(3)$	S_3	SO_3 (subreg.)	1	subreg.
§5.2.1(3a) §6.0.1(3c)	E/F ramified quadratic	$\mathbb{Z}/2$	SO_4	μ_2	subreg.
§5.2.3	supercuspidal L -parameter into $\mathrm{SL}_2(\mathbb{C})$	non-split extension of $\{\pm 1\}$ by \mathbb{Z}/n , $n \geq 4$ even	SL_2 (short)	μ_2	short
§5.2.3	supercuspidal L -parameter into $\mathrm{SL}_2(\mathbb{C})$	non-split extension of $\{\pm 1\}$ by \mathbb{Z}/n , $n \geq 4$ even	SL_2 (long)	μ_2	long

Note that the second row is indeed a special case of the last row when the L -parameter is tame, $n = 6$ and $|\varphi(I_F)| = 3$.

6. SUPERCUSPIDAL REPRESENTATIONS OF G_2

Our goal here is to attach, to every irreducible supercuspidal representation of $G_2(F)$, a cuspidal enhanced Langlands parameter into G^\vee . When the supercuspidal representation is unipotent, this has been done in [Lus95], and when the supercuspidal representation is non-singular, this has been done in [Kal19, Kal21]. In this section, we shall focus on the cases *not* covered in pre-existing literature. By [Kim07, Fin21], all supercuspidal representations of $\mathbf{G}(F)$ are *tame* supercuspidal representations in the sense of [Yu01] when $p \neq 2, 3$.

6.0.1. *Depth-zero supercuspidal representations.* As recalled in §4, there are three maximal parahoric subgroups with reductive quotient G_2 , SL_3 and SO_4 over $k_F = \mathbb{F}_q$. By (3.6.11), depth-zero irreducible supercuspidal representations are in bijection with irreducible cuspidal representations on these reductive quotients \mathbb{G}_x . By (3.6.4) and Proposition 3.6.1, any such cuspidal representation is labeled with

- (i) a semisimple conjugacy class (s) with s an *isolated* element of \mathbb{G}_x^\vee , i.e. $Z_{\mathbb{G}_x^\vee}(s)$ is not contained in a \mathbb{F}_q -rational Levi subgroup of any proper \mathbb{F}_q -rational parabolic subgroup of $Z_{\mathbb{G}_x^\vee}(s)$;
- (ii) a unipotent cuspidal representation of the group of the k_F -rational points of the reductive algebraic group $\mathbb{H} := Z_{\mathbb{G}_x^\vee}(s)^\vee$, where $k_F = \mathbb{F}_q$. We denote $\mathbb{H} := \mathbb{H}(k_F)$.

- (1) We begin with the case $\mathbb{G}_x = G_2$, i.e. $x = x_0$ and G_{x_0} is the hyperspecial maximal parahoric subgroup of G . By [Bon05, Prop. 4.9, Theorem 5.1 and Table I], the group \mathbb{H}^\vee can be G_2 , SL_3 , SU_3 , (split) SO_4 , U_2 or any elliptic maximal torus. By [CR74], when \mathbb{H}^\vee is of type A_2 ,

$$(6.0.1) \quad \mathbb{H}^\vee(\mathbb{F}_q) = (Z_{\mathbb{G}_{x_0}^\vee}(s))(\mathbb{F}_q) = \begin{cases} SL_3(q) & \text{if } q \equiv 1 \pmod{3}, \\ SU_3(q) & \text{if } q \equiv -1 \pmod{3}. \end{cases}$$

By the classification of unipotent cuspidal representations in [Lus78],

- neither $SL_3(\mathbb{F}_q)$ nor $SO_4(\mathbb{F}_q)$ has cuspidal unipotent representations. Indeed, for $m \geq 2$, the group $SO_{2m}(\mathbb{F}_q)$ has a (unique) cuspidal unipotent representation only if m is the square of an integer;
- $SU_n(\mathbb{F}_q)$, $n = k(k+1)/2$ are the only projective unitary groups that possess unipotent cuspidal representations, and each one of them has a unique unipotent cuspidal representation, which corresponds to the partition $(k, k-1, \dots, 1)$ of n .

Thus the only possibilities for $\mathbb{H}(\mathbb{F}_q)$ equipped with a cuspidal unipotent representation are $G_2(\mathbb{F}_q)$, $SU_3(\mathbb{F}_q)$, or torus.

- (a) In the first case, i.e. when $\mathbb{H} = G_2$, the corresponding supercuspidal and cuspidal representations are unipotent. The group $G_2(\mathbb{F}_q)$ has four cuspidal unipotent irreducible representations (see [Lus78, Theorem 3.28]), denoted as $G_2[1]$, $G_2[-1]$, $G_2[\zeta_3]$ and $G_2[\zeta_3^2]$. Their dimensions are computed in [Car93, p. 460] and recorded in Table 9. The corresponding

irreducible supercuspidal unipotent representations are

$$(6.0.2) \quad \pi[\zeta_3] := i_{\mathbb{G}_{x_0}}^G \mathbb{G}_2[\zeta] \quad \text{for } \zeta \in \{1, -1, \zeta_3, \zeta_3^2\}.$$

With Haar measure on G normalized as in (3.4.2), by 3.6.16 the representation $\pi[\zeta]$ has formal degree

$$(6.0.3) \quad \text{fdeg}(\pi[\zeta]) = \frac{q}{|\mathbb{G}_2(\mathbb{F}_q)|_{p'}} \cdot \dim \mathbb{G}_2[\zeta] = \frac{q}{(q^6 - 1)(q^2 - 1)} \cdot \dim \mathbb{G}_2[\zeta].$$

Representation $\mathbb{G}_2[\zeta]$	Dimension of $\mathbb{G}_2[\zeta]$	fdeg($\pi[\zeta]$)
$\mathbb{G}_2[1]$	$\frac{q(q-1)^2(q^3+1)}{6(q+1)}$	$\frac{q^2-q+1}{6(q+1)^2(q^4+q^2+1)}$
$\mathbb{G}_2[-1]$	$\frac{q(q-1)(q^6-1)}{2(q^3+1)}$	$\frac{q^2}{2(q^3+1)(q+1)}$
$\mathbb{G}_2[\zeta_3], \mathbb{G}_2[\zeta_3^2]$	$\frac{q(q^2-1)^2}{3}$	$\frac{1}{3(q^4+q^2+1)}$

TABLE 9. Unipotent supercuspidal irreducible representations of $\mathbb{G}_2(F)$.

Remark 6.0.1. Let $P(X) := \sum_{w \in W_G} X^{\ell(w)}$ be the Poincaré polynomial of W_G , with ℓ the length function on W_f . We have

$$(6.0.4) \quad P(X) = \frac{(X+1)(X^6-1)}{X-1}.$$

By (6.0.3) and (4.0.13), we have

$$(6.0.5) \quad \text{fdeg}(\pi[\zeta]) = \frac{q}{(q-1)^2 \cdot P(q)} \cdot \dim \mathbb{G}_2[\zeta] = \frac{\dim \mathbb{G}_2[\zeta]}{\text{Vol}(\mathcal{J}_{\mathbb{G}_2}) \cdot P(q)}.$$

- (b) In the case where \mathbb{H}^\vee is any elliptic maximal torus, the representation τ is k_F -non-singular. Since G_{x_0} is a hyperspecial maximal parahoric subgroup of G , the notion of k_F -non-singularity and F -non-singularity agree in this case. Thus the supercuspidal representation π is non-singular (equivalently it is also regular).
- (c) In the last remaining case where $\mathbb{H}^\vee = \text{SU}_3$. Note that $Z_{\text{SU}_3} = Z_{\mathbb{H}^\vee}$ needs to contain s as a non-zero rational point, which implies that $q \equiv -1(3)$. Conversely, if $q \equiv -1(3)$, the Coxeter torus of \mathbb{G}_x has two order-3 elements whose centralizers are SU_3 . In fact, all order-3 elements $s \in \mathbb{G}_2(\overline{\mathbb{F}}_q)$ with $Z_{\mathbb{G}_2}(s) \cong \text{SL}_3(\overline{\mathbb{F}}_q)$ lie in a single geometric orbit. Since SU_3 is connected (\mathbb{G}_2 is simply connected), all such rational order-3 elements lie in a rational orbit. Hence there is one choice for $\mathbb{H}^\vee \cong \text{SU}_3$ when $q \equiv -1(3)$, and no choice when $q \equiv 1(3)$. When $q \equiv -1 \pmod{3}$, the group $(Z_{\mathbb{G}_{x_0}^\vee}(s))(\mathbb{F}_q)$ is the special unitary group $\text{SU}3(\mathbb{F}_q)$, and τ_{unip} is the unique irreducible cuspidal unipotent representation of $\text{SU}3(\mathbb{F}_q)$,

which is parametrized by the partition $(2, 1)$ of 3. It is clear that in this case, the representation τ is k_F -singular, in particular, the representation π is singular by [Kal21, 3.1.4]. Consequently, there exists one singular non-unipotent cuspidal representation τ of $\mathbb{G}_x(\mathbb{F}_q)$ when $q \equiv -1(3)$, and none otherwise. Therefore, when $q \equiv -1(3)$, there is a singular depth-zero supercuspidal representation of G , and we need to find the non-supercuspidal representation in its L -packet. We do so by computing the formal degree of π . By [Car93, §13.7], $\dim(\tau_{\text{unip}}) = q(q-1)$. By (3.6.6)

$$(6.0.6) \quad \dim(\tau) = \frac{|\mathbb{G}_{x_0}|_{p'}}{|(Z_{\mathbb{G}_{x_0}^\vee}(s)^\vee(\mathbb{F}_q))|_{p'}} \dim(\tau_{\text{unip}}).$$

We have

$$(6.0.7) \quad |\text{SU}_3(\mathbb{F}_q)| = q^6(q^{3/2} + 1)(q-1)(q^{1/2} + 1).$$

By (3.4.2), we have

$$(6.0.8) \quad \text{Vol}(G_{x_0}) = q^{-\text{rk}(G)/2} |\mathbb{G}_{x_0}|_{p'} = q^{-1} |\mathbb{G}_{x_0}|_{p'}.$$

Since $\pi = i_{\mathbb{G}_{x_0}}^G \tau$, by combining (3.6.6), (6.0.7) and (6.0.8), we have

$$(6.0.9) \quad \text{fdeg}(\pi) = \frac{\dim \tau}{\text{Vol}(G_{x_0})} = \frac{\dim(\tau_{\text{unip}})}{|(Z_{\mathbb{G}_{x_0}^\vee}(s)^\vee(\mathbb{F}_q))|_{p'} \cdot q^{-1}}.$$

- (2) Next we look at the case where $\mathbb{G}_x = \text{SL}_3$, i.e. $x = x_1$. The center of SL_3 , which consists of the scalar matrices $a \text{Id}_3$ such that $a^3 = 1$, is finite, hence disconnected. By [Bon05, Proposition 5.2], the possibilities for \mathbb{H}^\vee are PGL_3 (i.e. when $d = 1$ and $s = \text{Id}$ *loc.cit.*) and $\mathbb{T}^\vee \rtimes \mu_3$ (i.e. when $d = 3$ and $s = \text{diag}(1, \zeta_3, \zeta_3^2) \pmod{\mathbb{F}_q^\times}$ for the primitive third root of unity ζ_3 , by *loc.cit.* in this case the Weyl group of $\mathbb{H}^\circ = Z_{\mathbb{G}_{x_1}^\vee}(s)^\circ$ is trivial, thus $\mathbb{H}^\circ = \mathbb{T}^\vee$) or \mathbb{T}^\vee is an elliptic maximal torus of PGL_3 (they are all rationally conjugate). Let us discuss these cases:

- (a) $\mathbb{H} = \text{SL}_3$, The group SL_3 does not admit cuspidal unipotent representations, so the case $\mathbb{H} = \text{SL}_3$ does not occur.
- (b) $\mathbb{H}^\vee = \mathbb{T}$. This happens if $s \in \mathbb{T}(\mathbb{F}_q)$ is not of order 1 or 3. In this case, one checks that $Z_{G^\vee}(s)$ is a torus, i.e. we get only regular supercuspidal representations.
- (c) $\mathbb{H}^\vee \cong \mathbb{T} \rtimes \mu_3$. This happens if s is an order-3 element in $\mathbb{T}(\mathbb{F}_q) \cong \mathbb{F}_{q^3}^\times / \mathbb{F}_q^\times$. Such an element exists if and only if $q \equiv 1(3)$. Note that while all the T 's are conjugate, the two order-3 elements are *not* rationally conjugate in $\text{PGL}_3(\mathbb{F}_q)$, because they map to the two different non-trivial classes in $\text{coker}(\text{SL}_3(\mathbb{F}_q) \rightarrow \text{PGL}_3(\mathbb{F}_q))$. Hence when $q \equiv 1(3)$, we have two choices of such s 's. The Deligne-Lusztig induction for each s is a direct sum of 3 cuspidal representations of $\text{SL}_3(\mathbb{F}_q)$ of the same dimension. Hence we get 2 families, of 3 singular supercuspidal representations all of the same formal degree.

Using (3.6.16), we compute the formal degree as follows (note that $p \neq 2, 3$): we have $|\mu_3|_{p'} = 3$ and $|\mathbb{T}| = \frac{q^3-1}{q-1} = q^2 + q + 1$, thus

$$(6.0.10) \quad \text{fdeg}(i_{\mathbb{G}_{x_1}}^G \tau) = \frac{q^{2/2} \cdot 1}{3(q^2 + q + 1)} = \frac{q}{3(q^2 + q + 1)}.$$

Hence we obtain

$$(6.0.11) \quad \text{fdeg}(i_{\mathbb{G}_{x_1}}^G \tau) = \frac{q}{3(q^2 + q + 1)}.$$

(3) Lastly, we look at the case $\mathbb{G}_x = \text{SO}_4$. A centralizer subgroup in $\mathbb{G}_x^\vee \cong \text{SO}_4$ has unipotent cuspidal representations only if its identity component is a torus. Hence we begin with $\mathbb{T}^\vee \subset \mathbb{G}_x^\vee$ any representative of the unique rational conjugacy class of elliptic maximal torus. Abstractly, $\mathbb{T}^\vee \cong \text{U}_1 \times \text{U}_1$ over \mathbb{F}_q so that $\mathbb{T}^\vee(\mathbb{F}_q) \cong (\mathbb{Z}/(q+1))^2$. We discuss three scenarios:

- (a) $s \in \mathbb{T}^\vee(\mathbb{F}_q)$ is not perpendicular to any coroot of G . In this case we have non-singular supercuspidals, and in fact regular ones because G^\vee is simply connected.
- (b) $s \in \mathbb{T}^\vee(\mathbb{F}_q)$ is perpendicular to a unique pair of coroots $\pm\alpha^\vee$ of G . The cocharacter lattice is generated by α^\vee and another λ . If α^\vee is a short coroot, then s is not perpendicular to any other coroots if and only if $(2\lambda)(s) = \lambda(s)^2 \neq 1$. If α^\vee is a long coroot, then s is not perpendicular to any other coroots if and only if $\lambda(s)^2 \neq 1 \neq \lambda(s)^3$. The rational conjugacy class of s is determined by $\{\lambda(s)^{\pm 1}\}$. Hence there are in general $\frac{q-1}{2}$ such s unless $q \equiv -1(3)$ and α^\vee is a long coroot, in which case there are $\frac{q-3}{2}$ such s . Each s gives a singular supercuspidal matched with those in §5.2.1(4), or rather the tamely ramified L -parameters in §5.2.3.
- (c) $s \in \mathbb{T}^\vee$ is perpendicular to two pairs of coroots of G , which correspond to $\text{SO}_4(\mathbb{C}) \subset \text{G}_2(\mathbb{C})$. This is the case when s is the unique (up to rational conjugacy) order 2 element in $\mathbb{T}^\vee(\mathbb{F}_q)$ with $Z_{\mathbb{G}_x^\vee}(s)^\circ = \mathbb{T}^\vee$. In this case $Z_{\mathbb{G}_x^\vee}(s) \cong \mathbb{T}^\vee \rtimes \mu_2$. Indeed,

$$(6.0.12) \quad (Z_{\mathbb{G}_{x_2}^\vee}(s))(\mathbb{F}_q) = \{(g_1, g_2) \in \text{O}_2^-(\mathbb{F}_q) \times \text{O}_2^-(\mathbb{F}_q) : \det(g_1) = \det(g_2)\},$$

where O_2^- denotes the non-split form of O_2 . The group $\text{SO}_2^-(\mathbb{F}_q)$ has a unique cuspidal unipotent representation, the trivial representation. Thus $\text{O}_2^-(\mathbb{F}_q)$ admits two cuspidal unipotent representations: the trivial one and the sign representation. So $(Z_{\mathbb{G}_{x_2}^\vee}(s))^\vee(\mathbb{F}_q)$ has two cuspidal unipotent irreducible representations: $1 \otimes 1$ and $\text{sgn} \otimes \text{sgn}$. Both have dimension one. In this case, one can check that $Z_{G^\vee}(\tilde{s})^\circ = Z_{G^\vee}(\tilde{s}) = \text{SO}_4(\mathbb{C})$, and thus $Z_{G^\vee}(\varphi|_{I_F}) = \text{SO}_4(\mathbb{C})$. In particular, $Z_{G^\vee}(\varphi|_{W_F})$ is not finite, and the corresponding supercuspidal representations of $\text{G}_2(F)$ are singular. The Deligne-Lusztig representation is the direct sum of two cuspidal representations of the same dimension. Hence we get two supercuspidal representations to be matched in the L -packets in 5.2.1(3a).

We have $|\mathcal{O}_2^-(\mathbb{F}_q)| = 2(q+1)$, which gives $|\mathcal{O}_2^-(\mathbb{F}_q) \times \mathcal{O}_2^-(\mathbb{F}_q)| = 4(q+1)^2$ by [Car93], and hence

$$(6.0.13) \quad |(Z_{\mathbb{G}_{x_2}^\vee}(s))(\mathbb{F}_q)| = \frac{|\mathcal{O}_2^-(\mathbb{F}_q) \times \mathcal{O}_2^-(\mathbb{F}_q)|}{2} = 2(q+1)^2.$$

By (3.6.16), the formal degree of $i_{\mathbb{G}_{x_2}}^G \tau$ is given as follows:

$$(6.0.14) \quad \text{fdeg}(i_{\mathbb{G}_{x_2}}^G \tau) = \frac{q^{4/2}}{2(q+1)^2} = \frac{q^2}{2(q+1)^2},$$

which agrees with the formal degree for $\pi(\eta_2)$ from Table 17.

6.0.2. *Positive-depth supercuspidals.* Since $p \neq 2, 3$ any supercuspidal representation π of G is constructed via a Yu datum \mathcal{D} . The latter includes a tamely ramified twisted Levi sequence in \mathbf{G} is a finite sequence $\vec{\mathbf{G}} = (\mathbf{G}^0 \subset \mathbf{G}^1 \subset \dots \subset \mathbf{G}^d = \mathbf{G})$ of twisted Levi subgroups of \mathbf{G} that splits over a tamely ramified extension E of F (i.e., $\mathbf{G}_E^i := \mathbf{G}^i \times_F E$ is a split Levi subgroup of \mathbf{G}_E^{i+1} for each i), such that $Z_{\mathbf{G}^0}$ is anisotropic (since $Z_{\mathbf{G}} = \{1\}$), a positive-depth character on $\mathbf{G}^0(F)$, and a depth-zero supercuspidal of $\mathbf{G}^0(F)$. If the depth of π is positive then we have $d \geq 1$. The group \mathbf{G} being F -split, we have $\mathbf{G}_E = \mathbf{G}$. The only possibilities for \mathbf{G}_E^0 are \mathbf{G} , $\text{GL}_2(E)$ or a maximal E -split torus \mathbf{S} . Thus the only possibilities for \mathbf{G}^0 are \mathbf{G} , a torus or a unitary group (a priori it could be either $\text{U}(1, 1)$ or U_2 , but since $Z_{\mathbf{G}^0}$ is anisotropic, only the compact unitary group $\text{U}_b(2)$, for the quadratic extension $F(\sqrt{b})$ is possible).

The representation π is regular (resp. non-singular) if and only if the depth-zero supercuspidal of $\mathbf{G}^0(F)$ is. Hence the question is largely about classifying such \mathbf{G}^0 and their depth-zero supercuspidal representations. We note that \mathbf{G}^0 is determined by $\mathbf{Z}_0 := Z_{\mathbf{G}^0}$, an anisotropic subtorus of \mathbf{G} satisfying $Z_{Z_{\mathbf{G}}(\mathbf{Z}_0)} = \mathbf{Z}_0$. Also since we need \mathbf{G}^0 to have some positive-depth character, $1 \leq \dim \mathbf{Z}_0 \leq \text{rank } \mathbf{G} = 2$. We now discuss the possibilities for \mathbf{Z}_0 .

