THE EXPLICIT LOCAL LANGLANDS CORRESPONDENCE FOR G_2 II: CHARACTER FORMULAS AND STABILITY

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Abstract. We write down character formulas for representations of G_2 considered in [AX22a], and show that stability for L-packets uniquely pins down the Local Langlands Correspondence constructed in [AX22a], thus proving unique characterization of the LLC *loc.cit.*

Contents

1. Introduction

In this article, we complete the unique characterization of the explicit local Langlands correspondence for p-adic G_2 constructed in [AX22a]. More precisely, we use stability property of L-packets to uniquely pin down the choices of twists in the L-packets from [AX22a].

The rough idea is as follows: we explicitly calculate Harish-Chandra characters for the representations (including non-supercuspidals) in certain neighborhoods of semisimples in G_2 (see for example §3.4, §3.5, §4.3 and §4.4). In particular, stability property 2.1.1 (as formulated by DeBacker and Kaletha) implies the stability of the sum of characters in an L-packet locally around each semisimple. Using [DK06] (which builds on some works of Waldspurger), we deduce that the sum of two specific characters (one for a non-supercuspidal and another one for a *singular* supercuspidal) are stable, thus pinning down the size 2 mixed packets in [AX22a] (see Theorem 3.5.2). The size 3 mixed packets are pinned down similarly (see Theorem 4.4.1 and Theorem 4.4.2). Our computations involve a refinement of Roche's Hecke algebra isomorphisms (see §2.3).

2. Preliminaries

Let π be an admissible representation of G_2, which gives rise to a distribution Ch_π on $C_c^\infty(G_2)$. Then [HC99, Theorem 16.3] shows that Ch_π can be represented by a locally constant function on G_2^{rss}, the regular semisimple locus in G_2.

2.1. Stability of L-packets.

Property 2.1.1 (DeBacker, Kaletha)**.** Let φ be a discrete L-parameter. There exists a non-zero $\mathbb{C}$-linear combination

$$\sum_{\pi\in\Pi_\varphi} \dim(\rho_\pi)\mathrm{Ch}_\pi, \quad \text{for } z_\pi \in \mathbb{C}, \tag{2.1.2}$$

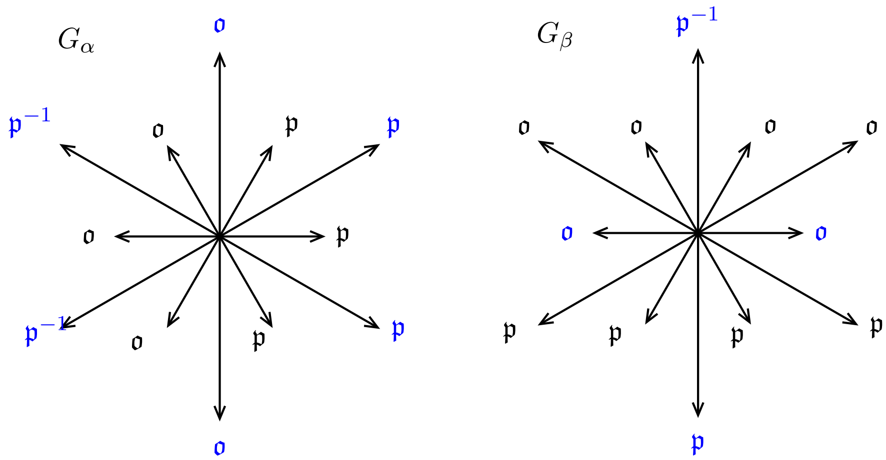

FIGURE 1. The parahoric subgroups G_α and G_β

which is stable. In fact, one can take $z_\pi = \dim(\rho_\pi)$ where ρ_π is the enhancement of the L-parameter. Moreover, no proper subset of Π_φ has this property.

2.2. **Parahoric subgroups.** We fix the choice of the following parahoric subgroups in $G_2(F)$, as in Diagram 1 where the blue nodes are the roots multiplied by $\mathfrak{p}$ in the unipotent radical G_{x+}.

Non-canonically (i.e., given a choice of uniformizer) there are isomorphisms $G_\alpha/G_{\alpha+} \cong \mathrm{SL}_3(\mathbb{F}_q)$ and $G_\beta/G_{\beta+} \cong \mathrm{SO}_4(\mathbb{F}_q)$,

More canonically, we can identify $G_\alpha/G_{\alpha+}$ the reductive quotient of the parahoric of SL_3:

$$H_\alpha := \left\{ g \in \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-1} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-1} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} \end{pmatrix} : \det g = 1 \right\}. \tag{2.2.1}$$

Similarly,

$$H_\beta := \left\{ (g,h) \in \begin{pmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} \end{pmatrix} \times \begin{pmatrix} \mathfrak{o} & \mathfrak{p}^{-1} \\ \mathfrak{p} & \mathfrak{o} \end{pmatrix} : \det(g) = \det(h) \right\} / \mathfrak{o}_F^\times \tag{2.2.2}$$

is a parahoric subgroup of $\mathrm{SO}_4(F)$, and there is a canonical isomorphism $H_\beta/H_{\beta+} \cong G_\beta/G_{\beta+}$ induced by the inclusion $\mathrm{SO}_4(F) \subset G_2(F)$.

2.3. **Refining Roche's isomorphism.** Let G be a connected split reductive group over F with maximal torus T, and let $T_0 \subset T$ be the maximal compact subgroup. Given a character $\chi\colon T_0 \to \mathbb{C}^\times$, let $\chi^\vee\colon \mathfrak{o}_F^\times \to T^\vee(\mathbb{C})$ be the dual, and let H be a split reductive group over F with maximal torus T such that $H^\vee = \mathrm{Z}_{G^\vee}(\mathrm{im}(\chi^\vee))$, where we assume $\mathrm{Z}_{G^\vee}(\mathrm{im}(\chi^\vee))$ is connected.

Roche [Roc98, Thm 8.2] produces a support-preserving isomorphism $\mathcal{H}(G/\!/I,\chi) \cong \mathcal{H}(H/\!/J,1)$ where I is an Iwahori subgroup of G and J is an Iwahori subgroup of H, but it is non-canonical. We make the isomorphism more canonical by slightly modifying the right-hand side:

Proposition 2.3.1. *There is a unique support preserving isomorphism $\mathcal{H}(G/\!/I,\chi) \cong \mathcal{H}(H/\!/J,\chi)$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{H}(T/\!/T_0,\chi) & = & \mathcal{H}(T/\!/T_0,\chi) \\ \downarrow t_u & & \downarrow t_u \\ \mathcal{H}(G/\!/I,\chi) & \xrightarrow{\sim} & \mathcal{H}(H/\!/J,\chi), \end{array}$$

where $t_u = t_{\delta_B^{-1/2}}$ is as in [Roc98, pg 399].

Unipotent pairs	Representations of $W \cong \mu_2^2$
$(00, \mathbb{C})$	$(1,1), 1$
$(0e, \mathbb{C})$	$1 \otimes \mathrm{sgn}$
$(e0, \mathbb{C})$	$\mathrm{sgn} \otimes 1$
$(ee, \mathbb{C})$	$\mathrm{sgn} \otimes \mathrm{sgn}$
$(ee, \mathcal{L})$	cuspidal

TABLE 1. Springer Correspondence for $\mathrm{SO}_4(\mathbb{C})$

Proof. Let $\overline{H}^\vee := H^\vee/\mathrm{Z}(H^\vee)$, so we have a cover $\overline{H} \xrightarrow{\pi} H$. Let $\overline{T}^\vee := T^\vee/\mathrm{im}(\chi^\vee)$ be a maximal torus of $\overline{H}^\vee$, which gives rise to a maximal torus $\overline{T} \subset \overline{H}$. For some finite discrete group g we have the exact sequence of algebraic groups

$$1 \to \mathrm{Z}_{\overline{H}} \to \overline{T} \xrightarrow{\pi} T \to 1$$

where since $\mathrm{im}(\chi^\vee) \subset \mathrm{Z}_{H^\vee}$ the composition $\pi^\vee \circ \chi^\vee \colon \mathfrak{o}_F^\times \to \overline{T}^\vee$ is trivial, we also have that $\chi \circ \pi = 1$. Thus, χ factors through $H^1_{\mathrm{gal}}(F, \mathrm{Z}_{\overline{H}})$, and so can be viewed as a character of H, since $H/\pi(\overline{H}) \cong H^1_{\mathrm{gal}}(F, \mathrm{Z}_{\overline{H}})$.

By [Roc98, Thm 6.3] there is a unique support-preserving homomorphism $\mathcal{H}(\overline{H}//\overline{J}, 1) \hookrightarrow \mathcal{H}(G//I, \chi)$, which extends[1] to a support-preserving isomorphism $i \colon \mathcal{H}(H//J, \chi) \xrightarrow{\sim} \mathcal{H}(G//I, \chi)$. The restriction of i to $\mathcal{H}(T//T_0, \chi)$ is then trivial on $\mathcal{H}(\overline{T}//\overline{T}_0, 1)$, so it is given by twisting by a character of $T/\pi(\overline{T})$. Since $T/\pi(\overline{T}) \cong H/\pi(\overline{H})$ such twists extend to the entire Hecke algebra $\mathcal{H}(H//J, \chi)$. Thus we have constructed an isomorphism $\mathcal{H}(G//I, \chi) \cong \mathcal{H}(H//J, \chi)$ satisfying the properties given.

Uniqueness is a general observation on automorphisms of Iwahori Hecke algebras $\mathcal{H}(H//J, 1)$ being determined by its restriction to $\mathbb{C}[T/T_0] = \mathcal{H}(T//T_0, 1)$. $\square$

3. SIZE 2 MIXED PACKETS

Recall the size 2 depth-zero mixed packets from [AX22a], where $\pi(\eta_2)$ is the principal series representation in Table 17 *loc.cit.*. It is the unique (tempered) sub-representation of the parabolic induction $I_B^{G_2}(\eta_2 \otimes \nu\eta_2)$, where η_2 is a ramified quadratic character of $F^\times$.

3.1. **Preliminaries on** $\mathrm{SO}_4(F)$**.** We let $\mathrm{SO}_4(F) := \{(g,h) \in \mathrm{GL}_2(F) \times \mathrm{GL}_2(F) : \det(g) = \det(h)\}/F^\times$, where $F^\times$ is diagonally embedded as $\{(aI_2, aI_2) : a \in F^\times\}$. It has a standard rank 2 maximal torus $T := \{(\mathrm{diag}(a_1,a_2), \mathrm{diag}(b_1,b_2)) : a_1a_2 = b_1b_2\}/F^\times$. Given characters $\chi_1, \chi_2, \varphi_1, \varphi_2$ of $F^\times$ such that $\chi_1\chi_2 = \varphi_1\varphi_2$, we let $\chi_1 \otimes \chi_2 \otimes \varphi_1 \otimes \varphi_2$ denote the character

$$\chi_1 \otimes \chi_2 \otimes \varphi_1 \otimes \varphi_2(\mathrm{diag}(a_1,a_2), \mathrm{diag}(b_1,b_2)) = \chi_1(a_1)\chi_2(a_2)\varphi_1(b_1)\varphi_2(b_2).$$

Note that for any character θ of $F^\times$, we have $\chi_1 \otimes \chi_2 \otimes \varphi_1 \otimes \varphi_2 = \theta\chi_1 \otimes \theta\chi_2 \otimes \theta\varphi_1 \otimes \theta\varphi_2$.

By abuse of notation, let $\widetilde{\det} \colon \mathrm{SO}_4(F) \to F^\times/(F^\times)^2$ be defined by $\widetilde{\det}(g,h) := \det(g) = \det(h)$. Thus, for any order 2 character η of $F^\times$, we obtain a character $\eta \circ \widetilde{\det}$ of $\mathrm{SO}_4(F)$. The same conventions apply for $\mathrm{SO}_4(\mathfrak{o}_F)$ and $\mathrm{SO}_4(\mathbb{F}_q)$.

The generalized Springer correspondence for SO_4 is given in Table 1 (see [CM93, §10.1, p. 166]), where e denotes the regular unipotent of SL_2, and $\mathcal{L}$ denotes the unique nontrivial cuspidal local system on the orbit of ee. Let $\mathcal{G}_{\mathrm{sgn}}$ denote the generalized Green function associated to the cuspidal local system $(ee, \mathcal{L})$, as in [DK06, §5.2.2].

3.2. **Calculating parahoric invariants for** $\pi(\eta_2)$**.**

[1] a priori the extension is non-canonical, but there is a unique choice making the diagram commute

3.2.1. *Calculating* $\pi(\eta_2)^{G_{\beta+}}$. By [Bon11, §4.3], there are two reducible Deligne-Lusztig inductions of $\mathrm{SL}_2(\mathbb{F}_q)$: the principal series representations $R_\pm(\alpha_0)$ and the cuspidal representations $R'_\pm(\theta_0)$, where α_0 and θ_0 are the unique order 2 character of $\mathbb{F}_q^\times$ and μ_{q+1}, respectively (in [Lus78, §2], $R'_\pm(\theta_0)$ is denoted H'_ϵ and H''_ϵ).

Remark 3.2.1. [Bon11, Table 5.4] gives the following, for $x \neq 0 \in \mathbb{F}_q$:

$$\mathrm{tr}(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}, R_\pm(\alpha_0)) = \frac{1}{2}(1 \pm \epsilon(x)\sqrt{q^*}) \tag{3.2.1}$$

$$\mathrm{tr}(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}, R'_\pm(\theta_0)) = \frac{1}{2}(-1 \pm \epsilon(x)\sqrt{q^*}), \tag{3.2.2}$$

where $q^* := (-1)^{\frac{q-1}{2}}q \equiv 1 \pmod 4$.

Definition 3.2.2. Let H_β be the parahoric defined in (2.2.2), which contains the index 2 subgroup

$$H_\beta^0 := \left\{(g,h) \in \begin{pmatrix}\mathfrak{o} & \mathfrak{o}\\ \mathfrak{o} & \mathfrak{o}\end{pmatrix} \times \begin{pmatrix}\mathfrak{o} & \mathfrak{p}^{-1}\\ \mathfrak{p} & \mathfrak{o}\end{pmatrix} : \det(g) = \det(h) = 1\right\}/\pm 1. \tag{3.2.3}$$

For a ramified quadratic character η_2 of $F^\times$, let $\varpi \in F$ be a uniformizer such that $\eta_2(\varpi) = 1$. We define the following irreducible representations of $G_\beta/G_{\beta+} \cong H_\beta/H_{\beta+}$:

$$\omega_{\mathrm{princ}}^{\eta_2} := \mathrm{Ind}_{G_\beta^0}^{G_\beta}(R_+(\alpha_0) \boxtimes R_+(\alpha_0)^{\mathrm{diag}(\varpi,1)}) \tag{3.2.4}$$

$$\omega_{\mathrm{cusp}}^{\eta_2} := \mathrm{Ind}_{G_\beta^0}^{G_\beta}(R'_+(\theta_0) \boxtimes R'_+(\theta_0)^{\mathrm{diag}(\varpi,1)}) \tag{3.2.5}$$

This is independent of the choice of the uniformizer ϖ.

