Frobenius manifolds and quantum groups

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Abstract

We introduce an isomonodromic Knizhnik–Zamolodchikov connection with respect to the quantum Stokes matrices, and prove that the semiclassical limit of the KZ type connection gives rise to the Dubrovin connections of semisimple Frobenius manifolds. This quantization procedure of Dubrovin connections is parallel to the quantization from Poisson Lie groups to quantum groups, and is conjecturally formulated as a deformation of Givental’s twisted loop group.

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1 Introduction

The concept of Frobenius manifolds was introduced by Dubrovin [10] as a geometrical manifestation of the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations [7, 41] governing deformations of 2D topological field theories. Examples include K. Saito’s Frobenius structures on unfolding spaces of singularities [32], quantum cohomology (see e.g. [10, 31]), the Barannikov-Kontsevich construction from Batalin-Vilkovisky algebras [11] and so on. The theory of Frobenius manifolds was investigated by many authors, and has been one of the principle tools in the study of Gromov-Witten theory, integrable hierarchies, mirror symmetry, quantum singularity theory, see e.g., [6, 12, 16, 25, 30, 35] and the literature cited there.

Several important developments in the direction are illuminated by the second arrow of the following diagram. They connect the theory of semisimple Frobenius manifolds with meromorphic connections, integrable hierarchy, symplectic geometry on loop spaces and Poisson Lie groups, following Dubrovin [10, 11], Dubrovin-Zhang [12], Givental [19, 20] and Boalch [2, 3] respectively. The main idea in the mentioned work of Dubrovin, Givental and Zhang is to construct all the building of a given 2D TFT starting from the corresponding Frobenius manifold, and two formalisms have been proposed: the Dubrovin-Zhang integrable hierarchy formalism and Givental’s quantization formalism. Both formalisms in turn rely on the

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*Keyword: Semisimple Frobenius manifolds, Dubrovin connections, Knizhnik–Zamolodchikov equations, Isomonodromy deformation, Stokes phenomenon*
theory of irregular singularities. From a different starting point, Boalch’s work on irregular singularities and Poisson/symplectic varieties enables him to identify the moduli space of semisimple Frobenius manifolds with Poisson Lie groups.

Thus there arise several natural questions. First, one hopes to deepen the connection among the works mentioned above. For example, to relate the Givental group symmetry to the Poisson group symmetry on the moduli space of semisimple Frobenius manifolds. On the other side, more interestingly, the Poisson Lie groups originate as the semiclassical limit of Drinfeld-Jimbo quantum groups. In other words, the moduli space of semisimple Frobenius manifolds has a natural quantization from the perspective of quantum groups and Poisson geometry (the irregular Atiyah-Bott construction \[3\]). Thus one expects the appearance of quantum groups in the theory of Frobenius manifolds. Given the second viewpoint, our goal is to first find the possible connections, and then apply the theory of the quantum group to study the quantization of the integrable hierarchy associated to semisimple Frobenius manifolds (which should be complement to the dispersion/quadratic Hamiltonian deformation studied in \[12, 19, 20\]). See the comments below.

In this paper, we make the first step towards the goal by quantizing Dubrovin connections via a Knizhnik-Zamolodchikov type connection. We then explore several important aspects of the quantization, including the quantum monodromy data, isomonodromicity and symplectic geometry on loop spaces, as quantum analogs of the theory in \[10, 11, 19, 20, 2\] (as seen in the first arrow of the diagram, where s.c.l standards for semiclassical limit). It is mainly motivated by our joint work with Toledano Laredo \[38\]. The second step, for example the quantization of Dubrovin-Zhang hierarchy, may be given by the interaction between the flat sections of isomonodromic KZ connections with quantum KdV systems, which we will not discuss here but hope to explore in a future work. We now explain our results and the arrows/connections in the following diagram in detail.

Arrow a: Semisimple Frobenius manifolds and Dubrovin connections. In \[10\], associated to a Frobenius manifold \(M\), Dubrovin introduced a connection \(\nabla\) on \(M \times \mathbb{C}\) to encode the Frobenius structure. Many applications of Frobenius manifolds are given via the study of the Dubrovin connection. Under a technical condition of semisimpleness, there exist local canonical coordinates, unique up to permutation, around any semisimple point. If one puts the canonical coordinates system as the diagonal entries of a matrix

\[\begin{array}{c}
\text{Symplectic geometry on } \mathbb{H}[\hbar] \\
h=0
\end{array}\]

\[\begin{array}{c}
\text{Isomonodromic KZ connections} \\
g: \text{Conj 1.4}
\end{array}\]

\[\begin{array}{c}
\text{Quantum groups} \\
e: \text{s.c.l Thm 4.7}
\end{array}\]

\[\begin{array}{c}
\text{Symplectic geometry on loop space } \mathbb{H} \\
f: \text{Flat sections}
\end{array}\]

\[\begin{array}{c}
\text{Dubrovin connections} \\
f: [19, 20]
\end{array}\]

\[\begin{array}{c}
\text{Stokes data} \\
a: [10]
\end{array}\]

\[\begin{array}{c}
\text{Stokes data} \\
b: [2]
\end{array}\]

\[\begin{array}{c}
\text{Poisson Lie groups} \\
e: \text{s.c.l}
\end{array}\]

\[\begin{array}{c}
\text{Semisimple} \\
\text{Frobenius manifolds}
\end{array}\]

1Quantum groups are closely related to quantum integrable systems, Yang-Baxter equations, representation theory, knot invariant and so on. See e.g., \[15\] for an introduction.
where $d_z$ and $d_u$ are the differentials along $C$ and $M$ direction, $V(u)$ is a $n$ by $n$ matrix and $\Lambda(z,u)$ a $n$ by $n$ matrix of 1-forms. See Section 2.3 for more details. Following [10], the connection $\nabla$ is flat and the flatness condition imposes a system of PDEs for $V(u)$. Any solution $V(u)$, in a neighbourhood $C$ of a generic point $u_0$, is determined by the initial data $V_0 = V(u_0)$, which in turn defines a semisimple Frobenius manifold on $C$. Thus the semisimple germ of Frobenius manifolds at $u_0$ is parametrized by the (skew-symmetric) matrices $V_0$ [10] Corollary 3.3, and we denote by $\nabla_{V_0}$ the corresponding Dubrovin connection.

Another parametrization is as follows. For any $u \in C$, the connection $\nabla_z$ has an irregular singularity around $z = \infty$, and thus can be assigned a pair of Stokes matrices $S_{\pm}(u)$ as the "internal" monodromy (see e.g. [39]). Following [10], the system of PDEs for $V(u)$ is then the isomonodromy deformation equation (22) of the meromorphic connection $\nabla_z$ with respect to the irregular data $u$ (or $U$). As a result, the Stokes matrices $S_{\pm}(u)$ don’t depend on $u \in C$, therefore they can be used to characterize the Frobenius manifold. One of the main results (Theorem 3.2) in [10] is the identification of the local moduli of semisimple Frobenius manifolds around $u$ with the entries of a Stokes matrix. This is expressed as the arrows a and b in the diagram.

**Arrow b: Stokes matrices and Poisson Lie groups.** It was then pointed out by Boalch [2] that the space of Stokes matrices, equipped with the natural Poisson structure from the irregular Atiyah-Bott construction (see Boalch [3] and Krichever [23]), is identified with the Poisson Lie group $G^*$ dual to $GL_n(C)$. One consequence is that, when restricted to the connections whose residue $V(u)$ is skew-symmetric, it induces the Dubrovin-Ugaglia Poisson structure on the space of Stokes matrices of semisimple Frobenius manifolds [39]. Another consequence is that Boalch’s dual exponential map (or the Stokes map)

$$\mu : g^* \rightarrow G^*$$

associating the Stokes matrices of (1) to any $V_0 = V(u_0) \in g^* \cong g$, is a local analytic Poisson isomorphism. This gives rise to a Ginzburg-Weinstein type diffeomorphism [21], which is one of the most important topics in Poisson geometry.

**Arrow c. Isomonodromic Knizhnik–Zamolodchikov connections.** Let us now put Frobenius manifolds aside for a while, and introduce our main construction. Let us take a complex semisimple Lie algebra $g$, with a root space decomposition $g = b \oplus_{\alpha \in \Phi} e_\alpha$. Let $U(g)$ be the universal enveloping algebra of $g$. Set $U = U(g)[h]$ and, for any $n \geq 1$, $U^{\otimes n} = U \otimes \cdots \otimes U$, where $\otimes$ is the completed tensor product of $\mathbb{C}[h]$-modules.

For any $u_0 \in \mathfrak{h}_{\text{reg}}$ (the set of regular elements in $\mathfrak{h}$), we introduce a $U^{\otimes 2}$-valued isomonodromic KZ connection on the product of $C$ and a neighbourhood $\tilde{C}$ of $u_0$ in $\mathfrak{h}_{\text{reg}}$ (with a minor imprecision, to be corrected in Section 3).

$$\nabla_{\text{KZ}} = dz - \left( u \otimes 1 + h \frac{\Omega(u)}{z} \right) dz,$$

$$\nabla_{\text{ISO}} = dh - \left( z(d_h u \otimes 1) + h \sum_{\alpha \in \Phi^+} \frac{d\alpha}{\alpha} T^{-1}(u)(e_\alpha \otimes e_{-\alpha} + e_{-\alpha} \otimes e_\alpha)T(u) \right).$$

The Dubrovin connection, its fundamental solutions and isomonodromicity are defined for any $V_0 \in g_{\text{reg}}$. Throughout this paper, we will allow any $V_0$. To relate to the theory of Frobenius manifold, we can then restrict $V_0$ to the subspace of skew-symmetric ones.
We postpone the study of these questions to our future work.

We refer the reader to Section 3 for the definition of the functions Ω(u) and T(u) ∈ Ω⊗2, and the flatness of the connection.

Let Λ′ = U(h[1,2]) ⊂ Λ be the Drinfeld dual Hopf algebra [8]. It is known that Λ′ is a flat deformation of the completed symmetric algebra S = \prod_{n \geq 0} S^n g, i.e., Λ′/hΛ′ = S(g). See e.g. [2, 13]. Given any A ∈ Λ⊗Λ′, we define the semiclassical limit of \( A \) as its image of in

\[ \Omega(\Lambda) / h(\Lambda \otimes \Lambda') = U(g) \otimes S(g) \]

(regarded as a formal function on \( g^* \) with values in \( U(g) \)).

In the case when \( g = gl_n \), and \( h \) is the set of diagonal matrices, the following theorem relates the semiclassical limit of the isomonodromic KZ connection to the Dubrovin connections. See Section 4 for the more precise statement.

