Introduction to algebraic D-modules

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CHAPTER 1

$\mathcal{D}$-modules on affine varieties

In this chapter we will develop the theory of $\mathcal{D}$-modules on affine varieties. Unless specified otherwise, we will work over a base field k of characteristic 0 (in most cases one can assume that $k = \mathbb{C}$).

1.1. Analytic continuation of distributions with respect to a parameter and $\mathcal{D}$-modules

1.1.1. An analytic problem. As a motivation for what is going to come, let us first look at the following analytic problem.

Let $p \in \mathbb{R}[x_1, \ldots, x_n]$ be a nonconstant real polynomial in n variables, viewed as a function $p : \mathbb{R}^n \to \mathbb{R}$. Let U be a connected component of $\{\mathbf{x} \mid p(\mathbf{x}) > 0\}$. Define

$$p_U(\mathbf{x}) = \begin{cases} p(\mathbf{x}) & \text{if } \mathbf{x} \in U \\ 0 & \text{otherwise.} \end{cases}$$

Let us take any $\lambda \in \mathbb{C}$ and consider the function $p_U(\mathbf{x})^\lambda$.

It is easy to see that if $\mathrm{Re}\lambda \geq 0$ then $p_U(\mathbf{x})^\lambda$ makes sense as a distribution on $\mathbb{R}^n$, i.e., $\int_{\mathbb{R}^n} p_U(\mathbf{x})^\lambda f(\mathbf{x}) d\mathbf{x}$ is convergent for any $f \in C_c^\infty(\mathbb{R}^n)$, where $C_c^\infty(\mathbb{R}^n)$ is the space of smooth functions on $\mathbb{R}^n$ with compact support.

Example 1.1.1. Let $n = 1$, $p(x) = x$ and $U = \mathbb{R}_+$. Then $\int_0^\infty x^\lambda f(x) dx$ is defined for $\mathrm{Re}\lambda \geq 0$. (Of course, this integral is actually well-defined for $\mathrm{Re}\lambda > -1$, but we do not need this). We will denote the corresponding distribution by x_+^λ.

It is easy to see that for $\mathrm{Re}\lambda \geq 0$, we have a holomorphic family of distributions $\lambda \mapsto p_U(\mathbf{x})^\lambda$. In fact, it is defined and holomorphic even in a slightly larger region, $\mathrm{Re}\lambda > -\delta$, for some $\delta > 0$. However, if p has zeros then when the real part of λ decreases, at some point the integral becomes divergent. This gives rise to the following natural question.

Question 1.1.2. (I. M. Gelfand, M. Sato): Can one extend this family meromorphically in λ to the whole $\mathbb{C}$?

Example 1.1.3. Let $n = 1$, $p(x) = x$, $U = \mathbb{R}_+$. We have a distribution x_+^λ defined for $\mathrm{Re}\lambda \geq 0$. We know that $\frac{d}{dx}(x_+^{\lambda+1}) = (\lambda + 1)x_+^\lambda$, and the left hand side is defined for $\mathrm{Re}\lambda \geq -1$. Hence the expression

$$x_+^\lambda = \frac{1}{\lambda + 1} \frac{d}{dx}(x_+^{\lambda+1})$$

gives us an extension of x_+^λ to $\mathrm{Re}\lambda \geq -1$, $\lambda \neq -1$. Continuing this process, we get

$$x_+^\lambda = \frac{1}{(\lambda + 1)...(\lambda + n)} \frac{d^n}{dx^n}(x_+^{\lambda+n})$$

and thus we have

PROPOSITION 1.1.4. *x_+^λ extends to the whole of $\mathbb{C}$ meromorphicaly with at most simple poles at the negative integers. In particular, for every $f \in C_c^\infty(\mathbb{R})$ the function*

$$J_f(\lambda) = \int_0^\infty x^\lambda f(x)dx$$

has a meromorphic continuation to the whole of $\mathbb{C}$ with at most simple poles at $-1, -2, -3, \ldots$.

EXAMPLE 1.1.5. The proposition works not only for functions f with compact support, but also for functions which are rapidly decreasing at $+\infty$ together with all derivatives. For example, we can take $f(x) = e^{-x}$. In this case we have $\int_0^\infty x^\lambda e^{-x}dx = \Gamma(\lambda+1)$. The proposition implies the well-known fact that $\Gamma(\lambda)$ has a meromorphic continuation with poles at $0, -1, -2\ldots$.

It is remarkable that a similar property holds in several variables, even though it is harder to prove. Namely, we have the following theorem.

THEOREM 1.1.6. *[M. Atiyah [**A**], J. Bernstein and S. Gelfand [**BG**]] The distribution p_U^λ has a meromorphic continuation to the whole of $\mathbb{C}$ with poles of bounded order in a finite number of semi-infinite arithmetic progressions of the form $a, a-1, a-2, \ldots,\ a \in \mathbb{C}$.*

The first proofs of Theorem 1.1.6 were based on Hironaka's theorem on resolution of singularities. We are going to give a completely algebraic proof of Theorem 1.1.6 which is due to J. Bernstein. Let us first formulate an algebraic statement that implies Theorem 1.1.6.

1.1.2. An algebraic reformulation. Let $\mathcal{D} = \mathcal{D}(\mathbb{A}^n)$ denote the algebra of differential operators with polynomial coefficients acting on $\mathcal{O} = \mathcal{O}(\mathbb{A}^n) = \mathbb{C}[x_1, \ldots, x_n]$. In other words, $\mathcal{D}$ is the subalgebra of $\operatorname{End}_{\mathbb{C}} \mathcal{O}$ generated by multiplication by x_i and $\frac{\partial}{\partial x_i}$, $i = 1, \ldots, n$.

Note that for any nonconstant polynomial $p \in \mathcal{O}$, we can formally apply any element of $\mathcal{D}$ to the formal power p^λ, where λ is a variable. More precisely, the space $\mathcal{O}[p^{-1}, \lambda]p^\lambda$ (which is really just $\mathcal{O}[p^{-1}, \lambda]$) is naturally a $\mathcal{D}[\lambda]$-module; for instance, $\frac{\partial}{\partial x_i} p^\lambda = \lambda \frac{\partial p}{\partial x_i} p^{\lambda-1}$, where $p^{\lambda-1} := p^{-1}p^\lambda$, etc.

THEOREM 1.1.7. *There exists $L \in \mathcal{D}[\lambda]$ and $b \in \mathbb{C}[\lambda]$ such that*

$$Lp^{\lambda+1} = b(\lambda)p^\lambda.$$

EXAMPLE 1.1.8. Let $n = 1$ and $p(x) = x$. Then we can take $L = \frac{d}{dx}$ and $b(\lambda) = \lambda + 1$.

Note that the set of possible b forms an ideal in $\mathbb{C}[\lambda]$. As $\mathbb{C}[\lambda]$ is a principal ideal domain, this ideal must be generated by a monic polynomial. This polynomial is called the *Bernstein-Sato polynomial* or the *b-function* of p.

REMARK 1.1.9. It is known that the zeros of the Bernstein-Sato polynomial are at negative rational numbers (see [**K**]), but we will not prove it here.

We claim now that Theorem 1.1.7 implies Theorem 1.1.6 (note that Theorem 1.1.7 is a completely algebraic statement). Indeed, suppose $Lp^{\lambda+1} = b(\lambda)p^\lambda$. Then $Lp_U^{\lambda+1} = b(\lambda)p_U^\lambda$, where as usual for a distribution $\mathcal{E}$ on $\mathbb{R}^n$ we define $\frac{\partial \mathcal{E}}{\partial x_i}(f) = -\mathcal{E}(\frac{\partial f}{\partial x_i})$. The left hand side is defined for $\mathrm{Re}\lambda \geq -1$, so the expression

$$p_U^\lambda = \frac{1}{b(\lambda)} Lp_U^{\lambda+1}$$

gives us a meromorphic continuation of p_U^λ to $\mathrm{Re}\lambda \geq -1$. Indeed, integrating by parts, we get

$$\int_{\mathbb{R}^n} p_U(\mathbf{x})^\lambda f(\mathbf{x})d\mathbf{x} = \frac{1}{b(\lambda)} \int_{\mathbb{R}^n} p_U(\mathbf{x})^{\lambda+1} L^* f(\mathbf{x})d\mathbf{x},$$

where L^* is the adjoint operator of L (defined by the equalities $x_i^* = x_i$, $\left(\frac{\partial}{\partial x_i}\right)^* = -\frac{\partial}{\partial x_i}$, and $(L_1L_2)^* = L_2^*L_1^*$). Arguing by induction similarly to the case of one variable, we see that the distribution p_U^λ can be meromorphically extended to the whole of $\mathbb{C}$, with poles at arithmetic progressions a, $a-1$, $a-2$, ... where a is a root of $b(\lambda)$. Moreover, it is clear that the orders of these poles are bounded by the degree of b.

We now want to reformulate Theorem 1.1.7 once again. Set $\mathcal{D}(\lambda) = \mathcal{D} \otimes_{\mathbb{C}} \mathbb{C}(\lambda)$, and algebra over the field $\mathbb{C}(\lambda)$. Denote by $M(p)$ the $\mathcal{D}(\lambda)$-module consisting of all formal expressions $qp^{\lambda-i}$ where $i \in \mathbb{Z}$ and $q \in \mathbb{C}(\lambda)[x_1, \ldots, x_n]$ subject to the relations $qp^{\lambda-i+1} = (qp)p^{\lambda-i}$ (the action of $\mathcal{D}(\lambda)$ is defined in the natural way). Note that $M(p)$ is isomorphic to $\mathbb{C}(\lambda)[x_1, \cdots, x_n, p^{-1}]$ as a $\mathbb{C}(\lambda)[x_1, \cdots, x_n]$-module; namely, we can write $M(p) = \mathbb{C}(\lambda)[x_1, ..., x_n, p^{-1}]p^\lambda$.

THEOREM 1.1.10. *$M(p)$ is finitely generated over $\mathcal{D}(\lambda)$.*

Let us show that Theorem 1.1.7 and Theorem 1.1.10 are equivalent.

Theorem 1.1.10 $\Rightarrow$ Theorem 1.1.7: Denote by M_i the submodule of $M(p)$ generated by $p^{\lambda-i}$. Then $M_i \subset M_{i+1}$ and

$$M(p) = \bigcup_i M_i. \tag{1.1}$$

Assume that $M(p)$ is finitely generated. Then (1.1) implies that there exists $j \in \mathbb{Z}$ such that $M(p) = M_j$. In other words, the module $M(p)$ is generated by $p^{\lambda-j}$. Hence there exists $\widetilde{L} \in \mathcal{D}(\lambda)$ such that $\widetilde{L}p^{\lambda-j} = p^{\lambda-j-1}$.

Let σ_j be the $\mathbb{C}$-linear automorphism of $\mathbb{C}(\lambda)$ sending λ to $\lambda + j + 1$. Then σ_j extends to an automorphism of the algebra $\mathcal{D}(\lambda)$ (which we will also denote by σ_j) and we have $\sigma_j(\widetilde{L})p^{\lambda+1} = p^\lambda$. But $\sigma_j(\widetilde{L})$ can be written as $\frac{L}{b(\lambda)}$, where $L \in \mathcal{D}[\lambda]$ and $b(\lambda) \in \mathbb{C}[\lambda]$. Thus we have $Lp^{\lambda+1} = b(\lambda)p^\lambda$.

Theorem 1.1.7 $\Rightarrow$ Theorem 1.1.10: By shifting λ we see that for every integer $i > 0$ there exists a differential operator $L_i \in \mathcal{D}[\lambda]$ such that

$$L_i p^\lambda = b(\lambda - 1) \ldots b(\lambda - i)p^{\lambda-i}.$$

This clearly implies that p^λ generates $M(p)$.

We now want to prove Theorem 1.1.10. To do this, we need to develop some machinery. The proof will be completed in Subsection 1.2.6.

1.1.3. Filtrations. Let V be a vector space over a field k. Recall that an *increasing filtration* on V is a collection of k-subspaces $F_iV \subset V$ for $i \geq 0$ such that

1) $F_iV \subset F_{i+1}V$ for $i \geq 0$;
2) $V = \cup_{i\geq 0} F_iV$.

We agree that $F_iV = 0$ for $i < 0$, and will drop the word "increasing" when no confusion is possible.

If a space V is equipped with a filtration F, we can define the *associated graded space* $\mathsf{gr}^F V$ of V in the following way:

$$\mathsf{gr}^F V = \bigoplus_{i=0}^{\infty} F_iV/F_{i-1}V.$$

We set $\mathsf{gr}_i^F V = F_iV/F_{i-1}V$.

If $V = A$ is a unital associative algebra, we say that a filtration F on A is an *algebra filtration* (or (A, F) is a *filtered algebra*) if $1 \in F_0A$ and $F_iA \cdot F_jA \subseteq F_{i+j}A$. Then $\mathsf{gr}^F A$ has the natural structure of a graded algebra (i.e., we have $\mathsf{gr}_i^F A \cdot \mathsf{gr}_j^F A \subseteq \mathsf{gr}_{i+j}^F A$). We will sometimes drop the super-script F when it does not lead to confusion.

Similarly, let M be a left module over a filtered algebra A (with filtration F). Then a *module filtration* on M is a filtration on M (which we denote by the same letter F) such that $F_iA \cdot F_jM \subseteq F_{i+j}M$. As before, one can define

$$\mathsf{gr}^F M = \bigoplus_{j=0}^{\infty} F_jM/F_{j-1}M,$$

which in the case of a module filtration is a graded $\mathsf{gr}\, A$-module in an obvious way. The module $\mathsf{gr}^F M$ is called *the associated graded module* of M under F.

In the sequel, unless specified otherwise, any filtration on an algebra or a module is assumed to be an algebra filtration, respectively a module filtration.

DEFINITION 1.1.11. (1) A filtration F on a module M over a filtered algebra A is called a *good filtration* if $\mathsf{gr}^F M$ is finitely generated as a $\mathsf{gr}\, A$-module.
(2) Two filtrations F and F' on M are called *equivalent* if there exist integers j_0 and j_1 such that

$$F'_{j-j_0}M \subseteq F_jM \subseteq F'_{j+j_1}M \text{ for all } j \geq 0.$$

It is clear that the relation of equivalence of filtrations is an equivalence relation, which justifies the terminology.

PROPOSITION 1.1.12. (1) *Let F be a good filtration on a left A-module M, and let $\overline{m}_1, \overline{m}_2, \cdots, \overline{m}_n$ be generators of $\mathsf{gr}^F M$ over $\mathsf{gr}\, A$ of degrees $d_1, d_2, \cdots, d_n$. Let $m_1, m_2, \cdots, m_n$ be any lifts of $\overline{m}_1, \overline{m}_2, \cdots, \overline{m}_n$ to M. Then for any $j \geq 0$, $F_jM = F_{j-d_1}A \cdot m_1 + \cdots + F_{j-d_n}A \cdot m_n$. In particular, M is a finitely generated A-module with generators $m_1, \cdots m_n$.*
(2) *Assume that we have two filtrations F and F' on M such that F is good. Then there exists an integer j_1 such that $F_jM \subseteq F'_{j+j_1}M$ for any j.*
(3) *Any finitely generated A-module M has a good filtration.*

PROOF. (1) We have $\mathsf{gr}^F_j M = \mathsf{gr}_{j-d_1} A \cdot \overline{m}_1 + \cdots + \mathsf{gr}_{j-d_n} A \cdot \overline{m}_n$, so the statement follows by induction in j.
(2) Let j_1 be such that for all i, $m_i \in F'_{j_1}M$. Then by (1), $F_jM \subseteq F'_{j+j_1}M$ for all j.
(3) If $m_1, ..., m_n$ are generators of M, then we can define a filtration on M by setting $F_jM := \sum_{i=1}^n F_jA \cdot m_i$. Then the images $\overline{m}_i$ of m_i in $\mathsf{gr}^F_0 M$ are generators of $\mathsf{gr}^F M$. □

COROLLARY 1.1.13. *Any two good filtrations on a left A-module M are equivalent.*

PROOF. This clearly follows from Proposition 1.1.12(2). □

EXAMPLE 1.1.14. A finitely generated module M may nevertheless have a filtration which is not good. Indeed, let $A = M = \mathbb{C}[x]$, the filtration of A be defined by the grading with $\deg(x) = 1$, and the filtration on M be defined by $F_iM = M$ for all $i \geq 0$. Then x acts by 0 on $\mathsf{gr}\, M$, so $\mathsf{gr}\, M$ is an infinitely generated $A = \mathsf{gr}\, A$-module.

LEMMA 1.1.15. *If A is a filtered algebra and $\mathsf{gr}\, A$ is left (resp. right) Noetherian, then so is A.*

PROOF. Assume $\mathsf{gr}\, A$ is left Noetherian. Let M be a finitely generated left A-module. We have to prove that any submodule N of M is also finitely generated. Since M is finitely generated, it admits a good filtration F. On N, we have the induced filtration $F_iN = F_iM \cap N$. Since $\mathsf{gr}\, A$ is left Noetherian and $\mathsf{gr}\, M$ is finitely generated, we conclude that $\mathsf{gr}\, N$ is finitely generated as well and thus N is finitely generated, as desired.

Here is another proof: if $I_0 \subset I_1 \subset ...$ is a strictly ascending chain of left ideals in A then so is $\mathsf{gr}\, I_0 \subset \mathsf{gr}\, I_1 \subset \dots$ in $\mathsf{gr}\, A$. If $\mathsf{gr}\, A$ is left Noetherian, the second chain must be finite, so the first one must be finite as well. Hence A is left Noetherian. □

1.1.4. Two filtrations on $\mathcal{D}$. Here is our main example. As before, let $\mathcal{D}$ be the algebra of polynomial differential operators in n variables (over a field k). Let's define two filtrations on $\mathcal{D}$:

Bernstein's (or arithmetic) filtration: $F_0\mathcal{D} = k$, $F_1\mathcal{D} = k + \mathrm{span}(x_j, \frac{\partial}{\partial x_j})$, $F_i\mathcal{D}$ is the image of $F_1\mathcal{D}^{\otimes i}$ under the multiplication map.

Geometric filtration (filtration by order of differential operator) denoted by $\mathcal{D}_0 \subset \mathcal{D}_1 \subset ...$:
$\mathcal{D}_0 = \mathcal{O} = k[x_1, \ldots, x_n]$.
$\mathcal{D}_1 = \mathrm{span}(f \in \mathcal{O}; g\frac{\partial}{\partial x_j}$ where $g \in \mathcal{O})$, $\mathcal{D}_i$ is the image of ${\mathcal{D}_1}^{\otimes i}$ under the multiplication map.

The following lemma describes the algebra $\mathcal{D}$ explicitly as a vector space. The proof is easy and left to the reader.

Lemma 1.1.16. *For any $L \in \mathcal{D}$ there exists a unique decomposition*

$$L = \sum_{m\geq 0} \sum_{1\leq i_1\leq \cdots \leq i_m \leq n} p_{i_1,\ldots i_m} \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_m}}$$

where $p_{i_1 \ldots i_m} \in k[x_1, \ldots, x_n]$.

This lemma immediately implies:

Proposition 1.1.17. *For both filtrations $\mathrm{gr}\,\mathcal{D} \cong k[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$. Here the x_i are the images of the x_i and the ξ_j are the images of the $\frac{\partial}{\partial x_j}$.*

Proof. Let us show that for both filtrations, the x_i's and ξ_j's commute in $\mathrm{gr}\,\mathcal{D}$. For each $i \neq j$, we have $x_i\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j}x_i$ in $\mathcal{D}$ and hence in $\mathrm{gr}\,\mathcal{D}$. For $i = j$ we have $\frac{\partial}{\partial x_i}x_i - x_i\frac{\partial}{\partial x_i} = 1 \in F_0\mathcal{D}$ and hence $x_i\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i}x_i = 0$ in $\mathrm{gr}\,\mathcal{D}$.

It now follows easily from Lemma 1.1.16 that $\mathrm{gr}\,\mathcal{D}$ is a polynomial algebra in the x_i's and ξ_j's. □

For Bernstein's filtration the above argument shows a little more – namely that for all i, j, we have $[F_i\mathcal{D}, F_j\mathcal{D}] \subset F_{i+j-2}\mathcal{D}$ (note that for the geometric filtration we only have $[\mathcal{D}_i, \mathcal{D}_j] \subset \mathcal{D}_{i+j-1}$). We will need this fact below.

Corollary 1.1.18. *$\mathcal{D}$ is both left and right Noetherian.*

Proof. This follows immediately from Lemma 1.1.15 and the Hilbert basis theorem, as $\mathrm{gr}\,\mathcal{D} \cong k[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$. □

1.1.5. Exercises.

Exercise 1.1.1. (a) Show that if the Bernstein-Sato polynomial of P is b then for any positive integer m, the Bernstein-Sato polynomial of P^m divides $b(m\lambda)b(m\lambda + 1) \ldots b(m\lambda + m - 1)$.