Remark 3.2.3. The representation $\omega_{\mathrm{princ}}^{\eta_2}$ is an irreducible constituent of the length two representation $R_T^{\mathrm{SO}_4}(\epsilon \circ \widetilde{\det})$, for $T \subset \mathrm{SO}_4$ a split torus. Similarly $\omega_{\mathrm{cusp}}^{\eta_2}$ is an irreducible constituent of the length two representation $R_{T'}^{\mathrm{SO}_4}(\epsilon \circ \widetilde{\det})$, where $T' \subset \mathrm{SO}_4$ is a maximal anistropic torus. There are multiple ways to characterize the representations $\omega_{\mathrm{princ}}^{\eta_2}$ and $\omega_{\mathrm{cusp}}^{\eta_2}$ in the Deligne-Lusztig inductions:

(1) By Remark 3.2.1, for a regular unipotent $u = (\begin{pmatrix}1 & x\\ & 1\end{pmatrix}, \begin{pmatrix}1 & y\\ & 1\end{pmatrix}) \in H_\beta$ with $x \in \mathfrak{o}\backslash\mathfrak{p}$ and $y \in \mathfrak{p}^{-1}\backslash\mathfrak{o}$, we have

$$\mathrm{tr}(u, \omega_{\mathrm{princ}}^{\eta_2}) = \mathrm{tr}(u, \omega_{\mathrm{cusp}}^{\eta_2}) = \frac{1}{2}(1 + \eta_2(xy)q^*). \tag{3.2.6}$$

(2) By [Bon11, pg 55], they are characterized as irreducible components of the Gelfand-Graev representation $\Gamma_{\beta,\mathcal{O}}$ (notation as in [BM97, Thm 4.5]) associated to the nilpotent orbit $\mathcal{O} = \mathcal{O}_1^+$ (notation as in [DK06, §7.1]).

We use the following Hecke algebra isomorphism from [AX22b, AX22a, Roc98]: consider two copies of $\mathrm{SO}_4(F)$ which are Weyl group conjugates to each other. Let $\mathrm{SO}_4^{(1)}$ have roots $\pm\alpha, \pm(3\alpha + 2\beta)$, and let $\mathrm{SO}_4^{(2)}$ have roots $\pm(\alpha+\beta), \pm(3\alpha+\beta)$. The following is a corollary of Proposition 2.3.1.

Corollary 3.2.4. *Let I be the standard Iwahori of G_2. There exist canonical support-preserving isomorphisms of Hecke algebras*

$$\mathcal{H}(G_2//I, \epsilon \otimes \epsilon) \cong \mathcal{H}(\mathrm{SO}_4^{(1)}//J^{(1)}, \epsilon \circ \widetilde{\det}) \tag{3.2.7}$$

$$\mathcal{H}(G_2//I, \epsilon \otimes 1) \cong \mathcal{H}(\mathrm{SO}_4^{(2)}//J^{(2)}, \epsilon \circ \widetilde{\det}), \tag{3.2.8}$$

under which the representation $\pi(\eta_2)$ corresponds to the representation $\eta_2 \operatorname{St}_{\mathrm{SO}_4}$, where $J^{(i)} := I \cap \mathrm{SO}_4^{(i)}$ is an Iwahori subgroup of $\mathrm{SO}_4^{(i)}(F)$. The isomorphisms are characterized by the following commutative diagrams

$$\begin{array}{ccc} \mathcal{H}(T//T_0, \epsilon\otimes\epsilon) & = & \mathcal{H}(T//T_0, \epsilon\circ\widetilde{\det}) \\ \downarrow t_u & & \downarrow t_u \\ \mathcal{H}(G_2//I, \epsilon\otimes\epsilon) & \xrightarrow{\sim} & \mathcal{H}(\mathrm{SO}_4^{(1)}//J^{(1)}, \epsilon\circ\widetilde{\det}), \end{array} \tag{3.2.9}$$

$$\begin{array}{ccc} \mathcal{H}(T//T_0, \epsilon\otimes 1) & = & \mathcal{H}(T//T_0, \epsilon\circ\widetilde{\det}) \\ \downarrow t_u & & \downarrow t_u \\ \mathcal{H}(G_2//I, \epsilon\otimes 1) & \xrightarrow{\sim} & \mathcal{H}(\mathrm{SO}_4^{(2)}//J^{(2)}, \epsilon\circ\widetilde{\det}), \end{array} \tag{3.2.10}$$

where $t_u = t_{\delta_B^{-1/2}}$ is as in [Roc98, pg 399].

Proof. For brevity we write down the proof for the first isomorphism; the proof for the second isomorphism is entirely analogous. By [Roc98, Thm 6.3 and Thm 8.2], there is a canonical injection

$$\mathcal{H}(\mathrm{SL}_2\times\mathrm{SL}_2(F)//J, 1) \hookrightarrow \mathcal{H}(G_2//I, \epsilon\otimes\epsilon)$$

which extends (a priori) non-canonically to an isomorphism $\mathcal{H}(\mathrm{SO}_4(F)//J, 1) \cong \mathcal{H}(G_2//I, \epsilon\otimes\epsilon)$. There is, however, a unique extension to $\mathcal{H}(\mathrm{SO}_4(F)//J, 1)$ which makes $\pi(\eta_2)$ correspond to $\eta_2 \operatorname{St}_{\mathrm{SO}_4}$ as in Proposition 2.3.1.

The commutative diagrams follow from looking at the Jacuqet modules: the representation $\pi(\eta_2)$ is identified with a homomorphism $\mathcal{H}(G_2//I, \epsilon\otimes\epsilon) \to \mathbb{C}$, and the (normalized) Jacquet restriction $\mathrm{r}_\emptyset\pi(\eta_2) = \nu\eta_2\otimes\eta_2 + \nu\otimes\eta_2 + \eta_2\otimes\nu$ by [AX22a, §9] (see also [Mui97, Prop 4.1]). By [Roc98, Thm 9.2], the restriction of the homomorphism to $\mathcal{H}(T//T_0, \epsilon\otimes\epsilon\otimes 1\otimes 1)$ corresponds to the $\epsilon\otimes\epsilon$-isotypic component $\nu\eta_2\otimes\eta_2$.

Analogously, the (un-normalized) Jacquet restriction of $\eta_2 \operatorname{St}_{\mathrm{SO}_4^{(i)}}$ is $\mathrm{r}_\emptyset(\eta_2 \operatorname{St}_{\mathrm{SO}_4^{(i)}}) = \nu^{-1/2}\eta_2 \otimes \nu^{1/2}\eta_2 \otimes \nu^{-1/2} \otimes \nu^{1/2}$. These two characters are equal as the maximal torus of G_2 and the maximal torus of $\mathrm{SO}_4^{(i)}$ are canonically identified. □

By the Mackey formula, we have an isomorphism of representations of $G_\beta/G_{\beta+} \cong \mathrm{SO}_4(\mathbb{F}_q)$,

$$I_B^{G_2}(\nu\eta_2\otimes\eta_2)^{G_{\beta+}} \cong \bigoplus_{w\in B\backslash G_2/G_\beta} \operatorname{Ind}_{G_\beta\cap wBw^{-1}/(G_{\beta+}\cap wBw^{-1})}^{G_\beta/G_{\beta+}} (\epsilon\otimes\epsilon)^w, \tag{3.2.11}$$

where

$$B\backslash G_2/G_\beta \cong W(G_2)/W(\mathrm{SO}_4) = W/\langle s_\alpha, s_{3\alpha+\beta}\rangle = \{1, s_\beta, s_{3\alpha+\beta}\}. \tag{3.2.12}$$

The intersections $G_\beta \cap wBw^{-1}$ are shown in the following diagram 1, where the blue nodes correspond to the reductive quotient of the parahoric. (Note that in $G_{\beta+}$, the blue nodes are multiplied by $\mathfrak{p}$.) Therefore, the $G_{\beta+}$-invariants of $I_B(\nu\eta_2\otimes\eta_2)^{G_{\beta+}}$ gives

$$I_B^{G_2}(\nu\eta_2\otimes\eta_2)^{G_{\beta+}} \simeq \operatorname{Ind}_B^{\mathrm{SO}_4}(\epsilon\otimes\epsilon\otimes 1\otimes 1) + \operatorname{Ind}_B^{\mathrm{SO}_4}(\epsilon\otimes 1\otimes\epsilon\otimes 1)^2 \tag{3.2.13}$$

Analogously, computing the $G_{\beta+}$-invariants of I_α (resp. I_β) from [AX22a, §9] gives us the following

$$I_\alpha(\nu^{1/2}\eta_2 \operatorname{St})^{G_{\beta+}} \simeq \operatorname{Ind}_P^{\mathrm{SO}_4}(\epsilon \operatorname{St}) + \operatorname{Ind}_B^{\mathrm{SO}_4}(\epsilon\otimes 1\otimes\epsilon\otimes 1) \tag{3.2.14}$$

$$I_\beta(\nu^{1/2}\eta_2 \operatorname{St})^{G_{\beta+}} \simeq \operatorname{Ind}_P^{\mathrm{SO}_4}(\epsilon \operatorname{St}) + \operatorname{Ind}_B^{\mathrm{SO}_4}(\epsilon\otimes 1\otimes\epsilon\otimes 1) \tag{3.2.15}$$

We pin down the $G_{\beta+}$-invariance of $\pi(\eta_2)$ in Corollary 3.2.6.

Proposition 3.2.5. *The I_+-invariants of $\pi(\eta_2)$ is*

$$\pi(\eta_2)^{I_+} \cong \epsilon\otimes\epsilon + 1\otimes\epsilon + \epsilon\otimes 1.$$

Proof. A priori we know that

$$\pi(\eta_2)^{I_+} \hookrightarrow I(\nu\eta_2\otimes\eta_2)^{I_+} = \bigoplus_{w\in W}(\epsilon\otimes\epsilon)^w = (\epsilon\otimes\epsilon)^4 + (1\otimes\epsilon)^4 + (\epsilon\otimes 1)^4.$$

By Lemma 3.2.4, the multiplicity of $\epsilon\otimes\epsilon$ in $\pi(\eta_2)$, which is the same as the multiplicity of $\epsilon\otimes\epsilon\otimes 1\otimes 1$ in the representation $\eta_2\operatorname{St}_{\mathrm{SO}_4}$, is one. Thus the same holds for all of the Weyl group orbits of the character. □

Corollary 3.2.6. *There is an isomorphism of $G_\beta/G_{\beta+}$-representations*

$$\pi(\eta_2)^{G_{\beta+}} \cong \epsilon\operatorname{St}_{G_\beta/G_{\beta+}} \oplus \omega_{\mathrm{princ}}^{\eta_2}$$

Proof. Let $N = I_+/G_{\beta+} \subseteq G_\beta/G_{\beta+}$ be a maximal unipotent subgroup of $\mathrm{SO}_4(\mathbb{F}_q)$. Let ω' and ω'' be the irreducible constituents of $\operatorname{Ind}_B^{\mathrm{SO}_4}(1\otimes\epsilon\otimes 1\otimes\epsilon)$. By Proposition 3.2.5, the $\mathrm{SO}_4(\mathbb{F}_q)$-representation $\pi(\eta_2)^{G_{\beta+}}$ has N-invariants $\epsilon\otimes\epsilon\otimes 1\otimes 1 + \epsilon\otimes 1\otimes\epsilon\otimes 1 + \epsilon\otimes 1\otimes 1\otimes\epsilon$. Thus

$$\pi(\eta_2)^{G_{\beta+}} = I_\alpha(\nu^{1/2}\eta_2\operatorname{St})^{G_{\beta+}} \cap I_\beta(\nu^{1/2}\eta_2\operatorname{St})^{G_{\beta+}} \tag{3.2.16}$$

$$\subseteq \epsilon\operatorname{St}_{\mathrm{SO}_4} + \omega' + \omega'' \tag{3.2.17}$$

must contain either just ω' or ω'' (but not both), since

$$(\omega')^N, (\omega'')^N \cong \epsilon\otimes 1\otimes\epsilon\otimes 1 + \epsilon\otimes 1\otimes 1\otimes\epsilon.$$

Thus either $\pi(\eta_2) = \epsilon\operatorname{St}_{\mathrm{SO}_4} + \omega'$ or $\pi(\eta_2) = \epsilon\operatorname{St}_{\mathrm{SO}_4} + \omega''$ as abstract representations of $\mathrm{SO}_4(\mathbb{F}_q)$.

To further pin down the choice, let $\widetilde{\mathcal{J}} := \mathcal{J} \rtimes \langle \begin{pmatrix} & 1 \\ \varpi & \end{pmatrix} \begin{pmatrix} & 1 \\ \varpi & \end{pmatrix} \rangle$ be the stabilizer of an alcove in the Bruhat-Tits building of $\mathrm{SO}_4(F)$. Then we have the following commutative diagram involving the support-preserving isomorphism of Lemma 3.2.4:

$$\begin{array}{ccc} \mathcal{H}(G_2/\!/\mathcal{I}, \epsilon\otimes 1) & \xrightarrow{\sim} & \mathcal{H}(\mathrm{SO}_4/\!/\mathcal{J}, \epsilon) \\ \uparrow & & \uparrow \\ \mathcal{H}(G_\beta/\!/\mathcal{I}, \epsilon\otimes 1) & \xrightarrow{\sim} & \mathcal{H}(\widetilde{\mathcal{J}}/\!/\mathcal{J}, \epsilon) \end{array} \tag{3.2.18}$$

Indeed, since (3.2.7) is support-preserving, the image of $\mathcal{H}(G_\beta/\!/\mathcal{I}, \epsilon\otimes 1)$ under the isomorphism consists of functions supported on $G_\beta\cap\mathrm{SO}_4(F)$. Certainly $\tilde{\mathcal{J}} \subset G_\beta\cap\mathrm{SO}_4(F)$, since elements of $\tilde{\mathcal{J}}$, which fixes an alcove of $\mathrm{SO}_4(F)$, must also fix the vertex β in the building of G_2. Equality follows from observing that both $\mathcal{H}(G_\beta/\!/\mathcal{I}, \epsilon\otimes 1)$ and $\mathcal{H}(\tilde{\mathcal{J}}/\!/\mathcal{J}, \epsilon)$ have dimension 2. By the characterization in Lemma 3.2.4, the restriction of $\eta_2\operatorname{St}_{\mathrm{GL}_2}$ to $\mathcal{H}(\tilde{\mathcal{J}}/\!/\mathcal{J}, \epsilon)$ is the representation $\eta_2\circ\det$ on $\tilde{\mathcal{J}}$. Via the bottom isomorphism, $\eta_2\circ\det$ corresponds to the representation $\omega_{\mathrm{princ}}^{\eta_2}$ of G_β.

