Abstract

Suppose $V$ is a finite-dimensional complex vector space. Hilbert’s Nullstellensatz says that any maximal ideal in the polynomial ring $S(V)$ consists of the polynomials vanishing at some point $\xi \in V^*$. This theorem is the beginning of relating the commutative algebra of ideals in $S(V)$ to the geometry of points in $V^*$. If you have an analytic turn of mind, you can think of it as the beginning of the theory of the Fourier transform (relating functions on $V$ to functions on $V^*$).

If $R$ is a noncommutative ring, then one important analogue of maximal ideals is primitive ideals. The goal of these lectures is to describe the primitive ideals in the enveloping algebra $U(g)$ of a reductive Lie algebra $g$. The answer is that they have something to do with points of $g^*$. I’ll explain exactly what the relationship is, and hint at what the result has to say about analysis on groups.

A bit more precisely, my plan is first to describe a certain category of representations (“Harish-Chandra bimodules”) more or less analogous to the Bernstein-Gelfand-Gelfand category $O$. I’ll explain the very close relationship between Harish-Chandra bimodules and ideals in the enveloping algebra. I’ll explain (with incomplete proofs) how to classify irreducible Harish-Chandra bimodules. The classification of primitive ideals is an easy consequence.

Finally I’ll explain how the Poincaré-Birkhoff-Witt theorem (which says that $\text{gr} U(g) = S(g)$) relates ideals in the enveloping algebra to ideals in the polynomial ring $S(g)$, and therefore to algebraic geometry in $g^*$.

Reading

Humphrey’s book *Introduction to Lie algebras and representation theory* should be more than enough background for these lectures.

Much more than the material in the lectures is explained in Jantzen’s book *Einhüllende Algebren halbeinfacher Lie-Algebren*. There is a survey in English by Borho