(b) Show that the Bernstein-Sato polynomial of x^m is $(\lambda + \frac{1}{m})...(\lambda + \frac{m-1}{m})(\lambda + 1)$.

(c) Compute the Bernstein-Sato polynomial of $x_1^{m_1}...x_n^{m_n}$, and deduce that the Bernstein-Sato polynomial may have roots of arbitrarily high multiplicity.

Exercise 1.1.2. Show that the Bernstein-Sato polynomial of $P := x_1^2 + x_2^2 + \cdots + x_n^2$ is $b(\lambda) = (\lambda + 1)(\lambda + \frac{n}{2})$. (Use rotational symmetry).

Exercise 1.1.3. Prove that the Bernstein-Sato polynomial of a non-constant polynomial P is always a multiple of $\lambda + 1$.

Hint. Look at a neighborhood of a generic point of the divisor $P = 0$.

EXERCISE 1.1.4. Show that if 0 is a regular value of P then the Bernstein-Sato polynomial of P is $b(\lambda) = \lambda + 1$.

Hint. Show that there exists a vector field v and a polynomial F such that $vP = 1 - FP$. For this purpose, for any point $\mathbf{x}$ such that $P(\mathbf{x}) = 0$, complete P to a local coordinate system near $\mathbf{x}$ (using that $dP(\mathbf{x}) \neq 0$), and use it to define a local vector field $v_{\mathbf{x}}$ near $\mathbf{x}$ such that $v_{\mathbf{x}}P = 1$ (namely, $v_{\mathbf{x}} = \frac{\partial}{\partial P}$). This vector field is regular in an affine neighborhood $U_{\mathbf{x}}$ of $\mathbf{x}$. These neighborhoods form a Zariski open cover of the zero set $Z(P)$ of P, which has a finite subcover $U_1, ..., U_n$. So we have vector fields v_i on U_i such that $v_iP = 1$. Suppose that the complement of U_i is defined by the equation $f_i = 0$. Then we have regular vector fields $v_i' := f_i^{n_i}v_i$ such that $v_i'P = f_i^{n_i}$. Since f_i and P have no common zeros, there exist polynomials F and β_i such that $FP + \sum_i \beta_i f_i^{n_i} = 1$. Then one can take $v = \sum \beta_i v_i'$.

EXERCISE 1.1.5. Let $P(z)$ be the complex polynomial $(z-r_1)^{e_1}(z-r_2)^{e_2}\cdots(z-r_n)^{e_n}$, where r_i are distinct and $e_i \geq 1$. Prove that the Bernstein-Sato polynomial of P is the least common multiple of the polynomials $\prod_{j=1}^{e_i}(\lambda + \frac{j}{e_i})$ for $i = 1, ..., n$.

EXERCISE 1.1.6. Let $X = (x_{ij})$ be an n-by-n matrix. (a) Show that

$$\int_{X>0} (\det X)^\lambda e^{-\operatorname{Tr}(X)} = C\Gamma(\lambda+1)\Gamma(\lambda+2)...\Gamma(\lambda+n),$$

where the integral is taken over positive definite Hermitian matrices, and C is a constant.

Hint. Let x_i be the eigenvalues of X. Write the integral in question as

$$\int_{0<x_1<\cdots<x_n} \prod_i x_i^\lambda \prod_{i<j}(x_i - x_j)^2 e^{-\sum x_i} d\mathbf{x}.$$

and compute it as a limiting case of the Selberg integral

$$\int_{0<x_1<\cdots<x_n<1} \prod_i x_i^\lambda (1-x_i)^\beta \prod_{i<j}(x_i - x_j)^2 d\mathbf{x},$$

making a change of variables $x_i \mapsto x_i/\beta$ and sending β to ∞ (using that $\lim_{\beta\to+\infty}(1-\frac{x}{\beta})^\beta = e^{-x}$).

(b) Deduce that the Bernstein-Sato polynomial $b(\lambda)$ of $P := \det X$ (as a polynomial of x_{ij}) is divisible by $(\lambda+1)(\lambda+2)\ldots(\lambda+n)$.

(c) Prove the Cauchy identity:

$$Lf^{\lambda+1} = (\lambda+1)\ldots(\lambda+n)f^\lambda,$$

where $L := \det(\partial/\partial x_{ij})$, and deduce that

$$b(\lambda) = (\lambda+1)(\lambda+2)\ldots(\lambda+n).$$

Hint. Use that the (multivalued) function $g := Lf^{\lambda+1}$ satisfies $g(AX) = (\det A)^\lambda g(X)$ for an invertible matrix A. Conclude that $Lf^{\lambda+1} = c(\lambda)f^\lambda$ for some monic polynomial c of degree n divisible by $b(\lambda)$. The use part (b) to show that $c(\lambda) = b(\lambda)$ and determine $b(\lambda)$.

EXERCISE 1.1.7. Let P be a real polynomial in n variables. Let U be a connected component of the region $P > 0$. Let f be a compactly supported smooth function on $\mathbb{R}^n$.

(a) Consider the function

$$H_f(t) := \int_{U\cap\{P<t\}} f(\mathbf{x})d\mathbf{x}$$

(note that $H_f(t) = 0$ when $t < 0$). Show that $H_f(t)$ is continuous, piecewise smooth, and $dH_f(t)$ has compact support.

(b) Show that the Mellin transform of the signed measure $dH_f(t)$, $\int_0^\infty t^\lambda dH_f(t)$, equals

$$J_f(\lambda) := \int_U P(\mathbf{x})^\lambda f(\mathbf{x})d\mathbf{x}$$

for $\mathrm{Re}\lambda \geq 0$.

(c) Use the Mellin inversion formula to show that

$$t\frac{dH_f(t)}{dt} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-\lambda} J_f(\lambda)d\lambda$$

for $c > 0$. Now move the contour to the left, and keep track of the residues picked up on the way. Applying Theorem 1.1.6 (constraining the poles of $J_f(\lambda)$), show that the function $t\frac{dH_f(t)}{dt}$ has an asymptotic expansion whose terms are of the form $t^{-r}(\log t)^j$ with some coefficients, where exponents r (which are negative rational numbers) belong to finitely many arithmetic progressions of the form $a, a-1, ...,$ and j takes finitely many nonnegative integer values. In other words, there exist $r_1, ..., r_p$, m such that we have an asymptotic (in general, non-convergent) expansion at $t = 0$:

$$t\frac{dH_f(t)}{dt} = \sum_{i=0}^{\infty}\sum_{j=0}^{m}\sum_{l=1}^{p} C_{ijl}t^{i-r_l}(\log t)^j, t \to 0.$$

Namely, $r_1, ..., r_p$ are the roots of the Bernstein-Sato polynomial of P.

(d) Consider the integral

$$I_f(\beta) := \int_U \exp(-\beta P(\mathbf{x}))f(\mathbf{x})d\mathbf{x},\ \beta > 0.$$

Show that as $\beta \to +\infty$, the function $I_f(\beta)$ has an asymptotic expansion

$$I_f(\beta) = \sum_{i=0}^{\infty}\sum_{j=0}^{m}\sum_{l=1}^{p} \widetilde{C}_{ijl}\beta^{r_l-i}(\log\beta)^j,\ \beta \to +\infty.$$

Hint. Show that $I_f(\beta)$ is the Laplace transform of $dH_f(t)$.

(e) Suppose that $P \geq 0$ and the sets $B_t := \{\mathbf{x} : P(\mathbf{x}) \leq t\}$ are compact for all $t \geq 0$. Show that the volume $\mathrm{Vol}(B_t)$ has an asymptotic expansion as in (c).

EXERCISE 1.1.8. (Oscillatory integrals) This is a version of the previous exercise when β is replaced by $i\beta$. Consider the integral

$$I_f^*(\beta) = \int_{\mathbb{R}^n} \exp(i\beta P(\mathbf{x}))f(\mathbf{x})d\mathbf{x}.$$

(a) Let I be the ideal in $\mathbb{C}[x_1, ..., x_n]$ generated by the partial derivatives of P. Show that if $f \in I^{2m-1}C_c^\infty(\mathbb{R}^n)$, then $I_f^*(\beta) = O(\beta^{-m})$ as $\beta \to +\infty$ (where $m \geq 1$).

Hint. Use that

$$\int_{\mathbb{R}^n} \exp(i\beta P)\frac{\partial P}{\partial x_j}fd\mathbf{x} = i\beta^{-1}\int_{\mathbb{R}^n} \exp(i\beta P)\frac{\partial f}{\partial x_j}d\mathbf{x}.$$

(b) Suppose 0 is a critical value of P, and P does not take other critical values in the support of f. Show that the function $H_f(t)$ from Exercise 1.1.7 is smooth for $t > 0$. Deduce that the Fourier transform of $dH_f(t)$ has an asymptotic expansion with terms $\beta^r(\log\beta)^j$ as in Exercise 1.1.7(d). Conclude that $I_f^*(\beta)$ also has such an expansion.

(c) Using (a) and (b), write an asymptotic expansion for $I_f^*(\beta)$ for any polynomial P and any f (the terms of this expansion should be $e^{i\beta P_l}\beta^r(\log\beta)^j$, where P_l are the critical values of P).

Hint. Use that the number of critical values is finite.

This result and its proof using resolution of singularities can be found in [**AGV**].

1.2. Bernstein's inequality and its applications

1.2.1. The Gelfand-Kirillov dimension of a module. Let A be a filtered algebra such that $\dim_k F_iA < \infty$ and $\mathsf{gr}\, A \cong k[y_1, \ldots, y_m]$ (with $\deg y_i = 1$). Let M be an A-module with a good filtration F. In this section we only consider nonzero modules. Define $h_F(M,j) = \dim_k F_jM$.

THEOREM 1.2.1. *There exists a polynomial $h_F(M)(t)$ (called the Hilbert polynomial of M with respect to F) such that $h_F(M,j) = h_F(M)(j)$ for any $j \gg 0$. This polynomial has the form $h_F(M)(t) = \frac{ct^d}{d!} + \{\text{lower order terms}\}$, where $d \le m$ and c is a positive integer.*

PROOF. This follows from the Hilbert syzygy theorem in commutative algebra (which asserts the same properties for a finitely generated graded module over a polynomial ring), as $h_F(M,j) = h_{\mathsf{gr}\,F}(\mathsf{gr}\, M, j)$, where $\mathsf{gr}\, F$ is the filtration induced by the grading on $\mathsf{gr}\, M$. $\square$

LEMMA 1.2.2. *The values of c and d in Theorem 1.2.1 depend only on M (and not on F).*

PROOF. Let F and F' be good filtrations. Then by Corollary 1.1.13 there exist j_0 and j_1 such that

$$F'_{j-j_0}M \subseteq F_jM \subseteq F'_{j+j_1}M$$

and hence $h_{F'}(j-j_0) \le h_F(j) \le h_{F'}(j+j_1)$. This can be true only if h_F and $h_{F'}$ have the same degree and the same leading coefficient. $\square$

DEFINITION 1.2.3. For a finitely generated module M, let $d_F(M)$ be the value of d in Theorem 1.2.1. We call $d_F(M)$ the *Gelfand-Kirillov* or *functional* dimension of M with respect to the filtration F (on A).

For the rest of this subsection, F will always refer to Bernstein's filtration, and we will write $d(M)$ rather than $d_F(M)$. We will also use the notation $c(M)$ for the integer such that the leading coefficient of the Hilbert polynomial of $\mathsf{gr}\, M$ is $c(M)/d(M)!$. It will be convenient for us to set $c(M) = 0$ if $M = 0$.

Let $\mathrm{char}(k) = 0$.

THEOREM 1.2.4. *[Bernstein's inequality] For any finitely generated nonzero module M over $\mathcal{D} = \mathcal{D}(\mathbb{A}^n)$ we have $d(M) \ge n$.*

Before proving Theorem 1.2.4, we want to derive some very important corollaries of it. In particular, we will explain how Theorem 1.2.4 implies the results stated in the previous subsection.

REMARK 1.2.5. This theorem was first proved in J. Bernstein's thesis, then a simple proof was given by A. Joseph. Then O. Gabber proved in **[G]** a very general theorem which we will discuss later (Gabber's theorem implies the corresponding inequality also for the geometric filtration).

1.2.2. Examples.

EXAMPLE 1.2.6. Let $n \ge 1$ and M be a finitely generated $\mathcal{D}$-module. Suppose $\dim_k M < \infty$. Then for any two operators on M the trace of their commutator must be 0. But $[\frac{\partial}{\partial x_1}, x_1] = 1$ in $\mathcal{D}$, and the trace of the identity equals the dimension of M. This implies that $M = 0$. Thus, if $M \neq 0$ then $\dim_k M = \infty$, i.e., $d(M) \ge 1$, which, in particular, implies Bernstein's inequality for $n = 1$.

EXAMPLE 1.2.7. Consider $\mathcal{O} = k[x_1, \ldots, x_n]$ with its natural structure of a $\mathcal{D}$-module. Let $F_i\mathcal{O}$ be the space of polynomials of degree less than or equal to i. Then $\dim_k F_i\mathcal{O} = \binom{n+i}{n}$ is a polynomial of degree n with leading coefficient $\frac{1}{n!}$. Thus $d(M) = n$ and $c(M) = 1$.

Example 1.2.8. Let $n = 1$ and fix $a \in k$. Define the $\mathcal{D}$-module δ_a of δ-functions as the module with basis $\{\delta_a^{(m)}\}_{m=0}^{\infty}$ and the following action of $\mathcal{D}$:

$$\frac{d}{dx}(\delta_a^{(m)}) = \delta_a^{(m+1)}$$
$$(x-a)\delta_a^{(m)} = -m\delta_a^{(m-1)}$$
$$(x-a)\delta_a^{(0)} = 0.$$

(if $k = \mathbb{C}$ and $a \in \mathbb{R}$, this $\mathcal{D}$-module can be realized inside distributions on the real line, by taking $\delta_a^{(m)} = \frac{d^m}{dx^m}\delta(x-a)$.) It is easy to see that $d(\delta_a) = 1$ and $c(\delta_a) = 1$.

Exercise 1.2.1. Show that the $\mathcal{D}$-modules $\mathcal{O}$ and δ_a are simple.

1.2.3. Holonomic $\mathcal{D}$-modules.

Let M be a $\mathcal{D}$-module.

Definition 1.2.9. If $d(M) = n$ or $M = 0$ then M is called holonomic.

For example, $\mathcal{O}$ and δ_a are holonomic.

Exercise 1.2.2. Show that every simple $\mathcal{D}$-module M has $d(M) \leq 2n-1$. In particular, for $n = 1$, any simple $\mathcal{D}$-module is holonomic.

Remark 1.2.10. For a long time, it was unknown whether all simple $\mathcal{D}$-modules are holonomic. Then T. Stafford found a counterexample in [**St**]. Later J. Bernstein and V. Lunts [**BL**] constructed a lot of non-holonomic simple $\mathcal{D}$-modules for any $n \geq 2$ (of $d(M) = 2n-1$). Namely, one can take $M = \mathcal{D}/\mathcal{D}L$, where L is a "random" element of $\mathcal{D}$ degree $m \geq 3$. The fact that such a module is simple is related to the fact that a generic level surface of the leading part $L_0 \in \mathrm{gr}_m\mathcal{D}$ of L has no proper invariant subvarieties under the Hamiltonian flow defined by L_0. For instance, it is known from classical mechanics that if $U(x_1, ..., x_n)$ is a "random" polynomial potential of degree $m \geq 3$, then the Hamiltonian flow defined by the Hamiltonian $H := \frac{1}{2}\sum p_i^2 + U$ is "chaotic", i.e., does not have invariant algebraic subvarieties on a generic level surface of H.

Let A be a filtered algebra such that $\mathsf{gr}\, A$ is Noetherian, and let

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of nonzero A-modules. Let $\{F_jM_2\}$ be a good filtration on M_2. Then define F_jM_1 to be $F_jM_2 \cap M_1$ and F_jM_3 to be the image of F_jM_2. We have a short exact sequence

$$0 \to \mathsf{gr}^F M_1 \to \mathsf{gr}^F M_2 \to \mathsf{gr}^F M_3 \to 0.$$

In fact, $\{F_jM_1\}$ and $\{F_jM_3\}$ are good filtrations ($\mathsf{gr}^F M_3$ is finitely generated because it is a quotient of the finitely generated module $\mathsf{gr}^F M_2$, while $\mathsf{gr}^F M_1$ is a $\mathsf{gr}\, A$- submodule of $\mathsf{gr}^F M_2$ and is thus finitely generated, as $\mathsf{gr}\, A$ is Noetherian).

Proposition 1.2.11. *Using the above notations*

(1) $d(M_2) = \max(d(M_1), d(M_3))$;
(2) *If* $d(M_1) = d(M_2) = d(M_3)$, *then* $c(M_2) = c(M_1) + c(M_3)$;
(3) *If* $d(M_1) > d(M_3)$, *then* $c(M_2) = c(M_1)$ *and if* $d(M_3) > d(M_1)$, *then* $c(M_2) = c(M_3)$.

Proof. Using the above exact sequence, we get

$$h_F(M_2, i) = h_F(M_1, i) + h_F(M_3, i).$$

All the statements of the proposition follow from this fact. $\square$

Corollary 1.2.12. *Let M be a holonomic module and let $c = c(M)$. Then the length of M is less than or equal to c.*

PROOF. Suppose we have an exact sequence

$$0 \to N \to M \to N' \to 0.$$

Then we have $d(M) = d(N) = d(N') = n$ (since by Bernstein's inequality we have $d(N)$, $d(N') \geq n$) and $c(M) = c(N)+c(N')$. More generally, the same argument shows that if there is a filtration on M with successive quotients $N_1, \cdots, N_m$, then each N_i is holonomic and $c(N_1)+c(N_2)+\cdots+c(N_m) = c(M)$. This shows that no strictly increasing filtration on M can have length more than $c(M)$, so M has length at most $c(M)$. □

REMARK 1.2.13. It is easy to see that the length of M can be strictly smaller than $c(M)$. For example, for each $\lambda \in k$ we can consider $M(x, \lambda) := k[x, x^{-1}]x^{\lambda}$ with the natural $\mathcal{D}(\mathbb{A}^1)$-module structure. For this module we have
1) $c(M(x, \lambda)) = 2$; and
2) $M(x, \lambda)$ is irreducible $\Leftrightarrow \lambda \notin \mathbb{Z}$.

1.2.4. A criterion of holonomicity. We now come to a key generalization of Corollary 1.2.12.

COROLLARY 1.2.14. *Let M be any module over $\mathcal{D}$, and let F_jM be a (not necessarily good) filtration on M. Assume that there exists a polynomial $h \in \mathbb{R}[t]$ such that $h(t) = \frac{ct^n}{n!}+$ lower order terms with $c \geq 0$ and $\dim_k F_jM \leq h(j)$ for $j \gg 0$. Then M is holonomic and length$(M) \leq c$.*

PROOF. Let N be any finitely generated submodule of M. Let's prove that N is holonomic and $c(N) \leq c$.

We may assume that $N \neq 0$. Consider the induced filtration $F_jN = F_jM \cap N$ on N. Let F'_jN be a good filtration on N such that $F'_jN \subset F_jN$ (to construct such a filtration, choose j such that F_jN generates N and set $F'_iN = F_iN$ for $i \leq j$ and $F'_iN = F_{i-j}\mathcal{D} \cdot F_jN$ for $i > j$). Then we have $\dim F'_jN \leq \dim F_jN$ and hence $h_{F'}(N)(j) \leq h(j) = \frac{cj^n}{n!}$ + lower order terms. By Theorem 1.2.4 $d(N) \geq n$, hence $h_{F'}(N)(j) = \frac{c'j^n}{n!}$ + lower order terms, where $c' \leq c$. Corollary 1.2.12 now implies that N is holonomic and has length less or equal to c.

Now let $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_m \subset M$ be such that M_i are finitely generated, and $M_{i+1} \neq M_i$. Then M_i are holonomic for all i, and $c(M_i) \geq i$. Thus implies that $m \leq c$. Thus, M is holonomic (in particular, finitely generated) and *length*$(M) \leq c$ (had M been infinitely generated, we could have made m arbitrarily large). □

1.2.5. Proof of Theorem 1.1.10. Let $p \in k[x_1, \dots x_n]$. Recall that in Subsection 1.1.2 we defined a $\mathcal{D}(\lambda)$-module $M(p)$ by setting

$$M(p) = \{qp^{\lambda-i} \mid (qp)p^{\lambda-i} = qp^{\lambda-i+1}\}.$$

The following result is a strengthening of Theorem 1.1.10 stated in Subsection 1.1.2.

THEOREM 1.2.15. *$M(p)$ is holonomic. In particular, it is finitely generated.*

PROOF. By Corollary 1.2.14, it is enough to find a filtration F on $M = M(p)$ for which we have $\dim_k F_jM \leq h(j)$, where $h(x)$ is a polynomial of degree n. Let

$$F_jM(p) = \{qp^{\lambda-j} \mid \deg q \leq j(m+1)\}$$

for any $j \geq 0$ (here $m = \deg p$).