**Theorem 1.1.** Let \( F_h(z, u) \) be a canonical solution of the system \( \nabla_{IKZ} F_h = 0 \) and \( \nabla_{ISO} F_h = 0 \) with values in \( \Lambda \otimes \Lambda' \). Then the semiclassical limit \( F(z, u) = \text{scl}(F_h(z, u)) \) ∈ \( U(g) \otimes S(g) \) is such that for any \( V_0 \in g^* \), \( F(z, u; V_0) \) is a canonical fundamental solution of the equation \( \nabla_{IKZ} F = 0 \). (Recall that \( \nabla_{IKZ} \) is the Dubrovin connection of the Frobenius manifold parametrized by \( V_0 \).)

Motivated by the theorem, in Section 4 we propose a notion of quantization of semisimple Frobenius manifolds as a natural lifting problem from a "commutative world" to a "non-commutative world". There are two questions about the quantization remained to be understood:

- in Corollary 4.5 we prove that the semiclassical limit of the function \( \Omega(u) \) in isomonodromic KZ connection is the function \( V(u) \) in Dubrovin connection. Thus geometrically \( \Omega(u) \) may produce an \( h \)-deformation of the Frobenius structure constructed by \( V(u) \). It may also correspond to certain quantization of the Darboux–Egoroff system;

- there are two approaches to the study of semisimple Frobenius manifolds, i.e., the Darboux–Egoroff picture and the WDVV picture. Although in this paper we take the first approach, it is very interesting to consider the WDVV picture of the isomonodromic KZ connection. In particular, it will produce an \( h \)-deformation of the Hamiltonians of the dispersionless limits of Dubrovin-Zhang integrable hierarchies [10] Lecture 6). We expect that it will coincide with the \( h \) modification of the Hamiltonians in the quantization of the integrable hierarchies.

We postpone the study of these questions to our future work.

**Arrow d. Stokes data, isomonodromicity and quantum R-matrices.** Now for any \( u \in C \), we concentrate on the \( z \)-dependence of the solution of the differential equation \( \nabla_{IKZ} F_h = 0 \). This equation has an irregular singularity at \( z = \infty \), associated to which there is a pair of quantum Stokes matrices \( S_{h,\pm}(u) \) (see Section 3). Then we have

**Theorem 1.2.** The quantum Stokes matrices \( S_{h,\pm}(u) \) of the connection \( \nabla_{IKZ} \) don’t depend on \( u \).

Therefore, the defining equation (11) in Section 3 of \( \Omega(u) \) describes an isomonodromic deformation with respect to the Stokes data. It is equivalent to the fact that the equation (11) is the integrability condition for the isomonodromic KZ connection. See the discussion in Section 3.3. It is similar to the classical case: the integrability condition of the Dubrovin connection imposes a system of PDEs’ for \( V(u) \) which describes the isomonodromic deformation of \( \nabla_z \).

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\[ ^3 \text{In the literature of quantum groups, this semiclassical limit has been used by Enriquez, Etingof among others to study the } h \text{-adic property of universal } R \text{-matrices, admissible quasi-Hopf algebras and Drinfeld twists, quantization of Alekseev-Meinrenken } r \text{-matrices, Ginzburg-Weinstein linearization, vertex-IRF gauge transformations and so on. See e.g., } [12, 14] \text{ and the the references cited there.} \]
When \( u = u_0 \), the residue \( \Omega(u_0) \) equals to the Casimir element \( \Omega \), and the connection \( \nabla_{IKZ} \) coincides with the dynamical KZ connection \( \nabla_{DKZ} \), which was introduced by Felder, Markov, Tarasov and Varchenko \([17]\) in the study of bispectral problems. The \( U(g) \)-valued Stokes phenomenon of \( \nabla_{DKZ} \) was used by Toledano Laredo \([37]\) to give a canonical transcendental construction of the Drinfeld–Jimbo quantum group \( U_h(g) \). In \([38]\), the authors prove that the quantum Stokes matrices \( S_{h\pm}(u) \) of \( \nabla_{DKZ} \) give the \( R \)-matrices of \( U_h(g) \).

It then follows from the isomonodromicity (the above theorem) that the quantum Stokes matrices of \( \nabla_{IKZ} \) also give \( R \)-matrices. In particular, they satisfy the Yang-Baxter equations. See Proposition \(3.9\)

**Corollary 1.3.** The elements \( R_+ = e^{\pi i h_0} S_{h-1}^{-1}(u) \) and \( R_- = e^{\pi i h_0} S_{h+1}^{-1}(u) \) satisfy the Yang-Baxter equation

\[
R_{12} R_{13} R_{23}^2 = R_{23} R_{13} R_{12}^2 \in \mathfrak{u}^\otimes 3.
\]

**Arrow e:** from quantum Stokes data (quantum groups and \( R \)-matrices) to Stokes data (Poisson Lie groups). By Drinfeld’s duality principle \([8, 18]\), there exists a functor \( (\cdot)' : QUEA \to QFSHA \) from the category of quantized universal enveloping algebra (QUEA) to the category of quantized formal series Hopf algebras (QFSHA). For example, we have seen that the dual Hopf algebra \( \mathfrak{u}' \) of \( \mathfrak{u} \) is a flat deformation of \( \mathcal{O}_G \). If we denote by \( U_h \) the quantum group arising from the monodromy data of \( \nabla_{DKZ} \) (see \([37]\)), then its dual Hopf algebra \( U_h' \) is a flat deformation of the algebra of regular functions on the formal group \( \mathcal{O}_G^* \).

Using the results of Enriquez-Etingof-Marshall \([13]\) and Enriquez-Halbout \([14]\), the quantum Stokes matrices (\( R \)-matrices) can be used to construct an algebra isomorphism \( S_h : \mathcal{O}_h' \to \mathcal{O}' \). It in turn induces, by taking classical limit, a commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}_G^* & \xrightarrow{S} & \mathcal{O}_G
\\
\downarrow{s.c.l} & & \downarrow{s.c.l}
\\
\mathfrak{u}'_h & \xrightarrow{S_h} & \mathfrak{u}'
\end{array}
\]

Here, by \([38]\), the induced formal map \( S : \mathcal{O}_G^* \to \mathcal{O}_G^* \) coincides with the Stokes map (Boalch’s dual exponential map) \( \mu : g^* \to G^* \). Thus the remarkable Poisson geometric nature of \( \mu \) found by Boalch \([2]\) follows immediately: \( S \) is a Poisson algebra isomorphism because \( S_h \) is an associative algebra isomorphism.

The above discussion verifies that the quantization in arrow \( e \) commutes with taking Stokes data, i.e.,

\[
\text{the arrows composition } b \circ c = e \circ d,
\]

and the Poisson/symplectic geometric nature of the Stokes map is understood from the perspective of quantum groups. In other words, the quantization of (Dubrovin connections of) the semisimple Frobenius manifolds is parallel to the quantization from Poisson Lie groups to quantum groups, which are the Stokes data of the Dubrovin connections and isomonodromic KZ connections respectively.

Furthermore, the braid group action on the space of Stokes matrices \([10]\) Appendix F) should be the semiclassical limit of the braid group action on the quantum Stokes matrices, arising from the monodromy of the isomonodromy deformations of \( \nabla_{DKZ} \) around the hyperplanes \( u^i = u^j \) for \( i \neq j \) in \( h \). The latter is closely related to the Kohno-Drinfeld theorem for quantum Weyl groups developed by Toledano Laredo \([26]\), and would explain the result of Boalch \([4]\) that the braid group action on the space of Stokes matrices is the classical limit of the action on the corresponding quantum group, due to Lusztig \([28]\) and independently Kirillov-Reshetikhin \([24]\) and Soibelman \([33]\).

**Arrow f. Symplectic geometry on loop spaces.** Let \( h \) be an \( n \) dimensional vector space equipped with a nondegenerate symmetric bilinear form. Let \( H \) denote the vector space \( h((z^{-1})) \) consisting of Laurent
polynomials with coefficients in $h$, equipped with a symplectic form given by residue $\omega(f(z), g(z)) := \text{Res}_{z=0}(f(-z)g(z))$. Then Coates–Givental [8] and Givental [20] give a reformation of Frobenius manifolds of the following observation due to Givental: the canonical fundamental solutions of the Witten–Kontsevich tau-function (up to a “central charge”). The operator

$$L^{(2)} GL(h) := \{ H \in \mathbb{H} \mid H(-z)^* H(z) = 1 \}$$

on the set of Frobenius manifolds of rank $n$ (the asterisk here means transposition with respect to the bilinear form on $h$, and thus the elements in the group are symplectic transformations on $\mathbb{H}$). Under the semi-simplicity assumption, the action is transitive. In particular, for any semi-simple Frobenius manifold, the flat sections of the Dubrovin’s connection are in the twisted loop group and provide a transformation $T$ from the associated Lagrangian cone $\mathcal{L}$ to the Cartesian product of $n$ copies of the cone $\mathcal{L}_{pt}$ corresponding to the genus 0 Gromov–Witten theory of a point. See [20] Theorem 5.

More precisely, let $\{u^i\}$ be the dual coordinates of an orthonormal basis of $h$. If a semi-simple Frobenius manifold at $u_0$ is parametrized by $V_0$, then canonical fundamental solution $F_{V_0}$ of the equation $\nabla_{V_0} F = 0$ at $z = \infty$ has an asymptotic expansion $F_{V_0} = H_{V_0}(z, u)e^{iU}$, where $H_{V_0}(z, u) = \text{Id} + O(\frac{1}{z})$ is a formal matrix power series and satisfies $H_{V_0}(-z, u)^* H_{V_0}(z, u) = \text{Id}$. Thus $H_{V_0}(z, u)$ and $e^{iU}$ are elements of $L^{(2)} GL(h)$. Similarly, the fundamental solution of $\nabla_{V_0} F = 0$ at $z = 0$ gives rise to another element in the twisted loop group. Roughly speaking, the composition of these transformations produces the transformation $T(V_0)$ relating the Lagrangian cones $\mathcal{L}_{V_0}$ and $\mathcal{L}_{pt} \times \cdots \times \mathcal{L}_{pt}$.

In a series of works, Givental [19, 20] defines a total descendant potential for the semi-simple Frobenius manifold parametrized by $V_0$ via the action of a differential operator $T(V_0)$ on a product of the Witten–Kontsevich tau-function (up to a “central charge”). The operator $T(V_0)$ is the Weyl quantization of $T(V_0)$, thought of as a symplectic transformation on $\mathbb{H}$. It gives a reconstruction approach towards a Gromov–Witten type theory, conjectured by Givental and proved by Teleman [35]. From the viewpoint of Dubrovin-Zhang’s bi-Hamiltonian hierarchies of hydrodynamic type, this can be seen as a reconstruction of a dispersive hierarchy from its dispersionless limit.

**Arrow g. Deformation of symplectic transformations.** Just to give some feeling of how the quantization in this paper is possibly related to the quantization of Dubrovin-Zhang integrable hierarchy and Givental’s twist loop group, we formulate here a naive conjecture (some hints indicate finally the connection should be given via vertex algebras. We will explore this in the future). The idea is to understand the “quantum” analog of the following observation due to Givental: the canonical fundamental solutions of the equation $\nabla_{V_0} F = 0$ can be understood as symplectic transformation on the symplectic vector space $\mathbb{H}$.