Thus, we conclude that $\omega_{\mathrm{princ}}^{\eta_2}$ is a constituent of $\pi(\eta_2)^{G_{\beta+}}$. □

3.2.2. *Calculating $\pi(\eta_2)^{G_{\alpha+}}$.* Analogous to (3.2.11), we have

$$\begin{aligned} I_B^{G_2}(\nu\eta_2\otimes\eta_2)^{G_{\alpha+}} &\cong \bigoplus_{w\in W/W(\mathrm{SL}_3)} \operatorname{Ind}_{G_\alpha\cap wBw^{-1}/(G_{\alpha+}\cap wBw^{-1})}^{G_\alpha/G_{\alpha+}}(\epsilon\otimes\epsilon)^w \\ &= \operatorname{Ind}_B^{\mathrm{SL}_3}(\epsilon)^2. \end{aligned} \tag{3.2.19}$$

Moreover, we have isomorphisms

$$I_\alpha(\nu^{1/2}\eta_2\operatorname{St}_{\mathrm{GL}_2})^{G_{\alpha+}} = \operatorname{Ind}_P^{\mathrm{SL}_3}(\epsilon\operatorname{St}_{\mathrm{GL}_2})^2 \tag{3.2.20}$$

$$I_\beta(\nu^{1/2}\eta_2\operatorname{St}_{\mathrm{GL}_2})^{G_{\alpha+}} = \operatorname{Ind}_B^{\mathrm{SL}_3}(\epsilon), \tag{3.2.21}$$

where $P \subset \mathrm{SL}_3$ is the parabolic subgroup with Levi GL_2. The intersection is

$$\pi(\eta_2)^{G_{\alpha+}} = \mathrm{Ind}_P^{\mathrm{SL}_3}(\epsilon\,\mathrm{St}_{\mathrm{GL}_2}). \tag{3.2.22}$$

3.2.3. *Calculating* $\pi(\eta_2)^{G_{\delta+}}$. Again by a Mackey theory calculation, we have:

$$I(\nu\eta_2 \otimes \eta_2)^{G_{\delta+}} \cong \mathrm{Ind}_{B(\mathbb{F}_q)}^{G_2(\mathbb{F}_q)}(\epsilon \otimes \epsilon) \tag{3.2.23}$$

$$I_\alpha(\nu^{1/2}\eta_2\,\mathrm{St}_{\mathrm{GL}_2})^{G_{\delta+}} \cong \mathrm{Ind}_{P_\alpha(\mathbb{F}_q)}^{G_2(\mathbb{F}_q)}(\epsilon\,\mathrm{St}_{\mathrm{GL}_2}) \tag{3.2.24}$$

$$I_\beta(\nu^{1/2}\eta_2\,\mathrm{St}_{\mathrm{GL}_2})^{G_{\delta+}} \cong \mathrm{Ind}_{P_\beta(\mathbb{F}_q)}^{G_2(\mathbb{F}_q)}(\epsilon\,\mathrm{St}_{\mathrm{GL}_2}), \tag{3.2.25}$$

where P_α and P_β denote parabolic subgroups of $G_2(\mathbb{F}_q)$. Thus, $\pi(\eta_2)^{G_{\delta+}}$ is the intersection of $\mathrm{Ind}_{P_\alpha(\mathbb{F}_q)}^{G_2(\mathbb{F}_q)}(\epsilon\,\mathrm{St}_{\mathrm{GL}_2})$ and $\mathrm{Ind}_{P_\beta(\mathbb{F}_q)}^{G_2(\mathbb{F}_q)}(\epsilon\,\mathrm{St}_{\mathrm{GL}_2})$, denoted $\omega_{\mathrm{princ}}^{\epsilon}$. In terms of Lusztig's equivalence [Lus84, Theorem 4.23], if $s \in G_2(\mathbb{F}_q)$ is of order 2 such that $\mathrm{Z}_{G_2(\mathbb{F}_q)}(s) = \mathrm{SO}_4(\mathbb{F}_q)$, we have

$$\mathcal{E}(G_2(\mathbb{F}_q), s) \cong \mathcal{E}(\mathrm{SO}_4(\mathbb{F}_q), 1), \tag{3.2.26}$$

and $\omega_{\mathrm{princ}}^{\epsilon}$ corresponds to $\mathrm{St}_{\mathrm{SO}_4(\mathbb{F}_q)}$ under (3.2.26). Thus we have the following:

Proposition 3.2.7. *Let $\pi(\eta_2)$ be the unique sub-representation of $I(\eta_2 \otimes \nu\eta_2)$. Then,*

$$\pi(\eta_2)^{G_{\delta+}} \cong \omega_{\mathrm{princ}}^{\epsilon} \tag{3.2.27}$$

$$\pi(\eta_2)^{G_{\alpha+}} \cong \mathrm{Ind}_P^{\mathrm{SL}_3}(\epsilon\,\mathrm{St}_{\mathrm{GL}_2}) \tag{3.2.28}$$

$$\pi(\eta_2)^{G_{\beta+}} \cong \epsilon\,\mathrm{St}_{G_\beta/G_{\beta+}} + \omega_{\mathrm{princ}}^{\eta_2}. \tag{3.2.29}$$

3.3. **The supercuspidal representation $\pi_{\mathrm{s.c.}}(\eta_2)$.**

We denote the following depth-zero supercuspidal representation of $G_2(F)$ as

$$\pi_{\mathrm{s.c.}}(\eta_2) := \text{c-}\mathrm{Ind}_{G_\beta}^{G_2}(\omega_{\mathrm{cusp}}^{\eta_2}). \tag{3.3.1}$$

We may readily calculate the G_{x+}-invariants of the supercuspidal representations $\pi_{\mathrm{s.c.}}(\eta_2)$, for various vertices x in the Bruhat-Tits building as follows:

Lemma 3.3.1. *Let $\pi_{\mathrm{s.c.}}(\eta_2)$ be as defined in (3.3.1). We have*

$$\pi_{\mathrm{s.c.}}(\eta_2)^{G_{\alpha+}} = 0 \tag{3.3.2}$$

$$\pi_{\mathrm{s.c.}}(\eta_2)^{G_{\beta+}} \cong \omega_{\mathrm{cusp}}^{\eta_2} \tag{3.3.3}$$

$$\pi_{\mathrm{s.c.}}(\eta_2)^{G_{\delta+}} = 0 \tag{3.3.4}$$

Proof. For each vertex x, by Mackey theory we have

$$\begin{aligned} \pi_{\mathrm{s.c.}}(\eta_2)^{G_{x+}} &\cong \bigoplus_{g \in G_\beta \backslash G_2 / G_x} \mathrm{Ind}_{G_x \cap g^{-1}G_\beta g}^{G_x}((\omega_{\mathrm{cusp}}^{\eta_2})^g)^{G_{x+}\cap g^{-1}G_\beta g} \\ &= \bigoplus_{g \in G_\beta \backslash G_2 / G_x} \mathrm{Ind}_{G_x \cap G_{g^{-1}\beta}}^{G_x}((\omega_{\mathrm{cusp}}^{\eta_2})^g)^{G_{x+}\cap G_{g^{-1}\beta}}. \end{aligned} \tag{3.3.5}$$

Here,

$$((\omega_{\mathrm{cusp}}^{\eta_2})^g)^{G_{x+}\cap G_{g^{-1}\beta}} \cong (\omega_{\mathrm{cusp}}^{\eta_2})^{G_\beta \cap G_{gx+}},$$

which is 0 unless $\beta = gx$ since otherwise $G_\beta \cap G_{gx+}$ will contain the unipotent radical of some parabolic subgroup of G_β, so $(\omega_{\mathrm{cusp}}^{\eta_2})^{G_\beta \cap G_{gx+}} = 0$ since $\omega_{\mathrm{cusp}}^{\eta_2}$ is cuspidal. □

3.4. **Characters on a neighborhood of** 1. In this section, we express $\pi(\eta_2)^{G_{x+}}$ in terms of generalized Green functions (notations as in [DK06]), for $x = \delta, \alpha, \beta$. To each Weyl group conjugacy class $[w] \in W(G)$, let S_w be the unique torus in G such that Frobenius acts as w (i.e. the image of w under the bijection of [Car93, Prop 3.3.3]). We denote $R_w^\theta := R_{S_w}^\theta$. Firstly, note that

$$\mathrm{Ch}(\mathrm{St}_{\mathrm{GL}_2}) = \frac{1}{2}(R_1^1 - R_{(12)}^1). \tag{3.4.1}$$

(1) When $F = F_{G_2}$ (i.e. corresponding to the vertex δ), we have that $\pi(\eta_2)^{G_{\delta+}} \cong \omega_{\mathrm{princ}}^\epsilon$ corresponds to $\mathrm{St}_{\mathrm{SO}_4(\mathbb{F}_q)}$ under Lusztig's equivalence (3.2.26). By (3.4.1), we have

$$\mathrm{Ch}_{\mathrm{St}_{\mathrm{SO}(4)}} = \frac{1}{4}(R^1_{A_1\times\widetilde{A}_1} - R^1_{A_1} - R^1_{\widetilde{A}_1} + R^1_1). \tag{3.4.2}$$

Since Lusztig's equivalence (3.2.26) preserves multiplicities, we have

$$\mathrm{Ch}_{\pi^\epsilon_{\mathrm{princ}}} = \frac{1}{4}(R^\epsilon_{A_1\times\widetilde{A}_1} - R^\epsilon_{A_1} - R^\epsilon_{\widetilde{A}_1} + R^\epsilon_1). \tag{3.4.3}$$

Restricting to the unipotent locus, for $u \in G_2(\mathbb{F}_q)$ unipotent we have

$$\mathrm{Ch}_{\pi^\epsilon_{\mathrm{princ}}}(u) = \frac{1}{4}(\mathcal{Q}^{F_{G_2}}_{A_1\times\widetilde{A}_1} - \mathcal{Q}^{F_{G_2}}_{A_1} - \mathcal{Q}^{F_{G_2}}_{\widetilde{A}_1} + \mathcal{Q}^{F_{G_2}}_1).$$

(2) When $F = F_{A_2}$ (i.e. corresponding to the vertex α), we have that $\pi(\eta_2)^{G_{\alpha+}} \cong \mathrm{Ind}_P^{\mathrm{SL}_3}(\epsilon\,\mathrm{St}_{\mathrm{GL}_2}) \in \mathcal{E}(\mathrm{SL}_3, \begin{pmatrix}-1 & & \\ & -1 & \\ & & 1\end{pmatrix})$ corresponds, under Lusztig's equivalence, to $\mathrm{St}_{\mathrm{GL}_2} \in \mathcal{E}(\mathrm{GL}_2, 1)$. By (3.4.1), we have

$$\mathrm{Ch}(\mathrm{Ind}_P^{\mathrm{SL}_3}(\epsilon\,\mathrm{St}_{\mathrm{GL}_2})) = \frac{1}{2}(R_1^\epsilon - R_{A_1}^\epsilon). \tag{3.4.4}$$

Restricting to the unipotent locus, we have

$$\mathrm{Ch}_{\mathrm{Ind}_P^{\mathrm{SL}_3}(\epsilon\,\mathrm{St}_{\mathrm{GL}_2})} = \frac{1}{2}(\mathcal{Q}_1^{F_{A_2}} - \mathcal{Q}_{A_1}^{F_{A_2}}).$$

(3) When $F = F_{A_1\times\tilde{A}_1}$ (i.e. corresponding to the vertex β), we have that $\pi(\eta_2)^{G_{F+}} = \epsilon\,\mathrm{St}_{\mathrm{SO}_4} + \omega_{\mathrm{princ}}^{\eta_2}$. On the unipotent locus of $\mathrm{SO}_4(\mathbb{F}_q)$ we have (in the notation of §3.1):

$$\begin{cases}\mathrm{Ch}(\omega^{\eta_2}_{\mathrm{princ}}) + \mathrm{Ch}(\omega^{\eta_2'}_{\mathrm{princ}}) = R_1^1 \\ \mathrm{Ch}(\omega^{\eta_2}_{\mathrm{princ}}) - \mathrm{Ch}(\omega^{\eta_2'}_{\mathrm{princ}}) = q^*\mathcal{G}_{\mathrm{sgn}}\end{cases},$$

where q^* is as defined in Remark 3.2.1. This implies that on the unipotents,

$$\mathrm{Ch}_{\omega^{\eta_2}_{\mathrm{princ}}} = \frac{1}{2}(\mathcal{Q}_1^{F_{A_1\times\tilde{A}_1}} \pm q^*\mathcal{G}_{\mathrm{sgn}}). \tag{3.4.5}$$

Together with (3.4.2), we obtain:

$$\mathrm{Ch}_{\pi(\eta_2)^{G_{F+}}} = \frac{1}{2}(\mathcal{Q}_1^{F_{A_1\times\widetilde{A}_1}} \pm q^*\mathcal{G}_{\mathrm{sgn}}) + \frac{1}{4}(\mathcal{Q}^{F_{A_1\times\widetilde{A}_1}}_{A_1\times\widetilde{A}_1} - \mathcal{Q}^{F_{A_1\times\widetilde{A}_1}}_{A_1} - \mathcal{Q}^{F_{A_1\times\widetilde{A}_1}}_{\widetilde{A}_1} + \mathcal{Q}^{F_{A_1\times\widetilde{A}_1}}_1). \tag{3.4.6}$$

(4) When $F = F_{A_1}$ or F'_{A_1}, we have $\pi(\eta_2)^{G_{F+}} = \frac{3}{2}Q_1^{F_{A_1}} - \frac{1}{2}Q_{A_1}^{F_{A_1}}$ on unipotents.

(5) When $F = F_{\tilde{A}_1}$, then again $\pi(\eta_2)^{G_{F+}} = \frac{3}{2}Q_1^{F_{\tilde{A}_1}} - \frac{1}{2}Q_{\tilde{A}_1}^{F_{\tilde{A}_1}}$ on unipotents.

(6) When $F = F_\emptyset$ then $\pi(\eta_2)^{G_{F+}} = \epsilon\otimes\epsilon + 1\otimes\epsilon + \epsilon\otimes 1$, so the character on unipotents is $3 = 3Q_1^{\{e\}}$.

Similarly, we have

$$\mathrm{Ch}(\omega_{\mathrm{cusp}}^{\eta_2}) = \frac{1}{2}(\mathcal{Q}_{A_1\times\tilde{A}_1}^{F_{A_1\times\tilde{A}_1}} \pm q^*\mathcal{G}_{\mathrm{sgn}}). \tag{3.4.7}$$

Therefore, we have the following:

Proposition 3.4.1. *For any ramified quadratic characters η_2 and η_2', the sum $\pi(\eta_2)+\pi_{\mathrm{s.c.}}(\eta_2')$ has a stable character on the topologically unipotent elements.*

Proof. From the discussion above, in the notation of [DK06, Table 4], we see that for some explicitly computable constants c_i,

$$\mathrm{Ch}_{\pi(\eta_2)} = \frac{1}{8}c_1(D^{\mathrm{st}}_{A_1\times\tilde{A}_1} + D^{\mathrm{unst}}_{A_1\times\tilde{A}_1}) \pm c_2 D^{\mathrm{st}}_{(F_{A_1\times\tilde{A}_1},\mathcal{G}_{\mathrm{sgn}})} + c_3 D^{\mathrm{st}}_{A_1} + c_4 D^{\mathrm{st}}_{\tilde{A}_1} + c_5 D^{\mathrm{st}}_{\{e\}}$$
$$\mathrm{Ch}_{\pi_{\mathrm{s.c.}}(\eta_2)} = \frac{1}{8}c_1(D^{\mathrm{st}}_{A_1\times\tilde{A}_1} - D^{\mathrm{unst}}_{A_1\times\tilde{A}_1}) \pm c_2 D^{\mathrm{st}}_{(F_{A_1\times\tilde{A}_1},\mathcal{G}_{\mathrm{sgn}})}.$$

Thus, by [DK06, Lemma 6.4.1] the sum is always stable. □

3.5. **Characters on a neighborhood of $s \in G_2$.** Let $s \in G_2$ be order 2 such that $\mathrm{Z}_{G_2}(s) = \mathrm{SO}_4$. By the construction in [AK07, §7], the distributions $\mathrm{Ch}_{\pi(\eta_2)}$ and $\mathrm{Ch}_{\pi_{\mathrm{s.c.}}(\eta_2)}$ on G_2 induce distributions $\Theta_{\pi(\eta_2)}$ and $\Theta_{\pi_{\mathrm{s.c.}}(\eta_2)}$ on $(\mathrm{SO}_4)_{0+}$, the topologically unipotent elements in SO_4, such that the attached locally constant functions are compatible (see [AK07, Lemma 7.5]). We hope to see when the sum $\Theta_{\pi(\eta_2)} + \Theta_{\pi_{\mathrm{s.c.}}(\eta_2')}$ is a stable distribution on $(\mathrm{SO}_4)_{0+}$.