- (1) $\dim \mathbf{Z}_0 = 2$ is a maximal torus of G . In this case, we obtain only regular supercuspidal representations of positive depths.
- (2) $\dim \mathbf{Z}_0 = 1$ such that $\mathbf{G}^0/\mathbf{Z}_0 \cong \text{PGL}_2$ is split. In this case, all depth-zero supercuspidals of G^0 coming from cuspidals of reductive quotients of \mathbf{G}^0 are non-singular.
 - (a) It is regular unless,
 - (b) \mathbf{Z}_0 is ramified such that $\mathbf{G}^0 \cong \text{U}_2$ is ramified for which the reductive quotient Sp_2 has a unique pair of non-singular (but not regular) supercuspidal representations. Since there are two ramified quadratic extensions, we obtain two possible G^0 and 4 such non-singular supercuspidal representations in the L -packet in §5.2.2(2d).
- (3) $\dim \mathbf{Z}_0 = 1$ such that $\mathbf{G}^0/\mathbf{Z}_0$ is anisotropic, i.e. a compact inner form of PGL_2 . In this case,
 - (a) $G^0(F)$ has regular depth-zero supercuspidal representations
 - (b) $G^0(F)$ also has singular depth-zero supercuspidal representations. Any such singular depth-zero supercuspidal representation of $G^0(F)$ has the

property that its restriction to the derived subgroup is the trivial representation. Since $\mathbf{G}_{\text{der}}^0 := (\mathbf{G}^0)_{\text{der}}$ is simply connected, such a singular supercuspidal representation is a character of $\mathbf{G}^0(F)/\mathbf{G}_{\text{der}}^0(F)$. They all arise from L -parameters of SL_2 -type as in §5.2.3 and live in a mixed packet together with another non-supercuspidal representation, as described in Paragraph 5.2.1.

7. NON-SUPERCUSPIDAL REPRESENTATIONS OF G_2

Let $\mathfrak{s} = [L, \sigma]_G$, where L is a proper Levi subgroup L of $G = G_2(F)$. Let σ be an irreducible supercuspidal representation of L . In the case of $G = G_2(F)$, the Levi L being isomorphic to either $\text{GL}_2(F)$ or $F^\times \times F^\times$, all of its irreducible supercuspidal representations are regular. Thus σ is a regular supercuspidal, and we can consider an L -parameter $\varphi_\sigma: W_F \rightarrow L^V$ for σ , as defined in [Kal19] or [BH06a]. Both constructions coincide as shown in [OT21].

Lemma 7.0.1.

- (1) If $M = T$, we have $W_G^\mathfrak{s} \neq \{1\}$ if and only if $\mathfrak{s} = [T, \xi \otimes \xi]_G$ or $\mathfrak{s} = [T, \xi \otimes 1]_G$ with ξ an irreducible character of F^\times .
- (2) If $M = M_\alpha$ or $M = M_\beta$, we have $W_G^\mathfrak{s} \neq \{1\}$ if and only if σ is self-dual.

Proof. (1) Recall the definition from 3.3.3, we have

$$(7.0.1) \quad W_G^\mathfrak{s} = \{w \in W : w \cdot (\xi_1 \otimes \xi_2) = \chi(\xi_1 \otimes \xi_2) \text{ for some } \chi \in \mathfrak{X}_{\text{un}}(T)\}.$$

Let $\sigma^\circ := \xi_1|_{\mathfrak{o}_F^\times} \otimes \xi_2|_{\mathfrak{o}_F^\times}$. Then we have

$$(7.0.2) \quad W_G^\mathfrak{s} = \{w \in W : w \cdot \sigma^\circ = \sigma^\circ\}.$$

Let $\xi_i^\circ := \xi_i|_{\mathfrak{o}_F^\times}$. By Table 1, it follows that we have $W_G^\mathfrak{s} = \{1\}$ if and only if

$$\xi_1^\circ \neq 1, \quad \xi_2^\circ \neq 1, \quad \xi_1^\circ \xi_2^\circ \neq 1, \quad \xi_1^\circ \neq \xi_2^\circ, \quad (\xi_1^\circ)^2 \xi_2^\circ \neq 1, \quad \xi_1^\circ (\xi_2^\circ)^2 \neq 1.$$

Hence we have $W_G^\mathfrak{s} \neq \{1\}$ if and only if we are in one of the following cases:

- (i) We have $\xi_1^\circ = \xi_2^\circ$. We may and do assume that $\xi_1 = \xi_2 = \xi$.
 - (ii) We have $\xi_2^\circ = 1$. We may and do assume that $\xi_1 = \xi$ and $\xi_2 = 1$.
- (2) This is clear. See for example [AX22, 4.1.9] (or [Sha89]). □

Proposition 7.0.2. *Suppose that $W_G^\mathfrak{s} \neq \{1\}$. Then the possibilities for \mathfrak{s} and $W_G^\mathfrak{s}$ are as follows:*

- (1) $\mathfrak{s} = [T, 1]_G$. Here $W_G^\mathfrak{s} = W$.
- (2) $\mathfrak{s} = [T, \xi \otimes 1]_G$ with ξ ramified non-quadratic. Here $W_G^\mathfrak{s} \simeq \mathbb{Z}/2\mathbb{Z}$.
- (3) $\mathfrak{s} = [T, \xi \otimes \xi]_G$ with ξ ramified, neither quadratic nor cubic. Here $W_G^\mathfrak{s} \simeq \mathbb{Z}/2\mathbb{Z}$.
- (4) $\mathfrak{s} = [T, \xi \otimes \xi]_G$ with ξ ramified quadratic. Here $W_G^\mathfrak{s} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
- (5) $\mathfrak{s} = [T, \xi \otimes \xi]_G$ with ξ ramified cubic. Here $W_G^\mathfrak{s} \simeq S_3$.
- (6) $\mathfrak{s} = [M_\beta, \sigma]_G$ with σ self-dual. Here $W_G^\mathfrak{s} \simeq \mathbb{Z}/2\mathbb{Z}$.
- (7) $\mathfrak{s} = [M_\alpha, \sigma]_G$ with σ self-dual. Here $W_G^\mathfrak{s} \simeq \mathbb{Z}/2\mathbb{Z}$.

Proof. We first consider the case $M = T$ and describe the W -orbits. We have

$$(7.0.3) \quad W \cdot (\xi \otimes \xi) = \{\xi \otimes \xi, \xi^{-1} \otimes \xi^{-1}, \xi^2 \otimes \xi^{-1}, \xi^{-1} \otimes \xi^2, \xi \otimes \xi^{-2}, \xi^{-2} \otimes \xi\}.$$

It follows that

$$(7.0.4) \quad W \cdot (\xi \otimes \xi) = \begin{cases} \xi \otimes \xi, \xi \otimes 1, 1 \otimes \xi & \text{if } \xi \text{ is quadratic,} \\ \xi \otimes \xi, \xi \otimes \xi^{-1}, \xi^{-1} \otimes \xi, \xi^{-1} \otimes \xi^{-1} & \text{if } \xi \text{ is cubic.} \end{cases}$$

We have

$$(7.0.5) \quad |W \cdot (\xi \otimes \xi)| = \begin{cases} 1 & \text{if } \xi \text{ is trivial,} \\ 3 & \text{if } \xi \text{ is quadratic,} \\ 4 & \text{if } \xi \text{ is cubic,} \\ 6 & \text{otherwise.} \end{cases}$$

On the other hand, we have

$$(7.0.6) \quad W \cdot (\xi \otimes 1) = \{\xi \otimes 1, 1 \otimes \xi, \xi \otimes \xi^{-1}, \xi^{-1} \otimes 1, 1 \otimes \xi^{-1}, \xi^{-1} \otimes \xi\}.$$

If ξ is quadratic, then we have

$$(7.0.7) \quad W \cdot (\xi \otimes 1) = \{\xi \otimes \xi, \xi \otimes 1, 1 \otimes \xi\}.$$

Thus we have

$$(7.0.8) \quad |W \cdot (\xi \otimes 1)| = \begin{cases} 1 & \text{if } \xi \text{ is trivial,} \\ 3 & \text{if } \xi \text{ is quadratic,} \\ 6 & \text{otherwise.} \end{cases}$$

This gives the following possibilities for \mathfrak{s} :

(2) If $\xi_2 = 1$ and $\xi_1 = \xi$ with ξ a ramified non-quadratic character, by (7.0.6) we have

$$(7.0.9) \quad \begin{aligned} \mathfrak{s} &= [T, \xi \otimes 1]_G = [T, 1 \otimes \xi]_G = [T, \xi \otimes \xi^{-1}]_G \\ &= [T, \xi^{-1} \otimes 1]_G = [T, 1 \otimes \xi^{-1}]_G = [T, \xi^{-1} \otimes \xi]_G. \end{aligned}$$

It follows from Table 1 that

$$(7.0.10) \quad W_G^{\mathfrak{s}} = \{e, b\} \cong \mathbb{Z}/2\mathbb{Z}.$$

(3) If $\xi_1 = \xi_2 = \xi$ with ξ a ramified character that is neither quadratic nor cubic, by (7.0.3) we have

$$(7.0.11) \quad \begin{aligned} \mathfrak{s} &= [T, \xi \otimes \xi]_G = [T, \xi^{-1} \otimes \xi^{-1}]_G = [T, \xi^2 \otimes \xi^{-1}]_G \\ &= [T, \xi^{-1} \otimes \xi^2]_G = [T, \xi \otimes \xi^{-2}]_G = [T, \xi^{-2} \otimes \xi]_G. \end{aligned}$$

It follows from Table 1 that

$$(7.0.12) \quad W_G^{\mathfrak{s}} = \{e, a\} \cong \mathbb{Z}/2\mathbb{Z}.$$

(4) If $\xi_1 = \xi_2 = \xi$ with ξ a ramified quadratic character, since $\mathfrak{s} = [T, \xi_1 \otimes \xi_2]_G = [T, w \cdot (\xi_1 \otimes \xi_2)]_G$, we have $\mathfrak{s} = [T, \xi \otimes \xi]_G = [T, \xi \otimes 1]_G = [T, 1 \otimes \xi]_G$. Then Table 1 gives

$$\begin{aligned} W^{\mathfrak{s}} &= \{e, a, babab, bababa\} = \{e, a, r^3, ar^3\} \\ &= \langle s_\alpha, s_{3\alpha+2\beta} \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

(5) If $\xi_1 = \xi_2 = \xi$, with ξ a ramified character of order 3, we have

$$(7.0.13) \quad \mathfrak{s} = [T, \xi \otimes \xi]_G = [T, \xi^{-1} \otimes \xi^{-1}]_G = [T, \xi \otimes \xi^{-1}]_G = [T, \xi^{-1} \otimes \xi]_G.$$

It follows from Table 1 that

$$(7.0.14) \quad W_G^{\mathfrak{s}} = \{e, a, bab, abab, baba, ababa\} \cong S_3.$$

(6)&(7) We now consider the cases $M = M_\alpha$ and $M = M_\beta$. We have $N_G(M)/M \simeq \mathbb{Z}/2\mathbb{Z}$. Since σ is self-dual, by [AX22, 4.1.9] we have $W_G^{\mathfrak{s}} = N_G(M)/M$, thus $W_G^{\mathfrak{s}} \simeq \mathbb{Z}/2\mathbb{Z}$. \square

8. THE INTERMEDIATE SERIES

Let $\mathfrak{s} = [M, \sigma]_G$, where σ is a supercuspidal irreducible representation of $M \in \{M_\alpha, M_\beta\}$. Let ω_σ denote the central character of σ . Let P be a parabolic subgroup of G with Levi subgroup M , and let $i_P^G(\sigma)$ denote the (normalized) parabolic induction of σ . Since $M \simeq \mathrm{GL}_2(F)$, the representation σ is a *regular* supercuspidal in the sense of Definition 3.6.9. Therefore, the L -parameter φ_σ has trivial restriction to $\mathrm{SL}_2(\mathbb{C})$. Thus the corresponding unipotent class in $\mathrm{GL}_2(\mathbb{C})$ is trivial, and any cuspidal local system on this class is trivial (see §2.1.1 for details).

8.0.1. *The short root case.* When $M \simeq M_\alpha$, by [Sha91, Proposition 6.2]:

- (1) When $\omega_\sigma \neq 1$, $i_P^G(\sigma)$ is reducible and there are no complementary series.
- (2) When $\omega_\sigma = 1$, $i_P^G(\sigma)$ is irreducible, and $i_P^G(\sigma \otimes \nu_F^s)$ is irreducible unless $s = \pm 1/2$.
 - $i_P^G(\sigma \otimes \nu_F^{1/2})$ has length 2: it has a unique generic discrete series subrepresentation $\pi(\sigma)$ and a unique irreducible pre-unitary non-tempered Langlands quotient, $J(\sigma)$.
 - All the representations $i_P^G(\sigma \otimes \nu_F^s)$ for $0 < s < 1/2$ are in the complementary series and $s = 1/2$ is the edge of complementary series.

Since $\pi(\sigma)$ is a discrete series, by [Sol22, Proposition 9.3], it corresponds to a discrete series of the Hecke algebra $\mathcal{H}^s(G)$. On the other hand, by [AX22, Tables 4.5.6 and 4.5.8], the Hecke algebra $\mathcal{H}^s(G)$ is isomorphic to an extended affine Hecke algebra which is either of type $\tilde{A}_1(q, q)$ or has trivial parameters. In the first case, we can apply [Ram03, Table 2.1]. Since the representation $\pi(\sigma)$ is tempered, by [Ram03, Table 2.1], it has unipotent class e_{α_1} , which corresponds to \tilde{A}_1 by Table 6. When $\mathcal{H}^s(G)$ has trivial parameters, we obtain the enhanced L -parameters as in Table 8 (using the explicit formula in Definition 2.5.1).

Representation	indexing triple	enhanced L -parameter
$\pi(\sigma)$	$(t_a, e_{\alpha_1}, 1)$	$(\varphi_{\sigma, \tilde{A}_1}, 1)$
$J(\sigma)$	$(t_a, 0, 1)$	$(\varphi_{\sigma, 1}, 1)$

TABLE 10. : LLC for the irred. constituents of $i_P^G(\sigma \otimes \nu_F^{1/2})$ attached to $\mathfrak{s} = [M_\alpha, \sigma]_G$ where ω_σ unramified.

8.0.2. *The long root case.* When $M \simeq M_\alpha$, by [Sha90, Theorem 8.1, Proposition 8.3]:

- (1) If $\varphi_\sigma = \text{Ind}_{W_K}^{W_F}(\chi)$, $[K : F] = 2$, $\chi \in \widehat{K}^*$, then $i_P^G(\sigma)$ is irreducible if and only if either $\chi|_{F^\times} = \eta$, or $\chi|_{F^\times} = 1$ and $\chi^3 = 1$.
- (2) The representation $I(\tilde{\alpha}/i, \sigma)$ is reducible with a unique χ -generic subrepresentation $\pi(\sigma)$ which is in the discrete series. Its Langlands quotient $J(\sigma)$ is never generic. It is a pre-unitary non-tempered representation.

Remark 8.0.1. Since p is odd, the case where σ is extraordinary (so-called *exceptional* in [BH06b]) mentioned in [Sha90] does not occur (see [BH06b, Theorem 34.1 and Theorem 44.1]).

Representation	indexing triple	enhanced L -parameter
$\pi(\sigma)$	$(t_a, e_{\alpha_1}, 1)$	$(\varphi_{\sigma, A_1}, 1)$
$J(\sigma)$	$(t_a, 0, 1)$	$(\varphi_{\sigma, 1}, 1)$

TABLE 11. : LLC for the irred. constituents of $i_P^G(\sigma)$ attached to $\mathfrak{s} = [M_\beta, \sigma]_G$.

Since the representation $\pi(\sigma)$ is a discrete series, by [Sol22, Proposition 9.3], it corresponds to a discrete series of the Hecke algebra $\mathcal{H}^s(G)$. On the other hand, by [AX22, Tables 4.5.2 and 4.5.4], the Hecke algebra $\mathcal{H}^s(G)$ is isomorphic to an extended affine Hecke algebra which is of type $\tilde{A}_1(q^3, q)$, $\tilde{A}_1(q, q)$ or has trivial parameters. In the $\tilde{A}_1(q, q)$ case, we can apply [Ram03, Table 2.1]. Since $\pi(\sigma)$ is tempered, by [Ram03, Table 2.1], it has unipotent class e_{α_1} , which corresponds to A_1 by Table 6.

When $\mathcal{H}^s(G)$ has trivial parameters, similarly, we obtain the enhanced L -parameters as in Table 8 (using the explicit formula in Definition 2.5.1). When $\mathcal{H}^s(G)$ is of type $\tilde{A}_1(q^3, q)$, we use Property 10.1.9 to find the *singular* supercuspidal representation that shares the same L -packet as this intermediate series representation, and then use (2.5.1) again to explicitly construct the enhanced L -parameter. This is done in Theorem 5.1.2.

8.0.3. Formal degrees for depth-zero intermediate series. Consider a depth-zero representation $\pi \in \text{Irr}_{\mathfrak{s}}(G)$ where $\mathfrak{s} = [M, \sigma]_G$ and $M \in \{M_\alpha, M_\beta\}$. Since the depth is preserved via parabolic induction, the supercuspidal representation σ of $M \simeq \text{GL}_2(F)$ has also depth-zero and we have $\sigma = \text{c-Ind}_{F^\times M_{y,0}}^M(\tau)$, where $M_{y,0} \simeq \text{GL}_2(\mathfrak{o}_F)$ and τ denotes the extension to $F^\times M_{y,0}$ of the inflation to $M_{y,0}$ of a cuspidal irreducible representation of $\mathbb{M}_{y,0} \simeq \text{GL}_2(\mathbb{F}_q)$. By (3.6.4) the representation τ corresponds to the unipotent representation $\tau_{\text{unip}} \in \mathcal{E}(\mathbb{Z}_{\mathbb{M}_{y,0}^\vee}(s), 1)$. In this case, $\mathbb{Z}_{\mathbb{M}_{y,0}^\vee}(s)$ is either $\text{GL}_2(\mathbb{F}_q)$ or a torus. Since $\text{GL}_2(\mathbb{F}_q)$ carries no cuspidal unipotent representations, $\mathbb{Z}_{\mathbb{M}_{y,0}^\vee}(s)$ can only be a torus, and thus $\tau_{\text{unip}} = 1$. In particular, $\dim(\tau_{\text{unip}}) = 1$. On the other hand, we have

$$(8.0.1) \quad |\mathbb{M}_{y,0}| = |\text{GL}_2(\mathbb{F}_q)| = q(q^2 - 1)(q - 1).$$

Thus, since $|\mathbb{T}| = (q - 1)^2$, we have

$$(8.0.2) \quad \dim(\tau) = \frac{|\mathbb{M}_{y,0}|_{p'}}{|\mathbb{T}|_{p'}} \cdot \dim(\tau_{\text{unip}}) = \frac{(q^2 - 1)(q - 1)}{(q - 1)^2} = q + 1.$$

By [KY17], the pair $(G_{y,0}, \tau)$ is an \mathfrak{s} -type in G , and a G -cover of $(M_{y,0}, \tau)$ (see also [Mor99] or [MP96b]). Since $\mathbb{G}_{y,0} = \mathbb{M}_{y,0}$, by (3.4.2) we have

$$(8.0.3) \quad \text{Vol}(G_y) = q^{-\text{rk}(G)/2} \cdot |\mathbb{G}_{y,0}|_{p'} = q^{-1}(q^2 - 1) = \frac{q^2 - 1}{q}.$$

Therefore by a generalized version of Proposition 3.5.1, we have

$$(8.0.4) \quad \text{fdeg}(\pi) = \frac{\dim \tau}{\text{Vol}(G_y)} \cdot d(\pi^{\mathcal{H}}) = \frac{(q + 1)q}{q^2 - 1} d(\pi^{\mathcal{H}}).$$

8.0.5. In the following table, we compute the quantity $d(\pi_s^{\mathcal{H}})$ and $d(\pi_0^{\mathcal{H}})$ as in [Ree97, §9]. Recall that if $q_\alpha \neq q_0$, we must have $|R(\mathcal{O})| = 1$, and the formal degrees of square-integral modules for $\mathcal{H}(q_\alpha, q_0)$ are given by

$$(8.0.6) \quad d(\pi_s^{\mathcal{H}}) = \frac{q_\alpha q_0 - 1}{(q_\alpha + 1)(q_0 + 1)} \quad d(\pi_0^{\mathcal{H}}) = \frac{|q_\alpha - q_0|}{(q_\alpha + 1)(q_0 + 1)}.$$

On the other hand, if the Hecke algebra $\mathcal{H}(q_\alpha, q_0)$ is of rank one, and $q_\alpha = q_0$, there are $|R(\mathcal{O})|$ square-integrable $\mathcal{H}(q_\alpha, q_0)$ -modules $\eta \otimes \pi_s$, one for each character η of $R(\mathcal{O})$. They all have the same formal degree

$$(8.0.7) \quad d(\eta \otimes \pi_s^{\mathcal{H}}) = |R(\mathcal{O})|^{-1} \frac{q_\alpha - 1}{q_\alpha + 1}.$$

We compute these quantities $d(\pi)$ in the following table 8.0.8, following [AX22, Table 4.5.2]. To distinguish between simple modules of Hecke algebras and irreducible representations, we use the notation $d(\pi^{\mathcal{H}})$ to denote formal degrees for simple modules for Hecke algebras $\mathcal{H}(q_\alpha, q_0)$.