Let us consider the group $L^{(2)} GL(h) := \{ H_h \in \text{End}(h)((z^{-1}))[[h]] \mid H_h(-z)^* H_h(z) = 1 \}$. Its element $H_h = H(z) + hH_1(z) + o(h)$ is a symplectic transformation on $\mathbb{H}[[h]]$ (the symplectic vector space over $\mathbb{C}[[h]]$), and can be seen as a deformation of the symplectic transformation on $\mathbb{H}$ given by the leading term $H$.

Set $g = g(h)$ and identify $g$ with $g^*$ via the canonical pairing. The holomorphic part $H_h(z, u)$ of the canonical solution $F_h(z, u)$ of $\nabla_{1kZ} F_h = 0$, as a formal function on $g^*$ valued in $U(g)[[h]]$, has a decomposition

$$H(z, u) + hH_1(z, u) + \cdots,$$

where each $H_i(z, u) \in U(g)\hat{\otimes}S(g)$ is a formal function on $g^*$ valued in $U(g)$. Theorem 1.2 states that the classical term $H(z, u)$ is (the formal Taylor series at $V_0 = 0$ of) the map

$$g^* \rightarrow \text{End}(\mathbb{H}); \ V_0 \mapsto H_{V_0}(z, u)$$

---

4The composition requires different power series completions of the space $\mathbb{H}$. See [19].

5S. Li, Private communication, May 2017.
associating the holomorphic part $H_V(z,u)$ of the canonical solution of the Dubrovin equation $\nabla_V F = 0$ to $V_0$. After composing (the asymptotics expansion in $z^{-1}$ of) $H_V(z,u) : g^* \to U(g)[z^{-1},h]$ with the natural representation of $g = gl(h)$ on $h$, we get a function $H_\hbar(u,z) : g^* \to \text{End}(h)[\hbar]$.

**Conjecture 1.4.** For any $u \in C$, the function $H_\hbar(z,u) = \tilde{H}(z,u) + O(\hbar)$ is valued in the group $L(2)GL(h)\hbar$.

Thus we have seen two deformations of the fundamental solutions of the equation $\nabla_V F = 0$ associated to the Frobenius manifold parametrized by $V_0$: one is the $\hbar$-deformation via the isomonodromic KZ connection; another is the $\varepsilon$-deformation via Givental’s quadratic Hamiltonian quantization. The conjecture enables us to combine these two approaches to obtain a quantization/deformation with two parameters. In terms of integrable hierarchies, the two parameters $\varepsilon$ and $\hbar$ may correspond respectively to the dispersion and quantization parameters. It may be related to the following prediction of Li [26, 27]. He observed that the topological string theory on the product of Calabi-Yau manifolds $X \times \mathbb{C}$ suggests that a dispersion deformation (with $\varepsilon$ as the dispersion parameter) and a further quantization of Frobenius manifolds should exist in general. Here the dispersion deformation, constructed by Dubrovin-Zhang [12] for semisimple Frobenius manifolds, is equivalent to Givental’s quantization formalism.

The organization of the paper is as follows. The next section gives the preliminaries of Frobenius manifolds including some basic notions, the Dubrovin connections and moduli spaces. Section 3 introduces the isomonodromic KZ connection, its canonical solutions and quantum Stokes data, and then proves the isomonodromicity with respect to the Stokes data. Section 4 proves that the semiclassical limit of the isomonodromic KZ connection is Dubrovin connections. Section 5 revisits the quantization in Section 4, and proposes a notion of quantization of semisimple Frobenius manifolds.

## 2 Preliminaries on Frobenius manifolds

### 2.1 Frobenius manifolds

Let $M = (M, O_M)$ be a complex manifold of dimension $n$. We denote by $T_M$ its holomorphic tangent sheaf.

**Definition 2.1 (Dubrovin).** A Frobenius manifold structure on $M$ is a tuple $(g, \circ, e, E)$, where $g$ is a non-degenerate $O_M$-symmetric bilinear form, called metric, $\circ$ is $O_M$-bilinear product on $T_M$, defining an associative and commutative algebra structure with the unit $e$, and $E$ is a holomorphic vector field on $M$, called the Euler vector field, which satisfy:

- $g(X \circ Y, Z) = g(X, Y \circ Z), \forall X, Y, Z \in T_M$;
- The Levi-Civita connection, denoted by $\nabla$, with respect to $g$ is flat;
- The tensor $C : T_M \to \text{End}_{O_M}(T_M)$ defined by $C_X Y = X \circ Y$ is flat;
- the unit element $e$ is flat.
- $\mathcal{L}_E g = Dg$ for some constant $D$ and $\mathcal{L}_E(\circ) = \circ$.

In particular, we have a structure of Frobenius algebra $(\circ, g, e)$ on the tangent spaces $T_m M$ depending analytically on the point $m$. This notion was introduced by Dubrovin as a geometrical/coordinate-free manifestation of the WDVV equations, see [10, 11, 31] for more details, examples and the relations with 2D topological field theories. It is also known as conformal Frobenius manifolds.
2.2 Dubrovin connections of Frobenius manifolds

Given a Frobenius manifold \((M, \circ, g, e, E)\), let us take the sheaf \(\operatorname{Pr}_M^*(TM)\) on \(M \times (\mathbb{P}^1 \setminus \{0, \infty\})\), where \(\operatorname{Pr} : M \times (\mathbb{P}^1 \setminus \{0, \infty\}) \to M\) is the projection. The following construction and proposition are known and can be found in different versions in [10, 31].

**Definition 2.2.** The Dubrovin connection \(\nabla\) on \(\operatorname{Pr}_M^*(TM)\) is defined for any local vector field \(X \in \Gamma(TM)\) and \(Y \in \Gamma(\operatorname{Pr}_M^*(TM))\) by

\[
\nabla_X Y = \hat{\nabla}_X Y + zX \circ Y, \quad (5)
\]

\[
\nabla_z Y = \frac{dY}{dz} + \frac{1}{z}(\hat{\nabla}_Y E - \frac{D}{2}Y) + E \circ Y. \quad (6)
\]

**Proposition 2.3.** The connection \(\nabla\) is flat.

**Proof.** It follows from the definition of Frobenius manifolds. For example, the flatness of the pencil of connections \(\nabla_X(z)(Y) = \hat{\nabla}_X Y + zX \circ Y\) for any \(X, Y \in \Gamma(TM)\) (viewed as parametrized by \(z\)) is equivalent to that the \((M, \circ, g)\) is associative and potential. ■

The connection \(\nabla\) is also known as the first structure connection in [31]. Many applications of Frobenius manifolds are given via the study of the corresponding Dubrovin connections.

2.3 Semisimple Frobenius manifolds

**Definition 2.4.** [10] A Frobenius manifold \((M, \circ, g, e, E)\) is called semisimple if for a generic point \(m \in M\), the algebra \((T_m M, \circ, e)\) is semisimple, i.e., isomorphic, as a \(\mathbb{C}\)-algebra, to \(\mathbb{C}^n\) with component-wise multiplication.

The books [10, 31] contain a complete review of these structures. Here we rewrite, without a proof, the Dubrovin connections of semisimple Frobenius manifolds in terms of the canonical coordinates.

**Proposition 2.5.** [10] In a neighborhood \(C\) of a semisimple point \(u_0\) of a Frobenius manifold \(M\), there exist coordinates \(u^1, \ldots, u^n\) such that

- \(\partial_i \circ \partial_j = \delta_{ij} \partial_i\), where \(\partial_i := \frac{\partial}{\partial u^i}\);
- the eigenvalues of \(E\circ\) at each point \(m \in C\) are \((u^1(m), \ldots, u^n(m))\).

They are unique up to reordering and are called canonical coordinates. Furthermore,

- the metric \(g\) is diagonal in the canonical coordinates, that is \(g(u) = \sum_i h_i(u) du^i\), for some nonzero functions \(h_1(u), \ldots, h_n(u)\);
- the unity vector field \(e\) in the canonical coordinates has the form \(e = \sum_i \partial_i\).

Then in the coordinates \((z, u^1, \ldots, u^n)\) and in the frame of normalized idempotents \(\{ \frac{1}{\sqrt{h_i}}, \frac{\partial}{\partial u^i} \}\), the Dubrovin connection of \(M\) on \(\mathbb{C} \times \mathbb{C}\) can be written as follows: set \(U = \text{diag}(u^1, \ldots, u^n)\) and put the \(n \times n\)-matrix \(V(u) := [(r_{ij}(u)), U]\), where \((r_{ij})\) is a \(n \times n\)-matrix with entries (the rotation coefficients for the canonical coordinates) \(r_{ij} := \frac{\partial_i \sqrt{h_j}}{\sqrt{h_i}}\), \(i \neq j\), then \(V(u)\) is skew-symmetric, and for the horizontal sections we arrive at
Proposition 2.6 (Lemma 3.2). The Dubrovin connection in Definition 2.2 takes the form

\[
\begin{align*}
\frac{dz}{F} &= \left( U + \frac{V(u)}{z} \right) Fdz, \quad (7) \\
\frac{dh}{F} &= (zd\Lambda + \Lambda(u)) F, \quad (8)
\end{align*}
\]

where \( \Lambda(u) \) is a matrix of 1-forms given by \( \Lambda(u) := \sum_{i=1}^{n} V_i du^i \). Here \( V_i := \text{ad}_{E_{ii}} \text{ad}_{U^{-1}} V(u) \) for \( E_{ii} \) being the elementary matrix \( (E_{ii})_{ab} = \delta_{ia} \delta_{ib} \).

2.4 Canonical solutions and Stokes matrices

For the moment, we will fix \( u \) and concentrate on the differential equation taking the form

\[
\frac{dF}{dz} = \left( U + \frac{V(z)}{z} \right) F. \quad (9)
\]

Here \( U = \text{diag}(u^1, \ldots, u^n) \) with distinct diagonal elements (the coordinates of \( u \)), and \( V \) is an arbitrary \( n \) by \( n \) matrix. This equation has a regular singularity at \( z = 0 \) and an irregular one (pole of order 2) at \( z = \infty \).

In this paper, we are only concerned with the fundamental solutions and the Stokes/monodromy data at \( \infty \). For the discussions about the fundamental solutions at \( 0 \) and a complete set of the monodromy data of the equation (including also connections matrices, monodromy around \( z = 0 \)), we refer the reader to [10].

The Stokes rays (also known as anti-Stokes directions) of the equation in the complex \( z \)-plane are the rays \( \mathbb{R}_{>0} \cdot (u_i - u_j) \subset \mathbb{C}^* \) for any \( i \neq j \). The Stokes sectors are the open regions of \( \mathbb{C}^* \) bounded by two consecutive Stokes rays.

Let us choose an initial Stokes sector \( \mathbb{H}_+ \) with boundary rays \( l, l' \) (listed in counterclockwise order), and denote by \( \mathbb{H}_- \) its opposite sector. We choose the determination of \( \log z \) with a cut along \( l \). The following result is well–known. See, e.g., [40], pp. 58–61.

