All (possibly noncommutative) rings are assumed to have an identity element.

1. **Suppose \( V \) is a finite-dimensional vector space over a field \( F \). Let \( R = \text{End}(V) \) be the ring of linear maps from \( V \) to \( V \). Find all left ideals in \( \text{End}(V) \).**

   By easy facts from linear algebra, \( V \) is a simple module for \( R \). If \( \lambda \) is a nonzero linear functional on \( V \), we get an inclusion
   \[
   \phi^\lambda: V \hookrightarrow R, \quad v \mapsto T_v^\lambda
   \]
   where \( T_v^\lambda \) is the linear transformation
   \[
   T_v^\lambda(w) = \lambda(w)v.
   \]
   It’s clear that
   \[
   V^\lambda = \phi^\lambda(V) \subset R
   \]
   is a left ideal isomorphic to \( V \), since
   \[
   r \cdot T_v^\lambda = T_{r \cdot v}^\lambda.
   \]
   If \( \lambda_1, \ldots \lambda_n \) is a basis of \( V^* \), then the identification of linear transformations with matrices shows that
   \[
   R = V^{\lambda_1} \oplus V^{\lambda_2} \oplus \cdots \oplus V^{\lambda_n}
   \]
   as a left \( R \)-module. This is just the usual isomorphism
   \[
   R \simeq V^* \otimes_F V;
   \]
   the point is to say that it identifies \( R \) as a direct sum of \( n \) copies of the simple module \( V \).

   This identification provides a description of all \( R \)-submodules (that is, left ideals) \( I \subset R \): they correspond bijectively to \( F \)-subspaces \( W \subset V^* \),
   \[
   I = W \otimes_F V.
   \]
   Of course this correspondence respects inclusions.

   Another way to say the same thing (given the description of the maps \( \phi^\lambda \)) is that to each left ideal \( L \subset \text{End}(V) \) you can attach a subspace
   \[
   W^\perp = \{ v \in V \mid L \cdot v = 0 \} = \{ v \in V \mid \lambda(v) = 0 \mid \lambda \in W \}.
   \]
   Obviously the map \( L \mapsto W^\perp \) reverses inclusion.

2. **Prove that if \( V \) is finite-dimensional and nonzero, then \( R = \text{End}(V) \) is a simple ring.**

   Same argument proves that right ideals \( J \) in \( R \) correspond to subspaces \( E \subset V \),
   \[
   J = V^* \otimes E = \{ r \in R \mid r \cdot V \subset E \}.
   \]
Problem (1) shows that every nonzero left ideal $I$ satisfies $I \cdot V = V$. If $J$ is a right ideal not equal to $R$, then this problem shows that $J \cdot V \neq V$. Conclusion is that a nonzero two-sided ideal must be all of $R$.

3. Let $A = \mathbb{C}[x, \frac{d}{dx}]$, the ring of differential operators with polynomial coefficients. If $p \in \mathbb{C}[x]$ is a nonzero polynomial, prove that

$$\text{Ann}_A(p) = \{D \in A \mid Dp = 0\}$$

is a maximal left ideal in $A$. Find a maximal left ideal not of this form.

It's easy or well known that $M = \mathbb{C}[x]$ is a simple $A$-module; so the annihilator of every nonzero polynomial is a maximal left ideal. (That's what it means to be simple.) Each of these maximal ideals obviously contains $\frac{d^N}{dx^n}$ all sufficiently large $N$, and does not contain $x^M$ for any $M$.

Among the linear functionals on $M$ are the values of $r$th derivatives (of a polynomial) at zero. The span of these linear functionals (which is most naturally a right $A$-module) can be made into a left $A$ module, using the antiautomorphism of $A$ defined by

$$\left(\frac{x^n d^m}{dx^m}\right)^\vee = \frac{d^m}{dx^m} x^n,$$

$$[D \cdot \lambda](p) = \lambda(D^\vee p).$$

That is, if $\lambda_r$ is the $r$th derivative,

$$\lambda_r(p) = \frac{d^r p}{dx^r}(0),$$

then

$$\left[\left(\frac{x^n d^m}{dx^m}\right) \cdot \lambda_r \right](p) = \lambda_r \left(\frac{d^m}{dx^m} x^n p\right) = \left(\frac{d^{m+r}}{dx^{m+r}} x^n p\right)(0),$$

which is some linear combination of $\lambda_q$s that I won’t try to compute.

The conclusion is that the $\lambda_q$ span a simple $A$ module. Therefore the annihilator of any nonzero derivative $\lambda$ in this simple module must be a maximal left ideal. The formulas I didn’t compute make it clear that such a maximal left ideal contains $x^M$ for all sufficiently large $M$ (greater than the order of the derivative $\lambda$), and does not contain $\frac{dx^N}{dx^n}$ for any $N$. So these ideals are different from those constructed in the problem.

4. Prove that every maximal two-sided ideal is primitive.

By Zorn’s lemma, the maximal two-sided ideal $I$ is contained in a maximal (proper) left ideal $J$. Then $M = R/J$ is a simple $R$-module, so $\text{Ann} M$ is primitive. Obviously $\text{Ann} M \supset I$ (since $I \subset J$), and $\text{Ann} M \neq R$ (since $J$ is proper). Since $I$ is maximal, it follows that $I = \text{Ann} M$.

5. Prove that if $A$ is commutative, then every primitive ideal is maximal.

We more or less discussed this in class: any simple $A$-module is $A/I$, with $I$ a maximal left ideal. If $A$ is commutative, then it follows immediately that $I = \text{Ann} M$, so a primitive ideal is maximal.