8.0.8. Table for long root essentially depth zero cases.

We refer the reader to [AX22] for the notations.

r	\mathcal{D}	ω_σ	$\chi^2\chi'^{-1}$	$R(\mathcal{O})$	$W_\mathcal{O}$	$\mathcal{H}(G, \rho)$ $q_F^{\lambda(\alpha)}, q_F^{\lambda^*(\alpha)}$	$d(\pi^{\mathcal{H}})$	$\text{fdeg}(\pi)$
$r = 0$	$((G, M), (y, \iota), (M_{y,0}, \rho_M))$	$= 1$	unramified χ cubic	$= 1$	$\neq 1$	non-comm, q_F^3, q_F	$\frac{q^4-1}{(q^3+1)(q+1)}$; $\frac{q^2}{(q^3+1)(q+1)}$	$\frac{(q^2+1)q}{q^3+1}$
		$\neq 1$	unramified χ cubic	$= 1$	$\neq 1$	non-comm, q_F^2, q_F^2	$\frac{q^2-1}{q^2+1}$	$\frac{q(q+1)}{q^2+1}$
		$= 1$	ramified χ cubic	$= 1$	$\neq 1$	non-comm, q_F, q_F	$\frac{q-1}{q+1}$	$\frac{q}{q+1}$
		$\neq 1$	ramified χ cubic	*	$= 1$	$\mathbb{C}[R(\mathcal{O})] \times \mathbb{C}[\mathcal{O}]$	0	0
		$= 1$	χ not cubic N/A	$= 1$	$\neq 1$	non-comm, q_F, q_F	$\frac{q-1}{q+1}$	$\frac{q}{q+1}$
		$\neq 1$		*	$= 1$	$\mathbb{C}[R(\mathcal{O})] \times \mathbb{C}[\mathcal{O}]$	0	0
$r \neq 0$	$((M, G), M), (y, \iota), (r, 0), (\phi, 1), (M_{y,0}, \rho_M)$	$\neq 1$	N/A	$= 1$	$= 1$	$\mathbb{C}[\mathcal{O}]$	0	0

TABLE 8.0.8.

8.0.9. As can be seen from the table, the first row has equal formal degree as in the singular non-unipotent supercuspidal representation from (6.0.9), i.e. they both have formal degree $\frac{q^2}{q^3+1}$. In particular, we have obtained an L -packet mixing a singular non-unipotent supercuspidal with an intermediate series representation with supercuspidal support being a supercuspidal representation associated to the long root $M \simeq M_\beta$.

Remark 8.0.2. Recall that the singular supercuspidal representation from (6.0.9) can only occur when $q \equiv -1 \pmod{3}$, which is also precisely the condition for this intermediate series representation in the first row of Table 8.0.8 to occur. Recall from [AX22, (4.2.6)] that

$$(8.0.10) \quad \omega_\sigma(\varpi_F) + \chi^2\chi'^{-1}(\varpi_L) = 0,$$

which can only occur when $q \equiv -1 \pmod{3}$.

9. THE PRINCIPAL SERIES

9.1. Notation and background. Let \mathbf{T} be an F -split maximal torus in \mathbf{G} , and let T^\vee denote its Langlands dual torus, which is a maximal torus in G^\vee . The principal series consists of all G -representations that occur in parabolic inductions of characters of $T := \mathbf{T}(F)$ to $G_2(F)$. Let T_0 denote the maximal compact subgroup of T . Since \mathbf{T} is F -split,

$$(9.1.1) \quad T \simeq F^\times \otimes_{\mathbb{Z}} X_*(T) \simeq (\mathfrak{o}_F^\times \times \mathbb{Z}) \otimes_{\mathbb{Z}} X_*(T) = T_0 \times X_*(T).$$

Since $W = W_{\mathbf{G}}$ (the Weyl group of \mathbf{G} with respect to \mathbf{T}) acts trivially on F^\times , these isomorphisms are W -equivariant if we endow the right hand side with the diagonal

W -action. Thus (9.1.1) determines a W -equivariant isomorphism of character groups

$$(9.1.2) \quad \text{Irr}(T) \cong \text{Irr}(T_0) \times \text{Irr}(X_*(T)) = \text{Irr}(T_0) \times X_{\text{nr}}(T).$$

How $\text{Irr}(T_0)$ is embedded depends on the choice of ϖ_F . However, the isomorphisms

$$(9.1.3) \quad \text{Irr}(T_0) \cong \text{Hom}(\mathfrak{o}_F^\times, T^\vee) \quad \text{and} \quad X_{\text{nr}}(T) \cong \text{Hom}(\mathbb{Z}, T^\vee) = T^\vee.$$

are canonical. Let φ_σ be the image of $\sigma \in \text{Irr}(T)$ under the canonical map

$$(9.1.4) \quad \text{Irr}(T) = \text{Hom}(F^\times \otimes_{\mathbb{Z}} X_*(T), \mathbb{C}^\times) \simeq \text{Hom}(F^\times, \mathbb{C}^\times \otimes_{\mathbb{Z}} X^*(T)) \simeq \text{Hom}(F^\times, T^\vee).$$

We define

$$(9.1.5) \quad c^\mathfrak{s} := \varphi_\sigma|_{\mathfrak{o}_F^\times} : \mathfrak{o}_F^\times \rightarrow T^\vee.$$

Note that unramified twists do not modify $\varphi_\sigma|_{\mathfrak{o}_F^\times}$, and conversely, (9.1.4) determines $c^\mathfrak{s}$ up to unramified twists. On the other hand, two elements of $\text{Irr}(T)$ are G -conjugate if and only if they are W -conjugate, so the W -orbit of $c^\mathfrak{s}$ contains the same amount of information as \mathfrak{s} . We define

$$(9.1.6) \quad \mathcal{J}^\mathfrak{s} := Z_{G^\vee}(\text{Im}(c_\mathfrak{s})).$$

By [Roc98, pp. 394–395], when \mathbf{G} has connected center, the group $\mathcal{J}^\mathfrak{s}$ is connected and its Weyl group is isomorphic to the group $W^\mathfrak{s}$ (see (3.3.3)). Let $J^\mathfrak{s}$ denote the group of F -rational points of the F -split reductive algebraic F -group whose Langlands dual is the group $\mathcal{J}^\mathfrak{s}$.

In [Roc98, §3], Roche constructed an \mathfrak{s} -type $(K^\mathfrak{s}, \tau^\mathfrak{s})$ in the sense of [BK98]. By [Roc98, Theorem 8.2], the algebra $\mathcal{H}(G, \tau^\mathfrak{s})$ and $\mathcal{H}(J^\mathfrak{s}, 1_J)$ are isomorphic via a family of $*$ -preserving, support-preserving (and hence inner-product preserving) isomorphisms. By [Roc98, proof of Theorem 10.7], these isomorphisms preserve square-integrability and, for corresponding $\pi \in \text{Irr}^2(G)$ and $\pi^1 \in \text{Irr}^2(J^\mathfrak{s})$, with the appropriate normalization of volume factors, we have

$$(9.1.7) \quad \text{fdeg}(\pi) = d^{\mathcal{H}}(\pi^1).$$

9.2. Principal series for G_2 . For $\gamma \in \{\alpha, \beta\}$, we set

$$(9.2.1) \quad I^\gamma(\xi_1 \otimes \xi_2) := i_{T(U \cap M_\gamma)}^{M_\gamma}(\xi_1 \otimes \xi_2).$$

By using (4.0.11) we obtain

$$(9.2.2) \quad I^\alpha(\xi_1 \otimes \xi_2) \otimes (\nu_F^\mathfrak{s} \circ \det) = I^\alpha(\xi_1 \nu_F^\mathfrak{s} \otimes \xi_2 \nu_F^\mathfrak{s}) \quad \text{and} \quad I^\beta(\xi_1 \otimes \xi_2) \otimes (\nu_F^\mathfrak{s} \circ \det) = I^\beta(\xi_2 \nu_F^\mathfrak{s} \otimes \xi_1 \xi_2^{-1}).$$

As recalled in [Mui97, Proposition 1.1], the principal series $I^\gamma(\xi_1 \otimes \xi_2)$ reduces if and only if $(\xi_1 \otimes \xi_2) \circ \gamma^\vee = \nu_F^{\pm 1}$. Let $\delta(\xi)$ denote the unique irreducible subrepresentation of $i_B^{\text{GL}_2(F)}(\nu_F^{1/2} \xi \otimes \nu_F^{-1/2} \xi)$, it is also the unique irreducible quotient of $i_B^{\text{GL}_2(F)}(\nu_F^{-1/2} \xi \otimes \nu_F^{1/2} \xi)$. Similarly, $\xi \circ \det$ is the unique irreducible quotient (resp. subrepresentation) of $i_B^{\text{GL}_2(F)}(\nu_F^{1/2} \xi \otimes \nu_F^{-1/2} \xi)$ (resp. $i_B^{\text{GL}_2(F)}(\nu_F^{-1/2} \xi \otimes \nu_F^{1/2} \xi)$). Thus, in the Grothendieck groups $R(M_\alpha)$, $R(M_\beta)$ we have

$$(9.2.3) \quad I^\alpha(\nu_F^{1/2} \xi \otimes \nu_F^{-1/2} \xi) = \delta(\xi) + \xi \circ \det \quad \text{and} \quad I^\beta(\nu_F^{-1/2} \xi \otimes \nu_F^{1/2} \xi) = \delta(\xi) + \xi \circ \det,$$

$$(9.2.4) \quad I^\alpha(\nu_F^{-1/2}\xi \otimes \nu_F^{1/2}\xi) = \delta(\xi) + \xi \circ \det \quad \text{and} \quad I^\beta(\nu_F^{1/2}\xi \otimes \nu_F^{-1}\xi) = \delta(\xi) + \xi \circ \det.$$

Let $s \in \mathbb{C}$. For an admissible representation π of $\mathrm{GL}_2(F)$, we write

$$(9.2.5) \quad I_\gamma(s, \pi) := i_{P_\gamma}^G(\pi \otimes (\nu_F^s \circ \det)), \quad \text{and} \quad I_\gamma(\pi) := I_\gamma(0, \pi).$$

When π a tempered irreducible representation and $s > 0$, the representation $I_\gamma(s, \pi)$ has a unique irreducible quotient denoted by $J_\gamma(s, \pi)$.

We denote by $I(\xi_1 \otimes \xi_2)$ the parabolically induced representation of $\sigma = \xi_1 \otimes \xi_2$. We will describe the irreducible components of the parabolically induced representation $I(\xi_1 \otimes \xi_2)$ following the methods developed in [Mui97, §3 and §4].

Remark 9.2.1. Recall (see [BZ77, Theorem 2.9]) that, for any $w \in W$, the Jordan-Hölder series of $I(\xi_1 \otimes \xi_2)$ and $I(w(\xi_1 \otimes \xi_2))$ have the same irreducible quotients.

If ω_1, ω_2 are unitary characters of F^\times and $s_1 > s_2 > 0$, then the representation $I(\nu_F^{s_1}\omega_1 \otimes \nu_F^{s_2}\omega_2)$ has a unique irreducible quotient denoted by $J(s_1, s_2, \omega_1, \omega_2)$.

Lemma 9.2.6. [Rod75, Theorem 7] We suppose that G is F -split. Let P be a parabolic subgroup of G with Levi subgroup L , and σ an irreducible smooth representation of L .

- (a) If σ is generic, then $i_P^G(\sigma)$ contains a unique generic irreducible component.
- (b) If σ is not generic, then no irreducible component of $i_P^G(\sigma)$ is generic.

By transitivity of the parabolic induction we have

$$(9.2.7) \quad I(\xi_1 \otimes \xi_2) = i_{P_\gamma}^G(I^\gamma(\xi_1 \otimes \xi_2)) = I_\gamma(s, I^\gamma(\xi_1 \otimes \xi_2) \otimes \nu_F^{-s} \circ \det).$$

By using (9.2.2), we get

$$(9.2.8) \quad I(\xi_1 \otimes \xi_2) = I_\alpha(s, I^\alpha(\xi_1 \nu_F^{-s} \otimes \xi_2 \nu_F^{-s})) = I_\beta(s, I^\beta(\xi_2 \nu_F^{-s} \otimes \xi_1 \xi_2^{-1})).$$

Lemma 9.2.9.

- (1) If $\xi_1 = \nu_F \xi_2$, then we have

$$(9.2.10) \quad \begin{aligned} I(\xi_1 \otimes \xi_2) &= I_\alpha(s, \delta(\nu_F^{1/2-s}\xi_2)) + I_\alpha(s, \nu_F^{1/2-s}\xi_2 \circ \det) \\ &= I_\beta(s, \delta(\nu_F^{1/2-s}\xi_2)) + I_\beta(s, \nu_F^{1/2-s}\xi_2 \circ \det). \end{aligned}$$

- (2) If $\xi_1 = \nu_F^{-1}\xi_2$, then we have

$$(9.2.11) \quad \begin{aligned} I(\xi_1 \otimes \xi_2) &= I_\alpha(s, \delta(\nu_F^{-1/2-s}\xi_2)) + I_\alpha(s, \nu_F^{-1/2-s}\xi_2 \circ \det) \\ &= I_\beta(s, \delta(\nu_F^{-1/2-s}\xi_2)) + I_\beta(s, \nu_F^{-1/2-s}\xi_2 \circ \det). \end{aligned}$$

Proof. The result follows from the combination of (9.2.8) with (9.2.3) and (9.2.4). \square

For $\gamma \in \{\alpha, \beta\}$, denote by $r_\gamma: R(G) \rightarrow R(M_\gamma)$ the normalized Jacquet restriction functor $r_{M_\gamma}^G$. For $\gamma, \gamma' \in \{\alpha, \beta\}$, let $W^{M_{\gamma'}, M_\gamma}$ be the subset of W defined in [BZ77, §2.11]. By [DM91, Lemmas 5.4-5.6], it consists in the elements $w \in W$ satisfying

$$(9.2.12) \quad \ell(s_\gamma w) = \ell(s_\gamma) + \ell(w) \quad \text{and} \quad \ell(ws_{\gamma'}) = \ell(w) + \ell(s_{\gamma'}).$$

Then, from Table 1, we obtain

$$(9.2.13) \quad W^{M_\alpha, M_\alpha} = \{1, s_\beta, s_{\alpha+\beta}, s_{3\alpha+\beta}\}, \quad W^{M_\beta, M_\alpha} = \{1, ba, baba\}$$

$$(9.2.14) \quad W^{M_\alpha, M_\beta} = \{1, ab, abab\}, \quad W^{M_\beta, M_\beta} = \{1, s_\alpha, s_{2\alpha+\beta}, s_{3\alpha+2\beta}\}$$

$$(9.2.15) \quad W^{M_\alpha, T} = \{1, s_\beta, ab, s_{\alpha+\beta}, abab, babab\}, \quad W^{T, M_\alpha} = \{1, s_\beta, ba, s_{\alpha+\beta}, baba, babab\}.$$

Lemma 9.2.16. Let $\xi \in \text{Irr}(F^\times)$. We have

$$(9.2.17) \quad r_\alpha(I_\alpha(s, \delta(\xi))) = \nu_F^s \delta(\xi) + \nu_F^{-s} \delta(\xi^{-1}) + I^\alpha(\nu_F^{2s} \xi^2 \otimes \nu_F^{1/2-s} \xi^{-1}) + I^\alpha(\nu_F^{s+1/2} \xi \otimes \nu_F^{-2s} \xi^{-2}).$$

$$(9.2.18) \quad r_\alpha(I_\beta(s, \delta(\xi))) = \delta(\nu_F^s \xi) + \nu_F^s \xi \circ \det + I^\alpha(\nu_F^{2s} \xi^2 \otimes \nu_F^{-s-1/2} \xi^{-1}) + I^\alpha(\nu_F^{s-1/2} \xi \otimes \nu_F^{-2s} \xi^{-2}).$$

Proof. (1) Since $I_\alpha(s, \delta(\xi)) = I_\alpha(\nu_F^s \delta(\xi))$ and $N_G(M_\alpha)/M_\alpha = \{1, s_{3\alpha+\beta}\}$, by [BZ77, Geometrical Lemma in §2.12], we get

$$r_\alpha(I_\alpha(s, \delta(\xi))) = \nu_F^s \delta(\xi) + s_{3\alpha+\beta}(\nu_F^s \delta(\xi)) + I^\alpha \circ s_{\alpha+\beta} \circ r_T^{M_\alpha}(\nu_F^s \delta(\xi)) + I^\alpha \circ s_\beta \circ r_T^{M_\alpha}(\nu_F^s \delta(\xi)).$$

By [Zel80, Proposition 1.11.(b)], we know that $\nu_F^s \delta(\xi)$ is the unique submodule of the length two representation $I^\alpha(\nu_F^{s+1/2} \xi \otimes \nu_F^{s-1/2} \xi)$, and $r_T^{M_\alpha}(\nu_F^s \delta(\xi)) = \nu_F^{s+1/2} \xi \otimes \nu_F^{s-1/2} \xi$. On the other hand, we have $s_{3\alpha+\beta}(\nu_F^s \delta(\xi)) = \nu_F^{-s} \delta(\xi^{-1})$. Hence, we get

$$r_\alpha(I_\alpha(s, \delta(\xi))) = \nu_F^s \delta(\xi) + \nu_F^{-s} \delta(\xi^{-1}) + I^\alpha \circ s_\beta(\nu_F^{s+1/2} \xi \otimes \nu_F^{s-1/2} \xi) + I^\alpha \circ s_{\alpha+\beta}(\nu_F^{s+1/2} \xi \otimes \nu_F^{s-1/2} \xi).$$

By Table 1, we have

$$s_\beta(\nu_F^{s+1/2} \xi \otimes \nu_F^{s-1/2} \xi) = \nu_F^{2s} \xi^2 \otimes \nu_F^{1/2-s} \xi^{-1} \quad \text{and} \quad s_{\alpha+\beta}(\nu_F^{s+1/2} \xi \otimes \nu_F^{s-1/2} \xi) = \nu_F^{s+1/2} \xi \otimes \nu_F^{-2s} \xi^{-2}.$$

Finally, we obtain (9.2.17).

(2) By (9.2.14) and [BZ77, Geometric Lemma], we have in $R(M_\alpha)$:

$$(9.2.19) \quad r_\alpha \circ I_\beta = I^\alpha \circ r_T^{M_\beta} + I^\alpha \circ ba \circ r_T^{M_\beta} + I^\alpha \circ baba \circ r_T^{M_\beta}.$$

Again by [Zel80, 1.11], we have

$$r_T^{M_\beta}(\nu_F^s \delta(\xi)) = \nu_F^{s+1/2} \xi \otimes \nu_F^{s-1/2} \xi.$$

By Table 1, we have $ba(\nu_F^{s+1/2} \xi \otimes \nu_F^{s-1/2} \xi) = \nu_F^{2s} \xi^2 \otimes \nu_F^{-s-1/2} \xi^{-1}$ and $baba(\nu_F^{s+1/2} \xi \otimes \nu_F^{s-1/2} \xi) = \nu_F^{s-1/2} \xi \otimes \nu_F^{-2s} \xi^{-2}$. Hence, we get

$$\begin{aligned} (ba \circ r_T^{M_\beta})(\nu_F^s \delta(\xi)) &= \nu_F^{2s} \xi^2 \otimes \nu_F^{2s+1/2} \xi^{-1} \\ (baba \circ r_T^{M_\beta})(\nu_F^s \delta(\xi)) &= \nu_F^{s-1/2} \xi \otimes \nu_F^{-2s} \xi^{-2} \end{aligned}$$

Then (9.2.19) gives

$$r_\alpha(I_\beta(s, \delta(\xi))) = I^\alpha(\nu_F^{s+1/2} \xi \otimes \nu_F^{s-1/2} \xi) + I^\alpha(\nu_F^{2s} \xi^2 \otimes \nu_F^{-s-1/2} \xi^{-1}) + I^\alpha(\nu_F^{s-1/2} \xi \otimes \nu_F^{-2s} \xi^{-2}).$$

Since $I^\alpha(\nu_F^{s+1/2}\xi \otimes \nu_F^{s-1/2}\xi) = \delta(\nu_F^s\xi) + \nu_F^s\xi \circ \det$ by 9.2.3, we have

$$r_\alpha(I_\beta(s, \delta(\xi))) = \delta(\nu_F^s\xi) + \nu_F^s\xi \circ \det + I^\alpha(\nu_F^{2s}\xi^2 \otimes \nu_F^{-s-1/2}\xi^{-1}) + I^\alpha(\nu_F^{s-1/2}\xi \otimes \nu_F^{-2s}\xi^{-2}).$$

□

Lemma 9.2.20. Notations as above. We have

$$(9.2.21) \quad r_\alpha I(\nu_F^{\mp 1} \otimes \xi_2) = I^\alpha(\nu_F^{\mp 1} \otimes \xi_2) + I^\alpha(\nu_F^{\pm 1} \otimes \nu_F^{\mp 1} \xi_2) + I^\alpha(\nu_F^{\mp 1} \xi_2 \otimes \xi_2^{-1}) + I^\alpha(\nu_F^{\mp 1} \otimes \nu_F^{\pm 1} \xi_2^{-1}).$$

Proof. By (9.2.8), we have

$$(9.2.22) \quad I(\nu_F^{\mp 1} \otimes \xi_2) = I_\alpha(s, I^\alpha(\nu_F^{\mp 1-s} \otimes \xi_2 \nu_F^{-s})).$$