**Theorem 2.7.** On each sector \( \mathbb{H}_\pm \), there is a unique holomorphic function \( H_\pm : \mathbb{H}_\pm \to \text{GL}_n \) such that the function

\[
F_\pm = H_\pm e^{z[V]} e^{U}
\]

satisfies \( \nabla_z F_\pm = 0 \), and \( H_\pm \) can be analytically continued to \( \hat{\mathbb{H}}_\pm \) and then is asymptotic to 1 within \( \hat{\mathbb{H}}_\pm \). Here \( [V] \) denotes the diagonal part of \( V \), and the supersectors \( \hat{\mathbb{H}}_\pm := \{ re^{i\phi} | r \in H_\pm, \phi \in (-\pi/2, \pi/2) \} \).

We will call \( F_\pm \) the canonical solutions (with respect to the Stokes sectors \( \mathbb{H}_\pm \)).

**Definition 2.8.** The Stokes matrices of the equation (9) (with respect to the sectors \( \mathbb{H}_\pm \)) are the matrices \( S_\pm \) determined by

\[
F_- = F_+ S_+ e^{2\pi i[V]}, \quad F_+ = F_- S_-
\]

where the first (resp. second) identity is understood to hold in \( \mathbb{H}_- \) (resp. \( \mathbb{H}_+ \)) after \( F_+ \) (resp. \( F_- \)) has been analytically continued counterclockwise.

**Remark 2.9.** The above theorem and definition were extended to the case of complex reductive groups in [4] Theorem 2.5, and to arbitrary algebraic groups (see [5] Theorem 2.6). We will need these two extensions in the proof of Propositions 4.4 and 4.6 respectively.
2.5 Isomonodromy deformations and Stokes data of semisimple Frobenius manifolds

Now let us specify the results from the last section to the Dubrovin connection, i.e., the systems (7) and (8) (recall that the matrix $V(u)$ in (7) is skew-symmetric.)

**Proposition 2.10** ([10]). In a neighborhood $C$ of a semisimple point $u_0$, the canonical solutions $F_\pm(z,u) = H_\pm(z) e^{\pm U}$ of the first equation (7) also satisfy the second equation (8). Furthermore, the matrix functions satisfy

$$H_+^T(u,-z) H_+(z,u) = 1.$$ 

Here $T$ denotes the matrix transposition.

**Proof.** For the first part, see e.g., [10] the proof of Proposition 3.11. The identity $H_+^T(u,-z) H_+(z,u) = 1$ is due to the fact that the matrix $V(u)$ is skew-symmetric. See e.g. [10] Proposition 3.10. and [2] Lemma 35.

The compatibility of the systems (7) and (8), or the flatness of the connection $\nabla$, imposes that

**Proposition 2.11** ([10] Proposition 3.7). For a semisimple Frobenius manifold, the function $V(u)$ in (7) satisfies the system of equations

$$\partial_k V(u) = [V(u), \text{ad}_{E_{kk}\text{ad}_U^{-1} V(u)}], \quad k = 1, \ldots, n \quad (10)$$

where $E_{kk}$ is the elementary matrix $(E_{kk})_{ij} = \delta_{ik}\delta_{kj}$.

In Section 4 we will write it as a time-dependent Hamiltonian system. Following Jimbo-Miwa-Ueno [22], this system of PDEs describes the isomonodromy deformation of the meromorphic differential equation (7), which means that

**Proposition 2.12** ([10] Proposition 3.11). Along the solution leaf $V(u)$ of (10), the Stokes matrices $S_\pm(u)$ of (7) are preserved (independent of $u$).

As a consequence, the Stokes matrices of a semisimple Frobenius manifold is locally constant.

2.6 Moduli space of semisimple Frobenius manifolds

2.6.1 Initial data of isomonodromy equations

The inverse of Proposition 2.11 is also true. That is a solution $V(u)$ of the system (10) determines locally a semisimple Frobenius manifold. See [10] Proposition 3.5 for the explicit construction for diagonalizable $V(u)$. Thus we have

**Theorem 2.13** ([10], Corollary 3.3). There exists a one-to-one correspondence between semisimple Frobenius manifolds (modulo certain transformations) and solutions of the system (10).

The solution $V(u)$ of (10) on a neighbourhood $C$ of $u_0$, and therefore the corresponding semisimple Frobenius structure, is determined by the initial data:

a skew-symmetric matrix $V_0(= V(u_0))$.

Thus the matrices $V_0$ parametrize the semisimple germs at $u_0$. 
2.6.2 Dubrovin monodromy data

As noted by Dubrovin, one has no “natural” point \( u_0 \) in the Frobenius manifold to specify the initial data. However, due to the isomonodromicity (Proposition 2.12, see also [11] Lecture 4), one can use Dubrovin monodromy data, including the Stokes matrices, connection matrices and monodromy at \( z = 0 \), to parametrize the Frobenius structure. The reconstruction of the Frobenius manifold from Dubrovin monodromy data is given by solving certain irregular Riemann-Hilbert problem.

We refer the reader to [31] for another description of moduli spaces: Manin’s classification data via the second structure connections.

2.6.3 Irregular Riemann-Hilbert correspondence and Poisson geometry

The above two descriptions are related by the Stokes map \( \mu \), which associates the Stokes matrices of the system \( \nabla V_0 F = 0 \) to \( V_0 \). Following Boalch, its remarkable relation with Poisson geometry is as follows. First, the space of matrices \( V_0 \) is equipped with Kirillov-Kostant-Souriau Poisson structure, and the space of Stokes matrices is also a Poisson variety by irregular Atiyah-Bott construction [4]. Then in [2], the latter is further identified with the dual Poisson Lie group, and the Stokes map is proven to be a local analytic Poisson isomorphism.

3 Isomonodromic KZ connections

In this section, we study the isomonodromy KZ connections. In Section 3.1, we recall the preliminaries on filtered algebras. In Section 3.2 and 3.3, we introduce the isomonodromy KZ connection. Then in Section 3.4–3.5, we study respectively its canonical solutions, quantum Stokes matrices and isomonodromy property.

3.1 Filtered algebras

In this section, we follow the notations in [38] closely. Let \( g \) be a simple complex Lie algebra and \( U(g) \) the corresponding universal enveloping algebra. Set \( A = U(g)^{\otimes n} \), a \( C \)–algebra endowed with the standard order filtration given by \( \deg(x^{(i)}) = 1 \) for \( x \in g \), where

\[
x^{(i)} = 1^{\otimes (i-1)} \otimes x \otimes 1^{\otimes (n-i)}.
\]

Choose a sequence \( o = \{ o_k \}_{k \in \mathbb{N}} \) of subadditive non–negative integers (i.e., \( o_k + o_l \leq o_{k+l} \) for any \( k, l \in \mathbb{N} \)), and such that \( o_1 \geq 2 \). Define

\[
A[\hbar]_o = \left\{ \sum_{k \geq 0} a_k \hbar^k \in A[\hbar] \mid a_k \in A_{o_k} \right\}
\]

Note that \( A[\hbar]_o \) is a (closed) \( C[\hbar] \)–submodule (resp. subalgebra) of \( A[\hbar] \) because \( o \) is increasing (resp. subadditive). Endow \( A[\hbar]_o \) with the decreasing filtration \( A[\hbar]_n = \hbar^n A[\hbar] \cap A[\hbar]_n \), with respect to which it is easily seen to be separated and complete. Note that each \( A[\hbar]_n \) is a two–sided, \( C[\hbar] \)–ideal in \( A[\hbar]_o \), and that the quotients

\[
A[\hbar]_n / A[\hbar]_m \cong A_{o_m} \oplus \hbar A_{o_1} \oplus \cdots \oplus \hbar^{n-1} A_{o_{n-1}}
\]

\( A[\hbar]_0 \) is then the \( \hbar \)–adic completion of the Rees construction of \( A \) corresponding to the filtration \( A_{o_0} \subset A_{o_1} \subset \cdots \)

\( A[\hbar]_n \) is separated (resp. complete) with respect to the filtration if the natural map \( V \to \lim V/V_n \) is injective (resp. surjective).
are finite dimensional if $A$ is filtered by finite dimensional subspaces.

We say that a map $F : X \to A[\hbar]^n$, where $X$ is a topological space (resp. a smooth or complex manifold) is continuous (resp. smooth or holomorphic) if its truncations $F_n : X \to A[\hbar]^n/A[\hbar]_n^n$ are. Since $A[\hbar]^n$ is separated and complete, giving such an $F$ amounts to giving continuous (resp. smooth or holomorphic) maps $F_n : X \to V_n$, such that $F_n = F_m \mod V_m / V_n$, for any $n \geq m$.

### 3.2 Isomonodromic Casimir element

Let $\mathfrak{g} = \hbar \oplus_{\alpha \in \Phi} \mathbb{C} e_\alpha$ be a root space decomposition, where the root vectors $e_\alpha$ are normalized so that $(e_\alpha, e_{-\alpha}) = 1$. Set $\mathfrak{U} = U(\mathfrak{g})[\hbar]$ and, for any $n \geq 1$, $\mathfrak{U}^{\otimes n} = \mathfrak{U} \otimes \cdots \otimes \mathfrak{U} = U(\mathfrak{g})^{\otimes n}[\hbar]$, where $\otimes$ is the completed tensor product of $\mathbb{C}[\hbar]$-modules. In what follows, we endow $\mathcal{A} = U(\mathfrak{g})^{\otimes 2}[\hbar] \subset \mathfrak{U}^{\otimes 2}$ with the filtration $\mathcal{A}_n = h^n U(\mathfrak{g})^{\otimes 2}[\hbar] \cap U(\mathfrak{g})^{\otimes 2}[\hbar]^n$, as described in Section 3.1.

Let us consider the equation

$$d_\hbar T(u) = \frac{\hbar}{2} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} (K_\alpha^{(1)} + K_\alpha^{(2)}) \cdot T(u)$$

(11)

where $K_\alpha := e_\alpha e_{-\alpha} + e_{-\alpha} e_\alpha \in \mathfrak{U}$ for any positive root $\alpha$, and $K_\alpha^{(1)} := \mathcal{K}_\alpha \otimes 1 \in \mathfrak{U}^{\otimes 2}$, $K_\alpha^{(2)} := 1 \otimes \mathcal{K}_\alpha \in \mathfrak{U}^{\otimes 2}$. Here given an element $a \in \mathfrak{U}^{\otimes 2}$, we abusively denote by $a$ the corresponding left multiplication operator $L(a)$.

Let $T(u) \in U(\mathfrak{g})^{\otimes 2}[\hbar]^n$ be the holomorphic solution of (11) defined on a neighbourhood $\mathcal{C}$ of $u_0$ with initial condition $T(u_0) = 1 \otimes 1$ (see Section 3.3 for an expression of $T(u)$). Let $\Omega$ be the Casimir element which has the form $\Omega := \sum e_\alpha \otimes e_\alpha$ for any orthogonal basis $\{e_\alpha\}$ of $\mathfrak{g}$.