We now look at the characters on an element of the form su for u topologically unipotent. They follow from computations in §3.4.

(1) When $F = F_{G_2}$, by (3.4.3) and [DL76, Thm 4.2], we have for $u \in \mathrm{SO}_4(\mathbb{F}_q)$ unipotent:

$$\begin{aligned}\mathrm{Ch}_{\pi^{\epsilon}_{\mathrm{princ}}}(su) =& \frac{1}{4}\left(R^{\epsilon}_{S_{A_1\times\tilde{A}_1}}(su) - R^{\epsilon}_{S_{A_1}}(su) - R^{\epsilon}_{S_{\tilde{A}_1}}(su) + R^{\epsilon}_{S_1}(su)\right)\\ =&\frac{1}{4|\mathrm{SO}_4(\mathbb{F}_q)|}\Bigg(\sum_{gsg^{-1}\in S_{A_1\times\tilde{A}_1}}\epsilon(gsg^{-1})\mathcal{Q}^{\mathrm{SO}_4}_{S_{A_1\times\tilde{A}_1}}(u) - \sum_{gsg^{-1}\in S_{A_1}}\epsilon(gsg^{-1})\mathcal{Q}^{\mathrm{SO}_4}_{S_{A_1}}(u)\\ &-\sum_{gsg^{-1}\in S_{\tilde{A}_1}}\epsilon(gsg^{-1})\mathcal{Q}^{\mathrm{SO}_4}_{S_{\tilde{A}_1}}(u) + \sum_{gsg^{-1}\in S_1}\epsilon(gsg^{-1})\mathcal{Q}^{\mathrm{SO}_4}_{S_1}(u)\Bigg)\\ =&\frac{1}{4}(\mathcal{Q}^{A_1\times\tilde{A}_1}_{A_1\times\tilde{A}_1}(u) - \mathcal{Q}^{A_1\times\tilde{A}_1}_{A_1}(u) - \mathcal{Q}^{A_1\times\tilde{A}_1}_{\tilde{A}_1}(u) + \mathcal{Q}^{A_1\times\tilde{A}_1}_{1}(u))\\ &+\frac{1}{2}(-1)^{\frac{q-1}{2}}\mathcal{Q}^{A_1\times\tilde{A}_1}_{1}(u) + \frac{1}{2}(-1)^{\frac{q+1}{2}}\mathcal{Q}^{A_1\times\tilde{A}_1}_{A_1\times\tilde{A}_1}(u),\end{aligned} \tag{3.5.1}$$

where the last equality folows from the observation that $gsg^{-1} \in S$ must be an order 2 element; there are 3 such elements for the tori $S_{A_1\times\tilde{A}_1}$ and S_1, while there is a unique such element for the tori S_{A_1} and $S_{\tilde{A}_1}$.

(2) When $F = F_{A_1\times\tilde{A}_1}$, since $s \in G_F$ is central, we simply have:

$$\mathrm{Ch}_{\pi(\eta_2)^{G_{F+}}}(su) = (-1)^{\frac{q-1}{2}}\frac{1}{2}(\mathcal{Q}_1^{F_{A_1\times\tilde{A}_1}} \pm q^*\mathcal{G}_{\mathrm{sgn}}) + \frac{1}{4}(\mathcal{Q}^{F_{A_1\times\tilde{A}_1}}_{A_1\times\tilde{A}_1} - \mathcal{Q}^{F_{A_1\times\tilde{A}_1}}_{A_1} - \mathcal{Q}^{F_{A_1\times\tilde{A}_1}}_{\tilde{A}_1} + \mathcal{Q}^{F_{A_1\times\tilde{A}_1}}_{1}). \tag{3.5.2}$$

Similarly, we have

$$\mathrm{Ch}_{\pi_{\mathrm{s.c.}}(\eta_2)^{G_{F+}}}(su) = (-1)^{\frac{q+1}{2}}\frac{1}{2}(\mathcal{Q}^{F_{A_1\times\tilde{A}_1}}_{A_1\times\tilde{A}_1} \pm q^*\mathcal{G}_{\mathrm{sgn}}). \tag{3.5.3}$$

Since we already know that the character of $\mathrm{St}_{\mathrm{SO}_4}$ is stable, we hope to see whether $\Theta_{\pi(\eta_2)} + \Theta_{\pi_{\mathrm{s.c.}}(\eta_2)} - \mathrm{Ch}_{\mathrm{St}_{\mathrm{SO}_4}}$ or $\Theta_{\pi(\eta_2)} + \Theta_{\pi_{\mathrm{s.c.}}(\eta_2')} - \mathrm{Ch}_{\mathrm{St}_{\mathrm{SO}_4}}$ is stable. Note that

$$\Theta_{\pi(\eta_2)}+\Theta_{\pi_{\mathrm{s.c.}}(\eta_2)}-\mathrm{Ch}_{\mathrm{St}_{\mathrm{SO}_4}} = c_1 D_{(F_{A_1\times\tilde{A}_1}, \mathcal{Q}^{F_{A_1\times\tilde{A}_1}}_{A_1\times\tilde{A}_1})} + c_2 D_{(F_{A_1\times\tilde{A}_1}, \mathcal{Q}_1^{F_{A_1\times\tilde{A}_1}})} \pm q^*\mathcal{G}_{\mathrm{sgn}} \pm q^*\mathcal{G}_{\mathrm{sgn}}, \tag{3.5.4}$$

where notations are as in [DK06, Definition 5.1.3].

Lemma 3.5.1. *The distribution $D_{(F_{A_1\times\tilde{A}_1},\mathcal{G}_{\mathrm{sgn}})}$ on $\mathrm{SO}_4(F)$ is not stable. Similarly, no linear combination of the distributions $D_{(F_{A_2},\mathcal{G}_{\chi'})}$ and $D_{(F_{A_2},\mathcal{G}_{\chi''})}$ on $\mathrm{SL}_3(F)$ are stable.*

Proof. A distribution on $\mathrm{SO}_4(F)$ is stable if and only if it is stable under conjugation by $\mathrm{PGL}_2(F)\times\mathrm{PGL}_2(F)$. Thus all stable distributions on SO_4 must be restricted from invariant distributions on $\mathrm{PGL}_2(F)\times\mathrm{PGL}_2(F)$. But the only invariant distributions on $\mathrm{PGL}_2(F)\times\mathrm{PGL}_2(F)$ are spanned by semisimple orbital integrals, and $D_{(F_{A_1\times\tilde{A}_1},\mathcal{G}_{\mathrm{sgn}})}$ is linearly independent from them (as can be seen by evaluating against $\mathcal{G}_{\mathrm{sgn}}$). An identical argument works for $D_{(F_{A_2},\mathcal{G}_{\chi'})}$ and $D_{(F_{A_2},\mathcal{G}_{\chi''})}$. □

Now, since $D_{(F_{A_1\times\tilde{A}_1},\mathcal{G}_{\mathrm{sgn}})}$ is not stable, the only linear combination of $\Theta_{\pi(\eta_2)}$ and $\Theta_{\pi_{\mathrm{s.c.}}(\eta_2)}$ that is stable are those for which $\pm q^*\mathcal{G}_{\mathrm{sgn}} \pm q^*\mathcal{G}_{\mathrm{sgn}} = 0$ (there are four possibilities). Remark 3.2.3 tells us the only such combinations are $\Theta_{\pi(\eta_2)} + \Theta_{\pi_{\mathrm{s.c.}}(\eta_2)} - \mathrm{Ch}_{\mathrm{St}_{\mathrm{SO}_4}}$ (one for η_2 and one for η_2'). Thus, we have:

Theorem 3.5.2. *For ramified quadratic characters η_2 and η_2', the character $\mathrm{Ch}_{\pi(\eta_2)} + \mathrm{Ch}_{\pi_{\mathrm{s.c.}}(\eta_2')}$ is stable in a neighborhood of s if and only if $\eta_2 = \eta_2'$. Thus, $\{\pi(\eta_2), \pi_{\mathrm{s.c.}}(\eta_2)\}$ is an L-packet, for each ramified quadratic character η_2.*

4. Size 3 mixed packets

Let ζ be an order 3 character of $\mathbb{F}_q^\times$. We will repeatedly use the following Hecke algebra isomorphisms, which is the analogue of Lemma 3.2.4.

Corollary 4.0.1. *Let I be the standard Iwahori of G_2. There exist a canonical support-preserving isomorphism of Hecke algebra*

$$\mathcal{H}(G_2//I, \zeta^{\pm1}\otimes\zeta^{\pm1}) \cong \mathcal{H}(\mathrm{PGL}_3//J, \zeta^{\pm1}\circ\det), \tag{4.0.1}$$

under which the representation $\pi(\eta_3)$ corresponds to the representation $\eta_3^{\pm1}\mathrm{St}_{\mathrm{PGL}_3}$, where J is an Iwahori subgroup of $\mathrm{PGL}_3(F)$. The isomorphism is characterized by the commutative diagram

$$\begin{array}{ccc} \mathcal{H}(T//T_0, \zeta^{\pm1}\otimes\zeta^{\pm1}) & = & \mathcal{H}(T//T_0, \zeta^{\pm1}\circ\det) \\ \Big\downarrow t_u & & \Big\downarrow t_u \\ \mathcal{H}(G_2//I, \zeta^{\pm1}\otimes\zeta^{\pm1}) & \xrightarrow{\sim} & \mathcal{H}(\mathrm{PGL}_3//J, \zeta^{\pm1}\circ\det), \end{array} \tag{4.0.2}$$

where $t_u = t_{\delta_B^{-1/2}}$ is as in [Roc98, pg 399].

Proof. Same proof as in Lemma 3.2.4. □

The lemma immediately gives:

Corollary 4.0.2. *Let I_+ be the pro-unipotent radical of the Iwahori subgroup I of G_2. Then*

$$\pi(\eta_3)^{I_+} = \zeta\otimes\zeta + \zeta^{-1}\otimes\zeta^{-1}.$$

4.1. Calculating parahoric invariants for $\pi(\eta_3)$.

4.1.1. *Calculating* $\pi(\eta_3)^{G_{\alpha+}}$. Similar to §3.2.1, we have an isomorphism of representations of $G_\alpha/G_{\alpha+}\cong \mathrm{SL}_3(\mathbb{F}_q)$,

$$I_B^{G_2}(\nu\eta_3\otimes\eta_3)^{G_{\alpha+}}\cong\bigoplus_{w\in W/W(\mathrm{SL}_3)}\mathrm{Ind}^{G_\alpha/G_{\alpha+}}_{G_\alpha\cap wBw^{-1}/(G_{\alpha+}\cap wBw^{-1})}(\zeta\otimes\zeta)^w, \tag{4.1.1}$$

Therefore, the $G_{\alpha+}$-invariants of $I_B^{G_2}(\nu\eta_3\otimes\eta_3)$ gives

$$I_B^{G_2}(\nu\eta_3\otimes\eta_3)^{G_{\alpha+}}\simeq \mathrm{Ind}_B^{\mathrm{SL}_3}(\zeta^{-1}\otimes 1\otimes\zeta)+\mathrm{Ind}_B^{\mathrm{SL}_3}(\zeta^{-1}\otimes 1\otimes\zeta). \tag{4.1.2}$$

Likewise, computing the $G_{\alpha+}$-invariants of I_α gives us the following

$$I_\alpha(\nu^{1/2}\eta_3\,\mathrm{St})^{G_{\alpha+}}\simeq \mathrm{Ind}_B^{\mathrm{SL}_3}(\zeta^{-1}\otimes 1\otimes\zeta) \tag{4.1.3}$$

$$I_\alpha(\nu^{1/2}\eta_3^{-1}\,\mathrm{St})^{G_{\alpha+}}\simeq \mathrm{Ind}_B^{\mathrm{SL}_3}(\zeta^{-1}\otimes 1\otimes\zeta). \tag{4.1.4}$$

The representation $\mathrm{Ind}_B^{\mathrm{SL}_3}(\zeta^{-1}\otimes 1\otimes\zeta)$ has length 3 and decomposes into three representations $\chi_{st'}(0)$, $\chi_{st'}(1)$, and $\chi_{st'}(2)$ in the notations of [SF73, Table 1b, §7]. These representations are conjugate under conjugation by $\mathrm{PGL}_3(\mathbb{F}_q)$. Similarly, the Deligne-Lusztig induction R_T^ζ, where $T\subset \mathrm{SL}_3(\mathbb{F}_q)$ is an anisotropic torus, decomposes into three cuspidal representations $\chi_{r^2s'}(0)$, $\chi_{r^2s'}(1)$, and $\chi_{r^2s'}(2)$ that form an orbit under conjugation by $\mathrm{PGL}_3(\mathbb{F}_q)$.

The representation $\chi_{st'}(0)$ (resp., $\chi_{r^2s'}(0)$) is characterized by the character value

$$\mathrm{Ch}_{\chi_{st'}(0)}\begin{pmatrix}1&\theta^\ell&\\&1&\theta^\ell\\&&1\end{pmatrix}=\mathrm{Ch}_{\chi_{r^2s'}(0)}\begin{pmatrix}1&\theta^\ell&\\&1&\theta^\ell\\&&1\end{pmatrix}=q\delta_{\ell 0}-\frac{q-1}{3},$$

where $\theta\in\mathbb{F}_q$ is such that $\theta^3\neq 1$.

Definition 4.1.1. Let η_3 be a ramified cubic character of $F^\times$. Then there is a uniformizer ϖ such that $\eta_3(\varpi)=1$. We let

$$\omega^{\eta_3}_{\mathrm{princ}}:=\chi_{st'}(0)^{\mathrm{diag}(1,1,\varpi)} \tag{4.1.5}$$

$$\omega^{\eta_3}_{\mathrm{cusp}}:=\chi_{r^2s'}(0)^{\mathrm{diag}(1,1,\varpi)} \tag{4.1.6}$$

be representations of $G_\alpha/G_{\alpha+}\cong H_\alpha/H_{\alpha+}$.