First note that $I_\alpha(s, I^\alpha(\nu_F^{\mp 1-s} \otimes \xi_2 \nu_F^{-s})) := I_\alpha(\nu_F^s I^\alpha(\nu_F^{\mp 1-s} \otimes \xi_2 \nu_F^{-s})) = I_\alpha(I^\alpha(\nu_F^{\mp 1} \otimes \xi_2))$. By [BZ77, Geometrical Lemma], we have in $R(M_\alpha)$:

$$(9.2.23)$$

$$(9.2.24) \quad r_\alpha(I_\alpha(s, I^\alpha(\nu_F^{\mp 1-s} \otimes \xi_2 \nu_F^{-s}))) = I^\alpha(\nu_F^{\mp 1} \otimes \xi_2) + s_{3\alpha+\beta} \circ I^\alpha(\nu_F^{\mp 1} \otimes \xi_2) + i_T^{M_\alpha} \circ s_\beta \circ r_T^{M_\alpha} I^\alpha(\nu_F^{\mp 1} \otimes \xi_2) + i_T^{M_\alpha} \circ s_{\alpha+\beta} \circ r_T^{M_\alpha} I^\alpha(\nu_F^{\mp 1} \otimes \xi_2).$$

The first term in (9.2.23) gives $s_{3\alpha+\beta} \circ I^\alpha(\nu_F^{\mp 1} \otimes \xi_2) = I^\alpha(\nu_F^{\pm 1} \otimes \nu_F^{\mp 1} \xi_2)$. Since $r_T^{M_\alpha} I^\alpha(\nu_F^{\mp 1} \otimes \xi_2) = \nu_F^{\mp 1} \otimes \xi_2$ and $s_\beta(\nu_F^{\mp 1} \otimes \xi_2) = \nu_F^{\mp 1} \xi_2 \otimes \xi_2^{-1}$, we have

$$(9.2.25) \quad i_T^{M_\alpha} \circ s_\beta \circ r_T^{M_\alpha} I^\alpha(\nu_F^{\mp 1} \otimes \xi_2) = I^\alpha(\nu_F^{\mp 1} \xi_2 \otimes \xi_2^{-1}).$$

To see the last term in (9.2.24), we have

$$(9.2.26) \quad i_T^{M_\alpha} \circ s_{\alpha+\beta} \circ r_T^{M_\alpha} I^\alpha(\nu_F^{\mp 1} \otimes \xi_2) = i_T^{M_\alpha} \circ s_{\alpha+\beta}(\nu_F^{\mp 1} \otimes \xi_2) = I^\alpha(\nu_F^{\mp 1} \otimes \nu_F^{\pm 1} \xi_2^{-1}).$$

Thus we have

$$r_\alpha(I_\alpha(s, I^\alpha(\nu_F^{\mp 1-s} \otimes \xi_2 \nu_F^{-s}))) = I^\alpha(\nu_F^{\mp 1} \otimes \xi_2) + I^\alpha(\nu_F^{\pm 1} \otimes \nu_F^{\mp 1} \xi_2) + I^\alpha(\nu_F^{\mp 1} \xi_2 \otimes \xi_2^{-1}) + I^\alpha(\nu_F^{\mp 1} \otimes \nu_F^{\pm 1} \xi_2^{-1}).$$

□

Remark 9.2.2. We observe that (9.2.17) coincides with the one computed in [Mui97, Proof of Proposition 4.1] in the particular case where $s = 1/2$ and ξ is quadratic. We have also noticed the following typos in [Mui97, §2]: in the computation of $r_\alpha \circ I_\alpha(\pi)$ the element $w_{2\alpha+\beta}$ must be replaced by $w_{\alpha+\beta}$, and moreover $w_{3\alpha+2\beta}$ must be replaced by $w_{3\alpha+\beta}$ in the computation of $r_\alpha \circ I_\beta(\pi)$, it should be I^α instead of I^β .

Lemma 9.2.27. Notations as above. We have

$$(9.2.28)$$

$$r_\emptyset I(\nu_F^{\mp 1} \otimes \xi_2) = \nu_F^{\mp 1} \otimes \xi_2 + \nu_F^{\mp 1} \xi_2 \otimes \xi_2^{-1} + \xi_2^{-1} \otimes \nu_F^{\mp 1} \xi_2 + \nu_F^{\mp 1} \otimes \xi_2^{-1} \nu_F^{\pm 1} + \nu_F^{\pm 1} \xi_2^{-1} \otimes \nu_F^{\mp 1} + \xi_2^{-1} \otimes \nu_F^{\pm 1}.$$

Proof. By (9.2.8) again, we have

$$(9.2.29) \quad r_\emptyset I(\nu_F^{\mp 1} \otimes \xi_2) = r_\emptyset(I_\alpha(\nu_F^s I^\alpha(\nu_F^{\mp 1-s} \otimes \xi_2 \nu_F^{-s}))) = r_\emptyset(I_\alpha(I^\alpha(\nu_F^{\mp 1} \otimes \xi_2))).$$

By [BZ77, Geometric Lemma] applied to (9.2.15), we have

$$r_\emptyset I(\nu_F^{\mp 1} \otimes \xi_2) = \nu_F^{\mp 1} \otimes \xi_2 + i_T^T \circ s_\beta \circ r_T^{M_\alpha} I^\alpha(\nu_F^{\mp 1} \otimes \xi_2) + i_T^T \circ ab \circ r_T^{\text{GL}_S^2} I^\alpha(\nu_F^{\mp 1} \otimes \xi_2) + i_T^T \circ abab \circ (\nu_F^{\mp 1} \otimes \xi_2) + i_T^T \circ babab \circ (\nu_F^{\mp 1} \otimes \xi_2) + i_T^T \circ s_{\alpha+\beta} \circ (\nu_F^{\mp 1} \otimes \xi_2)$$

Since $r_T^{\text{GL}_2^\alpha} I^\alpha(\nu_F^{\mp 1} \otimes \xi_2) = \nu_F^{\mp 1} \otimes \xi_2$, we have

$$\begin{aligned} r_{\mathcal{O}} I(\nu_F^{\mp 1} \otimes \xi_2) &= \nu_F^{\mp 1} \otimes \xi_2 + s_\beta(\nu_F^{\mp 1} \otimes \xi_2) + ab(\nu_F^{\mp 1} \otimes \xi_2) \\ &\quad + abab(\nu_F^{\mp 1} \otimes \xi_2) + babab(\nu_F^{\mp 1} \otimes \xi_2) + s_{\alpha+\beta}(\nu_F^{\mp 1} \otimes \xi_2) \\ &= \nu_F^{\mp 1} \otimes \xi_2 + \nu_F^{\mp 1} \xi_2 \otimes \xi_2^{-1} + \xi_2^{-1} \otimes \nu_F^{\mp 1} \xi_2 \\ &\quad + \nu_F^{\pm 1} \xi_2^{-1} \otimes \nu_F^{\mp 1} + \xi_2^{-1} \otimes \nu_F^{\pm 1} + \nu_F^{\mp 1} \otimes \xi_2^{-1} \nu_F^{\pm 1}. \end{aligned}$$

Thus we have the desired equality. \square

In the following, we will classify the principal series cases according to the possibilities for \mathfrak{s} . We will use the dual figure of Figure 1 in which the role of α and β is played by β^\vee and α^\vee , respectively to help us analyze the roots. Note that α^\vee (resp. β^\vee), which is long (resp. short), corresponds to the long (resp. short) root α_1 (resp. α_2) in the notations of [Ram03, § 6], as shown in [Ram03, Fig.6.1]. Since the root α^\vee (resp. β^\vee) is long (resp. short), $e_{\alpha^\vee} = e_{\alpha_1}$ (resp. $e_{\beta^\vee} = e_{\alpha_2}$) is a representative of A_1 (resp. \tilde{A}_1). Since $\alpha^\vee + \beta^\vee$ is short, $e_{\alpha^\vee + \beta^\vee}$ is a representative of \tilde{A}_1 (see [Ram03, Table 6.3]).

9.3. Case $\mathfrak{s} = [T, \xi \otimes \xi]_G$. We may write $\sigma = \chi_1 \xi \otimes \chi_2 \xi$, with ξ a **ramified** character of F^\times , and χ_1, χ_2 unramified characters of F^\times . We write $\chi_1 = \nu_F^{s_1}$ and $\chi_2 = \nu_F^{s_2}$ for $s_1, s_2 \in \mathbb{C}$.

–If both $\nu_F^{s_1} \xi$ and $\nu_F^{s_2} \xi$ are unitary, by [Key82, Theorem G₂], $I(\nu_F^{s_1} \xi \otimes \nu_F^{s_2} \xi)$ is reducible if and only if $\nu_F^{s_1} \xi$ and $\nu_F^{s_2} \xi$ are two distinct quadratic characters, in which case it is of multiplicity 1 and length 2. In this case, the two irreducible constituents are in the same L -packet and they are both tempered. One is generic and the other one is not. See Theorem 3.

–Otherwise (i.e., at least one of $\nu_F^{s_1} \xi, \nu_F^{s_2} \xi$ is non-unitary), by [Mui97, Proposition 3.1], $I(\nu_F^{s_1} \xi \otimes \nu_F^{s_2} \xi)$ is irreducible unless we are in one of the following situations:

- (1) $s_1 = \pm 1, \xi = 1, s_2$ arbitrary (resp. s_1 arbitrary, $s_2 = \pm 1, \xi = 1$);
- (2) $\nu_F^{s_1+s_2} \xi^2 = \nu_F^{\pm 1}$;
- (3) $s_1 - s_2 = \pm 1$;
- (4) $\nu_F^{2s_1+s_2} \xi^3 = \nu_F^{\pm 1}$;
- (5) $\nu_F^{s_1+2s_2} \xi^3 = \nu_F^{\pm 1}$.

We now explain how cases (1) and (2) are equivalent, using Remark 9.2.1 and Table 1: in case (1), $\xi_1 \otimes \xi_2 = \nu_F^{\pm 1} \otimes \nu_F^{s_2} = s_\beta(\nu_F^{s_2 \pm 1} \otimes \nu_F^{-s_2})$ by Table 1, which then falls under case (2) by Remark 9.2.1. Similarly, case (5) is equivalent to case (4) by using the action of s_α .

We will treat case (2), case (3) and case (4) separately.

We recall first that the character ξ admits a polar decomposition

$$(9.3.1) \quad \xi = \nu_F^t \omega, \quad \text{where } t \in \mathbb{R} \text{ and } \omega \text{ is unitary,}$$

(see for instance [Tat79, (2.2.1)]). Writing $s := s_2 + t \pm 1/2$, we obtain

$$(9.3.2) \quad \omega := \nu_F^{s_2 - s \pm 1/2} \xi.$$

9.3.1. **Case (2) within Case $\mathfrak{s} = [T, \xi \otimes \xi]_G$.** In case (2), we have

$$(9.3.3) \quad \nu_F^{s_1} \xi = \nu_F^{-s_2 \pm 1} \xi^{-1}.$$

Thus we have

$$(9.3.4) \quad I(\xi_1 \otimes \xi_2) = I(\nu_F^{-s_2 \pm 1} \xi^{-1} \otimes \nu_F^{s_2} \xi) \stackrel{ababa}{=} I(\nu_F^{\mp 1} \otimes \xi_2)$$

(a) Case $\nu_F^{\mp 1} \otimes \xi_2 \notin \{\nu_F \otimes 1, \nu_F \otimes \nu_F^2, \nu_F^{-1} \otimes 1, \nu_F^{-1} \otimes \nu_F^{-2}\}$: In this case, $I^\alpha(\nu_F^{\mp 1-s} \otimes \xi_2 \nu_F^{-s})$ is irreducible, but $I(\nu_F^{\mp 1} \otimes \xi_2)$ is reducible. We compute the irreducible constituents in the following.

(i) When ξ_2 is quadratic, this is covered in Tables 16 and 17 (depending on whether ξ_2 is unramified or not). In this case, $I(\nu_F^{\mp 1} \otimes \xi_2)$ has length 4.

(ii) When ξ_2 is non-quadratic, since $I(\nu_F^{\mp 1} \otimes \xi_2) \stackrel{bababa}{=} I(\nu_F^{\pm 1} \otimes \xi_2^{-1})$. By Lemma 9.2.20, we have

$$(9.3.5) \quad r_\alpha I(\nu_F^{\pm 1} \otimes \xi_2^{-1}) = I^\alpha(\nu_F^{\pm 1} \otimes \xi_2^{-1}) + I^\alpha(\nu_F^{\mp 1} \otimes \nu_F^{\pm 1} \xi_2^{-1}) + I^\alpha(\nu_F^{\pm 1} \xi_2^{-1} \otimes \xi_2) + I^\alpha(\nu_F^{\pm 1} \otimes \nu_F^{\mp 1} \xi_2).$$

Let $\pi(\nu_F^{\mp 1} \otimes \xi_2) \cong \pi(\nu_F^{\pm 1} \otimes \xi_2^{-1})$ be an irreducible subrepresentation of $I(\nu_F^{\mp 1} \otimes \xi_2) \cong I(\nu_F^{\pm 1} \otimes \xi_2^{-1})$. Comparing (9.3.5) with (9.2.21), we obtain that $r_\alpha I(\nu_F^{\mp 1} \otimes \xi_2)$ and $r_\alpha I(\nu_F^{\pm 1} \otimes \xi_2^{-1})$ share the terms $I^\alpha(\nu_F^{\mp 1} \otimes \nu_F^{\pm 1} \xi_2^{-1})$ and $I^\alpha(\nu_F^{\pm 1} \otimes \nu_F^{\mp 1} \xi_2)$. Thus we have

$$(9.3.6) \quad r_\alpha \pi(\nu_F^{\mp 1} \otimes \xi_2) = I^\alpha(\nu_F^{\mp 1} \otimes \nu_F^{\pm 1} \xi_2^{-1}) + I^\alpha(\nu_F^{\pm 1} \otimes \nu_F^{\mp 1} \xi_2) = r_\alpha \pi(\nu_F^{\pm 1} \otimes \xi_2^{-1}).$$

By [Rod81, Corollary, p.419], $S = \{s_{3\alpha+\beta}\}$ and $I(\nu_F^{\mp 1} \otimes \xi_2)$ has length $2^{|S|} = 2$. Let $J(\nu_F^{\mp 1} \otimes \xi_2)$ denote the irreducible quotient of $I(\nu_F^{\mp 1} \otimes \xi_2)$, and we have

$$(9.3.7) \quad I(\nu_F^{\mp 1} \otimes \xi_2) = \pi(\nu_F^{\mp 1} \otimes \xi_2) + J(\nu_F^{\mp 1} \otimes \xi_2).$$

In this case, we have the following two Tables depending on what ξ_2 is like.

(a) When ξ_2 is ramified cubic, $\mathcal{J}^s = \mathrm{SL}_3(\mathbb{C})$. By [ABP11, p. 145] this case corresponds to case t_b in [Ram03, Table 4.1]. By [Roc98, Theorem 9.4] the representation $\pi(\nu_F^{\mp 1} \otimes \xi_2)$ corresponds to an Iwahori-spherical, non-square-integrable, tempered representation of $\mathrm{PGL}_3(F)$ and hence is tempered. It is indexed in [Ram03, Table 4.2] by the triple $(t_b, e_{\alpha_2}, 1)$ with $t_b = s_2 s_1 t$. The list of irreducible constituents is summarized in Table 12.

Representation	enhanced L -parameter
$\pi(\nu_F^{\mp 1} \otimes \xi_2)$	$(\varphi_{\sigma, A_1}, 1)$
$J(\nu_F^{\mp 1} \otimes \xi_2)$	$(\varphi_{\sigma, 1}, 1)$

TABLE 12. : LLC for irred. constituents of $I(\nu_F^{\mp 1} \otimes \xi_2)$ where ξ_2 is ramified cubic.

(b) When ξ_2 is ramified neither quadratic nor cubic, $\mathcal{J}^s = \mathrm{GL}_2(\mathbb{C})$. The list of irreducible constituents is summarized in Table 13.

More precisely, this corresponds to case t_a in [Ram03, Table 2.1], where the tempered representation corresponds to the regular unipotent class e_{α_1} in $\mathrm{GL}_2(\mathbb{C})$. By Proposition 7.0.2, we have $W_G^s = \{1, s_\beta\}$. Thus by Table 6, this corresponds to the unipotent class \tilde{A}_1 in $G_2(\mathbb{C})$.

Representation	enhanced L -parameter
$\pi(\nu_F^{\mp 1} \otimes \xi_2)$	$(\varphi_{\sigma, \tilde{A}_1}, 1)$
$J(\nu_F^{\mp 1} \otimes \xi_2)$	$(\varphi_{\sigma, 1}, 1)$

TABLE 13. : LLC for irred. constituents of $I(\nu_F^{\mp 1} \otimes \xi_2)$ where ξ_2 is ramified neither cubic nor quadratic.

(c) When ξ_2 is unramified non-quadratic, $\mathcal{J}^s = G_2(\mathbb{C})$. The list of irreducible constituents is summarized in Table 14. We observe that

$$(9.3.8) \quad I(\nu^{1-s_2}\xi^{-1} \otimes \nu^{s_2}\xi) = I_\beta(1/2 - s_2, \xi^{-1} \otimes \mathrm{St}_{\mathrm{GL}_2}) \oplus I_\beta(1/2 - s_2, \xi^{-1} \circ \det).$$

It shows that $\pi(\nu_F^{\mp 1} \otimes \xi_2) := I_\beta(1/2 - s_2, \xi^{-1} \otimes \mathrm{St}_{\mathrm{GL}_2})$ is tempered. Thus this case corresponds to the case t_g in [Ram03, Table 6.1].

Representation	enhanced L -parameter
$\pi(\nu_F^{\mp 1} \otimes \xi_2)$	$(\varphi_{\sigma, A_1}, 1)$
$J(\nu_F^{\mp 1} \otimes \xi_2)$	$(\varphi_{\sigma, 1}, 1)$

TABLE 14. : LLC for irred. constituents of $I(\nu_F^{\mp 1} \otimes \xi_2)$ where ξ_2 is unramified non-quadratic.

(b) Case $\nu_F^{\mp 1} \otimes \xi_2 \in \{\nu_F \otimes 1, \nu_F \otimes \nu_F^2, \nu_F^{-1} \otimes 1, \nu_F^{-1} \otimes \nu_F^{-2}\}$: Since, by Table 1, we have $I(\nu_F \otimes 1) \stackrel{bababa}{=} I(\nu_F^{-1} \otimes 1)$ and $I(\nu_F \otimes \nu_F^2) \stackrel{bababa}{=} I(\nu_F^{-1} \otimes \nu_F^{-2})$, we are reduced to consider $\nu_F^{\mp 1} \otimes \xi_2 \in \{\nu_F \otimes 1, \nu_F \otimes \nu_F^2\}$. The LLC for the irreducible components of $I(\nu_F \otimes \nu_F^2)$ will be computed in Table 18.

By [Mui97, Proposition 4.3], the representation $I(\nu_F \otimes 1)$ contains exactly two irreducible subrepresentations $\pi(1)$ and $\pi(1)'$, which are square-integrable, and we

have in $R(G)$

$$(9.3.9) \quad I_\alpha(1/2, \delta(1)) = \pi(1)' + J_\alpha(1/2, \delta(1)) + J_\beta(1/2, \delta(1))$$

$$(9.3.10) \quad I_\beta(1/2, \delta(1)) = \pi(1) + \pi(1)' + J_\beta(1/2, \delta(1))$$

$$(9.3.11) \quad I_\alpha(1/2, 1_{\mathrm{GL}_2}) = \pi(1) + J_\beta(1/2, \delta(1)) + J_\beta(1/2, \pi(1, 1))$$

$$(9.3.12) \quad I_\beta(1/2, 1_{\mathrm{GL}_2}) = J_\beta(1, \pi(1, 1)) + J_\beta(1/2, \delta(1)) + J_\alpha(1/2, \delta(1)).$$

Hence $I(\nu_F \otimes 1)$ has length 6: its other irreducible constituents are $J_\alpha(1/2, \delta(1))$, $J_\beta(1/2, \delta(1))$, $J_\beta(1, \pi(1, 1))$, the latter occurring with multiplicity 2.

Lemma 9.3.13. The representation $\pi(1)'$ is generic.

Proof. By [Rod75, Theorem 8], the representation $\delta(1) = \mathrm{St}_{\mathrm{GL}_2}$ is generic. Hence by (9.3.9), both $I_\alpha(1/2, \delta(1))$ and $I_\beta(1/2, \delta(1))$ contain a generic irreducible component, and that none of the representations $\pi(1)$, $J_\beta(1/2, \delta(1))$, $J_\beta(1, \pi(1, 1))$ and $J_\alpha(1/2, \delta(1))$ is generic. Thus, $\pi(1)'$ is generic. \square

Using [Ram03, Table 6.1] Case t_e , we obtain the enhanced unipotent conjugacy classes, which give the enhanced L -parameters, and the dimensions of the involved irreducible representations, as in Table 15. More precisely, $J_\alpha(1/2, \delta(1))$ has dimension 1 and corresponds to e_{α^\vee} , which corresponds to the minimal unipotent class A_1 . On the other hand, $J_\beta(1/2, \delta(1))$ corresponds to trivial unipotent class. The two square-integrable representations $\pi(1)$ and $\pi(1)'$ correspond to $e_{\alpha^\vee} + e_{\alpha^\vee + 2\beta^\vee}$, which, as mentioned in [Ram03, Table 6.3], is a representative of the nilpotent class corresponding to $G_2(a_1)$. For any representative u of $G_2(a_1)$, we have $A_{G_2(\mathbb{C})}(u) \simeq S_3$, the symmetric group on three elements, which has irreducible representations indexed by the partitions (3), (21), (13) of 3, where (3) is the trivial one. The generic representation $\pi(1)'$ corresponds to $(t_e, G_2(a_1), (3))$, while $\pi(1)$ corresponds to $(t_e, G_2(a_1), (2, 1))$. There are five indexing triples, as in Table 15. Note that this table is consistent with the list for $\pi_{\sqrt{q}}^{-1}(t_e)$ in [ABP11, p.143].