**Definition 3.1.** We call $\Omega(u) := T(u)^{-1} \Omega T(u) \in \mathfrak{U}^{\otimes 2}$ the isomonodromic Casimir.

### 3.3 Isomonodromic KZ connections

Let $\mathcal{E} \subseteq \text{End}_\mathbb{C}(\mathcal{A})$ be the subalgebra defined by

$$\mathcal{E} = \{T \in \text{End}_\mathbb{C}(\mathcal{A}) | T(A_n) \subseteq A_n\},$$

and consider the filtration $\mathcal{E} = \mathcal{E}_0 \supseteq \mathcal{E}_1 \supseteq \cdots$ where $\mathcal{E}_n$ is the two-sided ideal of $\mathcal{E}$ given by $\mathcal{E}_n = \{T \in \mathcal{E} | \text{Im}(T) \subseteq V_n\}$. One checks $\mathcal{E}$ is separated and complete. Since the quotients $\mathcal{A}/\mathcal{A}_n$ are finite-dimensional, we may speak of holomorphic functions with values in $\mathcal{A}$ and $\mathcal{E}$, as explained in Section 3.1.

Given $u_0 \in \mathcal{C}$, and denote by $\Omega(u)$ the associated isomonodromic Casimir, we introduce

**Definition 3.2.** The isomonodromic KZ connection is the $\mathcal{E}$-valued connection on $\mathcal{C} \times \mathbb{C}$ given by

$$\nabla_{\text{IKZ}} = d_z - \left( \text{ad} u^{(1)} + \frac{h}{z} \frac{\Omega(u)}{z} \right) d\bar{z},$$

(12)

$$\nabla_{\text{ISO}} = d_\hbar - \left( z \text{ad}(d_\hbar u^{(1)}) + \hbar \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} T^{-1}(u)(e_\alpha \otimes e_{-\alpha} + e_{-\alpha} \otimes e_\alpha) T(u) \right).$$

(13)

### 3.4 Canonical solutions and quantum Stokes matrices

For the moment we will concentrate on the equation $\nabla_{\text{IKZ}} F_h = 0$, and construct its unique (therefore canonical) solution with prescribed asymptotics in a Stokes sector. These solutions are obtained via the canonical solutions of the dynamical KZ equation.
3.4.1 Canonical solutions of dynamical KZ equations

The dynamical KZ (DKZ) connection [17] with a parameter \( u \in \mathfrak{h} \) is the \( \Omega^{(2)} \)-valued connection on \( \mathbb{C} \) given by

\[
\nabla_{\text{DKZ}} = dz - (\text{ad} u^{(1)} \cdot h \frac{\Omega}{z}) dz,
\]

(14)

We first recall the canonical solutions of \( \nabla_{\text{DKZ}} F_h = 0 \) with prescribed asymptotics at \( \infty \). This equation has a pole of order two at \( \infty \), thus involves the Stokes phenomenon. We will assume the irregular data \( u \in \mathfrak{h}^R_{\text{reg}} \), in which case the Stokes sectors are the half planes \( \mathbb{H}_+ = \{ z \in \mathbb{C} | \Im(z) \geq 0 \} \). The following result is due to Toledano Laredo.

**Theorem 3.3.** [37]

1. For any \( u \in \mathfrak{h}^R_{\text{reg}} \), there are unique holomorphic functions \( H^o_{h \pm} : \mathbb{H}_+ \to \mathcal{A} \) such that \( H^o_{h \pm} (z, u) \) tends to 1 as \( z \to 0 \) in any sector of the form \( |\arg(z)| \in (\delta, \pi - \delta) \), \( \delta > 0 \) and, for any determination of \( \log z \), the \( \mathcal{E} \)-valued function

\[
F^o_{h \pm} (z, u) = F^o_{h \pm} (z, u) \cdot z^{\Omega_0} \cdot e^{z(\text{ad} u^{(1)})}
\]

satisfies \( \nabla_{\text{DKZ}} F^o_{h \pm} = 0 \). Here \( \Omega_0 := \sum_i h_i \otimes h_i \) for any orthogonal basis \( \{ h_i \} \) of \( \mathfrak{h} \).

2. \( H^o_{h \pm} \) and \( F^o_{h \pm} \) are smooth functions of \( u \), and \( F^o_{h \pm} \) satisfies

\[
\left( \frac{d}{d\alpha} - \frac{1}{2} \frac{dh}{h} \sum_{\alpha \in \Phi^+} \frac{d\alpha}{\alpha} \Delta(\kappa_\alpha) - z(d\alpha \text{ ad} u^{(1)}) \right) F^o_{h \pm} = F^o_{h \pm} \left( \frac{d}{d\alpha} - \frac{1}{2} \frac{dh}{h} \sum_{\alpha \in \Phi^+} \frac{d\alpha}{\alpha} (\kappa^{(1)}_\alpha + \kappa^{(2)}_\alpha) \right)
\]

where \( \Delta \) is the coproduct on \( \mathfrak{h} \) and recall that \( \kappa_\alpha = e_\alpha e_{-\alpha} + e_{-\alpha} e_\alpha \in \mathfrak{h} \).

3.4.2 Canonical solutions of isomonodromic KZ equations

Let us assume \( u_0 \in \mathfrak{h}^R_{\text{reg}} \) and let \( \mathcal{C} \subset \mathfrak{h}^R_{\text{reg}} \) be the fundamental Weyl chamber.

**Proposition 3.4.** For any \( u \in \mathcal{C} \), let \( H_{h \pm} : \mathbb{H}_+ \to \mathcal{A} \) be the holomorphic functions as in Theorem [3.3] Then \( H_{h \pm} (z, u) := T(u)^{-1} H^o_{h \pm} (z, u) T(u) \) are the unique holomorphic functions on \( \mathbb{H}_+ \) with valued in \( \mathcal{A} \) such that \( H_{h \pm} (z, u) \) tends to 1 as \( z \to 0 \) in any sector of the form \( |\arg(z)| \in (\delta, \pi - \delta) \), \( \delta > 0 \) and, for any determination of \( \log z \), the function

\[
F_{h \pm} (z, u) = H_{h \pm} (z, u) \cdot z^{\Omega_0} \cdot e^{z(\text{ad} u^{(1)})}
\]

satisfies the equation \( \nabla_{\text{IKZ}} F_h = 0 \).

**Proof.** Due to the fact \([T(u), z(\text{ad} u^{(1)})] = 0 \) and \([T(u), \Omega_0] = 0 \), we have

\[
F_{h \pm} (z, u) = T(u)^{-1} H^o_{h \pm} (z, u) T(u) z^{\Omega_0} e^{z(\text{ad} u^{(1)})} = T(u)^{-1} F^o_{h \pm} (z, u) T(u).
\]

Here \( F^o_{h \pm} = H^o_{h \pm} (z, u) \cdot z^{\Omega_0} \cdot e^{z(\text{ad} u^{(1)})} \) are the canonical solutions in Theorem [3.3]. It then follows from Theorem [3.3] and Equation (11) that for any \( u \in \mathcal{C} \), \( F_{h \pm} (z, u) \) satisfies the equation \( \nabla_{\text{IKZ}} F_h = 0 \).

The fact that \( H_{h \pm} (z, u) \) is valued in \( \mathcal{A} \) is a consequence of the \( h \)-adic property of \( T(u) \) given below in Section 5.4. Finally, the asymptotic behaviour and uniqueness of \( H_{h \pm} \) follows from those of \( H^o_{h \pm} \) .

Now let’s see how the functions \( F_{h \pm} (z, u) \) depend on the singularity data \( u \in \mathcal{C} \).
Proposition 3.5. The functions $F_{h \pm}(z, u)$ in Proposition 3.4 are smooth functions of $u$ and satisfy the system
\[ \nabla_{ISO} F_{h \pm} = 0. \]

Proof. Just use (2) in Theorem 3.3 and the defining equation (11) of $T(u)$. 

We call $F_{h \pm}(z, u)$ the canonical solutions of the KZ equations $\nabla_{IKZ} F_h = 0$ and $\nabla_{ISO} F_h = 0$.

3.4.3 Quantum Stokes matrices

Fix henceforth the branch of log $z$ with a cut along the negative real axis. Following Proposition 3.4 and 3.5, let $F_{h \pm}$ be the corresponding canonical solutions. For any fixed $u$, we shall consider the $F_{h \pm}(z, u)$ as (single-valued) holomorphic functions on $\mathbb{C} \setminus R \leq 0$.

Definition 3.6. For any $u \in \mathbb{C}$, the quantum Stokes matrices $S_{h \pm}(u) \in \mathbf{U} \otimes \mathbb{C}^2$ of the isomonodromy KZ connection (12), (13) are defined by
\[ F_{h +} = F_{h -} \cdot S_{h +}(u) \quad \text{and} \quad F_{h -} \cdot e^{\hbar \Omega_0} = F_{h +} \cdot S_{h -}(u) \]
where the first identity is understood to hold in $\mathbb{H}_-$ after $F_{h +}$ has been continued across the ray $R \geq 0$, and the second in $\mathbb{H}_+$ after $F_{h -}$ has been continued across $R \leq 0$.

3.5 Isomonodromy deformations

Theorem 3.7. The quantum Stokes matrices $S_{h \pm}(u)$ of the connection (12) stay constant in $\mathbb{C}$ (independent of $u$).

Proof. The ratio of the two solutions $F_{h \pm}$ of the common linear differential equation $\nabla_{ISO} F_h = 0$ doesn’t depend on $u$.

Actually, the construction of $\nabla_{IKZ}$ is motivated by solving the isomonodromy deformation problem of the connection $\nabla_{DKZ} = d - (\text{ad } u^{(1)} + \hbar \Omega)dz$ with respect to the singularity data $\mathcal{C}$, that is by searching a function $\Omega(u) \in \mathbf{U} \otimes \mathbb{C}^2$ on a neighbourhood $\mathcal{C}$ of $u_0$ with $\Omega(u_0) = \Omega$ such that the quantum Stokes matrices of
\[ \nabla_{IKZ} = d - (\text{ad } u^{(1)} + \hbar \frac{\Omega(u)}{z})F_h dz \]
are preserved.

Remark 3.8. Since $u$ is restricted to the subset $\mathcal{C} \subset \mathfrak{h}^0_{\text{reg}}$, the only two Stokes rays are positive and negative real axis. In general, the isomonodromy deformation requires more care since Stokes rays may split into distinct rays under arbitrarily small deformations of $u$. However, the isomonodromic KZ connection is defined for any $u$ in a neighbourhood of $u_0$ in $\mathfrak{h}^0_{\text{reg}}$, and has already encoded the isomonodromy deformation. See the discussion below.