Let us show that this is a filtration. It is clear that $F_{j-1}M \subset F_jM$ and $M = \bigcup F_jM$. As $\mathcal{D}$ is generated in degree 1, it only remains to show that $F_1\mathcal{D} \cdot F_jM \subset F_{j+1}M$.

For any $i \in \{1, \dots n\}$ we have

$$x_i \cdot (qp^{\lambda-j}) = (x_iqp)p^{\lambda-j-1} \in F_{j+1}M,$$

since $\deg(x_i qp) = \deg q + m + 1 \leq j(m+1) + m + 1 = (j+1)(m+1)$. We also have

$$\frac{\partial}{\partial x_i}(qp^{\lambda-j}) = \frac{\partial q}{\partial x_i}p^{\lambda-j} + (\lambda-j)qp^{\lambda-j-1}\frac{\partial p}{\partial x_i} =$$
$$= \left(p\frac{\partial q}{\partial x_i} + (\lambda-j)q\frac{\partial p}{\partial x_i}\right)p^{\lambda-j-1} \in F_{j+1}M,$$

since $\deg\left(p\frac{\partial q}{\partial x_i} + (\lambda-j)q\frac{\partial p}{\partial x_i}\right) = \deg q + m - 1 \leq (j+1)(m+1)$.

So F is really a filtration.

It is easy to see that $\dim_k F_jM = \binom{j(m+1)+n}{n}$. Thus, $\dim_k F_jM$ is a polynomial of degree n in j, and by Corollary 1.2.14, M is holonomic. □

1.2.6. Proof of Bernstein's inequality and end of proof of Theorem 1.1.6. Let us now prove Theorem 1.2.4. We begin with the following

LEMMA 1.2.16. *Let M be a module over $\mathcal{D}$ with a good filtration F. Assume F_0M is nonzero. Then the natural linear map*

$$F_i\mathcal{D} \longrightarrow \mathrm{Hom}(F_iM, F_{2i}M)$$

is an embedding for any i.

PROOF. We will prove Lemma 1.2.16 by induction on i.
1) For $i = 0$ the statement is clear, because $F_0\mathcal{D} = k$.
2) Suppose the statement is true for all $i' < i$. Let $a \in F_i\mathcal{D}$ be such that

$$a = \sum_m \sum_{i_1 \leq \cdots \leq i_m} p_{i_1,\dots i_m}\frac{\partial}{\partial x_{i_1}}\cdots\frac{\partial}{\partial x_{i_m}},\ p_{i_1,\dots,i_m} \in k[x_1,\dots,x_n].$$

We may assume that a is not constant. Suppose $\frac{\partial}{\partial x_m}$ occurs in the expression for a with a nonzero coefficient. Then $[a, x_m] \neq 0$. Similarly, if x_m occurs in the expression for a with a nonzero coefficient, then $[a, \frac{\partial}{\partial x_m}] \neq 0$. As Bernstein's filtration has the property that $[F_j\mathcal{D}, F_{j'}\mathcal{D}] \in F_{j+j'-2}\mathcal{D}$, $[a, x_m]$ and $[a, \frac{\partial}{\partial x_m}]$ must be in $F_{i-1}\mathcal{D}$.

Suppose that $[a, x_m] \neq 0$ (the other case is treated similarly). We have to show that $a(F_iM) \neq 0$. By the induction hypothesis, there exists $\alpha \in F_{i-1}M$ such that $[a, x_m](\alpha) \neq 0$. But if $a(F_iM) = 0$, then

$$[a, x_m](\alpha) = ax_m(\alpha) - x_m a(\alpha) = a(x_m(\alpha)) - x_m(a(\alpha)) = 0$$

and we get a contradiction. So $a(F_iM) \neq 0$ and the map

$$F_i\mathcal{D} \longrightarrow \mathrm{Hom}(F_iM, F_{2i}M)$$

is an embedding. □

It remains to explain how Lemma 1.2.16 implies Theorem 1.2.4.
Shift F so that F_0M is nonzero. We know that $\dim_k F_i\mathcal{D} = \frac{i^{2n}}{(2n)!}$ + lower order terms.
But by Lemma 1.2.16, $\dim_k F_i\mathcal{D} \leq \dim \mathrm{Hom}(F_iM, F_{2i}M) = h_F(M,i)h_F(M,2i)$, where $h_F(M,i) = \dim F_iM = \frac{ci^d}{d!}$ + lower order terms. Thus

$$\frac{i^{2n}}{(2n)!} + \text{lower order terms} \leq c^2\frac{i^d(2i)^d}{(d!)^2} + \text{lower order terms}$$

for sufficiently large i. This implies that $n \leq d$.

■

1.2.7. Exercises.

EXERCISE 1.2.3. For $\lambda \in k$, let $M(x,\lambda)$ denote the $\mathcal{D} = \mathcal{D}(\mathbb{A}^1)$-module with basis $x^{\lambda+i}$ for all $i \in \mathbb{Z}$ and the standard action of $\mathcal{D}$ (note that here λ is an actual element of k and not a variable). It is clear that $M(x,\lambda) = M(x,\lambda+1)$.

a) Show that $M(x,\lambda)$ is holonomic.

b) Show that $M(x,\lambda)$ is irreducible if and only if $\lambda \notin \mathbb{Z}$, and has length 2 if $\lambda \in \mathbb{Z}$.

c) There is an obvious homomorphism $\phi : \mathcal{D}(\mathbb{A}^1)/\mathcal{D}(\mathbb{A}^1)(x\partial - \lambda) \to M(x,\lambda)$ sending 1 to x^λ. For which λ is it an isomorphism?

d) Generalize b) to the case of an arbitrary polynomial p in n variables. Namely, show that $M(p,\lambda) := \mathbb{C}[x_1,...,x_n,p^{-1}]p^\lambda$ is irreducible if and only if λ is not an integer translate of a root of the Bernstein-Sato polynomial $b = b_p$ of p.

Hint. To prove the "if" direction, show that $M(p,\lambda)$ is generated by $p^{\lambda+r}$ for any r, and then show that any vector $fp^{\lambda+m} \in M(p,\lambda)$ (where $f \in k[x_1,...,x_n,p^{-1}]$) can be mapped to $f^{\lambda+r}$ for a suitable r by an element of $\mathcal{D}$.

To prove the "only if" direction, assume the contrary, and, by shifting λ by an integer, take it to be a root of $b(z)$ but not a root of $b(z-i)$ for $i = 1,2,3,\ldots$. Pick an element $K \in \mathcal{D}$ such that $Kp^{\lambda+1} = p^\lambda$, and for an indeterminate z write Kp^{z+1} as $p^z + (z-\lambda)fp^{z-m+1}$, where m is the order of K and $f \in k[x_1,...,x_n,z]$. On the other hand, pick $L_m \in \mathcal{D}[z]$ such that $L_m p^{z+1} = b(z)b(z-1)...b(z-m+1)p^{z-m+1}$. Now consider

$$Q := fL_m - (z-\lambda)^{-1}b(z)b(z-1)...b(z-m+1)K.$$

Show that $Qp^{z+1} = (z-\lambda)^{-1}b(z)b(z-1)...b(z-m+1)p^z$, and derive a contradiction.

EXERCISE 1.2.4. Let $f(x)$ be a nonzero rational function in one complex variable, and let N_f be the $\mathcal{D}$-module on the affine line generated by (a branch of) the multivalued analytic function

$$\exp\left(\int f(x)dx\right).$$

(A possible warmup is to first consider the case $f(x) = x^m$ for some integer m.)

a) Show that N_f is holonomic.

b) Write $f = P/Q$ where P, Q are relatively prime polynomials. Construct a natural homomorphism $\psi : \mathcal{D}(\mathbb{A}^1)/\mathcal{D}(\mathbb{A}^1)(Q\partial - P) \to N_f$. For which f is ψ an isomorphism?

c) Find the composition factors of N_f and the number $c = c(N_f)$.

d) For which f, g is N_f isomorphic to N_g?

1.3. More on $\mathcal{D}(\mathbb{A}^n)$-modules

1.3.1. $\mathcal{D}$-modules and systems of linear PDE. One of the motivations for the theory of $\mathcal{D}$-modules is the correspondence between $\mathcal{D}$-modules and systems of linear partial differential equations. Namely, assume that we have a system of differential equations of the form

$$\sum_{i=1}^{m} L_{ij}(f_i) = 0 \qquad j = 1,2,... \tag{1.1}$$

on m functions (or distributions) f_i on $\mathbb{R}^n$ with L_{ij} being differential operators with polynomial coefficients. Then we can consider the $\mathcal{D}$-module M generated by m elements $\xi_1,...,\xi_m$ with relations given by the same formulas as in (1.1). In this case solutions of the system (1.1) in the space $C^\infty(\mathbb{R}^n)$ (or the space of distributions $\mathrm{Dist}(\mathbb{R}^n)$, or any other similar space) are the same as elements of $\mathrm{Hom}_{\mathcal{D}}(M, C^\infty(\mathbb{R}^n))$. In other words, we can think about a system of linear differential equations (with polynomial coefficients) as a $\mathcal{D}$-module together with a choice of generators. In some sense, the main point of the theory of $\mathcal{D}$-modules is that the particular choice of generators is "irrelevant".

Note that $\mathcal{D}$ being Noetherian implies that it is always enough to consider finitely many equations in (1.1), as these equations are elements of the kernel of the natural map $\mathcal{D}^m \to M$ sending $(L_1, ..., L_m)$ to $L_1(\xi_1) + ... + L_m(\xi_m)$, and this kernel is finitely generated.

For example, consider the case when M is generated by one element ξ (such modules are called *cyclic*). In this case we have $M = \mathcal{D}/I$, where I is a left ideal. If I is generated by $L_1, \ldots, L_k$, then $d(M) \geq n - k$, and when $L_1, ..., L_k$ are in general position, we have an equality. Thus, Bernstein's inequality tells us that if a system of differential equations $L_1 f = 0, L_2 f = 0, ..., L_k f = 0$ with generic $L_1, ..., L_k$ is consistent (i.e., has nonzero solutions in some function space), then $k \leq n$.

1.3.2. Cyclicity of holonomic $\mathcal{D}$-modules. In some sense, there are a lot of cyclic modules over $\mathcal{D}$. For example, every holonomic module is cyclic. The proof is based on the following observation.

LEMMA 1.3.1. *$\mathcal{D}$ is a simple algebra, i.e., it has no proper two-sided ideals.*

PROOF. Assume that $I \subset \mathcal{D}$ is a two-sided ideal and $0 \neq L \in I$. Thus there exists $i \geq 0$ such that $L \in F_i\mathcal{D}$. We know that there exists x_α or $\frac{\partial}{\partial x_\beta}$ such that either $[L, x_\alpha] \neq 0$ or $[L, \frac{\partial}{\partial x_\beta}] \neq 0$. Both of these commutators are in I, since I is a two-sided ideal, and are in $F_{i-1}\mathcal{D}$, since Bernstein's filtration satisfies $[F_a, F_b] \subseteq F_{a+b-2}$. Repeating this process, we eventually get a nonzero $L' \in I$ that is also in $F_0\mathcal{D} = k$. Thus $I = \mathcal{D}$.

□

LEMMA 1.3.2. *Let A be a simple algebra which has infinite length as a left A-module. Then every A-module of finite length is cyclic.*

PROOF. By induction on the length of M it is enough to show that if we have an exact sequence of A-modules

$$0 \to K \to M \xrightarrow{\pi} N \to 0,$$

where $K \neq 0$ is simple and N is cyclic of finite length, then M is also cyclic.

Let n be a generator of N and let $I = \mathrm{Ann}_A(n)$ be the annihillator of n in A. If I were 0, then N would be a free module of rank 1 and thus would not have finite length. So I must be nonzero.

Choose some $m \in \pi^{-1}(n)$ and let $M' = A \cdot m$. Then $\pi : M' \to N$ is surjective. The kernel of this map is contained in the simple module K, so either the kernel is K or π is an isomorphism. If the kernel is K, then we must have $M' \cong M$, in which case M is cyclic. On the other hand, if π is an isomorphism, then $\mathrm{Ann}_A(m) = I$. Varying our choice of m by an element $v \in K$, we see that either M is cyclic or for any $v \in K$, we have $\mathrm{Ann}_A(m + v) = I$. In the latter case, we get that $I \cdot v = 0$ for any $v \in K$.

Now let $I' = \bigcap_{v \in K} \mathrm{Ann}_A(v)$. I' is a two-sided ideal in A containing I. Since A is simple, either $I' = A$ or $I' = 0$. As K is nonzero, we cannot have $I' = A$. But if $I' = 0$ then we also have $I = 0$, a contradiction. □

REMARK 1.3.3. The condition that A have infinite length as a left A-module is necessary. For example, if $A = k$, the module k^m is not cyclic for $m > 1$. More generally, if $A = \mathrm{Mat}_n(k)$, the module $(k^n)^m$ is not cyclic for $m > n$.

COROLLARY 1.3.4. *Any $\mathcal{D}$-module of finite length is cyclic.*

PROOF. It is easy to see that $\mathcal{D}$ has infinite length as a module over itself. Therefore, the result follows from Lemma 1.3.1 and Lemma 1.3.2. □

COROLLARY 1.3.5. *Any holonomic $\mathcal{D}$-module is cyclic.*

PROOF. We have proved that holonomic $\mathcal{D}$-modules have finite length. Hence by Corollary 1.3.4, holonomic $\mathcal{D}$-modules are cyclic. □

1.3.3. The singular support and singular cycle of a $\mathcal{D}$-module. Let N be a finitely generated module over $k[x_1, ..., x_m]$. Recall that in this case one can define the *support* $\mathrm{supp}N$, a closed subvariety of $\mathbb{A}^m$. Namely, $\mathrm{supp}N$ is the zero set of the annihilator $\mathrm{Ann}(N)$ of N.

Recall also that if Z is an irreducible component of $\mathrm{supp}N$, then we can define the *rank* of N on Z. To do so, let $J := \sqrt{\mathrm{Ann}(N)}$ be the radical of the annihilator, and consider the finite decreasing filtration $N \supset JN \supset J^2N \dots$. Then $\mathrm{gr}N$ is supported on $\mathrm{supp}N$ scheme-theoretically, i.e., any polynomial f vanishing on $\mathrm{supp}N$ annihilates $\mathsf{gr}\, N$. Then the rank of N on Z is, by definition, the dimension of the fiber of $\mathsf{gr}\, N$ over a generic point of Z.

Finally, recall that for any finitely generated $k[y_1, \cdots y_m]$-module N, we can canonically define the *support cycle* of N in $\mathbb{A}^m$. Namely, assume that $Z_1, ..., Z_p$ are the irreducible components of $\mathrm{supp}N$ of top dimension $d(N) := \dim(\mathrm{supp}N)$. Define m_i to be the rank of N on Z_i. Then we define the support cycle of N, $SC(N) := \sum m_i Z_i$. It is clear that $SC(M \oplus N) = SC(M) + SC(N)$, and, more generally, if P is an extension of M by N then $SC(P) = SC(M) \oplus SC(N)$. [1] Moreover, if N is graded then $SC(N)$ is invariant under the natural action of $\mathbb{G}_m$, and thus $SC(N)$ defines a cycle $SC_{\mathrm{proj}}(N)$ in $\mathbb{P}^{m-1}$. The degree of this cycle will be $c(N)$, the leading coefficient of the Hilbert polynomial of N multiplied by $d(N)!$.

The same definitions can be made in the more general case, when $k[y_1, ..., y_m]$ is replaced by $\mathcal{O}(Y)$, the algebra of regular functions on an irreducible affine algebraic variety Y (for the graded version, Y should be a conical variety). We will discuss this more general setting later.

Now let M be a $\mathcal{D}$-module.

DEFINITION 1.3.6.
(1) Let F be a good filtration on M with respect to Bernstein's filtration on $\mathcal{D}$. The variety $\mathrm{supp}(\mathsf{gr}^F M)$ is called the *arithmetic singular support* of M. It will be denoted by $SS^a(M)$.
(2) Let F be a good filtration on M with respect to the geometric filtration on $\mathcal{D}$. The variety $\mathrm{supp}(\mathsf{gr}^F M)$ is called the *geometric singular support* of M (or just the *singular support* of M). It will be denoted by $SS(M)$.

It is easy to see that the arithmetic singular support $SS^a(M)$ is invariant under the standard action of $\mathbb{G}_m$, dilating all coordinates by the same scalar. The geometric singular support $SS(M)$ is invariant under the following action of $\mathbb{G}_m$ on $\mathbb{A}^{2n}$: $\lambda(x_i) = x_i$ and $\lambda(\xi_j) = \lambda\xi_j$.

The following lemma shows that $SS^a(M)$ and $SS(M)$ are, in fact, independent on the filtration F (which is why we don't specify F in the notation).

LEMMA 1.3.7. *Let M be a $\mathcal{D}$-module, F a good filtration on it, and $\mathsf{gr}^F M$ the corresponding graded module. Let $I_F = \mathrm{Ann}(\mathsf{gr}^F M)$. Then $\sqrt{I_F}$ does not depend on F.*

PROOF. Let F and F' be good filtrations on M. Then by Corollary 1.1.13 they are equivalent, i.e, $F^{'}_{j-j_0}M \subset F_jM \subset F^{'}_{j+j_1}M$ for some j_0 and j_1. Let $t = j_0 + j_1 + 1$.

Let $\bar{f}$ be an element of $k[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$ such that $\deg \bar{f} = p$ and $\bar{f} \in \sqrt{I_F}$. Lift $\bar{f}$ to some $f \in F_p\mathcal{D}$. Since $\bar{f} \in \sqrt{I_F}$, there exists an integer q such that $f^q \cdot F_iM \subset F_{i+pq-1}M$. Then $f^{qt} \cdot F^{'}_iM \subset f^{qt} \cdot F_{i+j_0}M \subset F_{i+tpq-t+j_0}M \subset F^{'}_{i+tpq-1}M$. This means that $\bar{f}^{qt} \in I_{F'}$ and hence $\bar{f} \in \sqrt{I_{F'}}$. □

In fact, the same argument proves the following more general result:

PROPOSITION 1.3.8. *Let A be a filtered algebra such that $\mathsf{gr}\, A$ is commutative. Then for any finitely generated A-module M, $\mathrm{supp}(\mathsf{gr}^F M) \subset Spec(\mathsf{gr}\, A)$ is canonically defined as a closed algebraic subset, i.e., the ideal $\sqrt{\mathrm{Ann}(\mathsf{gr}^F M)}$ does not depend on a good filtration F.*

[1] If we take irreducible components not just of the top dimension, then $SC(M \oplus N)$ will not necessarily equal $SC(M) + SC(N)$, since components of the support of M may be embedded into components of the support of N, and vice versa.

EXAMPLE 1.3.9. While $\sqrt{I_F}$ does not depend on F, I_F may depend on F. For example, let $M = k[x]$ with its usual $\mathcal{D}(\mathbb{A}^1)$-module structure. If we take F_iM to be polynomials of degree at most i, then I_F is (ξ). But if we take F_0M to be constant multiples of x and F_iM to be polynomials of degree at most $i+1$, then x will have degree 0 and 1 will have degree 1 in $\mathsf{gr}^F M$, and therefore $\xi \notin I_F$. In fact, it can be easily computed that in this case $I_F = (\xi^2, x\xi)$.

EXERCISE 1.3.1. Let M be the $\mathcal{D}$-module generated by the function $e^{x^2/2}$, i.e., $M = \mathcal{D}/\mathcal{D}(\partial - x)$. Show that $SS^a(M)$ is the line $\xi = x$, while $SS(M)$ is the line $\xi = 0$.

Now let us define the arithmetic and geometric singular cycles of M. It is clear that for any good filtration F on M (for either filtration on $\mathcal{D}$) one has $d(\mathsf{gr}^F M) = d(M)$. Thus, using a good filtration F on M for Bernstein's filtration, we can attach to M its *arithmetic singular cycle*, $SC^a(M) := SC(\mathsf{gr}^F M)$. One can also consider its projectivization $SC^a_{\mathrm{proj}}(M) := SC_{\mathrm{proj}}(\mathsf{gr}^F M)$ of degree $c(M)$ (zero if $M = 0$). Similarly, using a good filtration F on M for the geometric filtration, we can attach to M its *geometric singular cycle* (or just *singular cycle*), $SC(M) := SC(\mathsf{gr}^F M)$. One can also consider its projectivization $SC_{\mathrm{proj}}(M) := SC_{\mathrm{proj}}(\mathsf{gr}^F M)$, a cycle in $\mathbb{A}^n \times \mathbb{P}^{n-1}$ (zero if $M = 0$). As before, the absence of F from the notation is justified by the fact that these objects don't actually depend on F, as we will show below.