Indexing triple	unipotent orbit	representation
$(t_e, 0, 1)$	1	$J_\beta(1, \pi(1, 1))$
$(t_e, e_{\alpha^\vee}, 1)$	A_1	$J_\alpha(1/2, \delta(1))$
$(t_e, e_{\beta^\vee + \alpha^\vee}, 1)$	\tilde{A}_1	$J_\beta(1/2, \delta(1))$
$(t_e, e_{\alpha^\vee} + e_{2\beta^\vee + \alpha^\vee}, (21))$	$G_2(a_1)$	$\pi(1)$
$(t_e, e_{\alpha^\vee} + e_{2\beta^\vee + \alpha^\vee}, (3))$	$G_2(a_1)$	$\pi(1)'$

Representation	enhanced L -parameter	dimension
$\pi(1)'$	$(\varphi_{\sigma, G_2(a_1)}, (3))$	3
$\pi(1)$	$(\varphi_{\sigma, G_2(a_1)}, (21))$	1
$J_\beta(1, \pi(1, 1))$	$(\varphi_{\sigma, 1}, 1)$	3
$J_\beta(1/2, \delta(1))$	$(\varphi_{\sigma, \tilde{A}_1}, 1)$	2
$J_\alpha(1/2, \delta(1))$	$(\varphi_{\sigma, A_1}, 1)$	1

TABLE 15. : LLC for the irreducible components of $I(\nu_F \otimes 1)$.

9.3.2. Case (3) within Case $\mathfrak{s} = [T, \xi \otimes \xi]_G$.

In case (3), $\xi_1 = \nu_F^{s_1} \xi = \nu_F^{s_1 - s_2} \xi_2 = \nu_F^{\pm 1} \xi_2$. By (9.2.10) we have

$$(9.3.14) \quad \begin{aligned} I(\xi_1 \otimes \xi_2) &= I_\alpha(s, \delta(\nu_F^{\pm 1/2 - s} \xi_2)) + I_\alpha(s, \nu_F^{\pm 1/2 - s} \xi_2 \circ \det) \\ &= I_\alpha(s, \delta(\nu_F^{s_2 - s \pm 1/2} \xi)) + I_\alpha(s, \nu_F^{s_2 - s \pm 1/2} \xi \circ \det). \end{aligned}$$

By [Mui97, Theorem 3.1 (i)], if the character ω defined in (9.3.2) is unitary, then $I_\alpha(s, \delta(\nu_F^{s_2 - s \pm 1/2} \xi))$ and $I_\alpha(s, \nu_F^{s_2 - s \pm 1/2} \xi \circ \det)$ are irreducible unless

$s = \pm 1/2$, $\nu_F^{s_2 - s \pm 1/2} \xi$ is quadratic, $s = \pm 3/2$, $\nu_F^{s_2 - s \pm 1/2} \xi = 1$ or $s = \pm 1/2$, $\nu_F^{s_2 - s \pm 1/2} \xi$ is cubic.

When they are irreducible, we compute the L -parameters for irreducible constituents and record them in Table 21 (when ξ is unramified) and Table 22 (when ξ is ramified).

When they are reducible, we have in $R(G)$:

(a) Case ($s = \pm 1/2$, $\nu_F^{s_2 - s \pm 1/2} \xi = \eta_2$ quadratic):

$$I(\xi_1 \otimes \xi_2) = I(\nu_F^{s \pm 1/2} \eta_2 \otimes \nu_F^{s \mp 1/2} \eta_2) = I(\nu_F^{\pm 1} \nu_F^{s \mp 1/2} \eta_2 \otimes \nu_F^{s \mp 1/2} \eta_2),$$

and thus Lemma 9.2.9 applies to this case, and we have

$$(9.3.15) \quad I(\xi_1 \otimes \xi_2) = I_\alpha(s, \delta(\eta_2)) + I_\alpha(s, \eta_2 \circ \det).$$

We now plug in $s = \pm 1/2$ and obtain

$$I(\xi_1 \otimes \xi_2) = I(\nu_F^{\pm 1} \eta_2 \otimes \eta_2) = I_\alpha(\pm 1/2, \delta(\eta_2)) + I_\alpha(\pm 1/2, \eta_2 \circ \det).$$

Note that by [BDK86, Lemma 5.4(iii)], we have

$$(9.3.16) \quad I_\alpha(-1/2, \delta(\eta_2)) \simeq I_\alpha(1/2, \delta(\eta_2)) \quad \text{and} \quad I_\alpha(-1/2, \eta_2 \circ \det) \simeq I_\alpha(1/2, \eta_2 \circ \det).$$

Therefore in the above we treat $\pm 1/2$ cases together.

Recall from (9.3.2), we have that $\eta_2 = \omega$ is unitary. Thus [Mui97, Proposition 4.1 (ii)] applies, and

$$(9.3.17) \quad I_\alpha(s, \delta(\eta_2)) = \pi(\eta_2) + J_\alpha(s, \delta(\eta_2))$$

$$(9.3.18) \quad I_\alpha(s, \eta_2 \circ \det) = J_\beta(1, \pi(1, \eta_2)) + J_\beta(s, \delta(\eta_2)).$$

Combined with (9.3.15), we have

$$(9.3.19) \quad I(\xi_1 \otimes \xi_2) = \pi(\eta_2) + J_\beta(1, \pi(1, \eta_2)) + J_\beta(1/2, \delta(\eta_2)) + J_\alpha(1/2, \delta(\eta_2)).$$

We observe that $I(\nu_F^{-2} \otimes \nu_F^1) \stackrel{babbaba}{=} I(\nu_F^2 \otimes \nu_F)$ in $R(G)$.

• When η_2 is *unramified* quadratic, $\mathcal{J}^s = G_2(\mathbb{C})$ as in [ABP11, § 4], and the L -parameters for irreducible constituents are recorded in Table 16. More precisely, we are in case t_d in [Ram03, Table 6.1] (see also [ABP11, p.17]), there are four indexing triples

Indexing triple	unipotent orbit	representation
$(t_d, 0, 1)$	1	$J_\beta(1, \pi(1, \eta_2))$
$(t_d, e_{\alpha^\vee}, 1)$	A_1	$J_\alpha(1/2, \delta(\eta_2))$
$(t_d, e_{2\beta^\vee + \alpha^\vee}, 1)$	\tilde{A}_1	$J_\beta(1/2, \delta(\eta_2))$
$(t_d, e_{\alpha^\vee} + e_{\alpha^\vee + 2\beta^\vee}, 1)$	$G_2(a_1)$	$\pi(\eta_2)$

Note that $(t_d, 0, 1)$ corresponds to the trivial unipotent in \mathcal{J}^s . In the Langlands classification (see for instance [SZ18]) $(s_1 t_d, \{1\})$ corresponds to the unique J_α term; since $(s_1 t_d, \{1\})$ corresponds to e_{α^\vee} , which is a representative of the minimal nilpotent orbit in \mathcal{J}^s . Likewise, the parameters $(t_d, \{2\})$ and $(s_2 s_1 s_2 t_d, \{2\})$ correspond to the two J_β terms. The element $e_{\alpha^\vee} + e_{\alpha^\vee + 2\beta^\vee}$ corresponds to the subregular unipotent orbit $G_2(a_1)$ in \mathcal{J}^s (see [Ram03, Table 6.3]).

Representation	enhanced L -parameter
$\pi(\eta_2)$	$(\varphi_{\sigma, G_2(a_1)}, 1)$
$J_\beta(1, \pi(1, \eta_2))$	$(\varphi_{\sigma, 1}, 1)$
$J_\beta(1/2, \delta(\eta_2))$	$(\varphi_{\sigma, \tilde{A}_1}, 1)$
$J_\alpha(1/2, \delta(\eta_2))$	$(\varphi_{\sigma, A_1}, 1)$

TABLE 16. LLC for the irreducible components of $I(\nu_F \otimes \eta_2)$ for η_2 unramified quadratic.

• When η_2 is *ramified* quadratic, by Proposition 7.0.2, we have $\mathcal{J}^s = \mathrm{SO}_4(\mathbb{C})$ as in [ABP11, § 8]. By [ABPS16a, Theorem 4.7], we are reduced to compute the Kazhdan-Lusztig triples for the F -split group $\mathrm{SO}_4(F)$. Since $\mathrm{SO}_4(\mathbb{C}) \simeq \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) / \{\pm 1\}$, we have $\mathcal{H}(\mathrm{SO}_4(\mathbb{C}), \mathcal{J}) \simeq (\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}))^{\mathbb{Z}/2\mathbb{Z}}$ by [Ree02, §1.5]. By [ABP11, proof

of Lemma 8.3], both $I(\nu_F \otimes \eta_2)$ and $I(\nu_F \eta_2 \otimes \eta_2)$ have length 4. The L -parameters for irreducible constituents are recorded in Table 17. More precisely:

Lemma 9.3.20. $\pi(\eta_2)$ is generic.

Proof. The proof is very similar to that of Lemma 9.3.13. Recall from (9.3.17) that $I_\alpha(s, \delta(\eta_2)) = \pi(\eta_2) + J_\alpha(s, \delta(\eta_2))$. Since $\delta(\eta_2)$ is generic, $I_\alpha(1/2, \delta(\eta_2))$ must contain a generic representation as an irreducible constituent. On the other hand, since $\eta_2 \circ \det$ is not generic, by Lemma 9.2.6, all of the irreducible constituents occurring in $I_\beta(1/2, \eta_2 \circ \det)$ are non-generic. In particular, by [Mui97, 4.1(ii)], $J_\alpha(1/2, \delta(\eta_2))$ is not generic. Therefore $\pi(\eta_2)$ must be generic. \square

Since $\pi(\eta_2)$ is generic, it has trivial enhancement as in the first row of Table 17. By [Roc98], we have

$$(9.3.21) \quad \text{Irr}(\mathcal{H}(G, \tau^s)) \xrightarrow{\sim} \text{Irr}(\mathcal{H}(J^s, 1)).$$

Since $\pi(\eta_2)$ is a discrete series, its image under (9.3.21) is also a discrete series, which by [ABP11, §8.0.3] corresponds to a Kazhdan-Lusztig triple with unipotent class in $\text{SO}_4(\mathbb{C})$ the regular unipotent class, i.e. by Table 4 the subregular unipotent $G_2(a_1)$ in G_2 . On the other hand, consistent with the tables in [Ram03] and our previous tables, $J_\beta(1, \pi(1, \eta_2))$ corresponds to the trivial unipotent class. It now remains to distribute $J_\beta(1/2, \delta(\eta_2))$ and $J_\alpha(1/2, \delta(\eta_2))$ among the two unipotent classes $(2, 2)'$ and $(2, 2)''$ in $\text{SO}_4(\mathbb{C})$. By the indexing triples in [Ram03, §2], $J_\beta(1/2, \delta(\eta_2))$ corresponds to e_{α_2} , which corresponds to e_{β^\vee} , which then corresponds to \tilde{A}_1 .

We now compute the formal degree of $\pi(\eta_2)$: by [Roc98, Theorem 10.7], up to normalization factors of volumes, we have

$$(9.3.22) \quad \text{fdeg}(\pi(\eta_2)) = d(\text{St}_{\text{SO}_4}^{\mathcal{H}}),$$

where St_{SO_4} is the Steinberg representation of $\text{SO}_4(F)$. Since $|Z(G^\vee)| = 2$ and $|A(s, u)| = 1$, by [CKK12, Theorem 4.1], we have

$$(9.3.23) \quad d(\text{St}_{\text{SO}_4}^{\mathcal{H}}) = \frac{1}{2} \cdot \frac{q-1}{q^2-1} \cdot \frac{q-1}{q^2-1} \cdot q^2 = \frac{q^2}{2(q+1)^2}.$$

Thus we have

$$(9.3.24) \quad \text{fdeg}(\pi(\eta_2)) = \frac{q^2}{2(q+1)^2},$$

which agrees with the formal degree for the singular supercuspidal computed in (6.0.14).

(b) Case $(s = \pm 3/2, \nu_F^{s_2-s\pm 1/2}\xi = 1)$: $\xi_2 = \nu_F^{s_2}\xi = \nu_F^{s\mp 1/2}$, and $\xi_1 = \nu_F^\pm \xi_2 = \nu_F^{\pm 1/2+s}$.

$$I(\xi_1 \otimes \xi_2) = I(\nu_F^{\pm 1/2+s} \otimes \nu_F^{s\mp 1/2}) = I(\nu_F^{\pm 1} \nu_F^{s\mp 1/2} \otimes \nu_F^{s\mp 1/2}),$$

and thus Lemma 9.2.9 applies to this case, and we have

$$(9.3.25) \quad I(\xi_1 \otimes \xi_2) = I_\alpha(s, \delta(\nu_F^{\pm 1/2-s}\xi_2)) + I_\alpha(s, \nu_F^{\pm 1/2-s}\xi_2 \circ \det).$$

Representation	enhanced L -parameter
$\pi(\eta_2)$	$(\varphi_{\sigma, G_2(a_1)}, 1)$
$J_\beta(1, \pi(1, \eta_2))$	$(\varphi_{\sigma, 1}, 1)$
$J_\beta(1/2, \delta(\eta_2))$	$(\varphi_{\sigma, \tilde{A}_1}, 1)$
$J_\alpha(1/2, \delta(\eta_2))$	$(\varphi_{\sigma, A_1}, 1)$

TABLE 17. LLC for the irreducible components of $I(\nu_F \otimes \eta_2)$ for η_2 ramified quadratic.

We now plug in $s = \pm 3/2$ and the explicit values for ξ_2 , and obtain

$$I(\xi_1 \otimes \xi_2) = I(\nu_F^{\pm 2} \otimes \nu_F^{\pm 1}) = I_\alpha(\pm 3/2, \delta(1)) + I_\alpha(\pm 3/2, 1_{GL_2}).$$

By [BDK86, Lemma 5.4(iii)], we have

$$(9.3.26) \quad I_\alpha(-3/2, \delta(1)) \simeq I_\alpha(3/2, \delta(1)) \quad \text{and} \quad I_\alpha(-3/2, 1_{GL_2}) \simeq I_\alpha(3/2, 1_{GL_2}).$$

Therefore it suffices to treat the $\pm 3/2$ cases together. When $s = 3/2$, by [Mui97, Proposition 4.4], we have the following decomposition into irreducible constituents:

$$(9.3.27) \quad I_\alpha(3/2, \delta(1)) = \text{St}_{G_2} + J_\alpha(3/2, \delta(1)) \quad \text{and} \quad I_\alpha(3/2, 1_{GL_2}) = 1_{G_2} + J_\beta(5/2, \delta(1)).$$

We thus obtain

$$(9.3.28) \quad I(\xi_1 \otimes \xi_2) = I(\nu_F^{\pm 2} \otimes \nu_F^{\pm 1}) = \text{St}_{G_2} + J_\alpha(3/2, \delta(1)) + 1_{G_2} + J_\beta(5/2, \delta(1)).$$

In the notation of [Ram03, Table 6.1] the central character of the irreducible components of $I(\nu_F^2 \otimes \nu_F)$ is denoted t_a . Thus two representations have dimension 1 (the trivial representation 1_{G_2} and the Steinberg representation St_{G_2}) and the two have dimension 5. These four irreducible representations are also listed in [ABP11, p.143] as the elements of $\pi_{\sqrt{q}}^{-1}(t_a)$. For each root γ^\vee in $R(G^\vee, T^\vee)$ let denote by e_{γ^\vee} an element of the root space $\mathfrak{g}_{\gamma^\vee}^\vee$. The representation St_{G_2} is square-integrable, hence in particular it is tempered. From [Ram03, Table 6.1], it has indexing triple $(t_a, e_{\alpha^\vee} + e_{\beta^\vee}, 1)$. Then the first row of [Ram03, Table 6.3] shows that it is attached to the regular unipotent class G_2 . By [Ram03, Table 6.1], the indexing triple of 1_{G_2} is $(t_a, 0, 1)$, and those of $J_\alpha(3/2, \delta(1))$ and $J_\beta(5/2, \delta(1))$ are $(t_a, e_{\alpha^\vee}, 1)$ and $(t_a, e_{\beta^\vee}, 1)$ or $(t_a, e_{\beta^\vee}, 1)$ and $(t_a, e_{\alpha^\vee}, 1)$. Also, by [Ram03, Table 6.3], since β^\vee is short, we have that e_{β^\vee} corresponds to the minimal unipotent class A_1 , while e_{α^\vee} corresponds to the subminimal one \tilde{A}_1 . In this case, $\mathcal{J}^s = G_2(\mathbb{C})$ as in [ABP11, §4], and the L -parameters for irreducible constituents are recorded in Table 18. More precisely, we are in case t_a in [Ram03, Table 6.1] and there are four indexing triples as in Table 18 (note that this table is consistent with the case $\pi_{\sqrt{q}}^{-1}(t_a)$ in [ABP11, p.143]). Table 18 shows that there are four L -packets. Each L -packet is a singleton, hence the attached group $\mathcal{G}_{\sigma, u}$ is connected.

Indexing triple	unipotent orbit	representation
$(t_a, 0, 1)$	1	1_{G_2}
$(t_a, e_{\beta^\vee}, 1)$	\tilde{A}_1	$J_\beta(5/2, \delta(1))$
$(t_a, e_{\alpha^\vee}, 1)$	A_1	$J_\alpha(3/2, \delta(1))$
$(t_a, e_{\alpha^\vee} + e_{\beta^\vee}, 1)$	G_2	St_{G_2}

Representation	enhanced L -parameter
St_{G_2}	$(\varphi_{\sigma, G_2}, 1)$
$J_\alpha(3/2, \delta(1))$	$(\varphi_{\sigma, A_1}, 1)$
$J_\beta(5/2, \delta(1))$	$(\varphi_{\sigma, \tilde{A}_1}, 1)$
1_{G_2}	$(\varphi_{\sigma, 1}, 1)$

TABLE 18. : LLC for the irreducible components of $I(\nu_F^2 \otimes \nu_F)$.

(c) Case ($s = \pm 1/2$, $\nu_F^{s_2 - s \pm 1/2} \xi = \eta_3$ cubic):

$$I(\xi_1 \otimes \xi_2) = I(\nu_F^{s \pm 1/2} \eta_3 \otimes \nu_F^{s \mp 1/2} \eta_3) = I(\nu_F^{\pm 1} \nu_F^{s \mp 1/2} \eta_3 \otimes \nu_F^{s \mp 1/2} \eta_3),$$

and thus Lemma 9.2.9 applies to this case, and we have the decomposition into irreducible constituents:

$$(9.3.29) \quad I(\xi_1 \otimes \xi_2) = I_\alpha(s, \delta(\eta_3)) + I_\alpha(s, \eta_3 \circ \det).$$

We now plug in $s = \pm 1/2$ and obtain

$$I(\xi_1 \otimes \xi_2) = I(\nu_F^{\pm 1} \eta_3 \otimes \eta_3) = I_\alpha(\pm 1/2, \delta(\eta_3)) + I_\alpha(\pm 1/2, \eta_3 \circ \det).$$

By [BDK86, Lemma 5.5(iii)], we have

$$I_\alpha(-1/2, \delta(\eta_3)) = I_\alpha(1/2, \delta(\eta_3^{-1})) \quad \text{and} \quad I_\alpha(-1/2, \eta_3 \circ \det) = I_\alpha(1/2, \eta_3^{-1} \circ \det).$$

The following essentially follows from [Mui97, Proposition 4.2(ii)], for the reader's convenience, we include a more detailed proof than *loc.cit.*. When ξ_2 is cubic, consider the following two cases.