One checks that the isomonodromic KZ connection $\nabla_{IKZ}$, $\nabla_{ISO}$ is flat: the equation (11) for $T(u)$ is nothing other than the integrability condition for the isomonodromic KZ connection. These are two equivalent approaches to the isomonodromy deformation problem in the spirit of Jimbo-Miwa-Ueno [22] Section 3:

1. To start from a family of functions $F_h(z, u)$, parametrized by some $u$, having the monodromy/Stokes data, independent of $u$, and to derive a system of linear differential equations in $(z, u)$ for $F_h(z, u)$.

2. To construct non-linear differential equations on the space of singularity data, so that each solution leaf (viewed as a family of ordinary differential equations) corresponds to one and the same partial monodromy data.
3.6 Quantum Stokes matrices and R-matrices

The Stokes phenomenon of the dynamical KZ connection $\nabla_{DKZ}$ was used by Toledano Laredo [37] to give a canonical transcendental construction of the Drinfeld–Jimbo quantum group $U_h(g)$. Later on in [38], the authors proved that the Stokes matrices $S_{h,\pm}(u_0)$ of $\nabla_{DKZ} = d_z - (ad u_0^{(1)} + h \Omega) dz$ give the R-matrices of the quantum group $U_h(g)$. In particular,

**Proposition 3.9.** The elements $R_{\pm} = e^{\pi i \Omega_0} \cdot S_{h,\pm}^{-1}(u)$ satisfy the Yang-Baxter equation (provided we replace the formal parameter $h$ by $h/2\pi i$ in the expression of $R_{\pm}$)

$$R_{\pm}^{12} R_{\pm}^{13} R_{\pm}^{23} = R_{\pm}^{23} R_{\pm}^{13} R_{\pm}^{12} \in U \otimes^3.$$

**4 Semiclassical limit of isomonodromic KZ connections as Dubrovin connections**

In this section, we show that the semiclassical limit of the isomonodromic KZ connection gives the Dubrovin connections of semisimple Frobenius manifolds. In Section 4.1 and 4.2, we recall the dual Hopf algebras and their semiclassical limit. In Section 4.3, we study the $h$-adic property of isomonodromic KZ connections. In Section 4.4 and 4.5, we prove that the semiclassical limit of isomonodromic KZ connections are Dubrovin connections.

**4.1 Drinfeld dual Hopf algebra $\mathfrak{U}'$**

In this section, we follow the notations in [38] closely. Let $\eta : C[h] \to \mathfrak{U}$ and $\epsilon : \mathfrak{U} \to C[[h]]$ be the unit and counit of $\mathfrak{U}$. Then $\mathfrak{U}$ splits as $\text{Ker}(\epsilon) \oplus C[[h]] \cdot 1$, with projection onto the first summand given by $\pi = 1 - \eta \circ \epsilon$. Let $\Delta^{(n)} : \mathfrak{U} \to \mathfrak{U} \otimes^n$ be the iterated coproduct recursively defined by $\Delta^{(0)} = \epsilon$, $\Delta^{(1)} = 1$, and $\Delta^{(n)} = \Delta \otimes \Delta^{(n-2)} \circ \Delta^{(n-1)}$ for $n \geq 2$.

Define $\mathfrak{U}' \subset \mathfrak{U}$ by

$$\mathfrak{U}' = \{ x \in \mathfrak{U} | n \otimes^n \circ \Delta^{(n)}(x) \in h^n \mathfrak{U} \otimes^n, n \geq 1 \}.$$ 

The algebra $\mathfrak{U}'$ has a natural Hopf algebra structure, and is known as a quantum formal series Hopf algebra. See e.g. [18]. In this paper, we only need the following well-known facts.

**Lemma 4.1.**

1. $\mathfrak{U}' = U(hg[[\hbar]])$. That is, $x = \sum_{n \geq 0} \hbar^n x_n$ lies in $\mathfrak{U}'$ if, and only if the filtration order of $x_n$ in $Ug$ is less than or equal to $n$.

2. $\mathfrak{U}'$ is a flat deformation of the completed symmetric algebra $\hat{S}g = \prod_{n \geq 0} S^n g$. 

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4.2 Admissibility and semiclassical limit

An element $A \in \mathfrak{U}^{\otimes n+1}$ is called admissible, if $A$ is further inside the subalgebra $\mathfrak{U}^{\otimes n} \hat{\otimes} \mathfrak{U}'$. Given an admissible $A \in \mathfrak{U}^{\otimes n} \hat{\otimes} \mathfrak{U}'$, the semiclassical limit of $A$, denoted by $\text{sc} (A)$ is the image of $A$ in

$$\mathfrak{U}^{\otimes n} \hat{\otimes} \mathfrak{U}' / h(\mathfrak{U}^{\otimes n} \hat{\otimes} \mathfrak{U}') = U(\mathfrak{g})^{\otimes n} \otimes \mathfrak{S} \mathfrak{g}$$

Given that $\mathfrak{S} \mathfrak{g} = C[\mathfrak{g}^+]$, we will regard $\text{sc} (A)$ as formal function on $\mathfrak{g}^+$ with values in $U(\mathfrak{g})^{\otimes n}$.

4.3 $h$-adic property of the isomonodromic KZ connections

Let us write the solution of the equation (11)

$$d_\hbar T(u) = \frac{h}{2} \sum_{\alpha \in \Phi_+} \frac{dk}{d\alpha} (K^{(1)}_\alpha + K^{(2)}_\alpha) \cdot T(u),$$

as $T(u) = e^{E(u)}$, where $E(u) = hE_1(u) + h^2E_2(u) + \cdots$ is the Magnus expansion [29]. Then each $E_i$ is given by an iterated integral as follows.

Let $I : [0,1] \to \mathcal{C}$ be a path from $u_0$ to $u$. We denote by $A(t) dt$ (a 1-form valued in $U(\mathfrak{g})^{\otimes 2}$) the pull back of the 1-form $\frac{1}{2} \sum_{\alpha \in \Phi_+} \frac{dk}{d\alpha} (K^{(1)}_\alpha + K^{(2)}_\alpha)$ on $\mathcal{C}$ by $I$. It follows from the continuous Baker-Campbell-Hausdorff formula (also known as generalized Baker-Campbell-Hausdorff-Dynkin, see, e.g., [34]) that

$$E_1 = \int_0^1 dt_1 A(t_1),$$
$$E_2 = \frac{1}{2} \int_0^1 dt_2 \int_0^{t_2} dt_1 [A(t_2), A(t_1)],$$
$$E_3 = \frac{1}{6} \int_0^1 dt_3 \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 ([A(t_3), [A(t_2), A(t_1)] + [A(t_1), [A(t_2), A(t_3)]]),$$

where the $i$-th order term $E_i$ is represented as an iterated integral of a linear combination of the nested commutators of $n A(t_i)$'s. In particular, because the standard order filtration degree of $A(t)$ in $U(\mathfrak{g})^{\otimes 2}$ is 2, the filtration degree of $E_i$ in $U(\mathfrak{g})^{\otimes 2}$ is less than or equal to $i + 1$. That is, if we write $E(u) = \sum_{i \geq 0} h^i x_i$, then the filtration order of $x_i$ is less than or equal to $i + 1$.

Now we show the $h$-adic property of the isomonodromic Casimir element $\Omega(u)$ (with respect to $u_0 \in h_{\text{reg}}$).

**Proposition 4.2.**

1. For any $X \in \mathfrak{U} \hat{\otimes} \mathfrak{U}'$ and $u \in \mathcal{C}$, we have $T(u)^{-1} XT(u) \in \mathfrak{U} \hat{\otimes} \mathfrak{U}'$. In particular, $h\Omega(u) = T(u)^{-1} (h\Omega) T(u) \in \mathfrak{U} \hat{\otimes} \mathfrak{U}'$;

2. the semiclassical limit $I(u)$ of $h\Omega(u)$ is a (formal) function on $\mathfrak{g}^+$ valued in $\mathfrak{g} \subset U(\mathfrak{g})$.

**Proof.**

1. Recall that $T(u) = e^{E(u)}$, where $E(u)$ is the Magnus expansion, and the filtration degree of $x_i \in U(\mathfrak{g})^{\otimes 2}$ in $E(u) = \sum_{i \geq 0} h^i x_i$ is less than or equal to $i + 1$. On the other hand, taking the Lie algebra $\mathfrak{U}^{\otimes 2}$ for the commutator, we have

$$T(u)^{-1} XT(u) = e^{-E(u)} X e^{E(u)} = X + [E(u), X] + \cdots + \frac{1}{n!} [E(u), [E(u), \cdots, [E(u), X] \cdots] + \cdots$$
Therefore by the above identity and Lemma 4.1, \( T(u)^{-1}XT(u) \in \Omega \).

2. From the discussion in part 1, we see that the nonzero contributions to the semiclassical limit \( I(u) \) of \( h\Omega(u) = e^{-E(u)}(h\Omega)e^{E(u)} \) are from the terms \( E^{(2)}(u) = hE^{(2)}_{1}(u) + h^2 E^{(2)}_{2}(u) + \ldots \) in \( E(u) \), where for example

\[
E^{(2)}_{1} = \int_{0}^{1} dt_{1} A^{(2)}(t_{1}),
\]
\[
E^{(2)}_{2} = \frac{1}{2} \int_{0}^{1} dt_{2} \int_{0}^{t_{2}} dt_{1} [A^{(2)}(t_{2}), A^{(2)}(t_{1})],
\]

for \( A(t)^{(2)}dt \) the pull back of the 1-form \( \frac{1}{2} \sum_{\alpha \in \Phi^{\circ}} d_{\alpha} (K_{\alpha}^{(2)}) \) on \( C \). In other words, \( e^{E^{(2)}(u)} \) is the solution of the equation \( d_{\hbar} T(u) = \frac{h}{2} \sum_{\alpha \in \Phi^{\circ}} d_{\alpha} (K_{\alpha}^{(2)}) \cdot T(u) \) with initial condition \( T(u_{0}) = 1 \).

Hence we deduce that

\[
(\Delta \otimes 1)(I(u)) = (\Delta \otimes 1) \text{scl} (e^{-E^{(2)}(u)}(h\Omega)e^{E^{(2)}(u)}) = \text{scl} (e^{-E^{(2)}(u)}(h(\Delta \otimes 1)(\Omega))e^{E^{(2)}(u)}).
\]

Here \( \Delta \) is the coproduct. It then follows from \( (\Delta \otimes 1)(\Omega) = \Omega^{(13)} + \Omega^{(23)} \) that

\[
(\Delta \otimes 1)(I(u)) = I(u)^{(13)} + I(u)^{(23)},
\]

which means that the image of \( I(u) \in U(\mathfrak{g}) \otimes S(\mathfrak{g}) \), though as formal function from \( \mathfrak{g}^{*} \) to \( U(\mathfrak{g}) \), is primitive, i.e., is valued in \( \mathfrak{g} \subset U(\mathfrak{g}) \). Here we use the standard convention that \( A^{(13)} := X_{a} \otimes 1 \otimes Y_{a}, A^{(23)} := 1 \otimes X_{a} \otimes Y_{a} \in U(\mathfrak{g})^{\otimes 3} \) for any \( A = X_{a} \otimes Y_{a} \in U(\mathfrak{g})^{\otimes 2} \).