Remark 4.1.2. Note that $\omega^{\eta_3}_{\mathrm{princ}}=\omega^{\eta_3^{-1}}_{\mathrm{princ}}$ and $\omega^{\eta_3}_{\mathrm{cusp}}=\omega^{\eta_3^{-1}}_{\mathrm{cusp}}$. These are the only overlaps in the definition above.

Remark 4.1.3. As in [DM20], the representations $\omega^{\eta_3}_{\mathrm{princ}}$ and $\omega^{\eta_3}_{\mathrm{cusp}}$ are common components of the reducible Deligne-Lusztig induction R_T^ζ and the Gelfand-Graev representation $\Gamma_{\beta,\mathcal{O}}$ (notation as in [BM97, Thm 4.5]) associated to the nilpotent orbit $\mathcal{O}=\mathcal{O}_1^1$ (notation as in [DK06, §7.1]).

Proposition 4.1.4. *There is an isomorphism of* $G_\alpha/G_{\alpha+}$*-representations*

$$\pi(\eta_3)^{G_{\alpha+}}\cong\omega^{\eta_3}_{\mathrm{princ}}.$$

Proof. Let $N=I_+/G_{\alpha+}\subseteq G_\alpha/G_{\alpha+}$ be a maximal unipotent subgroup. By Proposition 4.0.2, the $G_\alpha/G_{\alpha+}$-representation $\pi(\eta_2)^{G_{\alpha+}}$ has N-invariance $\zeta^{-1}\otimes 1\otimes\zeta+\zeta\otimes 1\otimes\zeta^{-1}$. Thus

$$\pi(\eta_2)^{G_{\beta+}}=I_\alpha(\nu^{1/2}\eta_3\,\mathrm{St})^{G_{\beta+}} \tag{4.1.7}$$

$$=\mathrm{Ind}_B^{\mathrm{SL}_3}(\zeta^{-1}\otimes 1\otimes\zeta) \tag{4.1.8}$$

must be of the form $\chi_{r^2s'}(u)$ for some u (as abstract representations of $\mathrm{SL}_3(\mathbb{F}_q)$), since

$$\chi_{r^2s'}(u)^N\cong\zeta^{-1}\otimes 1\otimes\zeta+\zeta\otimes 1\otimes\zeta^{-1}.$$

Consider the isomorphism Lemma 3.2.4

$$\mathcal{H}(G_2//\mathcal{I}, \zeta\otimes 1) \xrightarrow{\sim} \mathcal{H}(\mathrm{PGL}_3//\mathcal{J}, \zeta\circ\det), \tag{4.1.9}$$

which is support-preserving. Let $\widetilde{\mathcal{J}} := \mathcal{J} \rtimes \langle \begin{pmatrix} & 1 & \\ & & 1 \\ \varpi & & \end{pmatrix} \rangle$ be the stabilizer of an alcove in the building of $\mathrm{PGL}_3(F)$. Then we have the following commutative diagram,

$$\begin{array}{ccc} \mathcal{H}(G_2//\mathcal{I}, \zeta\otimes\zeta) & \xrightarrow{\sim} & \mathcal{H}(\mathrm{PGL}_3//\mathcal{J}, \zeta\circ\det) \\ \uparrow & & \uparrow \\ \mathcal{H}(G_\alpha//\mathcal{I}, \zeta\otimes\zeta) & \xrightarrow{\sim} & \mathcal{H}(\widetilde{\mathcal{J}}//\mathcal{J}, \zeta\circ\det) \end{array} \tag{4.1.10}$$

The representation $\pi(\eta_3)$ is viewed as a homomorphism $\mathcal{H}(G_2//\mathcal{I}, \zeta\otimes\zeta) \to \mathbb{C}$. Under the top isomorphism we obtain the representation $\eta_3\,\mathrm{St}_{\mathrm{PGL}_3}$, whose restriction to $\mathcal{H}(\widetilde{\mathcal{J}}/\mathcal{J}, \zeta\circ\det)$ is the character $\eta_3\circ\det$. Now under the bottom isomorphism we obtain $\omega^{\eta_3}_{\mathrm{princ}}$, so $\omega^{\eta_3}_{\mathrm{princ}}$ must be a constituent of $\pi(\eta_3)^{G_{\alpha+}}$.

In fact, by the discussion above, $\pi(\eta_3)^{G_{\alpha+}} \cong \omega^{\eta_3}_{\mathrm{princ}}$.

□

4.1.2. *Calculating* $\pi(\eta_3)^{G_{\beta+}}$. As usual, Mackey theory gives:

$$I_B^{G_2}(\eta_3\otimes\nu\eta_3)^{G_{\beta+}} = \mathrm{Ind}_B^{\mathrm{SO}_4}(\zeta\otimes\zeta^{-1}\otimes 1\otimes 1) + \mathrm{Ind}_B^{\mathrm{SO}_4}(\zeta\otimes 1\otimes\zeta\otimes 1)^2 \tag{4.1.11}$$

$$I_\alpha(\nu^{1/2}\eta_3\,\mathrm{St}_{\mathrm{GL}_2})^{G_{\beta+}} = \mathrm{Ind}_P^{\mathrm{SO}_4}(\zeta\otimes\zeta^{-1}\otimes\mathrm{St}_{\mathrm{GL}_2}) + \mathrm{Ind}_B^{\mathrm{SO}_4}(\zeta\otimes 1\otimes\zeta\otimes 1) \tag{4.1.12}$$

$$I_\alpha(\nu^{1/2}\eta_3^{-1}\,\mathrm{St}_{\mathrm{GL}_2})^{G_{\beta+}} = \mathrm{Ind}_P^{\mathrm{SO}_4}(\zeta^{-1}\otimes\zeta\otimes\mathrm{St}_{\mathrm{GL}_2}) + \mathrm{Ind}_B^{\mathrm{SO}_4}(\zeta^{-1}\otimes 1\otimes\zeta^{-1}\otimes 1). \tag{4.1.13}$$

Thus, as $\mathrm{SO}_4(\mathbb{F}_q) \cong G_\beta/G_{\beta+}$-representations, we have

$$\pi(\eta_3)^{G_{\beta+}} \subset \mathrm{Ind}_P^{\mathrm{SO}_4}(\zeta\otimes\zeta^{-1}\otimes\mathrm{St}_{\mathrm{GL}_2}) + \mathrm{Ind}_B^{\mathrm{SO}_4}(\zeta\otimes 1\otimes\zeta\otimes 1),$$

where now both summands are irreducible. Moreover, the invariants of these representation with respect to the standard maximal unipotent subgroup $N \subset \mathrm{SO}_4(\mathbb{F}_q)$ gives:

$$\mathrm{Ind}_P^{\mathrm{SO}_4}(\zeta\otimes\zeta^{-1}\otimes\mathrm{St}_{\mathrm{GL}_2})^N \cong \zeta\otimes\zeta^{-1}\otimes 1\otimes 1 + \zeta^{-1}\otimes\zeta\otimes 1\otimes 1 \tag{4.1.14}$$

$$\mathrm{Ind}_B^{\mathrm{SO}_4}(\zeta\otimes 1\otimes\zeta\otimes 1)^N \cong \zeta\otimes 1\otimes\zeta\otimes 1 + \zeta\otimes 1\otimes 1\otimes\zeta \tag{4.1.15}$$

$$+ 1\otimes\zeta\otimes\zeta\otimes 1 + 1\otimes\zeta\otimes 1\otimes\zeta. \tag{4.1.16}$$

Thus, by Lemma 4.0.2 we must have $\pi(\eta_3)^{G_{\beta+}} \cong \mathrm{Ind}_P^{\mathrm{SO}_4}(\zeta\otimes\zeta^{-1}\otimes\mathrm{St}_{\mathrm{GL}_2})$.

4.1.3. *Calculating* $\pi(\eta_3)^{G_{\delta+}}$. Mackey theory gives the isomorphism of $G_\delta/G_{\delta+} \cong G_2(\mathbb{F}_q)$:

$$I_B^{G_2}(\eta_3\otimes\nu\eta_3)^{G_{\delta+}} = \mathrm{Ind}_{B(\mathbb{F}_q)}^{G_2(\mathbb{F}_q)}(\zeta\otimes\zeta) \tag{4.1.17}$$

$$I_\alpha(\nu^{1/2}\eta_3^{\pm1}\,\mathrm{St}_{\mathrm{GL}_2})^{G_{\delta+}} = \mathrm{Ind}_{P_\alpha(\mathbb{F}_q)}^{G_2(\mathbb{F}_q)}(\zeta^{\pm1}\,\mathrm{St}_{\mathrm{GL}_2}). \tag{4.1.18}$$

Thus, $\pi(\eta_3)^{G_{\delta+}}$ is the intersection in $\mathrm{Ind}_{B(\mathbb{F}_q)}^{G_2(\mathbb{F}_q)}(\zeta\otimes\zeta)$ of the two sub-representations $\mathrm{Ind}_{P_\alpha(\mathbb{F}_q)}^{G_2(\mathbb{F}_q)}(\zeta\,\mathrm{St}_{\mathrm{GL}_2})$ and $\mathrm{Ind}_{P_\alpha(\mathbb{F}_q)}^{G_2(\mathbb{F}_q)}(\zeta^{-1}\,\mathrm{St}_{\mathrm{GL}_2})$, which we denote by $\omega^{\zeta}_{\mathrm{princ}}$. In terms of Lusztig's equivalence [Lus84, Thm 4.23], if $s \in G_2(\mathbb{F}_q)$ is of order 3 such that $\mathrm{Z}_{G_2(\mathbb{F}_q)}(s) = \mathrm{SL}_3(\mathbb{F}_q)$, we have

$$\mathcal{E}(G_2(\mathbb{F}_q), s) \cong \mathcal{E}(\mathrm{PGL}_3(\mathbb{F}_q), 1), \tag{4.1.19}$$

and $\omega^{\zeta}_{\mathrm{princ}}$ corresponds to $\mathrm{St}_{\mathrm{PGL}_3(\mathbb{F}_q)}$ under (4.1.19). Thus, in conclusion:

Proposition 4.1.5. *Let $\pi(\eta_3)$ be the unique sub-representation of $I(\eta_3 \otimes \nu\eta_3)$. Then,*

$$\pi(\eta_3)^{G_{\delta+}} = \omega_{\mathrm{princ}}^{\zeta} \tag{4.1.20}$$

$$\pi(\eta_3)^{G_{\alpha+}} = \omega_{\mathrm{princ}}^{\eta_3} \tag{4.1.21}$$

$$\pi(\eta_3)^{G_{\beta+}} = \mathrm{Ind}_P^{\mathrm{SO}_4}(\zeta \otimes \zeta^{-1} \otimes \mathrm{St}_{\mathrm{GL}_2}) \tag{4.1.22}$$

4.2. **The supercuspidal representation $\pi_{\mathrm{s.c.}}(\eta_3)$.** We consider the following depth-zero supercuspidal representation of $G_2(F)$:

$$\pi_{\mathrm{s.c.}}(\eta_3) := \text{c-Ind}_{G_\alpha}^{G_2}(\omega_{\mathrm{cusp}}^{\eta_3}). \tag{4.2.1}$$

By the same argument as in Lemma 3.3.1, we obtain

Lemma 4.2.1. *Let $\pi_{\mathrm{s.c.}}(\eta_3)$ be as defined in (4.2.1).*

$$\pi_{\mathrm{s.c.}}(\eta_3)^{G_{\delta+}} = 0 \tag{4.2.2}$$

$$\pi_{\mathrm{s.c.}}(\eta_3)^{G_{\alpha+}} = \omega_{\mathrm{cusp}}^{\eta_3} \tag{4.2.3}$$

$$\pi_{\mathrm{s.c.}}(\eta_3)^{G_{\beta+}} = 0. \tag{4.2.4}$$

4.3. **Characters on a neighborhood of 1.** Similar arguments as in §3.4 gives the following characters for $\pi(\eta_3)$ in terms of Green functions:

(1) For $F = F_{G_2}$, we have

$$\mathrm{Ch}_{\omega_{\mathrm{princ}}^{\zeta}} = \frac{1}{6}(R_1^{\zeta} - 3R_{A_1}^{\zeta} + 2R_{A_2}^{\zeta}),$$

thus for $u \in G_2(\mathbb{F}_q)$ unipotent, we have $\mathrm{Ch}_{\omega_{\mathrm{princ}}^{\zeta}}(u) = \frac{1}{6}(\mathcal{Q}_1^{F_{G_2}}(u) - 3\mathcal{Q}_{A_1}^{F_{G_2}}(u) + 2\mathcal{Q}_{A_2}^{F_{G_2}}(u))$.

(2) For $F = F_{A_2}$ we have, for $u \in G_F/G_{F+}$ unipotent,

$$\mathrm{Ch}_{\omega_{\mathrm{princ}}^{\eta_3}}(u) = \frac{1}{3}(\mathcal{Q}_1^{F_{A_2}}(u) + \omega\mathcal{G}_{\chi'}(u) + \omega^2\mathcal{G}_{\chi''}(u))$$

for some ω a cube root of unity (uniquely determined by η_3).

(3) For $F = F_{A_1 \times \tilde{A}_1}$, we have

$$\mathrm{Ch}_{\mathrm{Ind}_P^{\mathrm{SO}_4}(\zeta \otimes \zeta^{-1} \otimes \mathrm{St}_{\mathrm{GL}_2})} = \frac{1}{2}(R_1^{\zeta} - R_{\tilde{A}_1}^{\zeta}),$$

thus for $u \in G_F$ unipotent, we have

$$\mathrm{Ch}_{\mathrm{Ind}_P^{\mathrm{SO}_4}(\zeta \otimes \zeta^{-1} \otimes \mathrm{St}_{\mathrm{GL}_2})}(u) = \frac{1}{2}(\mathcal{Q}_1^{F_{A_1 \times \tilde{A}_1}}(u) - \mathcal{Q}_{\tilde{A}_1}^{F_{A_1 \times \tilde{A}_1}}(u)). \tag{4.3.1}$$

(4) For $F = F_{A_1}$, we have $\pi(\eta_3)^{G_{F+}} \cong \mathrm{Ind}_B^{\mathrm{GL}_2}(\zeta \otimes \zeta^{-1})$, so on unipotent elements, we have $\mathrm{Ch}_{\pi(\eta_3)^{G_{F+}}} = \mathcal{Q}_1^{A_1}$.

(5) For $F = F_{\tilde{A}_1}$, we have $\pi(\eta_3)^{G_{F+}} \cong \zeta\,\mathrm{St}_{\mathrm{GL}_2} + \zeta^{-1}\,\mathrm{St}_{\mathrm{GL}_2}$, so on unipotent elements, we have $\mathrm{Ch}_{\pi(\eta_3)^{G_{F+}}} = \mathcal{Q}_1^{\tilde{A}_1} - \mathcal{Q}_{\tilde{A}_1}^{\tilde{A}_1}$.

(6) Finally for $F = F_\emptyset$ we have $\pi(\eta_3)^{G_{F+}} = \zeta \otimes \zeta \oplus \zeta^{-1} \otimes \zeta^{-1}$ (as in Corollary 4.0.2), so the character on unipotent elements is $2\mathcal{Q}_{\{e\}}^{F_\emptyset}$.