1.3.4. Jantzen filtrations. Below we will need a piece of linear algebra called the Jantzen filtration.

THEOREM 1.3.10. *Let V and W be k-vector spaces, and $f : V[[t]] \to W[[t]]$ be a morphism of $k[[t]]$-modules. Suppose further that f is injective and $\operatorname{coker} f$ is annihilated by t^s for some positive integer s. Then f defines natural filtrations $V = V_0 \supset V_1 \supset \cdots \supset V_s \supset V_{s+1} = 0$ and $0 = W_{-1} \subset W_0 \subset \cdots \subset W_s = W$ and for all $0 \le i \le s$, an isomorphism between V_i/V_{i+1} and W_i/W_{i-1}.*

PROOF. For all i, f induces a map $f_i : V[[t]] \to W[[t]]/t^i$. Let V_i be the cokernel of the multiplication-by-t map $\ker f_{i-1} \to \ker f_i$, and let W_i be $W \cong t^i(W[[t]]/t^{i+1}) \cap \operatorname{im} f_{i+1}$. First of all, we obviously have $V_0 = V$ and $W_{-1} = 0$. Next we prove that $V_{s+1} = 0$ and $W_s = W$.

The condition that $\operatorname{coker} f$ is annihilated by t^s tells us that any multiple of t^s in $W[[t]]$ lies in the image of f. This immediately shows that $t^s(W[[t]]/t^{s+1})$ is in $\operatorname{im} f_{s+1}$, so $W_s = W$. We similarly want to show $V_{s+1} = 0$. If V_{s+1} is nonzero, then there is an element x of $V[[t]]$ with nonzero constant term and with $f(x)$ a multiple of t^{s+1}. But then there is an element y of $V[[t]]$ so that $f(y) = f(x)/t$, so $f(x - ty) = 0$ and $x - ty$ has nonzero constant term, a contradiction.

Finally, we define the inclusions $V_i \supset V_{i+1}$ and $W_i \subset W_{i+1}$. Note that f_{i+1} sends $\ker f_i$ to $t^i(W[[t]]/t^{i+1}) \cong W$. By the definition of W_i, the image of this map is in fact W_i and the kernel is $\ker f_{i+1}$. Defining the inclusion $W_{i-1} \to W$ by multiplication-by-t, we get a commutative diagram of complexes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker f_i & \longrightarrow & \ker f_{i-1} & \longrightarrow & W_{i-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker f_{i+1} & \longrightarrow & \ker f_i & \longrightarrow & W_i & \longrightarrow & 0
\end{array}$$

Now the Snake Lemma gives us an inclusion $V_i \to V_{i+1}$ satisfying $V_i/V_{i+1} \cong W_i/W_{i-1}$, as desired. ☐

The filtrations in Theorem 1.3.10 are called *Jantzen filtrations*. Jantzen filtrations play an important role in representation theory.

Recall that a $k[[t]]$ module Y is a *flat deformation* of $Y_0 := Y/tY$ if there is an isomorphism $Y \to Y_0[[t]]$ of $k[[t]]$-modules which reduces to the identity modulo t.

COROLLARY 1.3.11. *Suppose that A is a k-algebra which is a flat deformation over $k[[t]]$ of $\overline{A} = A/tA$ and V, W are A-modules which are flat deformations of $\overline{V} = V/tV$ and $\overline{W} = W/tW$. Suppose that $f : V \to W$ is an injective morphism of A-modules such that $t^s \operatorname{coker} f = 0$. Then $\overline{V}, \overline{W}$ admit finite filtrations (of length at most $s+1$) whose successive quotients are the same, but occur in opposite orders.*

PROOF. This follows from the fact that all the constructions in the proof of Theorem 1.3.10 respect the A-module structure. □

COROLLARY 1.3.12. *Suppose that A is a filtered algebra, and M is an A-module with filtrations F, F' such that $F'_{i-s}M \subset F_iM \subset F'_{i+s}M$ for some positive integer s. Then the $\mathsf{gr}\, A$-modules $\mathsf{gr}^F M$ and $\mathsf{gr}^{F'} M$ admit finite filtrations with the same quotients occuring in the opposite orders.*

PROOF. Let $\hat{A}$ be the completed Rees algebra of A, i.e., $\hat{A} = \{a_0 + ta_1 + \cdots | a_i \in F_iA\}$. Then $\hat{A}/t\hat{A} \cong \mathsf{gr}\, A$. Let $\hat{M}$ and $\hat{M}'$ denote the completed Rees modules of M with respect to F and F' (defined similarly). By construction, these deformations are flat. If $\alpha \in \hat{M}$, then $t^s\alpha \in \hat{M}'$, and vice versa, so we have multiplication-by-t^s morphisms $f : \hat{M} \to \hat{M}'$ and $g : \hat{M}' \to \hat{M}$ such that $f \circ g$ and $g \circ f$ are both multiplication by t^{2s}. We can now apply Corollary 1.3.11. □

1.3.5. Filtration-independence of the singular cycles. We now verify that $SC^a(M)$ and $SC(M)$ do not depend on filtrations.

THEOREM 1.3.13. *Let M be a finitely generated $\mathcal{D}$-module, and let Z be an irreducible component of $SS^a(M)$ (not necessarily of maximal dimension). Let F be a good filtration on M (for Bernstein's filtration). Then the rank of $\mathsf{gr}^F M$ on Z does not depend on F. In particular, $SC^a(M)$ does not depend on F.*

The same statement holds for the geometric filtration, implying that $SC(M)$ does not depend on F.

PROOF. Let F, F' be two good filtrations. Then they are equivalent, so by Corollary 1.3.12, $\mathsf{gr}^F M$ and $\mathsf{gr}^{F'} M$ have finite filtrations with the same successive quotients (occurring in opposite orders). Since the rank on a component of the support is additive on exact sequences, we find that the ranks of $\mathsf{gr}^F M$ and $\mathsf{gr}^{F'} M$ on Z are the same, which implies the desired statement. The statement for the geometric filtration is proved in a similar way. □

REMARK 1.3.14. Note that this argument gives another proof of the fact that for a finitely generated $\mathcal{D}$-module M, $SS(M)$ and $SS^a(M)$ do not depend on filtrations.

1.3.6. $\mathcal{O}$-coherent $\mathcal{D}$-modules. Let $\mathcal{O} = k[x_1, \ldots, x_n] \subset \mathcal{D}$. Let M be a $\mathcal{D}$-module.

DEFINITION 1.3.15. We say that M is *$\mathcal{O}$-coherent* if it is finitely generated over $\mathcal{O}$.

It turns out that all $\mathcal{O}$-coherent $\mathcal{D}$-modules are quite simple as $\mathcal{O}$-modules. Namely, we have the following

THEOREM 1.3.16. *If M is $\mathcal{O}$-coherent then M is locally free over $\mathcal{O}$ ($\Leftrightarrow$ M is the module of sections of a vector bundle on $\mathbb{A}^n$).*

PROOF. Let x be a closed point in $\mathbb{A}^n$ and let $\mathcal{O}_x$ be the local ring of x. Let $\mathfrak{m}_x \subset \mathcal{O}_x$ be the maximal ideal. As usual, define $M_x = \mathcal{O}_x \otimes_{\mathcal{O}} M$. Since M is $\mathcal{O}$-coherent, $\dim_k M_x/\mathfrak{m}_x M_x < \infty$. Let $\bar{s}_1, \ldots, \bar{s}_m$ be a basis of $M_x/\mathfrak{m}_x M_x$ and let $s_1, \ldots, s_m$ be lifts of $\bar{s}_1, ..., \bar{s}_m$ to M_x. By Nakayama's lemma, $s_1, \ldots, s_k$ generate M_x. We have to show that these elements are linearly independent over $\mathcal{O}_x$.

Assume the contrary, i.e., that $\sum_{i=1}^m \varphi_i s_i = 0$, with not all of the φ_i zero. Define ord_x by $\mathrm{ord}_x 0 = \infty$ and $\mathrm{ord}_x \varphi = n$ if $\varphi \in \mathfrak{m}_x^n$ and $\varphi \notin \mathfrak{m}_x^{n+1}$. Define $\nu = \min_i(\mathrm{ord}_x \varphi_i)$. Without loss of generality we may assume that $\mathrm{ord}_x \varphi_1 = \nu$. As the $\bar{s}_i$ are linearly independent, we must have $\nu \geq 1$, and we may assume that ν takes the smallest possible value.

It is clear that there exists a vector field η (defined locally around x) such that $\eta(\varphi_1) \neq 0$ and $\mathrm{ord}_x \eta(\varphi_1) < \nu$ (in fact, a generic vector field in a suitable sense has this property). Let us apply such an η to the expression $\sum_i \varphi_i s_i = 0$. We get

$$0 = \sum_i \eta(\varphi_i) s_i + \sum_i \varphi_i \eta(s_i).$$

Since s_i generate M_x, we have $\eta(s_i) = \sum_j a_{ij} s_j$ for some $a_{ij} \in \mathcal{O}_x$, and thus we have

$$0 = \sum_i \left(\eta(\varphi_i) + \sum_j \varphi_j a_{ji} \right) s_i.$$

The coefficient of s_1 in this sum is $\eta(\varphi_1) + \sum_j \varphi_j a_{j1}$. Since $\mathrm{ord}_x \eta(\varphi_1) < \nu$ and $\mathrm{ord}_x(\sum_j \varphi_j a_{ij}) \geq \nu$, this coefficient is non-zero and has order less than ν at x. This is a contradiction, since ν was chosen minimal possible. $\square$

Remark 1.3.17. By Quillen's theorem (formerly Serre's conjecture), a locally free finitely generated $k[x_1, ..., x_n]$-module is free. This implies that M is in fact a free $\mathcal{O}$-module (not just locally free). However, this property does not generalize to arbitrary smooth affine varieties and will not be used below.

Here is a very important corollary of Theorem 1.3.16.

Corollary 1.3.18. *Let M be a finitely generated $\mathcal{D}$-module. Then M is $\mathcal{O}$-coherent if and only if*

$$SS(M) = \{(x, 0) |\ x \in \mathbb{A}^n\}.$$

More canonically, if we identify $\mathrm{gr}\,\mathcal{D}$ (with respect to the geometric filtration) with $\mathcal{O}(T^*\mathbb{A}^n)$ then M is $\mathcal{O}$-coherent if and only if $SS(M)$ is equal to the zero section in $T^*\mathbb{A}^n$.

Proof. Assume that $SS(M)$ is as above. Then it follows that for every good filtration F on M (with respect to the geometric filtration on $\mathcal{D}$), the module $\mathrm{gr}^F M$ is finitely generated over $k[x_1, ..., x_n, \xi_1, ..., \xi_n]$, and all the ξ_i act locally nilpotently on $\mathrm{gr}^F M$. This implies that $\mathrm{gr}^F M$ is finitely generated over $k[x_1, ..., x_n] = \mathcal{O}$. Hence M is also finitely generated over $\mathcal{O}$.

Conversely, assume that M is $\mathcal{O}$-coherent. Define a filtration on M by setting

$$F_j M = M \text{ for every } j \geq 0.$$

This filtration is good since M is finitely generated over $\mathcal{O}$. Then $\mathrm{gr}^F M = M$ as an $\mathcal{O}$-module and all the ξ_i act on $\mathrm{gr}^F M$ by 0. By Theorem 1.3.16, we know that M is locally free over $\mathcal{O}$ and thus that $SS(M)$ is equal to the zero section. $\square$

Example 1.3.19. Here is an example of a $\mathcal{D}$-module that is not $\mathcal{O}$-coherent. Recall that δ_b denotes the module of δ-functions at some $b \in \mathbb{A}^1$. Then $SS(\delta_b) = \{(b, \xi)\}$. (Note that $SS^a(\delta_b) = \{(0, \xi)\}$, so $SS(\delta_b) \neq SS^a(\delta_b)$ for $b \neq 0$).

Exercise 1.3.2. Show that for any $\lambda \in k$, the geometric singular support of $M(x, \lambda) = \mathbb{C}[x, x^{-1}]x^\lambda$ is the set of (x, ξ) such that $x\xi = 0$ (the union of two lines). In particular, this shows that the singular support of an irreducible $\mathcal{D}$-module may be reducible.

Remark 1.3.20. We see that geometric singular support somehow "measures" singularities of M.

1.3.7. Flat connections. Since the algebra $\mathcal{D}$ contains $\mathcal{O}$, one can think about $\mathcal{D}$-modules as $\mathcal{O}$-modules (or, geometrically speaking, quasicoherent sheaves) on $\mathbb{A}^n$ with some additional structure. Let us now see what kind of additional structure we need to introduce on a quasicoherent sheaf so that it becomes a $\mathcal{D}$-module.

Let M be an $\mathcal{O}$-module. A $\mathcal{D}$-module structure on M gives us a map $\nabla : M \to M \otimes_{\mathcal{O}} \Omega^1(\mathbb{A}^n)$, where $\Omega^1(\mathbb{A}^n)$ is the module of differential 1-forms. Namely, for any vector field v on $\mathbb{A}^n$ and any $m \in M$ we have $\nabla(m)(v) = v(m)$. This map satisfies the condition $\nabla(fm) = m \otimes df + f \cdot \nabla(m)$ for any $f \in \mathcal{O}$ and $m \in M$. Such a map ∇ is called a *connection*.

To formulate which connections arise from a $\mathcal{D}$-module structure, let us define the notion of flat connection.

Given a connection ∇ define a map $\nabla^2 : M \to M \otimes_{\mathcal{O}} \Omega^2(\mathbb{A}^n)$ in the following way:

$$\nabla^2(m) = (\nabla \otimes 1)(\nabla(m)) + (1 \otimes d)\nabla(m),$$

where d denotes the standard de Rham differential.

EXERCISE 1.3.3. Show that ∇^2 is a well-defined $\mathcal{O}$-linear map. Note that the two terms in the definition of $\nabla^2(m)$ are not well defined separately, since, for instance, $(1\otimes d)(v\otimes f\omega) \neq (1\otimes d)(fv\otimes\omega)$ for $f \in \mathcal{O}$, $\omega \in \Omega^1$, $v \in M$, but their sum is well defined.

Thus, ∇^2 can be thought of as an element of $\mathrm{End}_{\mathcal{O}}(M) \underset{\mathcal{O}}{\otimes} \Omega^2$.

DEFINITION 1.3.21. The element ∇^2 is called the *curvature* of the connection ∇. The connection ∇ is called *flat* if $\nabla^2 = 0$.

LEMMA 1.3.22. *Let M be an $\mathcal{O}$-module. Then a flat connection on M is the same thing as a $\mathcal{D}$-module structure on M.*

EXERCISE 1.3.4. Prove Lemma 1.3.22.

Hint. if M is an $\mathcal{O}$-module with a flat connection ∇, define $\frac{\partial}{\partial x_i} m = (\mathrm{Id} \otimes \frac{\partial}{\partial x_i}, \nabla(m))$, and show that this gives a $\mathcal{D}$-module structure on M. Conversely, if M is a $\mathcal{D}$-module, use the same formula to define a connection on M and show that it is flat.

Let V be a finite dimensional vector space over k and assume $M \cong V \otimes_k \mathcal{O}$ as an $\mathcal{O}$-module. Let $d : M \to M \otimes_{\mathcal{O}} \Omega^1$ be the de Rham differential. Then any connection ∇ on M has the form $\nabla = d + \omega$, where $\omega \in \mathrm{End}(V) \otimes \Omega^1$. The map

$$\omega : M \to M \otimes_{\mathcal{O}} \Omega^1 = V \otimes_k \Omega^1$$

is defined by $\omega(m) = T(v) \otimes f\alpha$ for $\omega = T \otimes \alpha$, where T is a map $V \to V$, and $m = v \otimes f$.

LEMMA 1.3.23. *∇ is a flat connection iff $d\omega + [\omega, \omega] = 0$.*

EXERCISE 1.3.5. Prove Lemma 1.3.23.

The equation in Lemma 1.3.23 is called the *Maurer-Cartan equation*.

1.3.8. Poisson structures.

DEFINITION 1.3.24. Let R be a commutative algebra. A Poisson bracket $\{\cdot,\cdot\} : R \times R \to R$ on R is a Lie bracket satisfying the Leibniz rule

$$\{f_1 f_2, g\} = f_1\{f_2, g\} + f_2\{f_1, g\}$$

for every $f_1, f_2, g \in R$.

Let X be a smooth affine algebraic variety over k. If we forget about the Jacobi identity, then a Poisson bracket on $\mathcal{O}(X) = R$ corresponds to a bivector field $\eta \in \Gamma(\bigwedge^2 T_X)$. Any $\eta \in \Gamma(\bigwedge^2 T_X)$ defines a map $\eta : T_X^* \to T_X$. If this map is an isomorphism, then there exists a nondegenerate differential 2-form $\omega \in \Gamma(\bigwedge^2 T_X^*) = \Omega_X^2$ corresponding to η.

EXERCISE 1.3.6. Show that η satisfies the Jacobi identity if and only if ω is closed.

DEFINITION 1.3.25. A closed non-degenerate 2-form ω is called a *symplectic form*. A variety is called symplectic if it is endowed with a symplectic form.

Let Y be any smooth variety. Then $X := T^*Y$ is naturally a symplectic variety. Indeed, there is a canonical 1-form α on T^*Y defined as follows. Let $\pi : T^*Y \to Y$ be the natural projection. If $v \in T_{(y,\xi)}(T^*Y)$, with $y \in Y$ and $\xi \in T_y^*Y$, then $\alpha(v) := (\xi, d\pi(v))$. Then one defines the symplectic form ω by the formula $\omega = d\alpha$. This form is obviously closed. To see that it is nondegenerate, let us choose local coordinates $x_1, ..., x_n$ near some point y of Y. Let $\xi_1, ..., \xi_n$ be the dual basis of T_yY to the basis $dx_1, ..., dx_n$ of T_y^*Y. Then it is easy to see that $\alpha = \sum_i \xi_i dx_i$, $\omega = \sum_i d\xi_i \wedge dx_i$, which implies the nondegeneracy.

EXAMPLE 1.3.26. If $Y = \mathbb{A}^n$, then $X = \mathbb{A}^{2n}$ with coordinates $x_1, \ldots, x_n, \xi_1, \ldots \xi_n$. In this case the symplectic form is $\omega = \sum_i d\xi_i \wedge dx_i$ and for the Poisson bracket we have $\{\xi_i, \xi_j\} = 0$, $\{x_i, x_j\} = 0$ and $\{\xi_i, x_j\} = \delta_{i,j}$. The Poisson bracket of two arbitrary functions is then given by

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial \xi_i} \right).$$

The same statements hold in local coordinates on T^*Y for any smooth Y.

1.3.9. The Poisson structure on the associated graded algebra. Assume that A is a filtered algebra and for some $l > 0$ one has $[F_iA, F_jA] \subset F_{i+j-l}A$. Then $\mathsf{gr}\, A$ is commutative and is endowed with a canonical Poisson bracket of degree $-l$ (which may be zero). Namely, for $\bar{x} \in \mathsf{gr}_i\, A$ and $\bar{y} \in \mathsf{gr}_j\, A$ let $x \in F_iA$ and $y \in F_jA$ be their preimages. Define $\{\bar{x}, \bar{y}\}$ as the image $\bar{z}$ of $[x, y]$ in $\mathsf{gr}_{i+j-l}\, A$.

LEMMA 1.3.27. (1) *$\bar{z}$ depends only on $\overline{x}$ and $\overline{y}$ (i.e. it does not depend on the choice of x and y).*
(2) *The assignment $(\overline{x}, \overline{y}) \mapsto \overline{z}$ is a Poisson bracket on $\mathsf{gr}\, A$.*

EXERCISE 1.3.7. Prove Lemma 1.3.27.

EXAMPLE 1.3.28. (Bernstein's filtration.) In this case $[F_i\mathcal{D}, F_j\mathcal{D}] \subset F_{i+j-2}\mathcal{D}$ and $l = 2$. The Poisson bracket defined as above coincides with the standard Poisson bracket on

$$\mathbb{A}^{2n} = \mathrm{Spec}(k[x_1, \ldots x_n, \xi_1, \ldots, \xi_n])$$

coming from the identification $\mathbb{A}^{2n} \simeq T^*\mathbb{A}^n$.

EXAMPLE 1.3.29. (Geometric filtration.) In this case $[\mathcal{D}_i, \mathcal{D}_j] \subset \mathcal{D}_{i+j-1}$ and $l = 1$. The Poisson bracket on $\mathbb{A}^{2n}$ is again the standard bracket.

1.3.10. Coisotropic subvarieties.

DEFINITION 1.3.30. Assume X is a Poisson affine algebraic variety (i.e., the structure algebra $\mathcal{O}(X)$ is a Poisson algebra), and let $Z \subset X$ be a closed subvariety. Let $\mathcal{I}(Z) \subset \mathcal{O}(X)$ be the ideal of Z. Then Z is called *coisotropic* if

$$\{\mathcal{I}(Z), \mathcal{I}(Z)\} \subset \mathcal{I}(Z).$$

Let X be a Poisson affine algebraic variety and let $Z \subset X$ be a closed subvariety. Let z be a smooth point of X, and $\eta : T_z^*X \to T_zX$ be the linear map corresponding to the Poisson bivector. We have $T_zZ \subset T_zX$ and $T_zZ^\perp \subset T_z^*X$.