**Example 4.3.** By definition, \( h\Omega(u_{0}) = h\Omega \), thus \( I(u_{0}) = \text{scl}(h\Omega(u_{0})) \) coincides with the isomorphism \( \mathfrak{g}^{*} \cong \mathfrak{g} \) given by the invariant product on \( \mathfrak{g} \).

### 4.4 Casimir elements and isomonodromy deformation equations

Let us take \( \omega = \frac{1}{2} \sum_{\alpha \in \Phi^{\circ}} K_{\alpha} d_{\alpha} \in \Omega^{1}(C) \otimes U(\mathfrak{g}) \). We view \( \omega \) as an element in \( \Omega^{1}(C) \otimes \text{Sym}(\mathfrak{g}) \) via the PBW isomorphism, i.e., a one form on \( C \) whose coefficients are quadratic polynomials on the Poisson space \( \mathfrak{g}^{*} \) (with the Kirillov-Kostant-Souriau bracket). By taking the corresponding Hamiltonian vector field, \( \omega \) further gives rise to an element \( H_{\omega} \in \Omega^{1}(C) \otimes \mathfrak{X}(\mathfrak{g}^{*}) \).

By Proposition 4.2, the semiclassical limit \( I(u) \) of \( h\Omega(u) \) is a map \( I(u) : C \times \mathfrak{g}^{*} \to \mathfrak{g} \cong \mathfrak{g}^{*} \).

**Proposition 4.4.** For any \( V_{0} \in \mathfrak{g}^{*} \), the function \( I(u; V_{0}) : C \to \mathfrak{g}^{*} \) satisfies the equation

\[
d_{\hbar} I(u; V_{0}) = H_{\omega}(I(u; V_{0}))
\]

and the initial condition \( I(u_{0}; V_{0}) = V_{0} \) at \( u_{0} \in C \).

**Proof.** By Theorem 3.7, the quantum Stokes matrices \( S_{h, \pm}(u) \) of

\[
\nabla_{IKZ} \Delta z = d_{z} - (\text{ad}(u^{(1)}) + \hbar \frac{\Omega(u)}{z}) dz
\]

are preserved. Here we assume the initial condition \( \Omega(u_{0}) = \Omega \). Thus by taking semiclassical limit and evaluating \( V_{0} \) (see the proof of Theorem 4.6 for more details), we conclude that the Stokes matrices \( S_{\pm}(I(u; V_{0})) \) of

\[
\nabla_{z} \Delta z = d_{z} - \left( u + \frac{I(u; V_{0})}{z} \right)
\]

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are preserved, i.e., don’t depend on \( u \).

On the other hand, following [4], equation (15) describes exactly the isomonodromy deformation for \( G \)-valued Stokes phenomenon. Thus \( I(u; V_0) \) is a solution of (15) with an initial condition \( I(u_0; V_0) = V_0 \) at \( u_0 \in \mathcal{C} \).

In particular, when \( g = g|_u \) and \( h \subset g \) the set of diagonal matrices, equation (15) becomes equation (10).

As a corollary, we have

**Corollary 4.5.** For any \( V_0 \in \mathfrak{g}^* \), the function \( I(u; V_0) = \text{scl} \left( \mathfrak{m} \Omega(u) \right)(V_0) \) coincides with the solution \( V(u) \) of (10) on a neighbourhood \( \mathcal{C} \) of \( u_0 \) with initial data \( V(u_0) = V_0 \).

### 4.5 Semiclassical limit of isomonodromic KZ connections as Dubrovin connections

#### 4.5.1 Formal Taylor series groups

Let \( G \) be a complex reductive Lie group. We denote by \( \mathbb{C}[G] \) its ring of regular functions (when regarded as an affine algebraic group over \( \mathbb{C} \)). For \( S \) a unital \( \mathbb{C} \)-algebra with a maximal ideal \( m \subset S \), denote by \( G(S) = \text{Alg}_\mathbb{C}([\mathbb{C}[G]], S) \) the \( S \)-points of \( G \), and denote by \( G(S)_m \subset G(S) \) the normal subgroup

\[
G(S)_m = \{ \phi \in \text{Alg}_\mathbb{C}([\mathbb{C}[G]], S) \mid \phi(I) \subset m \}
\]

where \( I = \{ f \in \mathbb{C}[G] \mid f(1) = 0 \} \) is the augmentation ideal. In the following, we will be interested in the case \( S = \mathbb{C}[\mathfrak{g}^*]/m^p \), for \( \mathbb{C}[\mathfrak{g}^*] \) being the algebra of regular functions on \( \mathfrak{g}^* \), \( p \) a positive integer and \( m \) the ideal of \( 0 \in \mathfrak{g}^* \).

Now let \( \mathbb{C}[[G]] = \lim \mathbb{C}[G]/I^n \) be the completion of \( \mathbb{C}[G] \) at the identity, then \( U(\mathfrak{g}) \) is identified, as a Hopf algebra, with the continuous dual

\[
\mathbb{C}[[G]]^* = \{ \varphi \in \text{Hom}_\mathbb{C}([\mathbb{C}[G]], \mathbb{C}) \mid \varphi(I^n) = 0, n > 0 \}.
\]

Thus one can see that \( G(\mathbb{C}[\mathfrak{g}^*]/m^p)_m \) embeds into the Hopf algebra \( U(\mathfrak{g}) \otimes (\mathbb{C}[\mathfrak{g}^*]/m^p) \) over \( \mathbb{C}[\mathfrak{g}^*]/m^p \), and its image is those elements which are grouplike.

Therefore, the inverse limit \( G(\mathfrak{g}^*)_0 := \lim_{\leftarrow} G(\mathbb{C}[\mathfrak{g}^*]/m^p)_m \) embeds into the topological Hopf algebra \( U(\mathfrak{g}) \otimes \tilde{S}(\mathfrak{g}) = \text{lim}_p U(\mathfrak{g}) \otimes \mathbb{C}[\mathfrak{g}^*]/m^p \). Here \( \tilde{S}(\mathfrak{g}) = \text{lim}_p \mathbb{C}[\mathfrak{g}^*]/m^p \) is the completion of \( \mathbb{C}[\mathfrak{g}^*] \) at \( m \).

#### 4.5.2 Semiclassical limit of isomonodromic KZ connections

Let us focus on the case \( \mathfrak{g} = \mathfrak{sl}_n \), and \( \mathfrak{h} \subset \mathfrak{g} \) the set of diagonal matrices. The following theorem comes from the observation that if \( F_h \) is a solution of equation

\[
\frac{dF_h}{dz} = \left( \text{ad} u^{(1)} + \frac{\Omega(u)}{z} \right) F_h
\]

with values in \( \mathfrak{h} \otimes \mathfrak{h}' \), then the semiclassical limit \( F \) of \( F_h \), as a formal function on \( \mathfrak{g} \) with values in \( U(\mathfrak{g}) \), satisfies

\[
\frac{dF}{dz} = \left( \text{ad} u + \frac{I(u)}{z} \right) F
\]

which by Corollary 5.5 coincides with the equation (7) (provided the evaluation on an initial data \( V_0 \) and the replacement \( \text{ad} u \rightarrow u \)).

**Proposition 4.6.** For any fixed \( u \in \text{Re}_p \), let \( F_{h,k}(z, u) = H_{h,k}(z, u) \cdot z^{\omega_0} \cdot e^{z(\text{ad} u^{(1)})} \) be the canonical solutions of \( \nabla_{\text{KZ}} F_h = 0 \) corresponding to the half planes \( \mathbb{H}_\pm \) given in Proposition 3.4. Then
• the holomorphic components \( H_{h^\pm} \) are valued in \( U(\mathfrak{g}') \), and their the semiclassical limit \( H_{\pm} := \text{sc}(H_{h^\pm}) \) take values in \( G[[\mathfrak{g}^*]]_0 \subset U(\mathfrak{g}) \otimes \widehat{S}(\mathfrak{g}) \);

• for any \( V_0 \in \mathfrak{g}^* \), \( H_{\pm}(z, u; V_0) \) coincides with the holomorphic components \( H_{\pm}(z, u) \) of the fundamental solutions \( F_{\pm}(z, u) = H_{\pm}(z, u)z^{[V(u)]}e^{U} \) of the equation (10) given in Theorem 2.7.

**Proof.** By the \( \hbar \)-adic property given in Proposition 4.2, \( H_{h^\pm} = T(u)^{-1} H_{h^\pm}^0 T(u) \) are valued in \( U(\mathfrak{g}') \); since \( H_{h^\pm}^0 \) are (see \[38\]). By definition, \( H_{h^\pm} \) are group-like, i.e., take values in \( U(\mathfrak{g}) \); (a formal function on \( \mathfrak{g}^* \) valued in \( U(\mathfrak{g}) \)), satisfy

\[
\frac{dH_{h^\pm}}{dz} = \left( u^{(1)} + \hbar \frac{\Omega(u)}{z} \right) H_{h^\pm} - H_{h^\pm} \left( u^{(1)} + \hbar \frac{\Omega_0}{z} \right).
\]

Thus the semiclassical limit \( H_{\pm} \) of \( H_{h^\pm} \), as an element of \( U(\mathfrak{g}) \otimes \mathfrak{g}' = U(\mathfrak{g}) \otimes \widehat{S}(\mathfrak{g}) \) (a formal function on \( \mathfrak{g}^* \) valued in \( U(\mathfrak{g}) \)) satisfy

\[
\frac{dH_{\pm}}{dz} = \left( u + \frac{I(u)}{z} \right) H_{\pm} - H_{\pm} \left( u + \frac{I_0}{z} \right)
\]

where \( I(u) : \mathfrak{g}^* \to U(\mathfrak{g}) \) (resp. \( I_0 \)) is the semiclassical limit of \( \Omega(u) \) (resp. \( \Omega_0 \)). By Proposition 4.2 \( \Delta(I(u)) = I(u)^1 + I(u)^2 \). It follows that both \( \Delta(H_{\pm}) \) and \( H_{1} H_{2}^\pm \) satisfy

\[
\frac{dH}{dz} = (u^1 + u^2 + \frac{1}{z}) H - H(u^1 + u^2 + \frac{I_1^0 + I_2^0}{z})
\]

with the same initial condition. Thus \( H_{\pm} \) are group-like, i.e., take values in \( G[[\mathfrak{g}^*]]_0 \subset U(\mathfrak{g})[[\mathfrak{g}^*]] \).

Now the second part follows from the uniqueness statement of Theorem 2.7 for the affine algebraic groups \( G[\mathfrak{g}^*/m^p], p \geq 1 \). See \[38\] for a similar argument.