Similarly, for $\pi_{\mathrm{s.c.}}(\eta_3)$ we have

$$\mathrm{Ch}_{\omega_{\mathrm{cusp}}^{\eta_3}}(u) = \frac{1}{3}(\mathcal{Q}_{A_2}^{F_{A_2}}(u) + \omega\mathcal{G}_{\chi'}(u) + \omega^2\mathcal{G}_{\chi''}(u)) \tag{4.3.2}$$

where ω is a cube root of unity (uniquely determined by η_3) and $\mathcal{G}_{\chi'}, \mathcal{G}_{\chi''}$ are generalized Green functions as in [DK06, §5.2.2]. Let $\pi_{\mathrm{s.c.}}(\eta_3)^\vee$ denote the dual representation of $\pi_{\mathrm{s.c.}}(\eta_3)$. We have:

Proposition 4.3.1. *All combinations* $\pi(\eta_3)+\pi_{\text{s.c.}}(\eta_3')+\pi_{\text{s.c.}}(\eta_3'')^\vee$ *for any (possibly equal) ramified cubic characters* η_3*,* η_3'*, and* η_3'' *have stable Harish-Chandra characters on the topologically unipotent elements of* G_2*.*

Proof. From the discussion above, in the notation of [DK06, Table 4], we see that for some explicitly computable[2] constants c_i and some cube roots of unity ω_i (uniquely determined by η_3, η_3', and η_3'', respectively),

$$\mathrm{Ch}_{\pi(\eta_3)} = \frac{1}{9}c_1(D^{\mathrm{st}}_{A_2}+2D^{\mathrm{unst}}_{A_2})+c_2(\omega_1 D^{\mathrm{st}}_{(F_{A_2},\mathcal{G}_{\chi'})}+\omega_1^2 D^{\mathrm{st}}_{(F_{A_2},\mathcal{G}_{\chi''})})-c_3 D^{\mathrm{st}}_{\tilde{A}_1}+c_4 D^{\mathrm{st}}_{\{e\}}$$
$$\mathrm{Ch}_{\pi_{\text{s.c.}}(\eta_3')} = \frac{1}{9}c_1(D^{\mathrm{st}}_{A_2}-D^{\mathrm{unst}}_{A_2})+c_2(\omega_2 D^{\mathrm{st}}_{(F_{A_2},\mathcal{G}_{\chi'})}+\omega_2^2 D^{\mathrm{st}}_{(F_{A_2},\mathcal{G}_{\chi''})})$$
$$\mathrm{Ch}_{\pi_{\text{s.c.}}(\eta_3'')^\vee} = \frac{1}{9}c_1(D^{\mathrm{st}}_{A_2}-D^{\mathrm{unst}}_{A_2})+c_2(\omega_3 D^{\mathrm{st}}_{(F_{A_2},\mathcal{G}_{\chi'})}+\omega_3^2 D^{\mathrm{st}}_{(F_{A_2},\mathcal{G}_{\chi''})})$$

Thus, by [DK06, Lemma 6.4.1] the sum $\mathrm{Ch}_{\pi(\eta_3)}+\mathrm{Ch}_{\pi_{\text{s.c.}}(\eta_3')}+\mathrm{Ch}_{\pi_{\text{s.c.}}(\eta_3'')^\vee}$ is always stable. □

4.4. **Characters on a neighborhood of $s\in G_2$.** Let $s\in G_2$ be order 3 such that $\mathrm{Z}_{G_2}(s)=\mathrm{SL}_3$. The same construction as in §3.5 gives rise to invariant distributions $\Theta_{\pi(\eta_3)}$, $\Theta_{\pi_{\text{s.c.}}(\eta_3)}$, and $\Theta_{\pi_{\text{s.c.}}(\eta_3)^\vee}$ on the topologically unipotent elements of SL_3 such that they are represented by compatible locally constant functions (for each ramified cubic η_3). Similar calculations as in §3.5 gives:

Theorem 4.4.1. *For ramified cubic characters* η_3*,* η_3'*, and* η_3''*, the sum* $\mathrm{Ch}_{\pi(\eta_3)}+\mathrm{Ch}_{\pi_{\text{s.c.}}(\eta_3')}+\mathrm{Ch}_{\pi_{\text{s.c.}}(\eta_3'')^\vee}$ *is stable in a neighborhood of* s *if and only if* $\eta_3=\eta_3'=\eta_3''$*. Thus,* $\{\pi(\eta_3),\pi_{\text{s.c.}}(\eta_3),\pi_{\text{s.c.}}(\eta_3)^\vee\}$ *is an L-packet, for each ramified cubic character* η_3*.*

Proof. By Lemma 3.5.1 (together with [DK06, Lemma 6.4.1]), a character on the topologically unipotent locus $(\mathrm{SL}_3(F))_{0+}$ in $\mathrm{SL}_3(F)$ is stable if and only if it is in the span of semisimple orbital integrals. By [SF73, Table 1b], for $u\in H_\alpha/H_{\alpha+}$ unipotent, we have

$$(\omega^{\eta_3}_{\mathrm{princ}}+\omega^{\eta_3}_{\mathrm{cusp}}+(\omega^{\eta_3}_{\mathrm{cusp}})^\vee)(su)=\mathcal{Q}_1^{F_{A_2}}(u)+2\mathcal{Q}_{A_2}^{F_{A_2}}(u),$$

which is the only linear combination of $\omega^{\eta_3}_{\mathrm{princ}}$, $\omega^{\eta_3}_{\mathrm{cusp}}$, and $(\omega^{\eta_3}_{\mathrm{cusp}})^\vee$ for which the generalized Green functions $\mathcal{G}_{\chi'}$ and $\mathcal{G}_{\chi''}$ do not appear. Thus, by [DK06, Lemma 5.2.10], the sum $\mathrm{Ch}_{\pi(\eta_3)}+\mathrm{Ch}_{\pi_{\text{s.c.}}(\eta_3)}+\mathrm{Ch}_{\pi_{\text{s.c.}}(\eta_3)^\vee}$ is the only stable combination. □

In fact:

Theorem 4.4.2. *For a ramified cubic character* η_3*, the sum* $\mathrm{Ch}_{\pi(\eta_3)}+\mathrm{Ch}_{\pi_{\text{s.c.}}(\eta_3)}+\mathrm{Ch}_{\pi_{\text{s.c.}}(\eta_3)^\vee}$ *is stable. Similarly, for a ramified quadratic character* η_2*, the sum* $\mathrm{Ch}_{\pi(\eta_2)}+\mathrm{Ch}_{\pi_{\text{s.c.}}(\eta_2)}$ *is stable.*

Proof. We have calculated distributions $\mathrm{Ch}_{\pi(\eta_3)}$, $\mathrm{Ch}_{\pi_{\text{s.c.}}(\eta_3)}$, and $\mathrm{Ch}_{\pi_{\text{s.c.}}(\eta_3)^\vee}$ (resp., $\mathrm{Ch}_{\pi(\eta_2)}$ and $\mathrm{Ch}_{\pi_{\text{s.c.}}(\eta_2)}$) on topologically unipotent neighborhoods of 1 and s. A similar (but easier) calculation gives explicit formulae for the distributions on neighborhoods of other (thus arbitrary) topologically semisimple elements $\gamma\in G_2$.

These calculations are enough to prove stability of the characters of $\mathrm{Ch}_{\pi(\eta_2)}+\mathrm{Ch}_{\pi_{\text{s.c.}}(\eta_2)}$ and $\mathrm{Ch}_{\pi(\eta_3)}+\mathrm{Ch}_{\pi_{\text{s.c.}}(\eta_3)}+\mathrm{Ch}_{\pi_{\text{s.c.}}(\eta_3)^\vee}$ on compact elements. By [Cas77, Theorem 5.2] (by an argument similar to [DR09, Lemma 9.3.1]), we conclude full stability, i.e. Property 2.1.1. □

Appendix A. Character Table of $\mathrm{SO}_4(\mathbb{F}_q)$

A.1. **Classifying conjugacy classes in $\mathrm{SO}_4(\mathbb{F}_q)$.** We introduce the following notation:

- $c_1(x)=\begin{pmatrix} x & \\ & x\end{pmatrix}$ where $x\in\mathbb{F}_q^\times$

[2] They are calculable via formulae in [DK06]; for brevity we do not include them here.

- $c_2(x,\gamma) = \begin{pmatrix} x & \gamma \\ & x \end{pmatrix}$ where $x \in \mathbb{F}_q^\times$ and $\gamma \neq 0 \in \mathbb{F}_q^\times$. When $\gamma = 1$ let $c_2(x) := c_2(x,1)$
- $c_3(x,y) = \begin{pmatrix} x & \\ & y \end{pmatrix}$ where $x \neq y \in \mathbb{F}_q^\times$. When $xy = 1$ let $c_3(x) := c_3(x, x^{-1})$, where $x \neq \pm 1$.
- $c_4(z)$ for the matrix with eigenvalues z and z^q, for $z \in \mathbb{F}_{q^2} \backslash \mathbb{F}_q$.

Moreover, choose and element $\Delta \in \mathbb{F}_q^\times \setminus (\mathbb{F}_q^\times)^2$ and an element $\alpha \in \mathbb{F}_{q^2}^\times$ such that $\alpha^{q-1} = -1$, a choice of which is unique up to scaling by $\mathbb{F}_q^\times$.

Lemma A.1.1. *Let q be odd. The conjugacy classes in $\mathrm{SO}_4(\mathbb{F}_q)$ are one of:*

(1) $c_1(1) \times c_1(\pm 1)$. *There are 2 such conjugacy classes.*
(2) $c_1(1) \times c_2(\pm 1)$. *There are 2 such conjugacy classes.*
(3) $c_1(1) \times c_3(x_2)$ *for* $x_2 \neq \pm 1 \in \mathbb{F}_q^\times$. *Since* $c_3(x_2) = c_3(x_2^{-1})$ *in* $\mathrm{SL}_2(\mathbb{F}_q)$, *there are* $(q-3)/2$ *such conjugacy classes.*
(4) $c_1(1) \times c_4(z_2)$ *for* $z_2 \in \mathbb{F}_{q^2} \backslash \mathbb{F}_q$ *such that* $z_2^{q+1} = 1$. *Since* $c_4(z_2) = c_4(z_2^{-1})$ *in* $\mathrm{SL}_2(\mathbb{F}_q)$ *there are* $(q-1)/2$ *such conjugacy classes.*
(5) $c_2(\pm 1) \times c_1(1) = c_2(1) \times c_1(\pm 1)$. *There are 2 such conjugacy classes.*
(6) $c_2(1) \times c_2(\pm 1, \gamma_2)$ *for* $\gamma_2 \in \{1, \Delta\}$. *There are 4 such conjugacy classes.*
(7) $c_2(1) \times c_3(x_2)$ *for* $x_2 \neq \pm 1 \in \mathbb{F}_q^\times$. *Since* $c_3(x_2) = c_3(x_2^{-1})$ *in* $\mathrm{SL}_2(\mathbb{F}_q)$, *there are* $(q-3)/2$ *such conjugacy classes.*
(8) $c_2(1) \times c_4(z_2)$ *for* $z_2 \in \mathbb{F}_{q^2} \backslash \mathbb{F}_q$ *with* $z_2^{q+1} = 1$. *Since* $c_4(z_2) = c_4(z_2^{-1})$ *there are* $(q-1)/2$ *such conjugacy classes.*
(9) $c_3(x_1) \times c_1(1)$ *for* $x_1 \neq \pm 1 \in \mathbb{F}_q^\times$. *Since* $c_3(x_1) = c_3(x_1^{-1})$ *in* $\mathrm{GL}_2(\mathbb{F}_q)$ *there are* $(q-3)/2$ *such conjugacy classes.*
(10) $c_3(x_1) \times c_2(1)$ *for* $x_1 \neq \pm 1 \in \mathbb{F}_q^\times$. *Since* $c_3(x_1) = c_3(x_1^{-1})$ *in* $\mathrm{SL}_2(\mathbb{F}_q)$ *there are* $(q-3)/2$ *such conjugacy classes.*
(11) $c_3 \times c_3$. *There are the following cases:*
   (a) $c_3(x_1) \times c_3(x_2)$ *where* $x_1^2 \neq -1$ *or* $x_2^2 \neq -1$, *then since* $c_3(x_1) = c_3(x_1^{-1})$ *and* $c_3(x_2) = c_3(x_2^{-1})$ *in* $\mathrm{SL}_2(\mathbb{F}_q)$, *and* $c_3(x_1) \times c_3(x_2) = c_3(-x_1) \times c_3(-x_2)$ *there are*
$$\begin{cases} \frac{(q-3)^2-4}{8} & q \equiv 1 \pmod 4 \\ \frac{(q-3)^2}{8} & q \equiv -1 \pmod 4 \end{cases}$$
   such conjugacy classes.
   (b) $c_3(x_1, \Delta x_1^{-1}) \times c_3(x_2, \Delta x_2^{-1})$ *where* $x_1, x_2 \in \mathbb{F}_q^\times$ *and* $x_1^2 \neq -\Delta$ *or* $x_2^2 \neq -\Delta$. *Since* $c_3(x_1, \Delta x_1^{-1}) = c_3(\Delta x_1^{-1}, x_1)$ *and* $c_3(x_2) = c_3(\Delta x_2^{-1})$ *in* $\mathrm{SL}_2(\mathbb{F}_q)$ *there are*
$$\begin{cases} \frac{(q-1)^2}{8} & q \equiv 1 \pmod 4 \\ \frac{(q-1)^2-4}{8} & q \equiv -1 \pmod 4 \end{cases}$$
   such conjugacy classes.
   (c) $c_3(-1,1) \times c_3(-1,1)$. *There is one such conjugacy class.*
(12) $c_3 \times c_4$. *There are the following cases:*
   - $c_3(x_1) \times c_4(z_2)$ *for* $x_1 \in \mathbb{F}_q^\times$ *and* $z \in \mathbb{F}_{q^2} \backslash \mathbb{F}_q$ *such that* $z_2^{q+1} = 1$.
   - $c_3(x_1, \Delta x_1^{-1}) \times c_4(z_2)$ *for* $x_1 \in \mathbb{F}_q^\times$ *and* $z_2 \in \mathbb{F}_{q^2}$ *such that* $z_2^{q+1} = \Delta$. *Since* $c_3(x_1, \Delta x_1^{-1}) = c_3(\Delta x_1^{-1}, x_1)$ *and* $c_4(z_2) = c_4(\Delta z_2^{-1})$, *there are*
$$\begin{cases} \frac{q^2-1}{4} & q \equiv 1 \pmod 4 \\ \frac{(q-1)(q+3)}{4} & q \equiv -1 \pmod 4 \end{cases}$$
   such conjugacy classes.
(13) $c_4(z_1) \times c_1(1)$ *for* $z_1 \in \mathbb{F}_{q^2}^1 \backslash \{\pm 1\}$. *There are* $(q-1)/2$ *such conjugacy classes.*

(14) $c_4(z_1) \times c_2(1)$ *for* $x, y \in \mathbb{F}_q^\times$ *and* $z_1 \in \mathbb{F}_{q^2}$ *with* $z_1^{q+1} = 1$*. There are* $(q-1)/2$ *such conjugacy classes.*

(15) $c_4(z_1) \times c_3(x_2)$ *for* $x_2 \neq \pm 1 \in \mathbb{F}_q^\times$ *and* $z_1 \in \mathbb{F}_{q^2} \backslash \mathbb{F}_q$ *such that* $z_1^{q+1} = 1$*. There are* $(q-1)(q-3)/4$ *such conjugacy classes.*

(16) $c_4(z_1) \times c_3(x_2, \Delta x_2^{-1})$ *for* $x_2 \in \mathbb{F}_q^\times$ *and* $z_1 \in \mathbb{F}_{q^2}^\times$ *such that* $z_1^{q+1} = \Delta$*. There are*

$$\begin{cases} \frac{q^2-1}{4} & q \equiv 1 \pmod 4 \\ \frac{(q-1)(q+3)}{4} & q \equiv -1 \pmod 4 \end{cases}$$

such conjugacy classes.