LEMMA 1.3.31. *Z is coisotropic if and only if $\eta(T_zZ^\perp) \subset T_zZ$ for any smooth point $z \in Z$.*

EXERCISE 1.3.8. Prove Lemma 1.3.31.

COROLLARY 1.3.32. *If X is symplectic and $Z \subset X$ is coisotropic, then for any irreducible component Z_α of Z we have $\dim Z_\alpha \geq \frac{1}{2}\dim X$.*

1.3.11. Gabber's theorem. Let A be a filtered algebra such that $[F_iA, F_jA] \subset F_{i+j-l}A$ for some l. Let M be an A-module with a good filtration F. Let $I_F = \mathrm{Ann}(\mathsf{gr}^F M)$. We would like to study the relationship between I_F and the Poisson bracket on $\mathsf{gr}\, A$ defined above.

EXERCISE 1.3.9. Show that if $l = 1$ then $\{I_F, I_F\} \subset I_F$.

EXAMPLE 1.3.33. If $l > 1$, then the inclusion $\{I_F, I_F\} \subset I_F$ may fail. Indeed, consider $\mathcal{D} = \mathcal{D}(\mathbb{A}^2)$ with Bernstein's filtration. We take the module $M = \mathcal{O}$ with the standard action of $\mathcal{D}$ but with a nonstandard filtration. Namely, define the degree of a monomial in x_1, x_2 to be one less than its usual degree unless the monomial is a power of x_2. We will define the degree of x_2^i to be i for $i \geq 1$ and the degree of 1 to be 1. Now let F_i be the space of polynomials $P(x_1, x_2)$ such that all monomials appearing in P have degree at most i. It can be checked that $I_F{=}(\xi_1^2, \xi_1 x_1, \xi_1 x_2, \xi_2)$. As $\{\xi_2, \xi_1 x_2\} = \xi_1$, we see that $\{I_F, I_F\} \not\subset I_F$.

In any case, it is more interesting to consider $J(M) = \sqrt{I_F}$, which, unlike I_F, is independent on F. This leads us to a fundamental theorem, which plays a central role in the theory of $\mathcal{D}$-modules.

THEOREM 1.3.34. *(O. Gabber) If $\mathsf{gr}\, A$ is noetherian, then $\{J(M), J(M)\} \subset J(M)$.*

This theorem is highly non-trivial and will be proved below for $\mathsf{gr}\, A$ regular.

Let us derive some corollaries from Gabber's theorem.

COROLLARY 1.3.35. *Let A be a filtered algebra such that $[F_iA, F_jA] \subset F_{i+j-l}A$ for some $l > 0$. Assume that $\mathsf{gr}\, A$ is isomorphic to the algebra of functions on a smooth affine symplectic variety X (as a Poisson algebra). Let M be a finitely generated A-module endowed with a good filtration F. Then the dimension of every irreducible component of $\mathrm{supp}(\mathsf{gr}^F M)$ is greater or equal to $\frac{1}{2}\dim X$.*

EXAMPLE 1.3.36. Take $A = \mathcal{D}$ with Bernstein's filtration. We showed before that $\dim(\mathrm{supp}\, \mathsf{gr}^F M) \geq n = \frac{1}{2}\dim \mathbb{A}^{2n}$ (Bernstein's inequality). Corollary 1.3.35 is, however, clearly a stronger statement. Also, applying the same argument to the geometric filtration, we see that the dimension of every component of $SS(M)$ is also greater or equal to n.

REMARK 1.3.37. Gabber's theorem for the geometric filtration on $\mathcal{D}$ was proved earlier by Malgrange.

DEFINITION 1.3.38. Assume that X is a smooth symplectic variety , $Z \subset X$ is coisotropic and $\dim Z_\alpha = \frac{1}{2}\dim X$ where Z_α is any irreducible component of Z. Then Z is called *Lagrangian.*

COROLLARY 1.3.39. *Let M be a holonomic $\mathcal{D}$-module. Then both $SS(M)$ and $SS^a(M)$ are Lagrangian.*

1.3.12. Proof of Gabber's theorem. We now prove Theorem 1.3.34 when $\mathsf{gr}\, A$ is regular.

We define the Rees algebra $\mathrm{Rees}(A)$ to be the algebra with underlying k-vector space $\oplus_{i\geq 0} F_iA \cdot \epsilon^i$ and multiplication given by $(a \cdot \epsilon^i)(b \cdot \epsilon^j) = ab \cdot \epsilon^{i+j}$. Denote by $\mathrm{Rees}(M)$ the $\mathrm{Rees}(A)$-module $\oplus_{i\geq 1} F_iM \cdot \epsilon^i$. We also define A' to be the algebra $\mathrm{Rees}(A)/\epsilon^{l+1}\,\mathrm{Rees}(A)$ and M' to be the A'-module $\mathrm{Rees}(M)/\epsilon^{l+1}\,\mathrm{Rees}(M)$. By assumpion, $A'/\epsilon^l A'$ is a commutative flat deformation of $\mathsf{gr}\, A$ over $k[\epsilon]/(\epsilon^l)$, and is thus trivial, as $\mathsf{gr}\, A$ is regular. It thus suffices to prove the following theorem:

THEOREM 1.3.40. *Let $\overline{A}$ be a commutative finitely generated algebra and $\overline{M}$ a finitely generated $\overline{A}$-module. Let A' and M' be flat deformations of $\overline{A}$ and $\overline{M}$ over $k[\epsilon]/(\epsilon^{l+1})$ such that A' is trivial mod ϵ^l. For $\overline{a}_1, \overline{a}_2 \in \overline{A}$, define $\{\overline{a}_1, \overline{a}_2\}$ as the class in $A'/\epsilon A' \cong \overline{A}$ such that $[a_1, a_2] = \epsilon^l\{\overline{a}_1, \overline{a}_2\}$. Let I be the radical of $\mathrm{Ann}_{\overline{A}}\overline{M}$. Then $\{I, I\} = I$.*

The rest of the subsection is dedicated to the proof of this theorem. Let $\mathfrak{p}$ be a prime ideal corresponding to an irreducible component of the support of $\overline{M}$. As I is the intersection of the possible $\mathfrak{p}$, it suffices to prove $\{\mathfrak{p}, \mathfrak{p}\} = \mathfrak{p}$ for any such $\mathfrak{p}$. To this end, we first show that we can localize the problem. Define the algebra $A'[\frac{1}{f}]$ by the relations $f\frac{1}{f} = \frac{1}{f}f = 1$. Then the algebras $\overline{A}_f, A'[\frac{1}{f}]$ and the modules over them $\overline{M}_f, A'[\frac{1}{f}] \otimes_{A'} M'$ still satisfy the conditions of the theorem. (To see this, use that $[\frac{1}{f}, a] = \frac{1}{f}[a, f]\frac{1}{f}$ is divided by a higher power of ϵ than a is.) Furthermore, as $\{\cdot, \cdot\}$ is a biderivation, it suffices to check the theorem after localizing by f.

Now, take a generic projection $pr : \mathrm{Spec}\,\overline{A} \to \mathbb{A}^d$, where d is the dimension of $\overline{A}/\mathfrak{p}$. We can modify this projection so that $V(\mathfrak{p})$ is the only irreducible component of $V(I)$ not sent to the origin. (A map to $\mathbb{A}^d$ is defined by d functions, so we can achieve this just by multiplying the defining functions by a function vanishing on all the components of $V(I)$ except $V(\mathfrak{p})$.) Take a function g on $\mathbb{A}^d$ so that the following three conditions are satisfied:

(1) g vanishes at the origin. (This implies that $\overline{M}_{pr^{-1}g}$ is annihilated by a power of $p_{pr^{-1}g}$.)
(2) The restriction of pr to $V(\mathfrak{p})$ is finite outside of $V(g)$.
(3) For every i, $\mathfrak{p}^i_{pr^{-1}g}\overline{M}_{pr^{-1}g}/\mathfrak{p}^{i+1}_{pr^{-1}g}\overline{M}_{pr^{-1}g}$ is free over $(\overline{A}/\mathfrak{p})_{pr^{-1}g}$.

To see that we can do this, note that if these conditions are true for g, then they hold for any multiple of g. It thus suffices to check the conditions individually. The first and second conditions are obvious, and the third condition follows from generic freeness. Now let $f = pr^{-1}g$, and localize A' at f.

Now let $B = \overline{A}_f/\mathfrak{p}_f$. We have a map $k[\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_d]_g \to A$ that makes B is a free algebra over $k[\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_d]_g$. We know that $\mathfrak{p}^i_f\overline{M}_f/\mathfrak{p}^{i+1}_f$ must be a free module over B, and thus a free module over $k[\overline{x}_1, \overline{x}_2, \cdots \overline{x}_d]_g$. This implies that $\overline{M}_f$ itself is a free module over $k[\overline{x}_1, \overline{x}_2, \cdots \overline{x}_d]_g$. (This is why we need to work in $k[\overline{x}_1, \overline{x}_2, \cdots \overline{x}_d]_g$ instead of B: there is no natural B-module structure on $\overline{M}_f$ in general.) Choose a $k[\overline{x}_1, \overline{x}_2, \cdots \overline{x}_d]_g$-basis $\overline{m}_1, \overline{m}_2, \cdots, \overline{m}_n$ of $\overline{M}_f$ such that there are integers $a_1, a_2, \cdots$ such that $\overline{m}_j \in \mathfrak{p}^i_f M_f$ for $j > a_i$ and (the images of) $\overline{m}_{a_i+1}, \cdots, \overline{m}_{a_{i+1}}$ form a basis of $\mathfrak{p}^i_f\overline{M}_f/\mathfrak{p}^{i+1}_f\overline{M}_f$.

We now claim that for all $\overline{a}, \overline{b} \in \mathfrak{p}_f$, the trace of the $k[\overline{x}_1, \overline{x}_2, \cdots \overline{x}_d]_g$-linear map $\overline{M}_f \to \overline{M}_f$ given by $x \to \{\overline{a}, \overline{b}\}x$ has zero trace. First we show that this implies the theorem. Note that left multiplication preserves the filtration $\mathfrak{p}^{i+1}_f\overline{M}_f \subseteq \mathfrak{p}^i_f\overline{M}_f \subseteq \cdots \subseteq \overline{M}_f$, so

$$\mathrm{tr}_{\overline{M}_f}\{\overline{a}, \overline{b}\} = \sum_{i=0}^{\infty} \mathrm{tr}_{\mathfrak{p}^i_f\overline{M}_f/\mathfrak{p}^{i+1}_f\overline{M}_f}\{\overline{a}, \overline{b}\} = (\sum_{i=0}^{\infty} \dim_{k[x_1,x_2,\cdots,x_n]_f} \mathfrak{p}^i_f\overline{M}_f/\mathfrak{p}^{i+1}_f\overline{M}_f)\,\mathrm{tr}_B\{\overline{a}, \overline{b}\}$$

We thus must have that $\mathrm{tr}_B\{\overline{a}, \overline{b}\} = 0$ for all $\overline{a}, \overline{b} \in \mathfrak{p}_f$. Multiplying $\overline{a}$ by $\overline{c} \in \overline{A}_f$, we find that $0 = \mathrm{tr}_B\{\overline{ca}, \overline{b}\} = \mathrm{tr}_B(\overline{c}\{\overline{a}, \overline{b}\} + \{\overline{c}, \overline{b}\}\overline{a})$. As $\overline{a}$ is in $\mathfrak{p}_f$, it acts by 0 on B and thus $\{\overline{c}, \overline{b}\}\overline{a}$ has zero trace. We therefore see that $\mathrm{tr}_B\,\overline{c}\{\overline{a}, \overline{b}\} = 0$ for any $\overline{c}$.

By the Frobenius property of semisimple algebras, this implies that the image of $\{\overline{a}, \overline{b}\}$ in B is zero, or equivalently that $\{\overline{a}, \overline{b}\} \in \mathfrak{p}_f$. We are therefore reduced to proving the claim.

Let a, b be lifts of $\overline{a}, \overline{b}$ to A', and let $m_1, m_2, \cdots, m_n$ be lifts of $\overline{m}_1, \overline{m}_2, \cdots, \overline{m}_n$. We now want to find $n \times n$ matrices U^k, V^k for $k = 0, 1, \cdots l$ with entries in A' satisfying the following conditions:

(1) U^0 and V^0 are strictly upper triangular.

(2) By the assumptions of the theorem, we have $A'/\epsilon^l A' \cong \overline{A} \otimes_k k[\epsilon]/(\epsilon^l)$. Let i be the natural inclusion map $\overline{A} \to A'/\epsilon^l A'$. Then the entries of U^k and V^k mod ϵ^l lie in the image of i.
(3) The entries of U^k and V^k mod ϵ lie in the image of $k[\overline{x}_1, \overline{x}_2, \cdots \overline{x}_d]_g$.
(4) $am_i = \sum_{k=0}^l \sum_{j=1}^n \epsilon^k U_{ij}^k m_j$, and $bm_i = \sum_{k=0}^l \sum_{j=1}^n \epsilon^k V_{ij}^k m_j$.

We construct these matrices one at a time. We have an upper triangular matrix $\overline{U}^0$ such that $\overline{am}_i = \overline{U}_{ij}^0 \overline{m}_j$. Choose any lift U^0 of $\overline{U}^0$ satisfying the first three conditions. Then $am_i - \sum_{j=1}^n U_{ij}^0 m_j$ is a multiple of ϵ, and choose U^1 so that U^1 satisfies the first three conditions and so that $am_i - \sum_{j=1}^n (U_{ij}^0 + \epsilon U_{ij}^1) m_j$ is a multiple of ϵ^2. Continuing in this way, we can construct all the U^k (and the V^k).

The claim is now a matter of computation. Let m be the column vector whose ith entry is m_i. We compute

$$\begin{aligned} abm &= aV^0 m + \epsilon a V^1 m + \cdots + \epsilon^l a V^l m \\ &= V^0 am + \epsilon V^1 am + \cdots + \epsilon^l V a^l m + \epsilon^l \{a, V^0\} m \\ &= (V^0 U^0 m) + \epsilon (V^1 U^0 + V^0 u^1) m + \cdots + \epsilon^l (\sum_{i=0}^l V^{l-i} U^i) m + \epsilon^L \{a, V^0\} m \end{aligned}$$

and we thus have

$$[a,b]m = [V^0, U^0]m + \epsilon([V^1, U^0] + [V^0, U^1])m + \cdots + \epsilon^l (\sum_{i=0}^l [V^{l-i}, U^i])m + \epsilon^l \{a, V^0\} m$$

As we know that $[a,b]$ is a multiple of ϵ^l, we see that the right hand side must also be a multiple of ϵ^l. Taking mod ϵ, we see that $[V^0, U^0]$ is a multiple of ϵ. But as the entries of U^0 and V^0 mod ϵ^l are the images along i of their values mod ϵ, this implies that $[V^0, U^0]$ is a multiple of ϵ^l. Now taking mod ϵ^2, we see that $[V^1, U^0] + [V^0, U^1]$ is a multiple of ϵ, and thus a multiple of ϵ^l. But that implies that $\epsilon([V^1, U^0] + [V^0, U^1]) = 0$. Continuing in this way, we see that

$$[a,b]m = [V^0, U^0]m + \epsilon^l (\sum_{i=0}^l [V^{l-i}, U^i])m + \epsilon^l \{a, V^0\} m$$

where $[V^0, U^0]$ must be a multiple of ϵ^l. Now note that $\frac{[V^0, U^0]}{\epsilon^l}$ and $\{a, V^0\}$ are strictly upper triangular, and thus have zero trace. On the other hand, $\sum_{i=0}^l [V^{l-i}, U^i]$ is a sum of commutators and thus has zero trace, proving the claim.

Remark 1.3.41. This proof is a slightly modified version of a proof due to F. Knop which appears in [**Gi**].

1.3.13. Exercises.

Exercise 1.3.10. Let M be the $\mathcal{D}$-module on $\mathbb{A}^2$ (with coordinates (x, y)) "generated by the function $e^{x/y}$", i.e., $M = k[x, y, y^{-1}]e^{x/y}$. The action of differential operators is standard.
a) Show that M is holonomic and irreducible.
b) Compute the (geometric) singular support of M.

Exercise 1.3.11. Let $\widehat{\mathcal{D}}$ be the algebra of differential operators in one variable whose coefficients are formal power series (i.e., of operators of the form

$$L = a_n(x)\partial^n + ... + a_1(x)\partial + a_0(x),$$

where $a_i \in k[[x]]$). Show that for k of characteristic zero, any $\widehat{\mathcal{D}}$-module M which is finitely generated over $k[[x]]$ is of the form $k[[x]]^n$ for a unique n (with the standard componentwise action of differential operators). Is this true if you replace $k[[x]]$ with the ring of entire functions (over $k = \mathbb{C}$)? with the polynomial ring $k[x]$?

EXERCISE 1.3.12. In this and the next problem we work over $\mathbb{C}$. Let f be a smooth function on some line interval, and $M_f := \mathcal{D}f$ (where $\mathcal{D} = \mathcal{D}(\mathbb{A}^1)$).

a) Show that M_f is holonomic if and only if f satisfies a linear differential equation with rational coefficients and coefficient 1 at the highest derivative. Moreover, it is $\mathcal{O}$-coherent outside of the poles of these coefficients.

b) Suppose M_f is holonomic. Is it true that if f is an entire function then M_f is $\mathcal{O}$-coherent on the whole complex plane?

c) Compute the composition factors of $M_{\log(x)}$, with multiplicities. What is the length of this $\mathcal{D}$-module? Do the same for $M_{\log(x)^n}$.

d) Do the same as in (c) for the dilogarithm function

$$L(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}.$$

e) Compute the successive quotients of the socle filtration in (c) and (d) (Recall that if M is a finite length module over a ring, then the socle filtration on M is defined inductively by the condition that $F_iM/F_{i-1}M$ is the socle (i.e., maximal semisimple submodule) of $M/F_{i-1}M$). What is the length of this filtration?

EXERCISE 1.3.13. Let G denote the symplectic group $\mathrm{Sp}(2n, \mathbb{C})$. Note that G acts on $\mathcal{D} = \mathcal{D}(\mathbb{A}^n)$ by automorphisms (namely, it acts on the $2n$-dimensional vector space $\mathrm{span}(x_i, \frac{\partial}{\partial x_j})$ which generates $\mathcal{D}$, and symplectic transformations (with respect to the standard symplectic form) extend to automorphisms of $\mathcal{D}$). Thus if M is a $\mathcal{D}$-module and $g \in G$, we denote by M^g the same module but with $\mathcal{D}$-action twisted by g (i.e., $L|_{M^g} := gLg^{-1}|_M$).

Let M be a holonomic $\mathcal{D}$-module on $\mathbb{A}^n$ and let $c = c(M)$.

a) Show that for generic $g \in G$ the module M^g is $\mathcal{O}$-coherent of rank c (hint: think what the transformation $M \mapsto M^g$ does to the arithmetic singular support of M and then formulate a sufficient condition for $\mathcal{O}$-coherence in terms of the arithmetic singular support).

Note that this gives another definition of c.

b) Explain what a) says explicitly for $M = \delta_a$ (where $a \in \mathbb{C}$).

c) Let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz space of $\mathbb{R}^n$, i.e. the space of functions all of whose partial derivatives are rapidly decreasing at ∞. Let $\mathcal{S}^*(\mathbb{R}^n)$ denote the appropriate topological dual space (the so-called *space of tempered distributions*). It has a natural structure of $\mathcal{D}$-module. The Stone-von Neumann theorem says (in particular) that $(\mathcal{S}^*(\mathbb{R}^n))^g$ is isomorphic to $\mathcal{S}^*(\mathbb{R}^n)$ for every $g \in Sp(2n, \mathbb{R})$ (the corresponding isomorphisms were explictly written by Weil). Show that for every holonomic $\mathcal{D}$-module M we have

$$\dim \mathrm{Hom}_{\mathcal{D}}(M, \mathcal{S}^*(\mathbb{R}^n)) \le c(M).$$

In other words, the space of solutions of M in the space of tempered distributions has dimension $\le c(M)$

Hint: use a).

1.4. Functional dimension and homological algebra

Let us pass to a different subject. Let M be a finitely generated $\mathcal{D}(\mathbb{A}^n)$-module, and define $d_a(M)$ to be $\dim(SS^a(M))$ and $d_g(M)$ to be $\dim(SS(M))$.

THEOREM 1.4.1. $d_a(M) = d_g(M)$

In order to prove this theorem, we will need some results and constructions from homological algebra, which we now briefly recall. We will study these things in much more detail later, when we discuss derived categories.

1.4.1. Complexes. Let A be any ring. Recall that a *complex* of (left) A-modules $M^\bullet$ is the following data:

- A (left) A-module M^i for each $i \in \mathbb{Z}$
- A homomorphism $\partial_i : M^{i-1} \to M^i$ for each $i \in \mathbb{Z}$ such that for all i we have

$$\partial_i \circ \partial_{i-1} = 0.$$

Recall also that the *cohomology* of the complex $M^\bullet$ are $H^i(M^\bullet) = \mathrm{Ker}(\partial_{i+1})/\mathrm{Im}(\partial_i)$.