- (1) If $3s + 1/2 = \pm 1$, i.e. $s = 1/6$ or $-1/2$, then $I^\alpha(\nu_F^{2s} \xi^2 \otimes \nu_F^{-s-1/2} \xi^{-1})$ in (9.2.18) becomes

$$I^\alpha(\nu_F^{2s} \xi^2 \otimes \nu_F^{-s-1/2} \xi^{-1}) = \delta(\nu_F^{2s \mp 1/2} \xi^2) + \nu_F^{2s \mp 1/2} \xi^2 \circ \det,$$

and the term $I^\alpha(\nu_F^{s+1/2}\xi \otimes \nu_F^{-2s}\xi^{-2})$ in (9.2.17) becomes

$$I^\alpha(\nu_F^{s+1/2}\xi \otimes \nu_F^{-2s}\xi^{-2}) = \delta(\nu_F^{-2s\pm 1/2}\xi) + \nu_F^{-2s\pm 1/2}\xi \circ \det.$$

By [Mui97, Theorem 3.1], we only need to consider the case where $s = -1/2$. In this case, we have

$$(9.3.30) \quad r_\alpha(I_\alpha(s, \delta(\xi))) = \nu_F^s \delta(\xi) + \nu_F^{-s} \delta(\xi^{-1}) + \nu_F^{s+1} \delta(\xi) + \nu_F^{s+1} \xi \circ \det + I^\alpha(\nu_F^{-1} \xi^{-1} \otimes \nu_F \xi^{-1}).$$

Thus plugging $s = -1/2$ in the above and comparing with the analogous formula for $r_\alpha(I_\alpha(s, \delta(\xi^{-1})))$, we obtain that $r_\alpha(I_\alpha(-1/2, \delta(\xi)))$ and $r_\alpha(I_\alpha(-1/2, \delta(\xi^{-1})))$ share the common terms $\nu_F^{1/2} \delta(\xi) + \nu_F^{1/2} \delta(\xi^{-1})$ (note that this is only true for very specially chosen values of s), thus

$$(9.3.31) \quad r_\alpha(\pi(\xi)) = \nu_F^{1/2} \delta(\xi) + \nu_F^{1/2} \delta(\xi^{-1}),$$

and collecting the differing terms we obtain

$$(9.3.32) \quad r_\alpha(J_\alpha(-1/2, \delta(\xi))) = \delta(\nu_F^{-1/2}\xi) + \nu_F^{1/2}\xi \circ \det + I^\alpha(\nu_F^{-1}\xi^{-1} \otimes \nu_F\xi^{-1}).$$

In summary, we have

$$(9.3.33) \quad I_\alpha(-1/2, \delta(\xi)) = \pi(\xi) + J_\alpha(-1/2, \delta(\xi)).$$

(2) If $3s - 1/2 = \pm 1$, i.e. $s = 1/2$ or $-1/6$; similarly, by [Mui97, Theorem 3.1], we only need to consider the case where $s = 1/2$. In this case, we have

$$(9.3.34) \quad r_\alpha(I_\alpha(s, \delta(\xi))) = \nu_F^s \delta(\xi) + \nu_F^{-s} \delta(\xi^{-1}) + \nu_F^{1-s} \delta(\xi^{-1}) + \nu_F^{1-s} \xi^{-1} \circ \det + I^\alpha(\nu_F^{s+1/2}\xi \otimes \nu_F^{-2s}\xi).$$

Thus plugging $s = 1/2$ in the above and comparing with the analogous formula for $r_\alpha(I_\alpha(s, \delta(\xi^{-1})))$, we obtain that $r_\alpha(I_\alpha(1/2, \delta(\xi)))$ and $r_\alpha(I_\alpha(1/2, \delta(\xi^{-1})))$ share the common terms $\nu_F^{1/2} \delta(\xi) + \nu_F^{1/2} \delta(\xi^{-1})$, thus

$$(9.3.35) \quad r_\alpha(\pi(\xi)) = \nu_F^{1/2} \delta(\xi) + \nu_F^{1/2} \delta(\xi^{-1}),$$

and collecting the differing terms we obtain

$$(9.3.36) \quad r_\alpha(J_\alpha(1/2, \delta(\xi))) = \nu_F^{-1/2} \delta(\xi^{-1}) + \nu_F^{1/2} \xi^{-1} \circ \det + I^\alpha(\nu_F \xi \otimes \nu_F^{-1} \xi).$$

In summary, we still have

$$(9.3.37) \quad I_\alpha(1/2, \delta(\xi)) = \pi(\xi) + J_\alpha(1/2, \delta(\xi)).$$

The Jacquet restrictions for $J_\alpha(s, \xi \circ \det)$ can be computed similarly.

9.3.38. Returning to the specific setting involving η_3 , we obtain:

$$I_\alpha(1/2, \delta(\eta_3^{\pm 1})) = \pi(\eta_3^{\pm 1}) + J_\alpha(1/2, \delta(\eta_3^{\pm 1})) \quad \text{and} \\ I_\alpha(1/2, \eta_3^{\pm 1} \circ \det) = J_\beta(1, \pi(\delta_3^{\pm 1}, \delta_3^{\mp 1})) + J_\alpha(1/2, \delta(\eta_3^{\mp 1})).$$

In summary, we obtain the following decomposition into irreducible constituents:

$$(9.3.39) \quad I(\xi_1 \otimes \xi_2) = I(\nu_F^{\pm 1} \eta_3 \otimes \eta_3) = \pi(\eta_3^{\pm 1}) + J_\alpha(1/2, \delta(\eta_3^{\pm 1})) + J_\beta(1, \pi(\delta_3^{\pm 1}, \delta_3^{\mp 1})) + J_\alpha(1/2, \delta(\eta_3^{\mp 1})).$$

When η_3 is unramified, it follows from [ABP11, p.142] that we are in the case of infinitesimal character t_c of [Ram03, Table 6.1]. The representation $\pi(\eta_3^{\pm 1})$ is a 2-dimensional square-integrable representation, with non-real central character (see [Ram03, §6]). The three other irreducible representations are non-tempered. It has indexing triple $(t_c, e_{\alpha^\vee} + e_{3\beta^\vee + \alpha^\vee}, 1)$. As observed in [Ram03, p. 25], it follows from [Stu71, Satz 1'], that $e_{\alpha^\vee} + e_{3\beta^\vee + \alpha^\vee}$ is a representative of the subregular unipotent orbit $G_2(a_1)$. The representation $J_\beta(1, \pi(\delta_3^{\pm 1}, \delta_3^\mp))$ has dimension 2 and its indexing triple is $(t_c, 0, 1)$. The representations $J_\alpha(1/2, \delta(\eta_3^{\pm 1}))$ and $J_\alpha(1/2, \delta(\eta_3^\mp))$ are both of dimension 4, with indexing triples $(t_c, e_\alpha, 1)$ and $(t_c, e_{\alpha+3\beta}, 1)$.

Indexing triple	unipotent orbit	representation
$(t_c, e_{\alpha^\vee} + e_{3\beta^\vee + \alpha^\vee}, 1)$	$G_2(a_1)$	$\pi(\eta_3^\pm)$
$(t_c, e_{\alpha^\vee}, 1)$	A_1	$J_\alpha(1/2, \delta(\eta_3^\pm))$
$(t_c, e_{3\beta^\vee + \alpha^\vee}, 1)$	A_1	$J_\alpha(1/2, \delta(\eta_3^\mp))$
$(t_c, 0, 1)$	1	$J_\beta(1, \pi(\delta_3^{\pm 1}, \delta_3^\pm))$

In this case, $\mathcal{J}^\mathfrak{s} = G_2(\mathbb{C})$ as in [ABP11, § 4], and the enhanced L -parameters are as in Table 19:

Representation	enhanced L -parameter
$\pi(\eta_3^{\pm 1})$	$(\varphi_{\sigma, G_2(a_1)}, 1)$
$J_\alpha(1/2, \delta(\eta_3^{\pm 1}))$	$(\varphi'_{\sigma, A_1}, 1)$
$J_\alpha(1/2, \delta(\eta_3^\mp))$	$(\varphi''_{\sigma, A_1}, 1)$
$J_\beta(1, \pi(\delta_3^{\pm 1}, \delta_3^\mp))$	$(\varphi_{\sigma, 1}, 1)$

TABLE 19. : LLC for the irreducible components of $I(\nu_F^{\pm 1} \eta_3 \otimes \eta_3)$ for η_3 cubic unramified.

9.3.40. When η_3 is ramified, the group $\mathcal{J}^\mathfrak{s}$ is isomorphic to $SL_3(\mathbb{C})$ with simple roots α^\vee and $3\beta^\vee + 2\alpha^\vee$. We compute the L -parameters by combining the case t_a in [Ram03, Table 4.1] with Table 5. There are four indexing triples as in the following table.

Indexing triple	unipotent orbit	representation
$(t_a, 0, 1)$	1	$J_\beta(1, \pi(\delta_3^{\pm 1}, \delta_3^{\mp}))$
$(t_a, e_{\alpha^\vee}, 1)$	A_1	$J_\alpha(1/2, \delta(\eta_3^{\pm 1}))$
$(t_a, e_{3\beta^\vee + 2\alpha^\vee}, 1)$	A_1	$J_\alpha(1/2, \delta(\eta_3^{\mp}))$
$(t_a, e_{\alpha^\vee} + e_{3\beta^\vee + 2\alpha^\vee}, 1)$	$G_2(a_1)$	$\pi(\eta_3^{\pm 1})$

By [Ram03, Table 4.2], $e_{\alpha^\vee} + e_{3\beta^\vee + 2\alpha^\vee}$ corresponds to the regular nilpotent orbit in $SL_3(\mathbb{C})$, which gets sent to the subregular nilpotent orbit in $G_2(\mathbb{C})$ by Table 5. By [Ram03, Table 4.2] again, $e_{3\beta^\vee + 2\alpha^\vee}$ corresponds to the subregular nilpotent orbit in $SL_3(\mathbb{C})$, which gets sent to the subminimal nilpotent orbit in G_2 by Table 5 again. Here u' and u'' are unipotent such that their conjugacy class $[u'] = [u''] = A_1$ in G_2 , but the restriction $\varphi|_{W_F}$ depends on u' and u'' itself, rather than their conjugacy class; therefore we are able to arrive at different L -parameters φ'_{σ, A_1} and φ''_{σ, A_1} for the two irreducible constituents $J_\alpha(1/2, \delta(\eta_3^{\pm 1}))$ and $J_\alpha(1/2, \delta(\eta_3^{\mp}))$.

Representation	enhanced L -parameter
$\pi(\eta_3^{\pm 1})$	$(\varphi_{\sigma, G_2(a_1)}, 1)$
$J_\alpha(1/2, \delta(\eta_3^{\pm 1}))$	$(\varphi'_{\sigma, A_1}, 1)$
$J_\alpha(1/2, \delta(\eta_3^{\mp}))$	$(\varphi''_{\sigma, A_1}, 1)$
$J_\beta(1, \pi(\delta_3^{\pm 1}, \delta_3^{\mp}))$	$(\varphi_{\sigma, 1}, 1)$

TABLE 20. LLC for the irreducible components of $I(\nu_F^{\pm 1} \eta_3 \otimes \eta_3)$ for η_3 cubic ramified.

We now compute the formal degree of $\pi(\eta_3)$: by [Roc98, Theorem 10.7], up to normalization factors of volumes, we have

$$(9.3.41) \quad \text{fdeg}(\pi(\eta_3)) = d(\text{St}_{\text{PGL}_3}^{\mathcal{H}}),$$

where St_{PGL_3} is the Steinberg representation of $\text{PGL}_3(F)$. By [Opd16, Theorem 4.11],

$$(9.3.42) \quad d(\text{St}_{\text{PGL}_3}^{\mathcal{H}}) = 3^{-1}[3]_q^{-1} = 3^{-1} \cdot \frac{q^{1/2} - q^{-1/2}}{q^{3/2} - q^{-3/2}} = \frac{q}{3(q^2 + q + 1)}.$$

Thus we have

$$(9.3.43) \quad \text{fdeg}(\pi(\eta_3)) = \frac{q}{3(q^2 + q + 1)},$$

which agrees with the formal degree for the singular supercuspidal in (6.0.11).

9.3.3. Case (5) within Case $\mathfrak{s} = [T, \xi \otimes \xi]_G$.

In case (5), $\xi_1 = \nu_F^{s_1} \xi = \nu_F^{-2s_2} \xi^{-2} \nu_F^{\pm 1} = \xi_2^{-2} \nu_F^{\pm 1}$. Thus

$$(9.3.44) \quad I(\xi_1 \otimes \xi_2) = I(\nu_F^{\pm 1} \xi_2^{-2} \otimes \xi_2) \stackrel{ababa}{=} I(\nu_F^{\pm 1} \xi_2 \otimes \xi_2).$$

Thus Lemma 9.2.9 applies to this case, and we have

$$(9.3.45) \quad I(\xi_1 \otimes \xi_2) = I(\nu_F^{\pm 1} \xi_2 \otimes \xi_2) = I_\alpha(s, \delta(\nu_F^{\pm 1/2-s} \xi_2)) + I_\alpha(s, \nu_F^{\pm 1/2-s} \xi_2 \circ \det)$$

Recall from (9.3.2) that $\omega = \xi \nu_F^{-t} = \xi \nu_F^{s_2-s\pm 1/2} = \nu_F^{\pm 1/2-s} \xi_2$ is unitary. Therefore [Mui97, Theorem 3.1 (i)] applies to this case, and $I_\alpha(s, \delta(\nu_F^{\pm 1/2-s} \xi_2))$ and $I_\alpha(s, \nu_F^{\pm 1/2-s} \xi_2 \circ \det)$ are irreducible unless

$s = \pm 1/2$, $\nu_F^{\pm 1/2-s} \xi_2 = \xi_2$ is quadratic, $s = \pm 3/2$, $\nu_F^{\pm 1/2-s} \xi_2 = 1$ or $s = \pm 1/2$, $\nu_F^{\pm 1/2-s} \xi_2$ is cubic.

When they are irreducible, we compute the L -parameters for irreducible constituents and record them in Table 21 (when ξ_2 is unramified) and Table 22 (when ξ_2 is ramified).

• When ξ_2 is unramified, we are in case t_g of [Ram03, Table 6.1] (see also [ABP11, §4]) and $\mathcal{J}^s = G_{2, \mathbb{C}}$, and we compute the L -parameters in Table 21. Note that since $\delta(\nu_F^{\pm 1/2-s} \xi_2)$ is discrete series, the induced representation $I_\alpha(s, \delta(\nu_F^{\pm 1/2-s} \xi_2))$ is tempered, which thus corresponds to the e_{α_1} unipotent class, i.e. the minimal unipotent class A_1 in G_2 .

Representation	indexing triple	enhanced L -parameter
$I_\alpha(s, \delta(\nu_F^{\pm 1/2-s} \xi_2))$	$(t_g, e_{\alpha_1}, 1)$	$(\varphi_{\sigma, A_1}, 1)$
$I_\alpha(s, \nu_F^{\pm 1/2-s} \xi_2 \circ \det)$	$(t_g, 0, 1)$	$(\varphi_{\sigma, 1}, 1)$

TABLE 21. : LLC for irred. constituents of case (5) within $I(\xi \otimes \xi)$ where ξ is unramified.

- When ξ_2 is ramified, we compute the L -parameters in different cases.
 - (i) When ξ is neither quadratic nor cubic, we are in case (3) of Proposition 7.0.2, where $W_G^s = \mathbb{Z}/2\mathbb{Z} = \{1, s_\alpha\}$ and $\mathcal{J}^s = \mathrm{GL}_2(\mathbb{C})$. We are in case t_a of [Ram03, Table 2.1]. Again since $\delta(\nu_F^{\pm 1/2-s} \xi_2)$ is discrete series, the induced representation $I_\alpha(s, \delta(\nu_F^{\pm 1/2-s} \xi_2))$ is tempered, which thus corresponds to the e_{α_1} unipotent class, which corresponds to A_1 in G_2 by Table 6. Thus we obtain Table 22.

Representation	indexing triple	enhanced L -parameter
$I_\alpha(s, \delta(\nu_F^{\pm 1/2-s} \xi_2))$	$(t_a, e_{\alpha_1}, 1)$	$(\varphi_{\sigma, A_1}, 1)$
$I_\alpha(s, \nu_F^{\pm 1/2-s} \xi_2 \circ \det)$	$(t_a, 0, 1)$	$(\varphi_{\sigma, 1}, 1)$

TABLE 22. : LLC for irred. constituents of $I(\xi_1 \otimes \xi_2)$ attached to Case (5) of $\mathfrak{s} = [T, \xi \otimes \xi]_G$ where ξ is ramified.

- (ii) When ξ is quadratic, we are in case (4) of Proposition 7.0.2, where $W_G^{\mathfrak{s}} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\mathcal{J}^{\mathfrak{s}} = \mathrm{SO}_4(\mathbb{C})$. However, this case cannot happen here (although it'll occur elsewhere in our other subsections), because in this case ξ_2 is quadratic and $I_\alpha(s, \delta(\nu_F^{\pm 1/2-s} \xi_2))$ and $I_\alpha(s, \nu_F^{\pm 1/2-s} \xi_2 \circ \det)$ are reducible, contradicting our assumption on irreducibility in this current subsection. (In fact, we indeed expect to have a length 4 table rather than length 2.)
- (iii) When ξ is cubic, we are in case (5) of Proposition 7.0.2, where $W_G^{\mathfrak{s}} = S_3$ and $\mathcal{J}^{\mathfrak{s}} = \mathrm{SL}_{3, \mathbb{C}}$. Likewise, this case cannot happen here (although it'll occur elsewhere in our other subsections), because in this case ξ_2 is cubic and $I_\alpha(s, \delta(\nu_F^{\pm 1/2-s} \xi_2))$ and $I_\alpha(s, \nu_F^{\pm 1/2-s} \xi_2 \circ \det)$ are reducible, contradicting our assumption on irreducibility in this current subsection. (In fact, we indeed expect to have a length 4 table rather than length 2.)

When they are reducible, we have in $R(G)$:

(a) Case ($s = \pm 1/2$, $\nu_F^{\pm 1/2-s} \xi_2 = \eta_2$ quadratic):

$$I(\xi_1 \otimes \xi_2) = I(\nu_F^{s \pm 1/2} \eta_2 \otimes \nu_F^{s \mp 1/2} \eta_2) = I(\nu_F^{\pm 1} \nu_F^{s \mp 1/2} \eta_2 \otimes \nu_F^{s \mp 1/2} \eta_2),$$

and thus Lemma 9.2.9 applies to this case, and we have

$$(9.3.46) \quad I(\xi_1 \otimes \xi_2) = I_\alpha(s, \delta(\eta_2)) + I_\alpha(s, \eta_2 \circ \det).$$

We now plug in $s = \pm 1/2$ and obtain

$$I(\xi_1 \otimes \xi_2) = I(\nu_F^{\pm 1} \eta_2 \otimes \eta_2) = I_\alpha(\pm 1/2, \delta(\eta_2)) + I_\alpha(\pm 1/2, \eta_2 \circ \det).$$

Note that by [BDK86, Lemma 5.4(iii)], we have

$$(9.3.47) \quad I_\alpha(-1/2, \delta(\eta_2)) \simeq I_\alpha(1/2, \delta(\eta_2)) \quad \text{and} \quad I_\alpha(-1/2, \eta_2 \circ \det) \simeq I_\alpha(1/2, \eta_2 \circ \det).$$

Therefore in the above we treat $\pm 1/2$ cases together. Recall from (9.3.2), we have that $\eta_2 = \omega$ is unitary, thus we can apply [Mui97, Proposition 4.1 (ii)] and obtain

$$(9.3.48) \quad I(\xi_1 \otimes \xi_2) = \pi(\eta_2) + J_\beta(1, \pi(1, \eta_2)) + J_\beta(1/2, \delta(\eta_2)) + J_\alpha(1/2, \delta(\eta_2)).$$

The LLC for these irreducible constituents was already computed in Table 16 (when η_2 is unramified) and Table 17 (when η_2 is ramified).

(b) Case ($s = \pm 3/2$, $\nu_F^{s_2 - s \pm 1/2} \xi = 1$): we have $\xi_2 = \nu_F^{s_2} \xi = \nu_F^{s \mp 1/2}$, and $\xi_1 = \nu_F^{\pm} \xi_2 = \nu_F^{\pm 1/2 + s}$.⁴ Thus

$$I(\xi_1 \otimes \xi_2) = I(\nu_F^{\pm 1/2 + s} \otimes \nu_F^{s \mp 1/2}) = I(\nu_F^{\pm 1} \nu_F^{s \mp 1/2} \otimes \nu_F^{s \mp 1/2}),$$

and thus Lemma 9.2.9 applies to this case, and we have

$$(9.3.49) \quad I(\xi_1 \otimes \xi_2) = I_\alpha(s, \delta(\nu_F^{\pm 1/2 - s} \xi_2)) + I_\alpha(s, \nu_F^{\pm 1/2 - s} \xi_2 \circ \det).$$

We now plug in $s = \pm 3/2$ and the explicit values for ξ_2 , and obtain

$$I(\xi_1 \otimes \xi_2) = I(\nu_F^{\pm 2} \otimes \nu_F^{\pm 1}) = I_\alpha(\pm 3/2, \delta(1)) + I_\alpha(\pm 3/2, 1_{\text{GL}_2}).$$

By [BDK86, Lemma 5.4(iii)], we have

$$(9.3.50) \quad I_\alpha(-3/2, \delta(1)) \simeq I_\alpha(3/2, \delta(1)) \quad \text{and} \quad I_\alpha(-3/2, 1_{\text{GL}_2}) \simeq I_\alpha(3/2, 1_{\text{GL}_2}).$$

Therefore it suffices to treat the $\pm 3/2$ cases together. When $s = 3/2$, by [Mui97, Proposition 4.4], we have the following decomposition into irreducible constituents:

$$(9.3.51) \quad I_\alpha(3/2, \delta(1)) = \text{St}_{\text{G}_2} + J_\alpha(3/2, \delta(1)) \quad \text{and} \quad I_\alpha(3/2, 1_{\text{GL}_2}) = 1_{\text{G}_2} + J_\beta(5/2, \delta(1)).$$

We thus obtain

$$(9.3.52) \quad I(\xi_1 \otimes \xi_2) = I(\nu_F^{\pm 2} \otimes \nu_F^{\pm 1}) = \text{St}_{\text{G}_2} + J_\alpha(3/2, \delta(1)) + 1_{\text{G}_2} + J_\beta(5/2, \delta(1)).$$

The LLC for these irreducible constituents was already computed in Table 18.