(17) $c_4(z_1) \times c_4(z_2)$ *for* $z_1, z_2 \in \mathbb{F}_{q^2} \backslash \mathbb{F}_q$ *with* $(z_1 z_2)^{q+1} = 1$ *and* $z_1^{q-1} \neq -1$ *or* $z_2^{q-1} \neq -1$*. The since* $c_4(z_1) \times c_4(z_2) = c_4(az_1) \times c_4(az_2)$ *for any* $a \in \mathbb{F}_q^\times$*, and* $c_4(z_1) = c_4(z_1^q)$ *and* $c_4(z_2) = c_4(z_2^q)$ *in* $\mathrm{SL}_2(\mathbb{F}_q)$*.*

(18) $c_4(\alpha) \times c_4(\alpha^{-1})$*. There is a unique such conjugacy class.*

A.2. **Classifying representations in** $\mathrm{SO}_4(\mathbb{F}_q)$**.** Let $\mathrm{GL}_{2,2}(\mathbb{F}_q) := \{(g,h) \in \mathrm{GL}_2(\mathbb{F}_q) \times \mathrm{GL}_2(\mathbb{F}_q) : \det(g) = \det(h)\}$. Then there is an isomorphism $\mathrm{SO}_4(\mathbb{F}_q) \cong \mathrm{GL}_{2,2}(\mathbb{F}_q)/\mathbb{F}_q^\times$. Let $\mathbb{T}$ denote the split maximal torus of $\mathrm{GL}_2(\mathbb{F}_q)$.

Now, the centralizer of a semisimple element $(g,h) \in \mathrm{GL}_{2,2}(\mathbb{F}_q)$ in $\mathrm{SO}_4(\mathbb{F}_q)$ is

$$\begin{aligned} \mathrm{Z}_{\mathrm{SO}_4(\mathbb{F}_q)}(g,h) &= \{(s,t) \in \mathrm{GL}_{2,2}(\mathbb{F}_q) : (sgs^{-1}, tht^{-1}) = a(g,h) \text{ for some } a \in \mathbb{F}_q^\times\}/\mathbb{F}_q^\times \\ &= \{(s,t) \in \mathrm{GL}_{2,2}(\mathbb{F}_q) : (sgs^{-1}, tht^{-1}) = \pm(g,h)\}/\mathbb{F}_q^\times, \end{aligned}$$

where the last equality is by observing $\det(g) = \det(sgs^{-1}) = \det(ag) = a^2 \det(g)$, so $a = \pm 1$. Thus, the centralizer depends on whether $-g$ is conjugate to g and whether $-h$ is conjugate to h under $\mathrm{GL}_2(\mathbb{F}_q)$.

The conjugacy classes of semisimple elements $s = (g,h)$ of $\mathrm{SO}_4(\mathbb{F}_q)$ fall into one of the following possibilities:

(1) $c_1(1) \times c_1(1)$, then $\mathrm{Z}_{\mathrm{SO}_4}(s) = \mathrm{SO}_4(\mathbb{F}_q)$. Since unipotent representations are independent of isogenies by [DL76, Prop 7.10] we have

$$\mathcal{E}(\mathrm{SO}_4(\mathbb{F}_q), 1) \cong \mathcal{E}(\mathrm{PGL}_2(\mathbb{F}_q) \times \mathrm{PGL}_2(\mathbb{F}_q), 1) = \{1 \boxtimes 1, 1 \boxtimes \mathrm{St}_{\mathrm{PGL}_2}, \mathrm{St}_{\mathrm{PGL}_2} \boxtimes 1, \mathrm{St}_{\mathrm{PGL}_2} \boxtimes \mathrm{St}_{\mathrm{PGL}_2}\}.$$

The representation $1_{\mathrm{PGL}_2} \boxtimes 1_{\mathrm{PGL}_2}$ corresponds to the representation 1_{SO_4} and $\mathrm{St}_{\mathrm{PGL}_2} \boxtimes \mathrm{St}_{\mathrm{PGL}_2}$ corresponds to the representation $\mathrm{St}_{\mathrm{SO}_4}$. There are 4 such representations.

(2) $c_1(1) \times c_1(-1)$, then again $\mathrm{Z}_{\mathrm{SO}_4}(s) = \mathrm{SO}_4(\mathbb{F}_q)$. The representations in $\mathcal{E}(\mathrm{SO}_4, s)$ are of the form $\pi \otimes \zeta$ where $\pi \in \mathcal{E}(\mathrm{SO}_4, 1)$ and $\zeta(g,h) := \epsilon(\det(g))$ is the unique order 2 character of $\mathrm{SO}_4(\mathbb{F}_q)$. There are 4 such representations.

(3) $c_1(1) \times c_3(x_2)$ for $x_2 \neq \pm 1 \in \mathbb{F}_q^\times$, then $\mathrm{Z}_{\mathrm{SO}_4}(s) = (\mathrm{GL}_2(\mathbb{F}_q) \times \mathbb{T})^1/\mathbb{F}_q^\times \cong \mathrm{GL}_2(\mathbb{F}_q)$. Here, $\mathrm{GL}_2(\mathbb{F}_q)$ has two unipotent representations, 1 and the Steinberg $\mathrm{St}_{\mathrm{GL}_2(\mathbb{F}_q)}$, of dimensions 1 and q, respectively.

Letting $\mathbb{P} = (\mathrm{GL}_2 \times \mathbb{B})^1/\mathbb{F}_q^\times \subset \mathrm{SO}_4(\mathbb{F}_q)$ be the parabolic subgroup with Levi $(\mathrm{GL}_2(\mathbb{F}_q) \times \mathbb{T})^1/\mathbb{F}_q^\times$, the representations correspond to $\mathrm{Ind}_{\mathbb{P}}^{\mathrm{SO}_4}(\chi 1_{\mathrm{GL}_2})$ and $\mathrm{Ind}_{\mathbb{P}}^{\mathrm{SO}_4}(\chi \, \mathrm{St}_{\mathrm{GL}_2})$, for a character χ of $\mathbb{F}_q^\times$ with $\chi^2 \neq 1$.

Note that these are irreducible since the Weyl group action replaces χ with χ^{-1}. There are a total of $2 \cdot (q-3)/2 = q-3$ representations.

(4) $c_1(1) \times c_4(z_2)$ then $\mathrm{Z}_{\mathrm{SO}_4}(s) = (\mathrm{GL}_2(\mathbb{F}_q) \times R_{\mathbb{F}_{q^2}/\mathbb{F}_q}\mathbb{G}_m)^1/\mathbb{F}_q^\times$. This has two cuspidal unipotents, 1_{PGL_2} and $\mathrm{St}_{\mathrm{PGL}_2}$, inflated via $(\mathrm{GL}_2(\mathbb{F}_q) \times R_{\mathbb{F}_{q^2}/\mathbb{F}_q}\mathbb{G}_m)^1/\mathbb{F}_q^\times \to \mathrm{PGL}_2(\mathbb{F}_q)$.

They correspond to representations $1_{\mathrm{GL}_2} \boxtimes \rho_\theta$ of $\mathrm{GL}_2 \times \mathrm{GL}_2$, restricted to $\mathrm{GL}_{2,2}$ and factored through SO_4. Here, θ is a regular character of $\mathbb{F}_{q^2}^\times$ with $\theta|_{\mathbb{F}_q^\times} = 1$.

(5) $c_3(x_1,y_1)\times c_3(x_2,y_2)$ for $x_1\neq\pm y_1, x_2\neq\pm y_2\in\mathbb{F}_q^\times$ then $\mathrm{Z}_{\mathrm{SO}_4}(s)=(\mathbb{T}\times\mathbb{T})^1/\mathbb{F}_q^\times$, the maximal split torus of $\mathrm{SO}_4(\mathbb{F}_q)$. This has a unique unipotent, 1.

They correspond to induced representations $\mathrm{Ind}_{\mathbb{B}}^{\mathrm{SO}_4}(\chi_1\otimes\chi_2\otimes\chi_3\otimes\chi_4)$, where $\mathbb{B}$ is the split Borel subgroup of $\mathrm{SO}_4(\mathbb{F}_q)$, where χ_i are characters of $\mathbb{F}_q^\times$ with $\chi_1\chi_2\chi_3\chi_4=1$ and $\chi_1^2\neq\chi_2^2$ and $\chi_3^2\neq\chi_4^2$. Here,

$$\chi_1\otimes\chi_2\otimes\chi_3\otimes\chi_4(\begin{pmatrix}a' & \\ & b'\end{pmatrix},\begin{pmatrix}c' & \\ & d'\end{pmatrix}):=\chi_1(a')\chi_2(b')\chi_3(c')\chi_4(d').$$

These representations are irreducible since the Weyl group acts by swapping χ_1 with χ_2, and swapping χ_3 with χ_4. The number of such representations is:

$$\begin{cases}(q+1)^2+4 & q\equiv 1 \pmod 4\\ (q+1)^2 & q\equiv 3 \pmod 4).\end{cases}$$

(6) $c_3(1,-1)\times c_3(1,-1)$. This has two unipotents, 1 and sgn.

These are the irreducible components of the length 2 representation $\mathrm{Ind}_{\mathbb{B}}^{\mathrm{SO}_4}(1\otimes\epsilon\otimes 1\otimes\epsilon)$, where ϵ is the unique order 2 character of $\mathbb{F}_q^\times$ and $\chi_1^2\chi_2^2=1$. Explicitly, they are induced representations from the index 2 subgroup $\mathrm{SL}_2(\mathbb{F}_q)\times\mathrm{SL}_2(\mathbb{F}_q)/\pm 1\subset\mathrm{SO}_4(\mathbb{F}_q)$:

$$\omega^+_{\mathrm{princ}}:=\mathrm{Ind}^{\mathrm{SO}_4}_{(\mathrm{SL}_2\times\mathrm{SL}_2)/\pm 1}(\omega_e^+\boxtimes\omega_e^+),\omega^-_{\mathrm{princ}}:=\mathrm{Ind}^{\mathrm{SO}_4}_{(\mathrm{SL}_2\times\mathrm{SL}_2)/\mu_2}(\omega_e^+\boxtimes\omega_e^-),$$

in the notation of Remark A.2.2. In particular, the restriction to $\mathrm{SL}_2(\mathbb{F}_q)\times\mathrm{SL}_2(\mathbb{F}_q)/\pm 1$ is $\omega_e^+\boxtimes\omega_e^+\oplus\omega_e^-\boxtimes\omega_e^-$ and $\omega_e^+\boxtimes\omega_e^-\oplus\omega_e^-\boxtimes\omega_e^+$, respectively.

(7) $c_3(x_1,y_1)\times c_4(z_2)$ where $x_1,y_1\in\mathbb{F}_q^\times$ and $z_2\in\mathbb{F}_{q^2}\backslash\mathbb{F}_q$ with $x_1y_1=z_2^{q+1}$. Then $\mathrm{Z}_{\mathrm{SO}_4}(s)=(\mathbb{T}\times R_{\mathbb{F}_{q^2}/\mathbb{F}_q}\mathbb{G}_m)^1/\mathbb{F}_q^\times$. This has a unique unipotent, 1.

Let $\mathbb{P}=(\mathbb{B}\times\mathrm{GL}_2)^1/\mathbb{F}_q^\times\subset\mathrm{SO}_4(\mathbb{F}_q)$ be the parabolic subgroup with Levi $(\mathbb{T}\times\mathrm{GL}_2(\mathbb{F}_q))^1/\mathbb{F}_q^\times\cong\mathrm{GL}_2(\mathbb{F}_q)$. These are the induced representations $\mathrm{Ind}_{\mathbb{B}}^{\mathrm{GL}_2}(\chi_1\boxtimes\chi_2)\boxtimes\rho_\theta$ of $\mathrm{GL}_2(\mathbb{F}_q)\times\mathrm{GL}_2(\mathbb{F}_q)$, restricted to $\mathrm{GL}_{2,2}$ and factored through SO_4. Here, χ_1 and χ_2 are characters of $\mathbb{F}_q^\times$ with $\chi_1^2\neq\chi_2^2$ and θ is a regular character of $\mathbb{F}_{q^2}^\times$, where $\chi_1\chi_2\theta|_{\mathbb{F}_q^\times}=1$.

(8) $c_4(z_1)\times c_4(z_2)$ where $z_1^{q+1}=z_2^{q+1}$ and $z_1^{q-1}\neq -1$ or $z_2^{q-1}\neq -1$. Here. $\mathrm{Z}_{\mathrm{SO}_4}(s)=(R_{\mathbb{F}_{q^2}/\mathbb{F}_q}\mathbb{G}_m\times R_{\mathbb{F}_{q^2}/\mathbb{F}_q}\mathbb{G}_m)^1/\mathbb{F}_q^\times$. This has a unique unipotent, 1.

They correspond to representations $\rho_{\theta_1}\boxtimes\rho_{\theta_2}$ of $\mathrm{GL}_2(\mathbb{F}_q)\times\mathrm{GL}_2(\mathbb{F}_q)$, restricted to $\mathrm{GL}_{2,2}(\mathbb{F}_q)$ and inflated to $\mathrm{SO}_4(\mathbb{F}_q)$. Here, $\theta_1\theta_2|_{\mathbb{F}_q^\times}=1$ and θ_1^2 or θ_2^2 is nontrivial on $\mathbb{F}^1_{q^2}$.

(9) $c_4(\alpha)\times c_4(\alpha^{-1})$. Here $\mathrm{Z}_{\mathrm{SO}_4}(s)=(R_{\mathbb{F}_{q^2}/\mathbb{F}_q}\mathbb{G}_m\times R_{\mathbb{F}_{q^2}/\mathbb{F}_q}\mathbb{G}_m)^1/\mathbb{F}_q^\times\rtimes\mu_2$. This has two unipotents, 1 and sgn.

They correspond to the two induced representations

$$\omega^+_{\mathrm{cusp}}:=\mathrm{Ind}^{\mathrm{SO}_4}_{\mathrm{SL}_2\times\mathrm{SL}_2/\pm 1}(\omega_0^+\boxtimes\omega_0^+)\quad\text{and}\quad\omega^-_{\mathrm{cusp}}:=\mathrm{Ind}^{\mathrm{SO}_4}_{\mathrm{SL}_2\times\mathrm{SL}_2/\pm 1}(\omega_0^+\boxtimes\omega_0^-),\tag{A.2.1}$$

using the notation of Remark A.2.3.