When it does not lead to confusion, we will write ∂ instead of ∂_i.

One can also define a *bicomplex* as a collection M^{ij} $(i, j \in \mathbb{Z})$ of A-modules with differentials $\partial^1_{ij} : M^{ij} \to M^{i+1,j}$ and $\partial^2_{ij} : M^{ij} \to M^{i,j+1}$ satisfying $\partial^1_{ij} \circ \partial^1_{i-1,j} = 0$, $\partial^2_{ij} \circ \partial^2_{i,j-1} = 0$ and $\partial^2_{i+1,j} \circ \partial^1_{ij} = \partial^1_{i,j+1} \circ \partial^2_{i,j}$. In this case one can define the *total complex* $\mathrm{Tot}(M^\bullet)$ of the bicomplex $M^\bullet$ by setting

$$\mathrm{Tot}^k(M^\bullet) = \bigoplus_{i+j=k} M^{ij}, \quad \partial_k = \bigoplus_{i+j=k} \partial^1_{ij} + (-1)^j \partial^2_{ij}.$$

1.4.2. Left exact and right exact functors. Let A-mod denote the category of left A-modules. Let $F : A\text{-mod} \to \mathfrak{Ab}$ be an additive functor (here $\mathfrak{Ab}$ denotes the category of abelian groups).

DEFINITION 1.4.2. (1) A functor F is called *left exact* if for any short exact sequence of A-modules $0 \to M_1 \to M_2 \to M_3 \to 0$, the sequence $0 \to F(M_1) \to F(M_2) \to F(M_3)$ is exact. (For F contravariant, we instead require that for a short exact sequence of modules, the sequence $0 \to F(M_3) \to F(M_2) \to F(M_1)$ is exact.)
(2) A functor F is called *right exact* if for any short exact sequence of A-modules $0 \to M_1 \to M_2 \to M_3 \to 0$, the sequence $F(M_1) \to F(M_2) \to F(M_3) \to 0$ is exact.
(3) A functor F is called *exact* if it both left and right exact.

EXAMPLE 1.4.3. For any A-module M, the functor $N \mapsto \mathrm{Hom}_a(M, N)$ is left exact. Also, the functor $N \mapsto \mathrm{Hom}_A(N, M)$ is a contravariant left exact functor. Finally, if N is a right A-module, then the functor $M \mapsto N \otimes_A M$ is right exact. These functors are not exact, in general.

We see that many important functors are only left or right exact but not exact. In some sense, the main point of homological algebra is to "correct" non-exactness of such functors. This is usually done by taking a projective/injective resolution, which we will soon define.

DEFINITION 1.4.4. (1) An A-module P is called projective if $\mathrm{Hom}(P, \cdot)$ is an exact functor.
(2) An A-module I is called injective if $\mathrm{Hom}(\cdot, I)$ is exact.

It is easy to see that free modules are projective.

DEFINITION 1.4.5. A projective resolution of a module M is a complex $P^\bullet$ of projective modules

$$\cdots \to P^{-2} \xrightarrow{\partial_{-1}} P^{-1} \xrightarrow{\partial_0} P^0 \to 0$$

such that $P^0/\mathrm{Im}(\partial_0) = M$ and $H^{-i}(P^\bullet) = \mathrm{Ker}(\partial_{-i+1})/\mathrm{Im}(\partial_{-i}) = 0$ for any $i > 0$.

A projective resolution will always exist. To see this, note that for any module M, there will be a surjective map from a free module P^0. This will also apply to the kernel of the map $P^0 \to M$, so we can find a free module P^{-1} and a map $\partial_0 : P^{-1} \to P^0$ such that $\mathrm{Im}(\partial_0) = \ker(P^0 \to M)$. Continuing in this fashion, we arrive at a free resolution of M. As free modules are always projective, this construction gives us a projective resolution and we are done.

Recall that $\mathrm{Ext}^i(M, N)$ is defined as follows: let $P^\bullet$ be a projective resolution of M. Then $\mathrm{Hom}(P^\bullet, N)$ is also a complex

$$0 \to \mathrm{Hom}(P^0, N) \to \mathrm{Hom}(P^{-1}, N) \to \mathrm{Hom}(P^{-2}, N) \to \dots$$

By definition $\mathrm{Ext}^i(M, N)$ is the i-th cohomology of this complex.

THEOREM 1.4.6. (1) *$\mathrm{Ext}^i(M, N)$ is a covariant functor in N and a contravariant functor in M. (In particular, it does not depend on the choice of $P^\bullet$.)*

(2) *Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be a short exact sequence of modules. Then we have a long exact sequence of Ext groups:

$$\dots \to \mathrm{Ext}^i(M_3, N) \to \mathrm{Ext}^i(M_2, N) \to \mathrm{Ext}^i(M_1, N) \to \mathrm{Ext}^{i+1}(M_3, N) \to \dots$$

Similarly, for a short exact sequence

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

we have a long exact sequence

$$\dots \to \mathrm{Ext}^i(M, N_1) \to \mathrm{Ext}^i(M, N_2) \to \mathrm{Ext}^i(M, N_3) \to \mathrm{Ext}^{i+1}(M, N_1) \to \dots$$

One can think about the second assertion of Theorem 1.4.6 as the statement which "compensates" the non-exactness of the Hom functor.

In fact $\mathrm{Ext}(M, N)$ makes sense not only for modules but also for complexes. Namely, suppose $M^\bullet$ and $N^\bullet$ are bounded complexes (i.e. collections of $\{M^i\}$ $i \in \mathbb{Z}$ and maps $\partial_i : M^i \to M^{i+1}$, where $M^i = 0$ for $\mid i \mid\gg 0$ and $\partial^i\partial^{i-1} = 0$.). It is a theorem that there exists a complex of projective modules $P^\bullet$ and a map of complexes $\alpha : P^\bullet \to M^\bullet$ such that α induces isomorphism on cohomologies.

Define a bicomplex $\mathrm{Hom}(P^\bullet, N^\bullet)$:

$$\begin{array}{ccccccc}
 & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & \mathrm{Hom}(P^{-i}, N^{j+1}) & \longrightarrow & \mathrm{Hom}(P^{-i-1}, N^{j+1}) & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & \mathrm{Hom}(P^{-i}, N^{j}) & \longrightarrow & \mathrm{Hom}(P^{-i-1}, N^{j}) & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & \mathrm{Hom}(P^{-i}, N^{j-1}) & \longrightarrow & \mathrm{Hom}(P^{-i-1}, N^{j-1}) & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & &
\end{array}$$

We now define $\mathrm{Ext}^i(M^\bullet, N^\bullet)$ to be $H^i(\mathrm{Tot}(\mathrm{Hom}(P^\bullet, N^\bullet)))$. (For the definition of the total complex of a bicomplex, see Section 1.4.1.)

1.4.3. Homological characterization of functional dimension.

THEOREM 1.4.7. *Let A be a filtered algebra over k such that $gr\,A$ is a finitely generated commutative regular algebra of dimension m. (A commutative algebra is regular if it is the algebra of functions on a smooth affine algebraic variety). Let M be any finitely generated left A-module. Define $d(M)$ to be $\dim(\mathrm{supp}\, gr^F M)$, where F is any good filtration on M and $j(M)$ to be $\min(j \mid \mathrm{Ext}^j(M, A) \neq 0)$. Then*

(1) *$d(M) + j(M) = m$;*
(2) *$\operatorname{Ext}^j(M, A)$ is a finitely generated right A-module and $\dim(\operatorname{Ext}^j(M, A)) \le m - j$;*
(3) *for $j = j(M)$ we have $\dim(\operatorname{Ext}^j(M, A)) = m - j$.*

Applying this to $A = \mathcal{D}$ for both the geometric filtration and Bernstein's filtration, we get:

COROLLARY 1.4.8. *$d_a(M) = d_g(M) = 2n - j(M)$ for any finitely generated $\mathcal{D}$-module M.*

From now on we set $d(M) = d_a(M) = d_g(M)$.

COROLLARY 1.4.9. *M is holonomic $\Leftrightarrow \operatorname{Ext}^j(M, \mathcal{D}) \neq 0$ only for $j = n$.*

PROOF. If M is holonomic, then $d(M) = n$ and hence $j(M) = n$. So we must have $\operatorname{Ext}^j(M, \mathcal{D}) = 0$ for $j < n$ by the definition of $j(M)$. But for $j > n$, Theorem 1.4.7 tells us that $\operatorname{Ext}^j(M, \mathcal{D})$ has dimension less than or equal to $2n - j < n$, so by Bernstein's inequality we must have $\operatorname{Ext}^j(M, \mathcal{D}) = 0$. □

As we have seen before, there is a natural right $\mathcal{D}$-action on $\operatorname{Ext}^n(M, \mathcal{D})$. Furthermore, $\mathcal{D} = \mathcal{D}(\mathbb{A}^{2n})$ has a natural antiinvolution σ, namely $x_i \mapsto x_i$ and $\frac{\partial}{\partial x_j} \mapsto -\frac{\partial}{\partial x_j}$. (An antiinvolution $\sigma : \mathcal{D} \to \mathcal{D}$ is a k-linear map such that $\sigma(d_1 d_2) = \sigma(d_2)\sigma(d_1)$ and $\sigma(\sigma(d_1)) = d_1$.) Treating σ as an isomorphism between $\mathcal{D}^{op}$ and $\mathcal{D}$, we can naturally turn a right $\mathcal{D}$-module into a left one or vice versa. module.

REMARK 1.4.10. The existence of the above involution is an "accidental" fact, i.e., when we replace $\mathbb{A}^n$ by a general algebraic variety it will not exist anymore. However, we shall see later (when we discuss general varieties) that the existence of a canonical equivalence between the categories of left and right $\mathcal{D}$-modules is not accidental.

COROLLARY 1.4.11. *For any holonomic module M let $\mathbb{D}(M)$ be $\operatorname{Ext}^n(M, \mathcal{D})$ considered as a left module. Then $M \mapsto \mathbb{D}(M)$ is an exact contravariant functor from the category of holonomic modules to itself and $\mathbb{D}(\mathbb{D}(M)) = M$.*

PROOF. The fact that $\mathbb{D}$ is exact follows immediately from the long exact sequence of Ext's.

Let us show that $\mathbb{D}^2 \simeq \mathrm{Id}$. Let P be a finitely generated projective $\mathcal{D}$-module. Let $P^\vee = \operatorname{Hom}(P, \mathcal{D})$ be the dual module (considered as a left module as before). It is clear that $P^\vee$ is projective. If $P^\bullet$ is a complex of projective $\mathcal{D}$-modules then we shall denote by $(P^\vee)^\bullet$ the complex defined by

$$(P^\vee)^i = (P^{-i})^\vee$$

with the obvious differential.

Let M be a holonomic $\mathcal{D}$-module and let $P^\bullet$ be a projective resolution of M. Then it is clear that $(P^\vee)^\bullet[n]$ is a projective resolution of $\mathbb{D}(M)$. Thus $(P^\vee[n])^\vee[n] = P$ is a projective resolution of $\mathbb{D}(\mathbb{D}(M))$. Hence $\mathbb{D}(\mathbb{D}(M)) = M$. □

COROLLARY 1.4.12. *Let M be $\mathcal{O}$-coherent. Then $\mathbb{D}(M) = \operatorname{Hom}_{\mathcal{O}}(M, \mathcal{O}) = M^\vee$ as an $\mathcal{O}$-module (i.e. $\mathbb{D}(M)$ is a dual vector bundle). The dual connection is described in the following way: Let $\nabla_M : M \to M \otimes_{\mathcal{O}} \Omega^1$. Then $\nabla_{M^\vee} : M^\vee \to M^\vee \otimes_{\mathcal{O}} \Omega^1$ is given by $\nabla_{M^\vee}(\xi)(m) = -\xi(\nabla_M(m)) \in \Omega^1$ for every $\xi \in M^\vee$ and $m \in M$.*

The proof will be given next time.

EXAMPLE 1.4.13. Let $M = V \otimes \mathcal{O}$ be a trivial $\mathcal{O}$-module with ∇_M given by $\omega_M \in \operatorname{End}(V) \otimes \Omega^1$. Then $\omega_{M^\vee} \in \operatorname{End}(V^*) \otimes \Omega^1$ is $(-\omega_M)^T$, where the T-superscript denotes matrix transpose.

1.4.4. Exercises.

EXERCISE 1.4.1. Let $M = k[x, x^{-1}]$ (char$k = 0$) with the natural structure of a $\mathcal{D}(\mathbb{A}^1)$-module. Note that we have a short exact sequence $0 \to \mathcal{O} \to M \to \delta \to 0$ where δ is the module of delta-functions at 0. Compute explicitly (e.g. by generators and relations or by writing an action of $\mathcal{D}$ on some explicit vector space) the module $\mathbb{D}(M)$ and explain how to see from this description that there is a short exact sequence $0 \to \delta \to \mathbb{D}(M) \to \mathcal{O} \to 0$.

1.5. Proofs of 4.7 and 4.12

In this lecture we will prove Theorem 1.4.7 and Corollary 1.4.12. Let us start with the latter. First we need to develop some technology.

1.5.1. de Rham complex. Let M be any left $\mathcal{D}$-module. We form a complex (the de Rham complex $dR(M)$)

$$0 \to M \to M \otimes_{\mathcal{O}} \Omega^1 \to M \otimes_{\mathcal{O}} \Omega^2 \to \dots \to M \otimes_{\mathcal{O}} \Omega^n \to 0$$

with M in cohomological degree $-n$.

We need to define the differentials $d : M \otimes_{\mathcal{O}} \Omega^k \to M \otimes_{\mathcal{O}} \Omega^{k+1}$. Take our flat connection $\nabla : M \to M \otimes_{\mathcal{O}} \Omega^1$. For any $m \otimes \omega \in M \otimes_{\mathcal{O}} \Omega^k$ we set

$$d(m \otimes \omega) = m \otimes d(\omega) + \nabla(m) \wedge \omega.$$

The flatness of ∇ implies that this is really a complex, i.e., $d^2 = 0$.

REMARK 1.5.1. Let $n = 1$. Then for any M the corresponding de Rham complex $dR(M)$ has the form $0 \to M \xrightarrow{\nabla} M \otimes_{\mathcal{O}} \Omega^1 \to 0$.

Now consider the complex $dR(\mathcal{D})$. For any module M, $\mathrm{End}_{\mathcal{D}M}$ acts on $dR(M)$, so in particular $\mathcal{D}^{op}$ acts on $dR(\mathcal{D})$. Thus $dR(\mathcal{D})$ is a complex of right $\mathcal{D}$-modules.

LEMMA 1.5.2. $H^i(dR(\mathcal{D})) = \begin{cases} 0 & \textit{if } i \neq 0; \\ \Omega^n & \textit{if } i = 0. \end{cases}$

PROOF. For any $0 \le i \le n$, $(dR(\mathcal{D}))^{-i} = \mathcal{D} \otimes_{\mathcal{O}} \Omega^{n-i}$ and $\Omega^{n-i} = \mathcal{O} \otimes \Lambda^{n-i}(k^n)$. So the de Rham complex for $\mathcal{D}$ has the form

$$0 \to \mathcal{D} \to \mathcal{D} \otimes_k k^n \to \cdots \to \mathcal{D} \otimes_k \Lambda^j(k^n) \to \cdots \to \mathcal{D} \otimes_k \Lambda^n(k^n) \to 0$$

□

In the case of $\mathbb{A}^n$ this complex coincides with the Koszul complex of M. Let us briefly recall this notion.

1.5.2. Koszul complex. Let N be a module over $k[y_1, \dots, y_n]$. There exists a natural complex called the Koszul complex of N of the form

$$0 \to N \to N \otimes k^n \to \cdots \to N \otimes \Lambda^j(k^n) \to \cdots \to N \otimes \Lambda^n(k^n) \to 0.$$

with the map $N \to N \otimes k^n$ given by $n \mapsto \oplus_i y_i(n)$, the map $N \otimes k^n \to N \otimes \Lambda^2(k^n)$ given by $(p_1, \dots, p_n) \mapsto \oplus(y_i p_j - y_j p_i)$, and so on. We shall denote this complex by $Kos(M)$.

LEMMA 1.5.3. *If M is free over $\mathcal{O}$, then $Kos(M)$ is a free resolution of $M/(< y_1, \dots, y_n > \cdot M)$, i.e., for $i \neq 0$ $H^i(Kos(M)) = 0$ and $H^0(Kos(M)) = M/(< y_1, \dots, y_n > \cdot M)$.*

To see that $H^0(Koszul(M)) = M/(< y_1, \dots, y_n > \cdot M)$, examine the map $N \otimes \Lambda^{n-1}(k^n) \xrightarrow{d} N \otimes \Lambda^n(k^n) \to 0$. $\Lambda^{n-1}(k^n)$ is isomorphic to k^n and d maps $(p_1, \dots, p_n)$ to $\sum_{i=1}^n y_i p_i$. The desired statement follows.

REMARK 1.5.4. Let M be any $\mathcal{D}$-module. M has a natural $k[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}]$-module structure, given by the inclusions $\frac{\partial}{\partial x_j} \in \mathcal{D}$. The Koszul complex corresponding to this module structure is the same complex as $dR(M)$.

EXAMPLE 1.5.5. Let $n = 2$. The de Rham complex has the form

$$0 \to M \to M \otimes \Omega^1 \to M \otimes \Omega^2 \to 0.$$

where the first map is $m \mapsto \frac{\partial m}{\partial x_1} dx_1 + \frac{\partial m}{\partial x_2} dx_2$ and the second map is $p_1 dx_1 + p_2 dx_2 \mapsto \left(\frac{\partial p_2}{\partial x_1} - \frac{\partial p_1}{\partial x_2}\right) dx_1 \wedge dx_2$. It is easy to see that these maps coincide with the corresponding differentials in the Koszul complex.

COROLLARY 1.5.6. *If M is free over $k[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}]$, then $H^i(dR(M)) = 0$ for any $i \neq 0$ and $H^0(dR(M)) = M/(< \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} > \cdot M)$.*

In particular $H^0(dR(\mathcal{D})) = \mathcal{D}/(< \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} > \cdot \mathcal{D}) = \Omega^n = \mathcal{O}$ and $dR(\mathcal{D})$ is a free resolution of $\mathcal{O}$.

1.5.3. Projective resolution of any $\mathcal{O}$-coherent module. Let M and N be left $\mathcal{D}$-modules. Then $M \otimes_{\mathcal{O}} N$ has a natural left $\mathcal{D}$-module structure. Derivatives act on $M \otimes_{\mathcal{O}} N$ according to the Leibnitz rule: $\frac{\partial}{\partial x_i}(m \otimes n) = \frac{\partial m}{\partial x_i} \otimes n + m \otimes \frac{\partial n}{\partial x_i}$.

Given M, we can consider $M \otimes_{\mathcal{O}} dR(\mathcal{D})$. Since M is $\mathcal{O}$-coherent, M is also locally free and hence projective.

LEMMA 1.5.7. *(Fact from commutative algebra). Let R be a commutative ring and P a projective module over R. Then $N \mapsto P \otimes_R N$ is an exact functor.*

Using this fact, we see that $H^i(M \otimes_{\mathcal{O}} dR(\mathcal{D})) = 0$ for $i \neq 0$ and $H^0(M \otimes_{\mathcal{O}} dR(\mathcal{D})) = M \otimes_{\mathcal{O}} \mathcal{O} = M$. Thus $M \otimes_{\mathcal{O}} dR(\mathcal{D})$ is a projective (over $\mathcal{D}$) resolution of M. (The resolution is projective over $\mathcal{D}$ because if N is projective over $\mathcal{O}$, then $N \otimes \mathcal{D}$ is projective over $\mathcal{D}$ since $\mathrm{Hom}_{\mathcal{D}}(N \otimes_{\mathcal{O}} \mathcal{D}, N') = \mathrm{Hom}_{\mathcal{O}}(N, N')$ so $\mathrm{Hom}_{\mathcal{D}}(N \otimes_{\mathcal{O}} \mathcal{D}, -)$ is exact iff $\mathrm{Hom}_{\mathcal{O}}(N, -)$ is exact.)

We now prove Corollary 1.4.12.
Consider the complex $\mathrm{Hom}_{\mathcal{D}}(M \otimes dR(\mathcal{D}), \mathcal{D})$

$$0 \to \mathrm{Hom}_{\mathcal{D}}(M \otimes dR^0(\mathcal{D}), \mathcal{D}) \to \cdots \to \mathrm{Hom}_{\mathcal{D}}(M \otimes dR^{-n}(\mathcal{D}), \mathcal{D}) \to 0$$

We claim that $\mathrm{Hom}_{\mathcal{D}}(M \otimes dR(\mathcal{D}), \mathcal{D}) = M^\vee \otimes dR(\mathcal{D})[-n]$ (by $dR(\mathcal{D})[-n]$ we mean the complex $dR(\mathcal{D})$, shifted by n to the right).