(c) Case ($s = \pm 1/2$, $\nu_F^{s_2 - s \pm 1/2} \xi = \eta_3$ cubic):

$$I(\xi_1 \otimes \xi_2) = I(\nu_F^{s \pm 1/2} \eta_3 \otimes \nu_F^{s \mp 1/2} \eta_3) = I(\nu_F^{\pm 1} \nu_F^{s \mp 1/2} \eta_3 \otimes \nu_F^{s \mp 1/2} \eta_3),$$

and thus Lemma 9.2.9 applies to this case, and we have the decomposition into irreducible constituents:

$$(9.3.53) \quad I(\xi_1 \otimes \xi_2) = I_\alpha(s, \delta(\eta_3)) + I_\alpha(s, \eta_3 \circ \det).$$

We now plug in $s = \pm 1/2$ and obtain

$$I(\xi_1 \otimes \xi_2) = I(\nu_F^{\pm 1} \eta_3 \otimes \eta_3) = I_\alpha(\pm 1/2, \delta(\eta_3)) + I_\alpha(\pm 1/2, \eta_3 \circ \det).$$

By [BDK86, Lemma 5.5(iii)], we have

$$I_\alpha(-1/2, \delta(\eta_3)) = I_\alpha(1/2, \delta(\eta_3^{-1})) \quad \text{and} \quad I_\alpha(-1/2, \eta_3 \circ \det) = I_\alpha(1/2, \eta_3^{-1} \circ \det).$$

By [Mui97, Proposition 4.2(ii)], we have

$$\begin{aligned} I_\alpha(1/2, \delta(\eta_3^{\pm 1})) &= \pi(\eta_3^{\pm 1}) + J_\alpha(1/2, \delta(\eta_3^{\pm 1})) \quad \text{and} \\ I_\alpha(1/2, \eta_3^{\pm 1} \circ \det) &= J_\beta(1, \pi(\delta_3^{\pm 1}, \delta_3^{\mp 1})) + J_\alpha(1/2, \delta(\eta_3^{\mp 1})). \end{aligned}$$

In summary, we obtain the following decomposition into irreducible constituents:

$$(9.3.54) \quad I(\xi_1 \otimes \xi_2) = I(\nu_F^{\pm 1} \eta_3 \otimes \eta_3) = \pi(\eta_3^{\pm 1}) + J_\alpha(1/2, \delta(\eta_3^{\pm 1})) + J_\beta(1, \pi(\delta_3^{\pm 1}, \delta_3^{\mp 1})) + J_\alpha(1/2, \delta(\eta_3^{\mp 1})).$$

The LLC for these irreducible representations was already computed in Tables 19 and 20.

⁴Note that here ξ_1 should be thought of as the ‘‘new’’ ξ_1 after applying the Weyl group actions.

9.4. **Case $\mathfrak{s} = [T, \xi \otimes 1]_G$ with ξ ramified.** Consider the induced representation $I(\nu_F^{s_1} \xi \otimes \nu_F^{s_2})$.

–If $I(\nu_F^{s_1} \xi \otimes \nu_F^{s_2})$ is unitary, by [Key82, Theorem G₂], it is reducible if and only if $\nu_F^{s_2}$ and $\nu_F^{s_1} \xi$ are distinct quadratic and unitary. When reducible, it is of length 2, and we obtain Table 23 (notations as in [ABP11]).

Representation	Kazhdan-Lusztig triple
$\pi(\nu_F^{s_1} \xi \otimes \nu_F^{s_2})$	$([s_i, s_i], [1, 1], 1)$
$\pi'(\nu_F^{s_1} \xi \otimes \nu_F^{s_2})$	$([s_i, s_i], [1, 1], \text{sgn})$

TABLE 23. : LLC for the irreducible constituents of unitary $I(\nu_F^{s_1} \xi \otimes \nu_F^{s_2})$ attached to $\mathfrak{s} = [T, \xi \otimes 1]_G$ with ξ ramified.

–If $I(\nu_F^{s_1} \xi \otimes \nu_F^{s_2})$ is non-unitary, by [Mui97, Proposition 3.1], $I(\nu_F^{s_1} \xi \otimes \nu_F^{s_2})$ is irreducible unless we are in one of the following two cases:

- (1) $\nu_F^{s_2} = \nu_F^{\pm 1}$, i.e. $s_2 = \pm 1$ and s_1 is arbitrary
- (2) $\nu_F^{2s_1+s_2} \xi^2 = \nu_F^{\pm 1}$, i.e. ξ is ramified quadratic and $2s_1 + s_2 = \pm 1$.

Remark 9.4.1. The case $\mathfrak{s} = [T, \xi \otimes \xi']$ for $\xi \neq \xi'$ both ramified either reduces to the case for $[T, \xi \otimes 1]$ or the case for $[T, \xi \otimes \xi]$, so there is no need to discuss this case separately.

9.4.1. Case (1) within Case $\mathfrak{s} = [T, \xi \otimes 1]_G$ with ξ ramified.

In this case,

$$I(\xi_1 \otimes \xi_2) = I(\nu_F^{s_1} \xi \otimes \nu_F^{\pm 1}) = I(\xi_1 \otimes \nu_F^{\pm 1}).$$

- (1) When ξ_1 is quadratic, this is similar to the cases covered in Tables 16 and 17.
- (2) When ξ_1 is non-quadratic (already assumed to be ramified), this is similar to the cases covered in Table 12 and Table 13.

9.4.2. **Case (2) within Case $\mathfrak{s} = [T, \xi \otimes 1]_G$ with ξ ramified quadratic.** When ξ is ramified quadratic, we are in case (4) of Proposition 7.0.2 via the action of W , where $W_G^{\mathfrak{s}} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and

$$(9.4.1) \quad \mathcal{J}^{\mathfrak{s}} = \text{SO}_4(\mathbb{C}) \simeq \text{SL}_2^{\text{lr}}(\mathbb{C}) \times \text{SL}_2^{\text{sr}}(\mathbb{C}) / \{\pm 1\}.$$

In this case, let $\chi := \nu_F^{s_1} \xi$ where ξ is ramified quadratic, and we have

$$(9.4.2) \quad I(\xi_1 \otimes \xi_2) = I(\nu_F^{s_1} \xi \otimes \nu_F^{\pm 1 - 2s_1}) = I(\chi \otimes \chi^{-2} \nu_F^{\pm 1}) \stackrel{babab}{=} I(\chi^2 \nu_F^{\mp 1} \otimes \chi^{-1}) \stackrel{b}{=} I(\chi \nu_F^{\mp 1} \otimes \chi).$$

Thus Lemma 9.2.9 applies to this case, and we have

$$(9.4.3) \quad I(\xi_1 \otimes \xi_2) = I(\nu_F^{s_1 \mp 1} \xi \otimes \nu_F^{s_1} \xi) = I_{\alpha}(s, \delta(\nu_F^{\mp 1/2-s} \chi)) + I_{\alpha}(s, \nu_F^{\mp 1/2-s} \chi \circ \det).$$

Recall from (9.3.2) that $\omega = \xi\nu_F^{-t} = \xi\nu_F^{s_2-s\pm 1/2}$ is unitary, thus

$$(9.4.4) \quad \nu_F^{\mp 1/2-s} \chi = \nu_F^{-s+s_2\pm 1/2} \xi = \omega$$

is unitary. Therefore [Mui97, Theorem 3.1] applies to this case and $I_\alpha(s, \delta(\nu_F^{\mp 1/2-s} \chi))$ and $I_\alpha(s, \nu_F^{\mp 1/2-s} \chi \circ \det)$ are irreducible unless

$s = \mp 1/2$, $\nu_F^{\mp 1/2-s} \chi = \chi$ is quadratic, $s = \mp 3/2$, $\nu_F^{\mp 1/2-s} \chi = 1$ or $s = \mp 1/2$, $\nu_F^{\mp 1/2-s} \chi$ is cubic.

Clearly the second and third cases cannot happen (since ξ is ramified quadratic), thus we are left with only the first possibility, i.e. $I_\alpha(s, \delta(\nu_F^{\mp 1/2-s} \chi))$ and $I_\alpha(s, \nu_F^{\mp 1/2-s} \chi \circ \det)$ are irreducible unless

$$s = \mp 1/2, \omega = \nu_F^{\mp 1/2-s} \chi = \chi \text{ is quadratic.}$$

Now, since both $\chi = \nu_F^{s_1} \xi$ and ξ are quadratic, we have $1 = \nu_F^{2s_1}$, and thus $s_1 = 0$. Thus we have $\xi = \chi = \omega$ is unitary and quadratic.

- (1) When they are irreducible i.e. when $s \neq \pm 1/2$, we compute the L -parameters for irreducible constituents and record them in Table 24. By Lemma 9.2.6 (a), since the representation $\delta(\xi)$ of $\mathrm{GL}_2(F)$ is generic, $I_\alpha(s, \nu_F^{\mp 1/2-s} \delta(\xi)) = i_{P_\alpha}^G(\delta(\xi))$ is also generic. By Lemma 9.2.6 (b), since the representation $\xi \circ \det$ of $\mathrm{GL}_2(F)$ is not generic, $I_\alpha(s, \nu_F^{\mp 1/2-s} \xi \circ \det) = i_{P_\alpha}^G(\xi \circ \det)$ is also non-generic.

Representation	enhanced L -parameter
$i_{M_\alpha}^G(\delta(\xi))$	$(\varphi_{\sigma, A_1}, 1)$
$i_{M_\alpha}^G(\xi \circ \det)$	$(\varphi_{\sigma, 1}, 1)$

TABLE 24. : LLC for the irreducible constituents of $I(\xi\nu_F^{\mp 1} \otimes \xi)$ attached to Case (2) of $\mathfrak{s} = [T, \xi \otimes 1]_G$ with ξ ramified quadratic.

- (2) When they are reducible i.e. $s = \mp 1/2$ and $\omega = \nu_F^{\mp 1/2-s} \chi = \chi$ is quadratic, we have in $R(G)$:

(a) Case ($s = \pm 1/2$, $\nu_F^{\mp 1/2-s} \chi = \eta_2$ quadratic)⁵:

$$I(\xi_1 \otimes \xi_2) = I(\nu_F^{s\pm 1/2} \eta_2 \otimes \nu_F^{s\mp 1/2} \eta_2) = I(\nu_F^{\pm 1} \nu_F^{s\mp 1/2} \eta_2 \otimes \nu_F^{s\pm 1/2} \eta_2),$$

and thus Lemma 9.2.9 applies to this case, and we have

$$(9.4.5) \quad I(\xi_1 \otimes \xi_2) = I_\alpha(s, \delta(\eta_2)) + I_\alpha(s, \eta_2 \circ \det).$$

⁵Note that although we have used the same notation η_2 as in the previous sections, this η_2 is not necessarily the same character as the η_2 from the previous sections; here we are simply abusing notations.

We now plug in $s = \pm 1/2$ and obtain

$$I(\xi_1 \otimes \xi_2) = I(\nu_F^{\pm 1} \eta_2 \otimes \eta_2) = I_\alpha(\pm 1/2, \delta(\eta_2)) + I_\alpha(\pm 1/2, \eta_2 \circ \det).$$

Note that by [BDK86, Lemma 5.4(iii)], we have

$$(9.4.6) \quad I_\alpha(-1/2, \delta(\eta_2)) \simeq I_\alpha(1/2, \delta(\eta_2)) \quad \text{and} \quad I_\alpha(-1/2, \eta_2 \circ \det) \simeq I_\alpha(1/2, \eta_2 \circ \det).$$

Therefore in the above we treat $\pm 1/2$ cases together. As in (9.4.4), $\eta_2 = \omega$ is unitary, by [Mui97, Proposition 4.1 (ii)] we have

$$(9.4.7) \quad I(\xi_1 \otimes \xi_2) = \pi(\eta_2) + J_\beta(1, \pi(1, \eta_2)) + J_\beta(1/2, \delta(\eta_2)) + J_\alpha(1/2, \delta(\eta_2)).$$

The computation of LLC for these irreducible constituents was already done in Table 17.

(b) Case ($s = \pm 3/2$, $\nu_F^{\mp 1/2-s} \chi = 1$): recall that $\chi = \nu_F^{s_1} \xi$. Plugging it in we get

$$(9.4.8) \quad \nu_F^{\mp 1/2-s} \nu_F^{s_1} \xi = \nu_F^{s_1 \pm 1} \xi = 1.$$

Since ξ is ramified quadratic in this case, the above equation (9.4.8) cannot happen.

(c) Case ($s = \pm 1/2$, $\nu_F^{\mp 1/2-s} \chi = \eta_3$ cubic): This case also cannot happen since ξ is ramified quadratic.

10. MAIN RESULTS

10.1. List of properties of the LLC. In this subsection, we state several properties that are expected to be satisfied by the local Langlands correspondence. Recall from Definition 2.1.2 that the L -packet of irreducible representations of G attached to the L -parameter φ is denoted by $\Pi_\varphi(G)$.

Property 10.1.1. [Bor79, §10.3] Let φ be an L -parameter for G .

- (1) φ is bounded if and only if one element (equivalently any element) of $\Pi_\varphi(G)$ is tempered;
- (2) φ is discrete if and only if one element (equivalently any element) of $\Pi_\varphi(G)$ is square-integrable modulo center;
- (3) φ is supercuspidal if and only if all the elements of $\Pi_\varphi(G)$ are supercuspidal.

Property 10.1.2. ([Art06, §2], and [Kal16, Conjecture B]) The elements of $\Pi_\varphi(G)$ are in bijection with $\text{Irr}(S_\varphi)$.

Property 10.1.3. [SZ18, §7.2] Let (P, π, ν) be a standard triple for G . We have

$$(10.1.4) \quad \varphi_{J(P, \pi, \nu)} = \iota_{L^\vee} \circ \varphi_{\pi \otimes \chi_\nu},$$

where $J(P, \pi, \nu)$ is the Langlands quotient defined in (3.2.1) and $\iota_{L^\vee} : L^\vee \hookrightarrow G^\vee$ is the canonical embedding.

In general, the bijection mentioned in Property 10.1.2 depends on the choice of a Whittaker datum $\mathfrak{w} := (U, \chi)$ —up to conjugation by G —(see for instance the $\text{SL}_2(F)$ -example in [GR10, pp.484-485]). We will denote this bijection as

$$(10.1.5) \quad \iota_{\mathfrak{w}} : \Pi_\varphi(G) \rightarrow \text{Irr}(S_\varphi).$$

An L -packet is called \mathfrak{w} -generic if it contains an element which is \mathfrak{w} -generic in the sense of Definition 3.2.2.

Property 10.1.6. [Sha90, Conjecture 9.4] If φ is bounded, then the L -packet $\Pi_\varphi(G)$ is \mathfrak{w} -generic for some Whittaker datum \mathfrak{w} . Moreover, the conjectural bijection $\iota_{\mathfrak{w}}: \Pi_\varphi(G) \rightarrow \text{Irr}(S_\varphi)$ maps the \mathfrak{w} -generic representation to the trivial representation of S_φ .

Lemma 10.1.7. Supercuspidal L -packets satisfy Property 10.1.6.

Proof. The depth-zero case follows from [Kal21, Lemma 4.2.1]. For the reader's convenience, we recall the argument here. By [Kal21, Lemma H.1], \mathfrak{w} determines, uniquely up to $G(F)$ -conjugacy, an absolutely special vertex $x \in \mathcal{B}(G, F)$ such that χ has depth-zero at x for some $(U, \chi) \in \mathfrak{w}$. By \mathfrak{w} -genericity, the supercuspidal $\pi \in \Pi_\varphi$ is induced from an irreducible representation of $G(F)_x$ containing a χ_x -generic representation of $G(F)_{x,0}$. By [Kal19, Lemma 3.4.12], up to $G(F)$ -conjugacy, there exists exactly one admissible embedding $j: S \rightarrow G$ such that the vertex x corresponds to the maximal torus $j(S) \subset G$. A \mathfrak{w} -generic member of the L -packet Π_φ necessarily arises from the non-singular Deligne-Lusztig packet $[\pi_{(j(S), \theta_j)}]$ (as defined in [Kal21, p.35]). By [DLM92, Proposition 3.10], there exists a unique χ_x -generic irreducible component of the Deligne-Lusztig character $\kappa_{(j(S), \theta_j^\circ)}$ (as defined in [Kal21, Definition 2.6.8 and p.34]), and hence a unique irreducible representation of $G(F)_x$ containing it after taking the restriction. Its compact induction to $G(F)$ gives the unique \mathfrak{w} -generic element of the Deligne-Lusztig packet $[\pi_{(j(S), \theta_j)}]$. As remarked in *loc.cit.*, the positive-depth case can be deduced from the depth-zero case via the local character expansions of [Spi18]. \square

We now state a compatibility property of the LLC with supercuspidal supports. First we recall the following conjecture from [Vog93, Conjecture 7.18], or equivalently [Hai14, Conjecture 5.2.2].

Conjecture 10.1.8. Let P be a parabolic subgroup of G with Levi subgroup L and σ a smooth irreducible supercuspidal representation of L . For any irreducible constituent π of $i_P^G \sigma$, the infinitesimal L -parameters $\lambda_{i_{L^\vee} \circ \varphi_\sigma}$ and λ_{φ_π} are G^\vee -conjugate.

The following Property 10.1.9 generalizes Conjecture 10.1.8 (see Remark 2.3.1). Let $\mathcal{L}(G)$ be a set of representatives for the conjugacy classes of Levi subgroups of G . By [ABPS17a, Proposition 3.1], for any $L \in \mathcal{L}(G)$ there is a canonical isomorphism between $W_G(L)$ and $W_{G^\vee}(L^\vee)$.

Property 10.1.9. [AMS18, Conjecture 7.8] The following diagram is commutative

$$(10.1.10) \quad \begin{array}{ccc} \text{Irr}(G) & \xrightarrow[\text{1-1}]{\text{LLC}} & \Phi_e(G) \\ \text{Sc} \downarrow & & \downarrow \text{Sc} \\ \bigsqcup_{L \in \mathcal{L}(G)} \text{Irr}_{\text{scusp}}(L)/W_G(L) & \xrightarrow[\text{1-1}]{\text{LLC}} & \bigsqcup_{L \in \mathcal{L}(G)} \Phi_{e, \text{cusp}}(L)/W_G(L). \end{array} \quad .$$

Remark 10.1.1. When $L = G$, the diagram (10.1.10) collapses to the bottom horizontal line, and Property 10.1.9 states that the (isomorphism classes of) irreducible

supercuspidal representations of G correspond under the local Langlands correspondence to the (G^\vee -conjugacy classes of) cuspidal enhanced Langlands parameters (see also [AMS18, Conjecture 6.10] and [Aub19, Conjecture 5.2]):

$$(10.1.11) \quad \text{LLC: } \text{Irr}_{\text{scusp}}(G) \xrightarrow{1-1} \Phi_{e,\text{cusp}}(G).$$

Note that Property 10.1.9 is known to hold for unipotent representations by [FOS20, Theorem 2], for all representations of general linear groups and split classical p -adic groups by [Mou17], and all representations of special linear groups by [AMS18].

Conjecture 10.1.12. [AMS18, Conjecture 2] For any $\mathfrak{s} = [L, \sigma]_G \in \mathfrak{B}(G)$, the LLC for L given by $\sigma \mapsto (\varphi_\sigma, \rho_\sigma)$ induces a bijection

$$(10.1.13) \quad \text{Irr}^{\mathfrak{s}}(G) \xrightarrow{\sim} \Phi_e^{\mathfrak{s}^\vee}(G),$$

where $\mathfrak{s}^\vee = [L^\vee, (\varphi_\sigma, \rho_\sigma)]_{G^\vee}$.

Conjecture 10.1.12 is proved for split classical groups [Mou17, §5.3], for $\text{GL}_n(F)$ and $\text{SL}_n(F)$ [ABPS16b, Theorems 5.3 and 5.6], for principal series representations of split groups [ABPS17b, §16]. For the group G_2 , a bijection between $\text{Irr}^{\mathfrak{s}}(G)$ and $\Phi_e^{\mathfrak{s}^\vee}(G)$ has been constructed in [AX22, Theorem 3.1.19].

Recall from [AX22], we have an isomorphism

$$(10.1.14) \quad \text{Irr}^{\mathfrak{s}}(G) \xrightarrow{\sim} \Phi_e^{\mathfrak{s}^\vee}(G)$$

for each Bernstein series $\text{Irr}^{\mathfrak{s}}(G)$ of *intermediate series*. On the other hand, the bijection (10.1.14) holds for *principal series* blocks thanks to [Roc98, Ree02, ABPS16a, AMS18].

Moreover, we verify the following two properties for our LLC.

Property 10.1.15. [Sha90] The quantity $\frac{\text{fddeg}(\pi)}{\dim(\rho)}$ is constant in an L -packet.

By Harish-Chandra (see [Wal03, Proposition III.4.1]), any tempered non-discrete series irreducible representation π is a subrepresentation of $i_P^G(\delta)$, where P is a parabolic subgroup of G with Levi factor L and δ is a discrete series representation of L , and the G -conjugacy class of pair (L, δ) is uniquely determined.

The following is a standard expected property of LLC.

Property 10.1.16. The L -parameter of π is the composition of the L -parameter of δ with the natural inclusion $L^\vee \rightarrow G^\vee$ and $\rho_\pi = \rho_\delta$.