Remark A.2.1. The Steinberg representation of $\mathrm{GL}_2(\mathbb{F}_q)$ has character values:

$c_1(x)$	q
$c_2(x)$	0
$c_3(x,y)$	1
$c_4(z)$	-1

Remark A.2.2. The principal series representation $\mathrm{Ind}_{\mathbb{B}}^{\mathrm{SL}_2}(\epsilon\otimes 1)$ of $\mathrm{SL}_2(\mathbb{F}_q)$ has length two, and splits as $\omega_e^+\oplus\omega_e^-$, where as usual $\epsilon\neq 1$ is the unique order 2 character of $\mathbb{F}_q^\times$. The character tables are:

	ω_e^+	ω_e^-
I_2	$\frac{q+1}{2}$	$\frac{q+1}{2}$
$-I_2$	$\frac{q+1}{2}\epsilon(-1)$	$\frac{q+1}{2}\epsilon(-1)$
$c_2(\pm 1, \gamma), \gamma \in \{1, \Delta\}$	$\frac{1}{2}(\epsilon(\pm 1) + \epsilon(\gamma)\sqrt{\epsilon(-1)q})$	$\frac{1}{2}(\epsilon(\pm 1) - \epsilon(\gamma)\sqrt{\epsilon(-1)q})$
$c_3(x)$	$\epsilon(x)$	$\epsilon(x)$
$c_4(z), z^{q+1} = 1$	0	0

Remark A.2.3. Let $\theta_0 \neq 1$ be the unique order 2 character of $\mathbb{F}_{q^2}^1$, so the restriction of the cuspidal representation ρ_{θ_0} of $\mathrm{GL}_2(\mathbb{F}_q)$, restricted to $\mathrm{SL}_2(\mathbb{F}_q)$, splits as $\omega_0^+ \oplus \omega_0^-$. The character tables are:

	ω_0^+	ω_0^-
I_2	$\frac{q-1}{2}$	$\frac{q-1}{2}$
$-I_2$	$-\frac{q-1}{2}\epsilon(-1)$	$-\frac{q-1}{2}\epsilon(-1)$
$c_2(\pm 1, \gamma), \gamma \in \{1, \Delta\}$	$\pm\frac{1}{2}(-\epsilon(\pm 1) + \epsilon(\gamma)\sqrt{\epsilon(-1)q})$	$\pm\frac{1}{2}(-\epsilon(\pm 1) - \epsilon(\gamma)\sqrt{\epsilon(-1)q})$
$c_3(x)$	0	0
$c_4(z), z \in \mathbb{F}_{q^2}^1$	$-\theta_0(z)$	$-\theta_0(z)$

Now, we can calculate the character table for $\mathrm{SO}_4(\mathbb{F}_q)$. Here, we ignore twists of representations by outer automorphisms (coming from $\mathrm{SO}_4 \subset \mathrm{O}_4$), which swaps the two GL_2-factors:

Representations of $SO_4(\mathbb{F}_q)$, cases 1-3								
	1_{SO_4}	ζ	$1_{PGL_2} \boxtimes \mathrm{St}_{PGL_2}$	$(1_{PGL_2} \boxtimes \mathrm{St}_{PGL_2}) \otimes \zeta$	St_{SO_4}	$\mathrm{St}_{SO_4} \otimes \zeta$	$\mathrm{Ind}_{\mathbb{P}}^{SO_4}(\chi 1_{GL_2})$	$\mathrm{Ind}_{\mathbb{P}}^{SO_4}(\chi \, \mathrm{St}_{GL_2})$
$c_1(1) \times c_1(\pm 1)$	1	1	q	q	q^2	q^2	$q+1$	$q(q+1)$
$c_1(1) \times c_2(\pm 1)$	1	1	0	0	0	0	1	q
$c_1(1) \times c_3(x_2)$	1	1	1	1	q	q	$\chi^2(x_2) + \chi^{-2}(x_2)$	$q(\chi^2(x_2) + \chi^{-2}(x_2))$
$c_1(1) \times c_4(z_2)$	1	1	-1	-1	$-q$	$-q$	0	0
$c_2(1) \times c_1(\pm 1)$	1	1	q	q	0	0	$q+1$	0
$c_2(1) \times c_2(\pm 1, \gamma_2)$	1	1	0	0	0	0	1	0
$c_2(1) \times c_3(x_2)$	1	1	1	1	0	0	$\chi^2(x_2) + \chi^{-2}(x_2)$	0
$c_2(1) \times c_4(z_2)$	1	1	-1	-1	0	0	0	0
$c_3(x_1) \times c_1(1)$	1	1	q	q	q	q	$q+1$	$q+1$
$c_3(x_1) \times c_2(1)$	1	1	0	0	0	0	1	1
$c_3(x_1, y_1) \times c_3(x_2, y_2)$	1	$\epsilon(x_1 y_1)$	1	$\epsilon(x_1 y_1)$	1	$\epsilon(x_1 y_1)$	$\chi(x_2 y_2^{-1}) + \chi(x_2^{-1} y_2)$	$\chi(x_2 y_2^{-1}) + \chi(x_2^{-1} y_2)$
$c_3(x_1, y_1) \times c_4(z_2)$	1	$\epsilon(x_1 y_1)$	-1	$-\epsilon(x_1 y_1)$	-1	$-\epsilon(x_1 y_1)$	0	0
$c_4(z_1) \times c_1(1)$	1	1	q	q	$-q$	$-q$	$q+1$	$-(q+1)$
$c_4(z_1) \times c_2(1)$	1	1	0	0	0	0	1	-1
$c_4(z_1) \times c_3(x_2, y_2)$	1	$\epsilon(x_2 y_2)$	1	$\epsilon(x_2 y_2)$	-1	$-\epsilon(x_1 y_1)$	$\chi(x_2 y_2^{-1}) + \chi(x_2^{-1} y_2)$	$-\chi(x_2 y_2^{-1}) - \chi(x_2^{-1} y_2)$
$c_4(z_1) \times c_4(z_2)$	1	$\epsilon(z_1^{q+1})$	-1	$-\epsilon(z_1^{q+1})$	1	$\epsilon(z_1^{q+1})$	0	0

Here, the representations $\mathrm{St}_{PGL_2} \boxtimes 1_{PGL_2}$ and $(\mathrm{St}_{PGL_2} \boxtimes 1_{PGL_2}) \otimes \zeta$ are twists of $1_{PGL_2} \boxtimes \mathrm{St}_{PGL_2}$ and $(1_{PGL_2} \boxtimes \mathrm{St}_{PGL_2}) \otimes \zeta$, respectively, under the unique outer automorphism.

Representations of $\mathrm{SO}_4(\mathbb{F}_q)$, cases 4-6

	$1_{\mathrm{GL}_2} \boxtimes \rho_\theta$	$\mathrm{Ind}_{\mathbb{B}}^{\mathrm{SO}_4}(\chi_1 \otimes \chi_2 \otimes \chi_3 \otimes \chi_4)$	ω^+_{princ}	ω^-_{princ}
$c_1(1) \times c_1(\pm 1)$	$q-1$	$(q+1)^2\chi_1\chi_2(\pm 1)$	$\frac{(q+1)^2}{2}\epsilon(\pm 1)$	$\frac{(q+1)^2}{2}\epsilon(\pm 1)$
$c_1(1) \times c_2(\pm 1)$	-1	$(q+1)\chi_1\chi_2(\pm 1)$	$\frac{q+1}{2}\epsilon(\pm 1)$	$\frac{q+1}{2}\epsilon(\pm 1)$
$c_1(1) \times c_3(x_2)$	0	$(q+1)(\chi_3^{-1}\chi_4(x_2) + \chi_3\chi_4^{-1}(x_2))$	$(q+1)\epsilon(x_2)$	$(q+1)\epsilon(x_2)$
$c_1(1) \times c_4(z_2)$	$-\theta(z_2) - \theta(z_2^q)$	0	0	0
$c_2(1) \times c_1(\pm 1)$	$q-1$	$(q+1)\chi_1\chi_2(\pm 1)$	$\frac{q+1}{2}\epsilon(\pm 1)$	$\frac{q+1}{2}\epsilon(\pm 1)$
$c_2(1) \times c_2(\pm 1, \gamma_2)$	-1	$\chi_1\chi_2(\pm 1)$	$\frac{1}{2}(\epsilon(\pm 1) + \epsilon(-\gamma_2)q)$	$\frac{1}{2}(\epsilon(\pm 1) - \epsilon(-\gamma_2)q)$
$c_2(1) \times c_3(x_2)$	0	$\chi_3^{-1}\chi_4(x_2) + \chi_3\chi_4^{-1}(x_2)$	$\epsilon(x_2)$	$\epsilon(x_2)$
$c_2(1) \times c_4(z_2)$	$-\theta(z_2) - \theta(z_2^q)$	0	0	0
$c_3(x_1) \times c_1(1)$	$q-1$	$(q+1)(\chi_1^{-1}\chi_2(x_1) + \chi_1\chi_2^{-1}(x_1))$	$(q+1)\epsilon(x_1)$	$(q+1)\epsilon(x_1)$
$c_3(x_1) \times c_2(1)$	1	$\chi_1^{-1}\chi_2(x_1) + \chi_1\chi_2^{-1}(x_1)$	$\epsilon(x_1)$	$\epsilon(x_1)$
$c_3(x_1, y_1) \times c_3(x_2, y_2)$	0	$(\chi_1^{-1}(x_1)\chi_2(y_1) + \chi_1(x_1)\chi_2^{-1}(y_1))(\chi_3^{-1}(x_2)\chi_4(y_2) + \chi_3(x_2)\chi_4^{-1}(y_2))$	$\begin{cases} 2\epsilon(x_1x_2) & x_1y_1 \in (\mathbb{F}_q^\times)^2 \\ 0 & x_1y_1 \notin (\mathbb{F}_q^\times)^2 \end{cases}$	$\begin{cases} 2\epsilon(x_1x_2) & x_1y_1 \in (\mathbb{F}_q^\times)^2 \\ 0 & x_1y_1 \notin (\mathbb{F}_q^\times)^2 \end{cases}$
$c_3(x_1, y_1) \times c_4(z_2)$	$-\theta(z_2) - \theta(z_2^q)$	0	0	0
$c_4(z_1) \times c_1(1)$	$q-1$	0	0	0
$c_4(z_1) \times c_2(1)$	-1	0	0	0
$c_4(z_1) \times c_3(x_2, y_2)$	0	0	0	0
$c_4(z_1) \times c_4(z_2)$	$-\theta(z_2) - \theta(z_2^q)$	0	0	0

Representations of $\mathrm{SO}_4(\mathbb{F}_q)$, cases 7-9

	$\mathrm{Ind}_{\mathbb{B}}^{\mathrm{GL}_2}(\chi_1 \boxtimes \chi_2) \boxtimes \rho_\theta$	$\rho_{\theta_1} \boxtimes \rho_{\theta_2}$	ω^+_{cusp}	ω^-_{cusp}
$c_1(1) \times c_1(\pm 1)$	$(q^2-1)\theta(\pm 1)$	$(q-1)^2\theta_1(\pm 1)$	$\pm\frac{(q-1)^2}{2}\epsilon(\pm 1)$	$\pm\frac{(q-1)^2}{2}\epsilon(\pm 1)$
$c_1(1) \times c_2(\pm 1)$	$-(q+1)\theta(\pm 1)$	$-(q-1)\theta_1(\pm 1)$	$\mp\frac{q-1}{2}\epsilon(\pm 1)$	$\mp\frac{q-1}{2}\epsilon(\pm 1)$
$c_1(1) \times c_3(x_2)$	0	0	0	0
$c_1(1) \times c_4(z_2)$	$-(q+1)(\theta(z_2) + \theta(z_2^q))$	$-(q-1)(\theta_2(z_2) + \theta_2(z_2^q))$	$-(q-1)\theta_0(z_2)$	$-(q-1)\theta_0(z_2)$
$c_2(1) \times c_1(\pm 1)$	$(q-1)\theta(\pm 1)$	$-(q-1)\theta_1(\pm 1)$	$\mp\frac{q-1}{2}\epsilon(\pm 1)$	$\mp\frac{q-1}{2}\epsilon(\pm 1)$
$c_2(1) \times c_2(\pm 1, \gamma_2)$	$-\theta(\pm 1)$	$\theta_1(\pm 1)$	$\pm\frac{1}{2}(\epsilon(\pm 1) + \epsilon(-\gamma_2)q)$	$\pm\frac{1}{2}(\epsilon(\pm 1) - \epsilon(-\gamma_2)q)$
$c_2(1) \times c_3(x_2)$	0	0	0	0
$c_2(1) \times c_4(z_2)$	$-(\theta(z_2) + \theta(z_2^q))$	$\theta_2(z_2) + \theta_2(z_2^q)$	$\frac{1}{2}\theta_0(z)(1 - \sqrt{q^*})$	$\frac{1}{2}\theta_0(z)(1 + \sqrt{q^*})$
$c_3(x_1) \times c_1(1)$	$(q-1)(\chi_1^{-1}\chi_2(x_1) + \chi_1\chi_2^{-1}(x_1))$	0	0	0
$c_3(x_1) \times c_2(1)$	$\chi_1^{-1}\chi_2(x_1) + \chi_1\chi_2^{-1}(x_1)$	0	0	0
$c_3(x_1, y_1) \times c_3(x_2, y_2)$	0	0	0	0
$c_3(x_1, y_1) \times c_4(z_2)$	$-(\chi_1(x_1)\chi_2(y_1) + \chi_2(x_1)\chi_1(y_1))(\theta(z_2) + \theta(z_2^q))$	0	0	0
$c_4(z_1) \times c_1(1)$	0	$-(q-1)(\theta_1(z_1) + \theta_1(z_1^q))$	$-(q-1)\theta_0(z_2)$	$-(q-1)\theta_0(z_2)$
$c_4(z_1) \times c_2(1)$	0	$\theta_1(z_2) + \theta_1(z_2^q)$	$\frac{1}{2}\theta_0(z_1)(1 - \sqrt{q^*})$	$\frac{1}{2}\theta_0(z_1)(1 + \sqrt{q^*})$
$c_4(z_1) \times c_3(x_2, y_2)$	0	0	0	0
$c_4(z_1) \times c_4(z_2)$	0	$(\theta_1(z_1) + \theta_1(z_1^q))(\theta_2(z_2) + \theta_2(z_2^q))$	$\begin{cases} 0 & z_1^{(q+1)/2} \in \mathbb{F}_q^\times \\ 2\theta_0((z_1z_2)^{(q-1)/2}) & z_1^{(q+1)/2} \notin \mathbb{F}_q^\times \end{cases}$	$\begin{cases} 0 & z_1^{(q+1)/2} \in \mathbb{F}_q^\times \\ 2\theta_0((z_1z_2)^{(q-1)/2}) & z_1^{(q+1)/2} \notin \mathbb{F}_q^\times \end{cases}$

Here, we let $q^* := \epsilon(-1)q \equiv 1 \pmod 4$. The last three representations are cuspidal.

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