$\mathrm{Hom}_{\mathcal{D}}(M \otimes dR(\mathcal{D}), \mathcal{D}) = \mathrm{Hom}_{\mathcal{D}}(M \otimes_{\mathcal{O}} \mathcal{D} \otimes \Omega^{n-i}, \mathcal{D}) = M^\vee \otimes_{\mathcal{O}} \mathcal{D} \otimes \mathrm{Hom}(\Omega^{n-i}, \mathcal{O})$.
But $\mathrm{Hom}(\Omega^{n-i}, \mathcal{O}) \cong \Omega^i$, if an identification of $\mathcal{O}$ and Ω^n is chosen (as in our case). And the differential of this complex is exactly the one of $M^\vee \otimes dR(\mathcal{D})$.

Corollary 1.4.12 now follows from that $\mathbb{D}(M)$ is exactly the nth cohomology of $\mathrm{Hom}_{\mathcal{D}}(M \otimes dR(\mathcal{D}))$.

EXAMPLE 1.5.8. Let $n = 1$ and let M be an $\mathcal{O}$-coherent sheaf of rank 1 (then $M \cong \mathcal{O}$ as an $\mathcal{O}$-module). Let α be a one form such that the flat connection corresponding to the $\mathcal{D}$-module structure has the form $\nabla(f) = df + f\alpha$.

The corresponding de Rham complex has the form $0 \to \mathcal{D} \to \mathcal{D} \to 0$, where the map $\mathcal{D} \to \mathcal{D}$ is given by right multiplication by $d - \alpha$, where $d - \alpha$ means $\frac{d}{dx} - g(x)$ if $\alpha = g(x)dx$.

As $\mathcal{O}$-modules, M and $M^\vee$ are isomorphic, while as $\mathcal{D}$-modules they correspond to forms α and $-\alpha$. Examining the above complex shows that $\mathbb{D}(M) = M^\vee$.

EXERCISE 1.5.1. Let $M = \delta_a$, where $a \in k$. Prove that $\mathbb{D}(\delta_a) = \delta_a$.

Let us now prove Theorem 1.4.7.

1.5.4. Proof of Theorem 1.4.7. We start with a fact from commutative algebra. For a proof, see [**E**],18.4 and 18.7.

THEOREM 1.5.9. *[Serre] Let R be a regular algebra of dimension m and let M be a finitely generated R-module. Set $d(M) = \dim(\operatorname{supp} M)$ and $j(M) = \min(j \mid \operatorname{Ext}^j(M,R) \neq 0)$. Then*
(1) *$d(M) + j(M) = m$;*
(2) *$\operatorname{Ext}^j(M,R)$ is a finitely generated R-module and $d(\operatorname{Ext}^j(M,R)) \leq m - j$;*
(3) *For $j = j(M)$ we have equality in (2).*

We now apply this theorem with $R = \mathsf{gr}\, A$.

LEMMA 1.5.10. *For any filtered A-module M there exists a filtered resolution $P^\bullet$ of M such that $\mathsf{gr}\, P^\bullet$ is a free resolution of $\mathsf{gr}\, M$.*

PROOF. Choose homogeneous generators $(\bar{m}_i)_{i\in I}$ of $\mathsf{gr}\, M$ (so that each m_i is of some degree k_i). Define a grading on R^I by $\deg(rx_i)$ $=\deg(r)k_i$, where x_i is the element corresponding to 1 in the ith component. Then there is a natural graded surjection $R^I \to \mathsf{gr}\, M$ given by sending x_i to $\bar{m}_i$.

Lift $\bar{m}_1, \dots, \bar{m}_k$ to $m_1, \dots, m_k$, where $m_j \in F_{i_j} M$. By Proposition 1.1.12, the $\{m_i\}$ generates M. We thus get a map of filtered A-modules $A^k \to M$ where the filtration on the jth component on A^k is shifted by i_j.

We have two surjective maps $\alpha_0 : A^I \twoheadrightarrow M$ and $\beta_0 : R^I \twoheadrightarrow \mathsf{gr}\, M$. By construction, $\mathsf{gr}\ \alpha_0 = \beta_0$. Let N be the kernel of α_0. We can repeat the same construction for N to get $\alpha_1 : A^J \twoheadrightarrow N$ such that $\mathsf{gr}\ \alpha_1 = \beta_1 : R^J \twoheadrightarrow N$. We will then have two exact sequences $A^l \xrightarrow{\alpha_1} A^k \xrightarrow{\alpha_0} M \to 0$ and $B^l \xrightarrow{\beta_1} B^k \xrightarrow{\beta_0} \mathsf{gr}\, M \to 0$.

Repeating this construction, we get a filtered resolution of M whose associated graded complex is a free resolution of $\mathsf{gr}\, M$.

□

Let $K^\bullet$ be a complex with a filtration $\{F_i K^\bullet\}$ (i.e. $d(F_i K^p) \subset F_i K^{p+1}$). Then we have a natural graded complex $\mathsf{gr}^F K^\bullet$ with $gr^F K^\bullet = \oplus gr^F_j K^\bullet$, where $gr^F_j K^\bullet$ is the complex $\{\dots \to gr^F_j K^p \to gr^F_j K^{p+1} \to \dots\}$. So we have a set of groups $H^i(\mathsf{gr}^F_j K^\bullet)$.

On the other hand, there is a natural filtration on the homologies of the original complex $\{\dots \to K^{i-1} \xrightarrow{d} K^i \xrightarrow{d} K^{i+1} \to \dots\}$. Taking the associated graded, we obtain another set of groups $\mathsf{gr}^F_j(H^i(K^\bullet))$.

The following lemma is well-known in homological algebra.

LEMMA 1.5.11. (1) *$\mathsf{gr}^F_j(H^i(K^\bullet))$ is a subquotient (i.e., a quotient of a subspace) of $H^i(\mathsf{gr}^F_j K^\bullet)$.*
(2) *Assume that there exists some $n \in \mathbb{Z}$ such that $H^i(\mathsf{gr}\, K^\bullet) = 0$ for $i > n$. Then $\mathsf{gr}^F_j(H^n(K^\bullet)) = H^n(\mathsf{gr}_j K^\bullet)$ for all j.*

It is clear that Lemma 1.5.11 and Theorem 1.5.9 together imply Theorem 1.4.7.

1.6. $\mathcal{D}$-modules on general affine varieties

Let X be a smooth irreducible affine algebraic variety over a field k of characteristic 0. We start by defining the algebra of differential operators on X, which we will denote by $\mathcal{D}_X$.

1.6.1. Definition of $\mathcal{D}_X$.

DEFINITION 1.6.1. (1) Let R be a commutative ring. We define a subalgebra $\mathcal{D}_X$ of $\operatorname{End}_k(R,R)$ inductively. Define $\mathcal{D}_0(R) = R$, and embed it into $\operatorname{End}_k(R,R)$ by sending r to multiplication by r. Assume we have defined $\mathcal{D}_{n-1}(R)$. Then we define $\mathcal{D}_n$ by the condition that a map $d : R \to R$ is in $\mathcal{D}_n(R)$ if and only if for any element $r \in \mathcal{D}_0(R)$,

$[d, r] \in \mathcal{D}_{n-1}(R)$. Now we take $\mathcal{D}(R)$ to be $\cup \mathcal{D}_n(R)$. We will call an element of $\mathcal{D}_n(R)$ a differential operator of order less or equal to n.

(2) Let M and N be two R-modules. Define the space of differential operators $Diff(M, N) = \cup Diff_n(M, N)$ from M to N in the following way:
- $Diff_0(M, N) = \mathrm{Hom}_R(M, N)$
- $d \in Diff_n(M, N)$ iff for any $r \in \mathcal{D}_0(R)$ we have $[d, r](m) = d(rm) - r(dm) \in Diff_{n-1}(M, N)$.

We now set $\mathcal{D}_X$ to be $\mathcal{D}(\Gamma(\mathcal{O}_X))$. Although this definition makes sense for arbitrary X, it works badly when X is singular, so we will work with it only for X smooth.

It will be desirable to have $\mathcal{D}_X$ as a quasicoherent sheaf, as our proofs will often start by reducing to a suitable affine covering. Furthermore, it is necessary to work at the level of sheaves to extend the notion of $\mathcal{D}$-module to a non-affine variety. We define $\mathcal{D}_X(U)$ for an open subset U by $\mathcal{D}(\mathcal{O}_U)$. That this gives a quasicoherent sheaf follows immediately from the following lemma. We will use $\mathcal{D}_X$ to refer both to the algebra and the sheaf.

LEMMA 1.6.2. *We have isomorphisms $\mathcal{D}_U \cong Diff(\mathcal{O}_X, \mathcal{O}_U) \cong \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{O}_U$ which are functorial with respect to restriction maps.*

PROOF. We have natural maps $\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{O}_U \to D_U$ and $D_U \to Diff(\mathcal{O}_X, \mathcal{O}_U)$, both of which can easily be seen to be injective and functorial with respect to restriction maps. It remains to show that their composition is surjective.

Let L be an element of $Diff(\mathcal{O}_X, \mathcal{O}_U)$. We want to show that there is an element f of $\mathcal{O}_X$ such that $fL \in \mathcal{D}_X$. Choose generators $x_1, x_2, \cdots x_m$ of $\mathcal{O}_X$ and choose f so that $fL(x_i) \in \mathcal{O}_X$. Then fL applied to any product of x_is lies in $\mathcal{O}_X$, and as the x_i generate O_X, we have $fL \in \mathcal{D}_X$. ☐

REMARK 1.6.3.
- For $X = \mathbb{A}^n$, our new definition of $\mathcal{D}_X$ coincides with our previous definition. The filtration by order of differential operator coincides with the geometric filtration.
- We will define $\mathcal{D}$-modules on a singular variety X in the next lecture.

We will now prove that $\mathsf{gr}(\mathcal{D}_x) \cong \mathcal{O}_{T^*X}$, where $\mathcal{D}_X$ is equipped with the order filtration. Before doing so, we need the following lemma:

LEMMA 1.6.4. *Let z be a point in an open set U of X and let L be an element of $\mathrm{End}(\mathcal{O}_X, \mathcal{O}_U)$. We have $Lf(z) = 0$ for all $f \in \mathcal{O}_X$ such that $f(z) = df(z) = \cdots = d^n f(z) = 0$ if and only if $L \in \mathcal{D}_{X,n}(U)$.*

PROOF. We induct on n. When $n = 0$, the statement is obvious. Assume we know the statement for $n - 1$. The condition on f is equivalent to f being in m^{n+1}, where m is the maximal ideal corresponding to U. Therefore, f is writable as a linear combination of elements $f_1 f_2$, where f_1 is in m and f_2 is in m^n, and it suffices to check the statement for such elements. We have $Lf_1 f_2 = f_1 L f_2 + [L, f_1] f_2$. As $f_1 \in m$, $f_1 L f_2(z) = 0$, so we have $L f_1 f_2(z) = [L, f_1] f_2(z)$. This immediately implies the desired statement. ☐

REMARK 1.6.5. This lemma in fact shows that $\mathcal{D}_{X,n}$ is, as an $\mathcal{O}_X$ module, the sections on the n-th nilpotent neighborhood of the diagonal in $X \times X$. In other words, letting m be the ideal of the diagonal, we have $\mathcal{D}_{X,n} \cong \frac{\mathcal{O}_{X\times X}}{m^{n+1}}$. The multiplication on $\mathcal{D}_X$ is given by convolution.

Thus for every element of $\mathcal{D}_{X,n}(U)/\mathcal{D}_{X,n-1}(U)$, we get a map from m_z^{n-1}/m_z^n to k for every element z of U. As the fiber of $S^n T$ at z is $(m_z^n/m_z^{n+1})^*$, we get a map from $\mathsf{gr}\, D$ to $\mathcal{O}_{T^*X} = S^\bullet T$. The only if direction of the above lemma implies that this map is injective, and now we want to prove that it is surjective. But this follows from $\mathcal{D}_{X,0}$ containing $\mathcal{O}_X$ and $\mathcal{D}_{X,1}$ containing T_X, as

$S^\bullet T$ is generated by $\mathcal{O}_X$ and $\mathcal{T}_X$. (Both of these containments are equivalences, but we do not need this.)

In short, we have proved:

THEOREM 1.6.6. *The symbol map gives an isomorphism* $\mathsf{gr}(\mathcal{D}_x) \cong \mathcal{O}_{T^*X}$.

As a corollary, we get

COROLLARY 1.6.7. $\mathcal{D}_X$ *is generated by* $\mathcal{O}_X$ *and* d_v *for* $v \in \operatorname{Vect} X$, *with the relations*

(1) $[d_v, f] = v(f)$, *where* $v \in \operatorname{Vect} X$ *and* $f \in \mathcal{O}_X$
(2) $[d_v, d_w] = d_{[v,w]}$, *where* $v, w \in \operatorname{Vect} X$ *and the second commutator is the Lie bracket*
(3) $d_{fv} = f d_v$ *where* $v \in \operatorname{Vect} X$ *and* $f \in \mathcal{O}_X$

PROOF. These generators and relations define an algebra D'_X. As these relations are satisfied in D_X, we get a natural map $D'_X \to D_X$. Giving D'_X the "order filtration", we see that the map $\mathsf{gr}\, D'_X \to D_X \cong S^\bullet T_X$ is an isomorphism, as desired. $\square$

We will now define a notion of "local coordinate system". Let $U \subset X$ be an open subset of X and let $\dim X = n$.

DEFINITION 1.6.8. A coordinate system on U is a set of functions $x_1, \dots, x_n, \in \mathcal{O}_U$ and a set of vector fields $\partial_1, \dots, \partial_n$ such that $\partial_i(x_j) = \delta_{i,j}$ (i.e., an etale map $U \to \mathbb{A}^n$).

LEMMA 1.6.9. *For any point* $x \in X$, *there exists a neighborhood* $U \ni x$ *that can be equipped with a coordinate system.*

PROOF. Choose functions $f_1, f_2, \cdots f_n$ with linearly independent differentials at x. Take U to be the open subset of X where $df_1, \cdots df_n$ are linearly independent. Then we can define a coordinate system on U by taking $f_1, f_2, \cdots f_n$ to be our functions and $df_1, df_2, \cdots df_n$ to be our vector fields. $\square$

LEMMA 1.6.10. *If we have a coordinate system on* U, *then* $\mathcal{D}_U = \oplus_\alpha \mathcal{O}_U \partial^\alpha$, *where* $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \mathbb{Z}_+$ *and* $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$.

PROOF. This follows immediately from Theorem 1.6.6. $\square$

As an example of the use of coordinate systems, we note that the symbol map is a map of Poisson algebras. We defined Poisson algebras in Definition 1.3.24 and equipped $\mathcal{O}_{T^*X}$ and $\mathcal{D}$ with Poisson structures in the discussion immediately following Definition 1.3.24. The symbol map is obviously a map of Poisson algebras when we have a coordinate system on X; the general result can be deduced by reducing to an affine open cover.

1.6.2. Left and right modules over $\mathcal{D}_X$. Let $\mathcal{M}^l(\mathcal{D}_X)$ denote the category of left $\mathcal{D}_X$-modules and $\mathcal{M}^r(\mathcal{D}_X)$ the category of right $\mathcal{D}_X$-modules.

Let M be an $\mathcal{O}_X$-module. A right $\mathcal{D}_X$-module structure on M is the same as an action of $\operatorname{Vect} X$ satisfying the following conditions:

(1) $(\partial_1 \cdot \partial_2 - \partial_2 \cdot \partial_1)(m) = -[\partial_1, \partial_2](m)$ for any vector fields ∂_1 and ∂_2
(2) $(f\partial)(m) = \partial(fm)$ for any $f \in \mathcal{O}_X$ and any $\partial \in \mathcal{D}_X$
(3) $(\partial(fm) - f\partial(m)) = -(\partial(f))m$ for any $f \in \mathcal{O}_X$ and any $\partial \in \mathcal{D}_X$

LEMMA 1.6.11. $\Omega^n(X)$ *has a canonical right* $\mathcal{D}_X$*-module structure.*

PROOF. We claim that the action of vector fields on $\Omega^n(X)$ by $-Lie_\partial$ satisfies the properties listed above.

By the Cartan formula $Lie_\partial(\omega) = d(\iota_\partial(\omega)) + \iota_\partial(d(\omega))$. For $\omega \in \Omega^n(X)$ the last term is equal to 0 and so we have $Lie_\partial(\omega) = d(\iota_\partial(\omega))$. Now checking that this satisfies the three properties above is a straightforward exercise. □

EXERCISE 1.6.1. Check that the three conditions are indeed satisfied.

We can similarly check that

LEMMA 1.6.12. *Let M be a left $\mathcal{D}_X$-module. Then $M \otimes \Omega^n(X)$ has a natural right $\mathcal{D}_X$-module structure given by*

$$\partial(m \otimes \omega) = \partial m \otimes \omega - m \otimes Lie_\partial(\omega).$$

Similarly if M is a right $\mathcal{D}_X$-module, then $M \otimes (\Omega^n(X))^{-1}$ has a structure of left $\mathcal{D}_X$-module.

EXERCISE 1.6.2. Describe the action of ∂ on $M \otimes (\Omega^n(X))^{-1}$.

COROLLARY 1.6.13. *The categories $\mathcal{M}^l(\mathcal{D}_X)$ and $\mathcal{M}^r(\mathcal{D}_X)$ are canonically equivalent. The equivalence is given by $M \mapsto M \otimes \Omega^n(X)$.*

We shall use the notation $\mathcal{M}(\mathcal{D}_X)$ for this category.

Let $M \in \mathcal{M}(\mathcal{D}_X)$ be finitely generated. There exist a good filtration on M and $SS(M) = supp(\mathrm{gr}\, M)$ is a closed subset of T^*X which does not depend on the filtration. By Gabber's theorem, $\dim SS(M) \geq n$. As in the case of $\mathbb{A}^n$, a $\mathcal{D}_X$-module M is called holonomic if $\dim SS(\mathrm{gr}\, M) = n$.

Assume that M is holonomic. In that case, we can define a cycle $s.c.(M)$ (the singular cycle of M) in the following way: assume that $Z_1, ..., Z_k$ are the irreducible components of $supp(grM)$ (note that Gabber's theorem implies that the dimension of each Z_i is equal to n). Define m_i to be the rank of $\mathrm{gr}\, M$ at the generic point of Z_i. Then we define $s.c.(M) = \sum m_i Z_i$. It is easy to see that $s.c.(M)$ doesn't depend on the choice of a good filtration. Moreover, if $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of holonomic modules, then $s.c.(M_2) = s.c.(M_1) + s.c.(M_3)$.

LEMMA 1.6.14. *Holonomic modules have finite length.*

PROOF. It is easy to see by induction that if $s.c.(M) = \sum m_i Z_i$ then the length of M does not exceed $\sum m_i$. □

Inverse and direct image functors.

1.6.3. Inverse Image and Tensor Product. Let $\pi : X \to Y$ be a morphism of affine algebraic varieties. Then for any $f \in C^\infty(Y)$ we can consider the function $f \circ \pi \in C^\infty(X)$. We would like to have an analogous construction for $\mathcal{D}$-modules.

DEFINITION 1.6.15. Let M be a left $\mathcal{D}_Y$-module. The inverse image of M is a left $\mathcal{D}_X$-module $\pi^0(M)$ with underlying $\mathcal{O}_X$-module structure $\mathcal{O}_X \otimes_{\mathcal{O}_Y} M$. The Vect_X action is given by $\partial(f \otimes m) = \partial(f) \otimes m + f \otimes \pi_*(\partial)m$ for any $\partial \in Vec_X$.

Let $\mathcal{D}_{X\to Y}$ be the inverse image of $\mathcal{D}_Y$. This has a natural left $\mathcal{D}_X$-module structure and a natural right $\mathcal{D}_Y$-module structure.

LEMMA 1.6.16. $\pi^0 M = \mathcal{D}_{X\to Y} \otimes_{\mathcal{D}_Y} M$.

PROOF. We have

$$\mathcal{D}_{X\to Y} \otimes_{\mathcal{D}_Y} M = (\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{D}_Y) \otimes_{\mathcal{D}_Y} M = \mathcal{O}_X \otimes_{\mathcal{O}_Y} M.$$

□

Suppose we have a coordinate system $y_1, \ldots y_m, \partial_1, \ldots, \partial_m$ on Y. Then $\mathcal{D}_Y = \oplus_\alpha \mathcal{O}_Y \partial^\alpha$ and $\mathcal{D}_{X\to Y} = \oplus_\alpha \mathcal{O}_X \partial^\alpha$, where α ranges over multi-indexes.

For functions $f, g \in C^\infty(X)$ we can consider their product. The corresponding operation on modules is the tensor product: let M and N be left $\mathcal{D}_X$-modules, then $M \otimes N = M \otimes_{\mathcal{O}_X} N$ with the action of vector fields given by the Leibnitz rule.