10.2. Main result. Construction of the Local Langlands Correspondence

$$(10.2.1) \quad \begin{aligned} \text{LLC: } \text{Irr}(G) &\xrightarrow{1-1} \Phi_e(G) \\ \pi &\mapsto (\varphi_\pi, \rho_\pi). \end{aligned}$$

Recall from (3.3.2) and (2.4.3) that we have

$$\text{Irr}^{\mathfrak{s}}(G) = \bigsqcup_{\mathfrak{s} \in \mathfrak{B}(G)} \text{Irr}^{\mathfrak{s}}(G) \quad \text{and} \quad \Phi_e(G) = \bigsqcup_{\mathfrak{s}^\vee \in \mathfrak{B}^\vee(G)} \Phi_e^{\mathfrak{s}^\vee}(G).$$

When $\pi \in \text{Irr}(G)$ is not supercuspidal, we have $\mathfrak{s} = [L, \sigma]_G$ where L is a proper Levi subgroup of G . Hence, L is isomorphic to either $F^\times \times F^\times$ or $\text{GL}_2(F)$. Let $\varphi_\sigma: W'_F \rightarrow L^\vee$ be the L -parameter attached to σ by the Local Langlands Correspondence for L (see [BH06b]). The L -packet $\Pi_{\varphi_\sigma}(L)$ is always a singleton (in particular, the enhancement ρ_σ is trivial). The L^\vee -conjugacy class of φ_σ is uniquely determined by σ , and we have $\varphi_{(\chi \circ \det) \otimes \sigma} = \varphi_\sigma \otimes \varphi_\chi$, i.e. [AX22, Property 3.12(1)] holds. This allows us to define

$$(10.2.2) \quad \mathfrak{s}^\vee := [L^\vee, (\varphi_\sigma, 1)]_{G^\vee}.$$

Let $\pi \mapsto (\varphi_\pi, \rho_\pi)$ be the bijection

$$(10.2.3) \quad \text{Irr}^{\mathfrak{s}}(G) \xrightarrow{\sim} \Phi_e^{\mathfrak{s}^\vee}(G),$$

established in [AX22, Main Theorem] (for intermediate series) and in [ABPS16a] (for principal series). We have given explicit Kazhdan-Lusztig triples and L -packets in §9 and §8 (see also Tables 7 and 8).

We consider now the case where π is supercuspidal. Hence we have $\mathfrak{s} = [G, \pi]_G$ for π an irreducible supercuspidal representation of G .

- (a) When π is non-singular supercuspidal, we define (φ_π, ρ_π) to be the enhanced L -parameter constructed in [Kal21]. In particular, when π is regular (in the sense of Definition 3.6.9), the enhanced L -parameter (φ_π, ρ_π) coincides with the one constructed in [Kal19].

When F has characteristic zero and $p \geq 7(e(F|\mathbb{Q}_p) + 2)$, where $e(F|\mathbb{Q}_p)$ is the ramification index of F/\mathbb{Q}_p (see the proof of [FKS19, Proposition 4.3.2]), the G^\vee -conjugacy class of (φ_π, ρ_π) is uniquely determined by π [FKS19, Theorem 4.4.4], since we have $S_{\varphi_\pi}^+ = S_{\rho_\pi}$ as G_2 is adjoint.

- (b) When π is a unipotent supercuspidal representation of G , we define (φ_π, ρ_π) to be the enhanced L -parameter constructed in [Lus95] and [Mor96, § 5.6].
- (c) Let π be a non-unipotent depth-zero *singular* supercuspidal representation of G . As recalled in (3.6.15), we have $\pi = i_{\mathbb{G}_x}^G \tau$, where x is a vertex of the Bruhat-Tits building of G and $\tau \in \mathcal{E}(\mathbb{G}_x, s)$ with $s \neq 1$. We have three cases:
- $x = x_1$: From §6.0.1(1)(c), when $q \equiv -1(3)$, we have one rational Lusztig series $\mathcal{E}(\mathbb{G}_{x_1}, s_1)$ giving a singular depth-zero non-unipotent supercuspidal representation π_1 , which by Property 10.1.9 lives in the same L -packet as an intermediate series representation $\pi(\sigma)$ given precisely in Table 8.0.8, as explained in 5.2.1(2b). The L -packet in this case is given by

$$(10.2.4) \quad \Pi_\varphi(G) = \{\pi_1, \pi(\sigma)\}.$$

Property 10.1.15 is verified in this case by comparing (6.0.9) and the quantities in Table 8.0.8, as explained in Paragraph 8.0.9.

- $x = x_2$: From §6.0.1(2), when $q \equiv 1(3)$, we have two possible rational Lusztig series $\mathcal{E}(\mathbb{G}_{x_2}, s_2[\zeta_3])$ and $\mathcal{E}(\mathbb{G}_{x_2}, s_2[\zeta_3^2])$ for the primitive third root of unity ζ_3 , where $s_2[\zeta] = \text{diag}(1, \zeta, \zeta^2) \pmod{\mathbb{F}_q^\times}$ for $\zeta \in \{\zeta_3, \zeta_3^2\}$. Each Lusztig series $\mathcal{E}(\mathbb{G}_{x_2}, s[\zeta])$ contains three cuspidal representations: $\tau_2^1[\zeta], \tau_2^2[\zeta], \tau_2^3[\zeta]$. Let $\zeta \in \{\zeta_3, \zeta_3^2\}$. Let $i \in \{1, 2, 3\}$, and let $\pi_2^i(\zeta)$ denote the representation $i_{\mathbb{G}_{x_2}}^G(\tau_2^i[\zeta])$.

There are three (depth-zero) ramified cubic characters $\eta_3^1, \eta_3^2, \eta_3^3$ of F^\times , corresponding to three ramified cubic extensions E_3^1, E_3^2, E_3^3 over F . For each $i \in \{1, 2, 3\}$, let $\sigma_3^i := \nu_F^{\pm 1} \eta_3^i \otimes \eta_3^i$ (a character of $T \simeq F^\times \times F^\times$), and let $\varphi(\eta_3^i): W_F \times \mathrm{SL}_2(\mathbb{C}) \rightarrow G^\vee$ be the L -parameter for G defined by

$$(10.2.5) \quad \varphi(\eta_3^i) := \varphi_{\sigma_3^i, u},$$

using the formula (2.5.1), where the G^\vee -conjugacy class of u is $G_2(a_1)$ (the subregular unipotent class in $G_2(\mathbb{C})$). The restriction of $\varphi(\eta_3^i)$ to W_F factors through $\mathrm{Gal}(E_3^i/F)$. We define

$$(10.2.6) \quad \varphi_{\pi_2^i(\zeta_3)} = \varphi_{\pi_2^i(\zeta_3^2)} := \varphi(\eta_3^i).$$

We have $\mathcal{G}_\varphi \simeq \mathrm{SL}_3(\mathbb{C})$, the unipotent element u is regular in \mathcal{G}_φ (see Table 5), and $S_\varphi \simeq \mu_3$. From Table 20, we have $\varphi(\eta_3^i) = \varphi_{\pi(\eta_3^i)}$, where $\pi(\eta_3^i)$ is a discrete series representation in $\mathrm{Irr}^\mathfrak{s}(G)$ for $\mathfrak{s} = [T, \sigma_3^i]_G$.

Thus we obtain three L -packets of size 3, for each $i = 1, 2, 3$,

$$(10.2.7) \quad \Pi_{\varphi_2^i}(G) := \{\pi_2^i(\zeta_3), \pi_2^i(\zeta_3^2), \pi(\eta_3^i)\}.$$

Each L -packet $\Pi_{\varphi(\eta_3^i)}(G)$ contains the two depth-zero *singular* supercuspidal representations $\pi_2^i(\zeta)$ for $\zeta \in \{\zeta_3, \zeta_3^2\}$, and a depth-zero discrete series $\pi(\eta_3^i)$ in the principal series such that $\mathcal{J}^\mathfrak{s} = \mathrm{SL}_3(\mathbb{C})$. Here $\pi(\eta_3^i)$ is given precisely in Table 20 with η_3^i ramified of depth-zero. Property 10.1.15 is verified in this case by comparing (9.3.43) and (6.0.11).

- $x = x_3$: From §6.0.1(3), we have one rational Lusztig series $\mathcal{E}(\mathbb{G}_{x_3}, s_3[\zeta_2])$ for the primitive square root of unity ζ_2 , where $s_3[\zeta_2] = \mathrm{diag}(1, \zeta_2, \zeta_2, 1) \bmod \mathbb{F}_q^\times$. The Lusztig series $\mathcal{E}(\mathbb{G}_{x_3}, s_3[\zeta_2])$ contains two cuspidal representations: $\tau_3^1[\zeta_2]$ and $\tau_3^2[\zeta_2]$. Let $i \in \{1, 2\}$, and let $\pi_3^i(\zeta_2)$ denote the representation $i_{\mathbb{G}_{x_3}}^G(\tau_3^i[\zeta_2])$.

There are two (depth-zero) ramified quadratic characters η_2^1, η_2^2 of F^\times , corresponding to two ramified cubic extensions E_2^1, E_2^2 over F . For each $i \in \{1, 2\}$, let $\sigma_2^i := \nu_F \otimes \eta_2^i$ (a character of the torus $T \simeq F^\times \times F^\times$), and let $\varphi(\eta_2^i): W_F \times \mathrm{SL}_2(\mathbb{C}) \rightarrow G^\vee$ be the L -parameter for G defined by

$$(10.2.8) \quad \varphi(\eta_2^i) := \varphi_{\sigma_2^i, u},$$

using the formula (2.5.1), where the G^\vee -conjugacy class of u is $G_2(a_1)$ (the subregular unipotent class in $G_2(\mathbb{C})$). The restriction of $\varphi(\eta_2^i)$ to W_F factors through $\mathrm{Gal}(E_2^i/F)$. We define

$$(10.2.9) \quad \varphi_{\pi_3^i(\zeta_2)} := \varphi(\eta_2^i).$$

We have $\mathcal{G}_\varphi \simeq \mathrm{SO}_4(\mathbb{C})$, the unipotent element u is regular in \mathcal{G}_φ (see Table 4), and $S_\varphi \simeq \mu_2$. From Table 17, we have $\varphi(\eta_2^i) = \varphi_{\pi(\eta_2^i)}$, where $\pi(\eta_2^i)$ is a discrete series representation in $\mathrm{Irr}^\mathfrak{s}(G)$ for $\mathfrak{s} = [T, \sigma_2^i]_G$.

Thus we obtain two L -packets of size 2, for each $i = 1, 2$,

$$(10.2.10) \quad \Pi_{\varphi(\eta_2^i)}(G) := \{\pi_3^i(\zeta_2), \pi(\eta_2^i)\}.$$

Each L -packet $\Pi_{\varphi(\eta_2^i)}(G)$ contains one depth-zero *singular* supercuspidal representations $\pi_3^i(\zeta_2)$ and a depth-zero discrete series $\pi(\eta_2^i)$ in the principal series such that $\mathcal{J}^s = \mathrm{SO}_4(\mathbb{C})$. Here $\pi(\eta_2^i)$ is given precisely in Table 17 with η_2^i ramified of depth-zero. Property 10.1.15 is verified in this case by comparing (9.3.24) and (6.0.14).

- (d) Let π be a positive-depth singular supercuspidal representation of G . As in §6.0.2(3b), such a singular supercuspidal representation necessarily arises from an L -parameter of SL_2 -type as in Definition 5.2.2, and lives in a mixed L -packet together with another non-supercuspidal representation described as follows. In this case, $Z_{G^v}(\varphi) = Z_{\mathrm{SL}_2(\mathbb{C})}(\varphi(W_F)) = \mu_2$ by Lemma 5.2.4. Hence the L -packet $\Pi_{\varphi}(G)$ consists of a non-supercuspidal representation and a supercuspidal representation by Lemma 5.2.1. Let P be a parabolic subgroup of G with Levi subgroup isomorphic to $\mathrm{GL}_2(F)$. Here the $\mathrm{GL}_2(F)$ corresponds to the short (resp. long) root if φ is of short (resp. long) SL_2 -type. By Property 10.1.9, the non-supercuspidal member has the following supercuspidal support: $\varphi|_{W_F}: W_F \rightarrow \mathrm{SL}_2(\mathbb{C})$, which gives us a supercuspidal τ of $\mathrm{PGL}_2(F)$. We consider $\tau_s := \tau \otimes (\nu_F \circ \det)^s$ as a representation of $\mathrm{GL}_2(F)$. Then the non-supercuspidal of G in this L -packet is the generic constituent $\pi(\tau_{\pm\frac{1}{2}})$ in $i_P^G \tau_{\pm\frac{1}{2}}$ (both $s = \pm\frac{1}{2}$ work), as detailed in Tables 10 and 8.0.8. Our L -packets are given by

$$(10.2.11) \quad \Pi_{\varphi}(G) = \{\pi^{\mathrm{sing}}, \pi(\tau_{\pm\frac{1}{2}})\}$$

Let G be the group of F -rational points of the exceptional group G_2 . We suppose that the residual characteristic of F is different from 2 and 3.

Theorem 10.2.1. *The Local Langlands Correspondence defined in (10.2.1) satisfies Properties 10.1.1, 10.1.2, 10.1.3, 10.1.9, 10.1.6 and 10.1.16, and satisfies Property 10.1.15 for depth-zero L -packets⁶.*

Moreover, Properties 10.1.1, 10.1.2, 10.1.3, 10.1.9 and 10.1.16 uniquely determine the bijection (10.2.1) (up few choices⁷).

Proof. By Property 10.1.3, the L -parameter φ_{π} of each irreducible non-tempered representation π of G is uniquely determined by (10.1.4). Since the L -packets of the representations of the proper Levi subgroups of G are all singletons, the L -packet $\Pi_{\varphi_{\pi}}(G)$ is a singleton (see also Theorem 5.1.2). Hence, by Property 10.1.2, we have $\rho_{\pi} = 1$. Thus the map (10.2.1) is uniquely characterized for non-tempered representations. This finishes the case of non-discrete series tempered representations.

Property 10.1.6 holds for supercuspidal L -packets by Lemma 10.1.7. For the mixed L -packets, this can be seen directly from paragraph (10.2) and the tables *loc.cit.*, where we specify which member in a given L -packet is generic.

Since we have already treated the discrete series in (10.2), we are done. \square

10.3. Summary of L -packets.

⁶we certainly expect this property to hold for positive-depth L -packets as well.

⁷See (10.2).

10.3.1. *Unipotent L -packets.* For unipotent irreducible representations, a Local Langlands Correspondence was constructed by Lusztig in [Lus95] for any simple p -adic reductive group \mathbf{G} of adjoint type. In the special case where $G = G_2(F)$, when the representation is also supercuspidal, it follows from [Mor96]. Moreover, a set of enhanced L -parameters

$$(10.3.1) \quad \{(\varphi[\zeta], \rho[\zeta]) : \zeta \in \{1, -1, \zeta_3, \zeta_3^2\}\}$$

was described by Morris in [Mor96, § 5.6]. Writing $s'[\zeta] := s_{\varphi[\zeta]}$ and $u[\zeta] := u_{\varphi[\zeta]}$, we have

- (1) $Z_{G^\vee}(s'[1]) \simeq G_2(\mathbb{C})$, $u[1] \in G_2(a_1)$, $S_{\varphi[1]} \simeq S_3$, and $\rho[1]$ is the sign representation of S_3 (i.e., it sends every element to its sign: for example, $(12) \mapsto -1$ and $(132) \mapsto 1$);
- (2) $Z_{G^\vee}(s'[-1]) \simeq SO_4(\mathbb{C})$, $u[-1] \in G_2(a_1)$, $S_{\varphi[-1]} \simeq \mathbb{Z}/2\mathbb{Z}$, and $\rho[-1]$ is the non-trivial representation of $\mathbb{Z}/2\mathbb{Z}$;
- (3) $Z_{G^\vee}(s'[\zeta]) \simeq SL_3(\mathbb{C})$, $u[\zeta] \in G_2(a_1)$, $S_{\varphi[\zeta]} \simeq \mathbb{Z}/3\mathbb{Z}$, and $\rho[\zeta]$ is a non-trivial representation of $\mathbb{Z}/3\mathbb{Z}$ for $\zeta \in \{\zeta_3, \zeta_3^2\}$.

As remarked in [Mor96, §5.6], the unipotent class is always the subregular ones $G_2(a_1)$. However, since $s'[1] \neq s'[-1]$, $s'[1] \neq s'[\zeta_3]$, and $s'[-1] \neq s'[\zeta_3]$, cases (1), (2) and (3) give different L -packets. We describe unipotent packets in these sections:

- §6.0.1(1)(1a),
- §5.2.1(1).

The unipotent discrete series of $G_2(F)$ belong to the following L -packets (see also [Ree94, p. 482] and [CFZ21, Table 2.7.1]):

- (1) $\Pi_{\varphi_1}(G) = \{\text{St}_{G_2}\}$,
- (2) $\Pi_{\varphi_2}(G) = \{\pi(1)', \pi(1), \pi[1]\}$,
- (3) $\Pi_{\varphi_3}(G) = \{\pi(\eta_2), \pi[-1]\}$, with η_2 unramified,
- (4) $\Pi_{\varphi_4}(G) = \{\pi(\eta_3), \pi[\zeta_3], \pi[\zeta_3^2]\}$, with η_3 unramified,

where St_{G_2} is the Steinberg representation of G , and the representations St_{G_2} , $\pi(1)'$, $\pi(1)$, $\pi(\eta_2)$ and $\pi(\eta_3)$ are in the principal series of G (these representations are Iwahori-spherical, i.e. they have non-zero Iwahori invariant vectors). Their enhanced L -parameters are also computed in [Ree94], using the Kazhdan-Lusztig parametrization established in [KL87]. Note that $\{\text{St}_{G_2}\}$ from Table 18 is a singleton L -packet, and coincides precisely with the L -packet labeled as $\tilde{\mathcal{E}}_2(\tau_1)$ in [Ree94, p.480]. The enhanced L -parameters for the representations St_{G_2} , $\pi(1)'$, $\pi(1)$, $\pi(\eta_2)$ and $\pi(\eta_3)$ belong to the same series $\Phi_e^{\mathfrak{s}^\vee}(G)$, where $\mathfrak{s}^\vee = [T^\vee, (\varphi_0, 1)]_{G^\vee}$. Here $\varphi_0: W_F/I_F \rightarrow T^\vee$, and 1 is the trivial representation of $S_{\varphi_0} = \{1\}$.

10.3.2. *Explicit Supercuspidal L -packets.*

- (1) *Depth-zero supercuspidal L -packets*
 - 5.2.1(5a): regular supercuspidal L -packets of sizes 4,3,1
 - 6.0.1(1)(1b): regular supercuspidal L -packets
 - 6.0.1(2)(2b): regular supercuspidal L -packets
 - 6.0.1(3)(3a): regular supercuspidal L -packets

(2) *Positive-depth supercuspidal L -packets*

- 5.2.2(1), 5.2.2(2b), 5.2.2(2c): regular supercuspidal L -packets of size 1, 2, 3, 4.
- 6.0.2(1): regular supercuspidal L -packets
- 6.0.2(2a): regular supercuspidal L -packets
- 5.2.2(2d) \longleftrightarrow 6.0.2(2b): non-singular (non-regular) positive-depth supercuspidal L -packet of size 4.
- 6.0.2(3a): regular supercuspidal L -packets

10.3.3. *Non-unipotent non-supercuspidal depth-zero packets.* We only list the non-singleton packets here. For a complete description of the singleton packets, see §5.

- 6.0.1(1c) \longleftrightarrow 5.2.1(2b): In this case, we obtain one L -packet, consisting of one singular supercuspidal representation coming from reductive quotient $\mathbb{G}_{x_0} \simeq G_2(\mathbb{F}_q)$ and $Z_{\mathbb{G}_{x_0}^\vee}(s) \simeq \mathrm{SU}_3(\mathbb{F}_q)$, and an intermediate series representation $\pi(\sigma)$ whose cuspidal support lives in $\mathrm{GL}_2^{\mathrm{lf}}$ and corresponds to a simple module of a Hecke algebra $\mathcal{H}^s(G)$ with unequal parameters $\{q^3, q\}$. In this case the formal degree for the L -packet should be $\frac{q^2}{q^3+1}$. This case only occurs when $q \equiv -1 \pmod{3}$.
- 6.0.1(2c) \longleftrightarrow 5.2.1(2a): In this case, we obtain three L -packets, each L -packet consisting of two singular supercuspidal representations coming from reductive quotient $\mathbb{G}_{x_1} \simeq \mathrm{SL}_3(\mathbb{F}_q)$ and $Z_{\mathbb{G}_{x_1}^\vee}(s) = \mathbb{T}^\vee \rtimes \mu_3$, and a generic principal series representation $\pi(\eta_3)$ from Table 20, where η_3 is a ramified cubic character. The unipotent class attached to these three L -packets is the subregular unipotent class $G_2(a_1)$, which comes from the regular unipotent of $\mathrm{SL}_3(\mathbb{C})$.
- 6.0.1(3c) \longleftrightarrow 5.2.1(3a): In this case, we obtain two L -packets, each L -packet consisting of one singular supercuspidal representation coming from reductive quotient $\mathbb{G}_{x_2} \simeq \mathrm{SO}_4(\mathbb{F}_q)$ and $Z_{\mathbb{G}_{x_2}^\vee}(s) \simeq \mathrm{S}(\mathrm{O}_2 \times \mathrm{O}_2)(\mathbb{F}_q)$ (here we take the non-split form of O_2), and a generic principal series representation $\pi(\eta_2)$ from Table 17, where η_2 is a ramified quadratic character.

10.3.4. *Non-unipotent non-supercuspidal (i.e. singular) positive-depth packets.*

- 5.2.2(2a) \longleftrightarrow 6.0.2(3b): size two L -packet as described in Paragraph 5.2.1, mixing an intermediate series representation with a singular supercuspidal.

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