Now for any skew-symmetric \( V_0 \), we denote by \( \nabla_{V_0} \) the Dubrovin connection

\[
\nabla_z = dz - \left( U + \frac{V(u)}{z} \right) dz,
\]

\[
\nabla_u = dh - (zd\hbar U + \Lambda(u)),
\]

where \( V(u) \) is the solution of (10) satisfying the initial condition \( V(u_0) = V_0 \). It is the connection of the Frobenius manifold parametrized by \( V_0 \) (see Section 2.6). As an immediate result of Propositions 4.6, 2.10 and 3.5, we have that

**Theorem 4.7.** Let \( F_{h^\pm} = H_{h^\pm}(z, u) \cdot e^{H_{h^\pm}(z, u)} \) be the canonical solutions of \( \nabla_{IKZ} \) \( F_h = 0 \) and \( \nabla_{ISO} \) \( F_0 = 0 \) corresponding to the half-planes \( \mathbb{H}_\pm \) (provided \( u \in \mathfrak{C} \in \mathfrak{K}_{\text{reg}} \)), then for any \( V_0 \in \mathfrak{so}_n \) the functions \( F_{\pm}(z, u; V_0) = \text{sc}(H_{h^\pm})(z, u; V_0) e^{U} \) are the canonical fundamental solutions of the equations \( \nabla_{V_0} F = 0 \) corresponding to \( \mathbb{H}_\pm \).

In summary, the semiclassical limit of the canonical solutions of the isomonoromnic KZ systems \( \nabla_{IKZ} \) and \( \nabla_{ISO} \) recovers the canonical fundamental solutions of the systems (7) and (8). Furthermore, they have the same isomonodromicity (Theorem 4.7 and Proposition 2.12), and their canonical solutions and monodromy/Stokes data are related by quantization/semiclassical limit. In other words, the isomonodromic KZ connection can be seen as a quantization of the Dubrovin connections.
5 Quantization revisited

In this section, we explain the quantization of semisimple Frobenius manifolds as a lifting problem. In Section 5.1, we encode the semisimple germs of Frobenius manifolds at \( u_0 \in h_{reg} \) by a \( GL_n[[\mathfrak{g}^*]]_0 \)-valued function \( \tilde{F} \). In Section 5.2, we define a quantization of \( \tilde{F} \) as a lifting function valued in the quantized algebra \( \mathcal{U} \otimes \mathcal{M}' \) with the same monodromy property as \( \tilde{F} \). In particular, such a quantization is given via isomonodromy KZ connections.

5.1 Semisimple germs of Frobenius manifolds

Let us study the semisimple germs at \( u_0 \), once fixed the only variable is the initial data \( V_0 = V(u_0) \). Denote by \( \nabla_{V_0} \) the corresponding Dubrovin connection. As in Section 2.4, let us take the canonical fundamental solution \( F_{V_0}(z, u) = H_{V_0}(z, u)z^{[V(u)]}e^{zU} \) of \( \nabla_{V_0} F = 0 \) on \( \mathbb{C} \times \mathbb{C} \) (for chosen Stokes sectors smoothly depending on \( u \)). Then the monodromy property of \( F_{V_0} \) (on the universal covering of \( [\mathbb{P}^1 \setminus \{0, \infty\}] \)) in turn determines the connection \( \nabla_{V_0} \), see e.g. [22] Proposition 2.5 and 2.6 and [11] Lecture 4 for more details. Hence, we can collect the canonical solutions \( \tilde{F}_{V_0} \) for all \( V_0 \) to encode the semisimple germs at \( u_0 \). We thus obtain a map (which depends on \( u, z \))

\[
\tilde{F}(z, u) : \mathfrak{g}^* \cong \mathfrak{g} \rightarrow GL_n; \ V_0 \mapsto F_{V_0}(z, u).
\]

Here \( \mathfrak{g} : = gl_n \) and is identified with \( \mathfrak{g}^* \). Taking its Taylor expansion around \( V_0 = 0 \in \mathfrak{g} \), we get a function of \((z, u)\) valued in the formal Taylor series group \( GL_n[[\mathfrak{g}^*]]_0 \), i.e.,

\[
\tilde{F} : \mathbb{C} \times \mathbb{C} \rightarrow GL_n[[\mathfrak{g}^*]]_0.
\]

Similarly, by considering the holomorphic components, we get the map

\[
\tilde{H}(z, u) : \mathfrak{g}^* \cong \mathfrak{g} \rightarrow GL_n; \ V_0 \mapsto H_{V_0}(z, u)
\]

whose formal Taylor series is

\[
\tilde{H} : \mathbb{C} \times \mathbb{C} \rightarrow GL_n[[\mathfrak{g}^*]]_0.
\]

5.2 Quantization of semisimple germs

Recall that \( GL_n[[\mathfrak{g}^*]]_0 = \lim_{\longrightarrow} G(\mathbb{C}[[\mathfrak{g}^*]])/I^n \) embeds into the topological Hopf algebra \( U(\mathfrak{g}) \otimes \hat{S}(\mathfrak{g}) \) as grouplike elements. Composing with the function \( \tilde{F}(z, u) \) (which encodes the semisimple germs at \( u_0 \)), we get

\[
\tilde{F} : \mathbb{C} \times \mathbb{C} \rightarrow GL_n[[\mathfrak{g}^*]]_0 \hookrightarrow U(\mathfrak{g}) \otimes \hat{S}(\mathfrak{g}).
\]

Since the semiclassical limit of \( \mathcal{U} \otimes \mathcal{M}' \) is \( U(\mathfrak{g}) \otimes \hat{S}(\mathfrak{g}) \), it motivates

**Definition 5.1.** A quantization of \( \tilde{F}(z, u) \) is a holomorphic function \( H_h(z, u) \) on \( \mathbb{C} \times \mathbb{C} \) with values in \( \mathcal{A} \cap \mathcal{U} \otimes \mathcal{M}' \) such that

- \( \text{sc}1(H_h) = \tilde{H} \), the holomorphic component of \( \tilde{F} \).
- In other words, \( H_h \) is a lift

\[
\begin{array}{ccc}
\mathbb{C} \times \mathbb{C} & \xrightarrow{\text{sc}1} & \mathcal{A} \\
\downarrow{\tilde{F}} & & \downarrow{\tilde{H}} \\
\mathcal{U} \otimes \mathcal{M}' & \xrightarrow{H_h} & \mathcal{U} \otimes \mathcal{M}'/s.c.l
\end{array}
\]

*Recall Section 3.2 for the definition of \( \mathcal{A} = U(\mathfrak{g}) \otimes \mathfrak{d}_{[a]}^o \subset \mathfrak{g} \otimes \mathfrak{d}^2 \)
• \( F_h(z, u) := H_h(z, u) \cdot z^{\hbar \Omega_0} \cdot e^{z \text{ad } u(1)} \in \mathcal{E} \) has the same monodromy property as \( \widetilde{F} \).

Here by the monodromy property, we refer to [22] Proposition 2.5. It requires that, for example, there exists a constant element \( S_h \in \mathfrak{U} \otimes \mathfrak{U}' \), such that for any \( u \), the functions \( F_h(z, u) \) and \( F_h(z, u) \cdot S_h \) on \( z \)-plane have the same asymptotic expansion in the Stokes sector defining \( \widetilde{F} \) and its opposite Stokes sector respectively. Taking semiclassical limit, it implies that \( \widetilde{F} \) and \( \widetilde{F}_{\text{sc}}(S_h) \) have the same asymptotic expansion in the two opposite Stokes sectors. It then follows from definition that \( \text{sc}l(S_h) \) should be (one component of) the Stokes map of Dubrovin connections, i.e., for any \( V_0 \in \mathfrak{g}^* \), \( \text{sc}l(S_h)(V_0) \) is one of the Stokes matrices of the connection \( \nabla_{V_0} \). Therefore, the quantization of \( \widetilde{F} \) also encodes a deformation of the Stokes/monodromy data of the Frobenius manifolds.

Note that any element \( A \in U(\mathfrak{g}) \otimes \widehat{S}(\mathfrak{g}) \), viewed as a formal function on \( \mathfrak{g}^* \), has a natural lift

\[
\hat{A} \in U(\mathfrak{g}) \otimes \widehat{S}(\mathfrak{h}g) \subset \mathfrak{U} \otimes \mathfrak{U}'
\]
given by

\[
\hat{A}(x) := A(hx), \quad \forall x \in \mathfrak{g}^*.
\]

Because the product in \( \mathfrak{U}' \) is not commutative anymore, the natural lift of \( \widetilde{F} \) in general violates the second condition of Definition [5.1]. Thus a quantization can be seen as a nontrivial correction of the natural lift.

### 5.3 Quantization via KZ connections

Let \( \hat{F}_\pm : \mathbb{C} \times \mathbb{C} \to \text{GL}_n[\mathfrak{g}^*]_0 \) be the formal Taylor series, of the canonical fundamental solutions w.r.t the Stokes sectors \( \mathbb{H}_\pm \), of the Dubrovin connections at a semisimple point \( u_0 \in \mathbb{C} \subset \mathfrak{h}^\text{reg} \).

Theorem [4.7] implies that the holomorphic components of the canonical solutions \( F_{h, \pm} : \mathbb{C} \times \mathbb{C} \to \mathcal{E} \) of the equation \( \nabla_{\text{IKZ}} F = 0 \) are lifts \( \hat{F}_\pm \) of \( F_{h, \pm} \) with the same monodromy property at \( \infty \) on the \( z \)-plane. In this paper, we only consider the Stokes phenomenon of the isomonodromic KZ connections and Dubrovin connections at \( z = \infty \). Similarly, one can consider the monodromy data at \( z = 0 \) and the (quantum) connection matrix of the connection \( \nabla_{\text{IKZ}} \), and show that \( F_{h, \pm} \) has the same monodromy property with \( \hat{F}_\pm \) for the full monodromy data. See [38] for the case of dynamical KZ equations. Thus according to Definition [5.1] we have

**Proposition 5.2.** The holomorphic components \( H_{h, \pm} \) of the canonical solutions \( F_{h, \pm} \) of the equation \( \nabla_{\text{IKZ}} F = 0 \) given in Proposition [3.4] are the quantization of the functions \( \hat{F}_\pm : \mathbb{C} \times \mathbb{C} \to \text{GL}_n[\mathfrak{g}^*]_0 \).

Hence we have explained the (flat sections of) isomonodromic KZ connection as a natural deformation of the (flat sections of) Dubrovin connections in the sense of Definition [5.1].

As in Section [5.2], the quantization of \( \hat{F}_\pm \) encodes a deformation of the Stokes/monodromy data of the Frobenius manifolds. Following Section [3.6] and arrow e in the introduction, it corresponds respectively to the quantization of classical \( r \)-matrices to quantum \( R \)-matrices, and the quantization of the Stokes map \( \hat{S} \) to the Hopf algebras isomorphism \( S_h \).

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\(^*\text{Strictly speaking Theorem [4.7] only gives a lift on a real part of } \hbar \text{ at } u_0. \text{ However, it admits an extension.}\)
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