For modules there also exists an operation called the exterior product: for $M \in \mathcal{M}(\mathcal{D}_X)$ and $N \in \mathcal{M}(\mathcal{D}_Y)$ we can construct $M \boxtimes N \in \mathcal{M}(\mathcal{D}_{X\times Y})$. This is defined to be $M \underset{k}{\otimes} N$ with the natural $\mathcal{D}_{X\times Y} = \mathcal{D}_X \otimes_k \mathcal{D}_Y$-module structure.

LEMMA 1.6.17. *$M \otimes N = \Delta^0(M \boxtimes N)$, where $\Delta : X \to X \times X$ is the diagonal embedding.*

PROOF. This follows from the corresponding theorem for $\mathcal{O}$-module and an explicit computation. □

1.6.4. Direct Image. Let π be a morphism $X \to Y$. For every distribution f with compact support on X, we can construct a distribution on Y by integrating f over the fibers of π. We now define an analogous operation on modules. Since distributions intuitively correspond to right $\mathcal{D}$-modules, it will be easier to spell out the definition of direct image for right modules. Since we have canonical equivalence between the categories of left and right modules, the definition will make sense for left modules as well.

DEFINITION 1.6.18. For any $\pi : X \to Y$ we define the functor $\pi_0 : \mathcal{M}^r(\mathcal{D}_X) \to \mathcal{M}^r(\mathcal{D}_Y)$ defined by $\pi_0(M) = M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$. The right $\mathcal{D}_Y$-module structure is induced from the right $\mathcal{D}_Y$ action on $\mathcal{D}_{X\to Y}$.

REMARK 1.6.19. "0" means that the functors are not yet derived.

EXAMPLE 1.6.20. Let $\pi : \operatorname{Spec} k \hookrightarrow \mathbb{A}^1$ be the inclusion of the origin. Consider k as a $\mathcal{D}_{\operatorname{Spec} k}$-module. We have:

$$\pi_0 k = \mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{D}_Y = \bigoplus_{n\geq 0} k \left(\frac{\partial}{\partial x}\right)^n = \mathcal{D}_{\mathbb{A}^1}/(x \cdot \mathcal{D}_{\mathbb{A}^1}) = \delta_0$$

as a right $\mathcal{D}_Y$-module.

Since the categories of right and left $\mathcal{D}$-modules are canonically equivalent, we can define direct and inverse images for both left and right modules. For example, let M be a left $\mathcal{D}_X$-module. Then

$$\pi_0(M) = (M \otimes_{\mathcal{O}_X} \Omega^n(X) \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \otimes_{\mathcal{O}_Y} (\Omega^m(Y))^{-1}.$$

EXAMPLE 1.6.21. Let $X = \mathbb{A}^1$ and $Y = pt$. Take π to be the constant map $X \to Y$. Then for any right $\mathcal{D}_X$-module M, we have $\pi_0 M = M/\left(M \cdot \frac{\partial}{\partial x}\right)$. Indeed, $\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_k k = \mathcal{O}_X$, and $\mathcal{O}_X$ is generated by 1 with relations $\partial(1) = 0$ for any $\partial \in Vec_X$. Hence $\pi_0 M = M \otimes_{\mathcal{D}_X} \mathcal{O}_X = M/\left(M \cdot \frac{\partial}{\partial x}\right)$.

More generally we have the following lemma, which is proven in exactly the same way.

LEMMA 1.6.22. *Let X be a smooth affine variety and let π be the constant map $X \to pt$. Then for every right $\mathcal{D}_X$-module M, $\pi_0 M = M/\operatorname{span}(M \cdot \partial)$ (coinvariants of Vec_X on M).*

We now statement some results about inverse and direct images which we will prove in the next two sections.

THEOREM 1.6.23. *Let $X \xrightarrow{\pi} Y \xrightarrow{\tau} Z$ be morphisms of affine algebraic varieties. Then*

1. $(\tau \cdot \pi)_0 = \tau_0 \cdot \pi_0$ *and* $(\tau \cdot \pi)^0 = \pi^0 \cdot \tau^0$.
2. *The functors π_0 and π^0 map holonomic modules to holonomic ones. The same is true for their derived functors (as π_0 and π^0 are defined via tensor products, they are right exact and so $L^i\pi_0$ and $L^i\pi^0$ are defined).*

THEOREM 1.6.24. *[Kashiwara] Let $\pi : X \to Y$ be a closed embedding. Then π_0 is an equivalence between $\mathcal{M}(\mathcal{D}_x)$ and $\mathcal{M}_X(\mathcal{D}_Y)$, where $\mathcal{M}_X(\mathcal{D}_Y)$ is the category of $\mathcal{D}_Y$-modules which are set-theoretically supported on X.*

Recall that a module M is set-theoretically supported on X if every element $f \in \mathcal{I}_X \subset \mathcal{O}_Y$ acts locally nilpotently on M.

EXAMPLE 1.6.25. Let $X = 0$ and $Y = \mathbb{A}^1$. Let $\delta_0 = \mathcal{D}_Y/(x \cdot \mathcal{D}_Y)$ and let M be a $\mathcal{D}_Y$-module supported at 0. Then $\mathrm{Hom}(\delta_0, M) = \{m \in M \mid xm = 0\}$. Since x acts locally nilpotently, there exists a nonzero $m \in M$ such that $x(m) = 0$ and $\mathrm{Hom}(\delta_0, M)$ is nonzero. Thus if M is irreducible, then $M = \delta_0$. Kashiwara's theorem in this case says that any module supported at 0 is a direct sum of δ_0's, which is equivalent to saying that $\mathrm{Ext}^1(\delta_0, \delta_0) = 0$. This may be computed explicitly.

1.6.5. Exercises.

EXERCISE 1.6.3. Let X be a smooth irreducible affine variety over an algebraically closed field k of characteristic zero.

(a) Show that the algebra $D(X)$ of differential operators on X is simple.

(b) Show that $D(X)^\times = \mathcal{O}(X)^\times$, where $A^\times$ denotes the group of invertible elements of a ring A. (Hint: look at the associated graded algebra).

(c) Let $K_X = \Omega_X^n$ be the canonical line bundle of X. Show that $D(X)^{\mathrm{op}}$ is canonically isomorphic as a filtered algebra to $D(K_X)$, the algebra of differential operators from K_X to K_X.

(d) Deduce that $D(X) \cong D(X)^{\mathrm{op}}$ as a filtered algebra if and only if K_X admits a flat connection. (Use that any automorphism of $\mathcal{O}(X)$ lifts to an automorphism of $D(X)$). Deduce that if X is a curve, then $D(X) \cong D(X)^{\mathrm{op}}$ as a filtered algebra.

(e) Show that the canonical equivalence of categories between left and right $D(X)$-modules comes from an antiautomorphism of $D(X)$ if and only if X is a Calabi-Yau variety, i.e., has a nonvanishing volume form. Give an example of an affine smooth curve which does not have a nonvanishing volume form (consider a hyperelliptic curve of genus 2 with a generic missing point).

(f) Give an example of a smooth affine surface X where K_X does not admit a flat connection, and hence where $D(X)$ and $D(X)^{\mathrm{op}}$ are not isomorphic as filtered algebras. (take $Y = \mathbb{P}^1 \times \mathbb{P}^1$, embed it in $\mathbb{P}^5$ using the very ample line bundle $\mathcal{O}(1) \boxtimes \mathcal{O}(2)$, and let $X = Y \cap L^c$, where $L \subset \mathbb{P}^5$ is a generic hyperplane, and L^c is the complement of L).

(g) Show that in (f), one can ensure that X is a closed subvariety of $(\mathbb{C}^*)^n$. (Pick a generic linear coordinate system in $\mathbb{P}^5$ and redefine X to be obtained from Y by deleting the coordinate hyperplanes; show that K_X still does not have a flat connection).

(h) Use (b) to show that in the situation of (g), $D(X)$ is not isomorphic to $D(X)^{\mathrm{op}}$ as an algebra, without regard for the filtration. (Use that an antiautomorphism of an algebra A has to map the centralizer of $A^\times$ to itself).

EXERCISE 1.6.4. a) Let X be given by equations $xy = 0$ in $\mathbb{A}^2$. What is $D(X)$? What is $\mathrm{gr}D(X)$? Is it Noetherian?

b)* Answer the same questions for the quadratic cone $xy = z^2$.

1.7. Proof of Kashiwara's theorem and its corollaries

We now want to prove Theorem 1.6.24. Theorem 1.6.23 will be proven in the next lecture.

In this lecture, we will always work with right $\mathcal{D}$-modules.

1.7.1. Kashiwara's theorem. First of all, let us show that the image of i_0 is contained in $\mathcal{M}_X(\mathcal{D}_Y)$. By definition, $i_0(M) = M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$, where $\mathcal{D}_{X\to Y} = \mathcal{D}_Y/\mathcal{J}\mathcal{D}_Y$, with $\mathcal{J}$ the ideal of X. It suffices to show that every element of $\mathcal{D}_{X\to Y}$ is killed by a large power of $\mathcal{J}$, i.e., that for every

$d \in \mathcal{D}_Y$ there exists an integer n such that $d\mathcal{J}^n \subset \mathcal{J}\mathcal{D}_Y$. To prove this, repeat the proof of the if component of Lemma 1.6.4 for $\mathcal{J}$ instead of m (the if component does not use that m is maximal.)

In order to prove Kashiwara's theorem, we construct a functor $i^! : \mathcal{M}(\mathcal{D}_Y) \to \mathcal{M}(\mathcal{D}_X)$, which will be the inverse of i_0 when restricted on $\mathcal{M}_X(\mathcal{D}_Y)$. For every $M \in \mathcal{M}(\mathcal{D}_Y)$, define $i^!M = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_X, M) = \{m \text{ in } M | \mathcal{J}m = 0\}$. The $\mathcal{D}_X$-action is given as follows: any vector field $\partial \in Vec_X$ can be extended locally to a vector field $\tilde{\partial} \in Vec_Y$ preserving $\mathcal{J}$. For any $m \in i^!M$, define $\partial m = \tilde{\partial} m$. This is an element of $i^!M$, since $\tilde{\partial}$ preserves $\mathcal{J}$. Let us show that this definition does not depend on the choice of the extension. Suppose we have two such extensions $\tilde{\partial}$ and $\tilde{\partial}'$. Then $\tilde{\partial} - \tilde{\partial}' = 0$ on X, i.e., $v = \tilde{\partial} - \tilde{\partial}' \in \mathcal{J} \cdot Vec_X$. It thus suffices to prove that $v(m) = 0$ for v of the form $f \cdot v'$, where $v' \in Vec_X$ and $f \in J$. But in that case, as M is a right $\mathcal{D}$-module, $v(m) = v'(fm) = v'(0) = 0$, as desired.

Recall the following definition.

Definition 1.7.1. Suppose we have two functors $F : \mathcal{C}_1 \to \mathcal{C}_2$ and $G : \mathcal{C}_2 \to \mathcal{C}_1$. Then F is called left adjoint to G (or G right adjoint to F) if there exists an isomorphism $\alpha_{A,B} : \mathrm{Hom}(F(A), B) \to \mathrm{Hom}(A, G(B))$ functorial in $A \in \mathcal{C}_1$ and $B \in \mathcal{C}_2$.

For any pair of adjoint functors we have canonical maps

$$FG \to \mathrm{Id}_{\mathcal{C}_1}, \quad \mathrm{Id}_{\mathcal{C}_2} \to GF,$$

called adjunction morphisms.

Remark 1.7.2. For a given F, if a functor G right adjoint to F exists, it is unique up to canonical isomorphism.

Theorem 1.7.3. *Let $i : X \hookrightarrow Y$ be a closed embedding.*

(1) *$i^!$ is right adjoint to i_0.*

(2) *The functors $\mathcal{M}(\mathcal{D}_X) \underset{i^!}{\overset{i_0}{\rightleftarrows}} \mathcal{M}_X(\mathcal{D}_Y)$ are mutually inverse.*

It is clear that Theorem 1.7.3 implies Theorem 1.6.24.

Proof. We want to prove that $i^!$ is right adjoint to i^0, i.e., that for any $\mathcal{D}_X$-module M and $\mathcal{D}_Y$-module N we have $\mathrm{Hom}(i_0M, N) \cong \mathrm{Hom}(M, i^!N)$.

There exists a map of k-vector spaces $M \hookrightarrow i_0M = M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$, given by $m \mapsto m \otimes 1$. Given $f \in \mathrm{Hom}(i_0M, N)$, restrict it to M. It is easy to check that the image of this map will lie in $i^!N$ and that this map is $\mathcal{D}_X$-linear. Now that we have a map from $\mathrm{Hom}(i_0M, N) \to \mathrm{Hom}(M, i^!N)$, we need to construct a map in the other direction. Given $g \in \mathrm{Hom}(M, i^!N)$, we want to construct $\tilde{g} : M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y} \to N$. We know that $\mathcal{D}_{X\to Y} = \mathcal{D}_Y / \mathcal{J}\mathcal{D}_Y$. So the map $m \otimes d \mapsto g(m)d$ is well-defined, since $g(m)$ is killed by $\mathcal{J}$, and we can define $\tilde{g}$ to be this map. Checking that our two operations are inverses is straightforward. The first part of Theorem 1.7.3 is proved.

Since i_0 and $i^!$ are adjoint, we have canonical adjunction morphisms $i_0 i^! \to \mathrm{Id}$ and $\mathrm{Id} \to i^! i_0$. In order to prove the second part of the theorem, we have to prove that these morphisms become isomorphisms when restricted to $\mathcal{M}_X(\mathcal{D}_Y)$. It is enough to show this locally.

By induction on the codimension of X in Y, we can assume that X is a smooth hypersurface in Y, given by an equation $f = 0$. Locally on Y, we can choose a coordinate system given by functions $y_1, \ldots, y_m$ and vector fields $\partial_1, \ldots, \partial_m$, such that $y_m = f$. We will use ∂ to denote ∂_m.

We claim that

$$\mathcal{D}_{X\to Y} = \bigoplus_n \mathcal{D}_X \partial^n,$$

Indeed,

$$\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes \mathcal{D}_Y = \bigoplus \mathcal{O}_X \partial_1^{\alpha_1} \ldots \partial_m^{\alpha_m}$$

and $\bigoplus_{\alpha_1,\dots,\alpha_{m-1}} \mathcal{O}_X \partial_1^{\alpha_1} \dots \partial_{m-1}^{\alpha_{m-1}}$ is $\mathcal{D}_X$.

We can now prove that $\mathrm{Id} \to i^! i_0$ is isomorphism. Let M be $\mathcal{D}_X$-module. Then

$$i_0 M = M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y} = M \otimes_{\mathcal{D}_X} \left(\bigoplus_j \mathcal{D}_X \partial^j \right) = \bigoplus_j M \partial^j .$$

By definition, $i^! i_0 M$ is the kernel of the multiplication-by-f map on $i_0 M$. We will abuse notation and call this map f as well. f sends $M\partial^j \to M\partial^{j-1}$. Under the natural identification of $M\partial^k$ with M, this corresponds to the map $M \to M$ given by $x \mapsto jx$. It immediately follows that $\mathrm{Ker} f = M$, and thus that $i^! i_0 M = M$.

Now we have to show that $i_0 i^! \to \mathrm{Id}$ is an isomorphism.

Let N be a $\mathcal{D}_Y$ module supported on X and let $S = i^! N$. We will prove that $N = i_0 S$, i.e., that $N = \bigoplus S\partial^j$.

Consider $\tilde{N} = \sum_j S\partial^j \subset N$ and $d = f\partial$. The $S\partial^j$ are eigenspaces of d with eigenvalue j. This can be proven by induction: if $nd = \lambda n$, then $n\partial d = n(f\partial + 1)\partial = (nd)\partial + n\partial = (\lambda+1)(n\partial)$. This shows that $\tilde{N}$ is in fact the direct sum of the $S\partial^j$.

Now consider $L = N/\tilde{N}$. To show that $L = 0$, it is enough to show that f has zero kernel on L (as we know that f acts locally nilpotently on L).

Let $l \in N$ be such that $lf \in \tilde{N}$. We want to show that $l \in \tilde{N}$. Since f is surjective, there exists $n \in \tilde{N}$ such that $nf = lf$. This implies that $(n-l)f = 0$ and thus that $n - l$ is an element of S. Since $n \in \tilde{N}$ and $n - l \in S \subset \tilde{N}$, $l \in \tilde{N}$. $\square$

As an immediate application of Kashiwara's theorem, we define the notion of a $\mathcal{D}$-module on a singular variety. We will see more applications in the future.

1.7.2. $\mathcal{D}$-modules on singular varieties. Let X be any affine variety. There exists a closed embedding $X \hookrightarrow Y$ with Y a smooth affine variety. Define $\mathcal{M}(\mathcal{D}_X)$ to be $\mathcal{M}_X(\mathcal{D}_Y)$. By Kashiwara's theorem, for smooth varieties this definition coincides with our previous definition. We claim that even for singular varieties, this definition does not depend on the embedding.

Suppose $i_1 : X \hookrightarrow Y_1$ and $i_2 : X \hookrightarrow Y_2$ are two such embeddings. Then there exists a smooth affine variety Y_3 and embeddings $Y_1 \to Y_3, Y_2 \to Y_3$ making the diagram

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[d, "i_2"'] & Y_1 \arrow[d, "j_1"'] \\ Y_2 \arrow[r, "j_2"] & Y_3 \end{tikzcd}$$

commutative. To construct such a Y_3, note that such a datum is equivalent to an embedding of $Y_1 \amalg_X Y_3$ into Y_3.

It clearly follows from Kashiwara's theorem that for $k = 1$ or 2 we have an equivalence $\mathcal{M}_X(\mathcal{D}_{Y_k}) \to \mathcal{M}_X(\mathcal{D}_{Y_3})$. This in turn defines an equivalence $\mathcal{M}_X(\mathcal{D}_{Y_1}) \simeq \mathcal{M}_X(\mathcal{D}_{Y_2})$. Let us check that that this equivalence does not depend on the choice of Y_3.

For any embedding $X \to Y$ with corresponding ideal $\mathcal{J}$, we have a functor $F : \mathcal{M}_X(\mathcal{D}_Y) \to \mathcal{M}(\mathcal{O}_X)$ defined by $M \mapsto \{m \in M \ \mid \mathcal{J} \cdot m = 0\}$. It is easy to see that this functor is faithful. If we have two embeddings $i_1 : X \hookrightarrow Y_1$, $i_2 : X \hookrightarrow Y_2$ then it is easy to see that the above equivalence commutes with the corresponding functors $F_i : \mathcal{M}_X(\mathcal{D}_{Y_i}) \to \mathcal{M}(\mathcal{O}_X)$. This implies that this equivalence does not depend on the choice of Y_3. Moreover, this shows that we have a well-defined faithful functor $\mathcal{M}(\mathcal{D}_X) \to \mathcal{M}(\mathcal{O}_X)$ for any affine variety. We can thus think of an object of $\mathcal{M}(\mathcal{D}_X)$, defined above, as an object of $\mathcal{M}(\mathcal{O}_X)$ plus some additional structure. If $M \in \mathcal{M}(\mathcal{O}_X)$ is a given a "$\mathcal{D}_X$-module" structure (i.e., an element of $\mathcal{M}(\mathcal{D}_X)$ with image in $\mathcal{M}(\mathcal{O}_X)$ M), then one can construct an action of $\mathcal{D}_X$ on M, but a "$\mathcal{D}_X$-module" structure cannot be recovered from a $\mathcal{D}_X$-action.

REMARK 1.7.4. Kashiwara proved his theorem before Bernstein's inequality was stated. In fact, the first proof of Berstein's inequality, which appeared in Bernstein's thesis, used Kashiwara's theorem.

1.7.3. The $\mathcal{D}$-module $\widetilde{\mathcal{D}}_X$ on a singular variety and its endomorphism ring. Let X be an affine algebraic variety over k (possibly singular). We want to define a canonical right $\mathcal{D}$-module $\widetilde{\mathcal{D}}_X$ on X, which in the case of smooth X coincides with $\mathcal{D}_X$.

Let $i : X \to V$ be an embedding of X into an affine space V. Let $I_X \subset \mathcal{O}(V)$ be the ideal of X. Consider the right $\mathcal{D}$-module $\widetilde{\mathcal{D}}_{X,i} := \mathcal{D}(V)/I_X\mathcal{D}(V)$ on V. By Kashiwara's equivalence, this $\mathcal{D}$-module corresponds to a $\mathcal{D}$-module $\widetilde{\mathcal{D}}_{X,i}$ on X.

EXERCISE 1.7.1. (I) Show that $\widetilde{\mathcal{D}}_{X,i}$ does not depend on the embedding i, in the sense that for any two embeddings $i_1 : X \to V_1$ and $i_2 : X \to V_2$ there is a canonical isomorphism $\phi_{i_1i_2} : \widetilde{\mathcal{D}}_{X,i_1} \to \widetilde{\mathcal{D}}_{X,i_2}$ such that $\phi_{i_1i_2} \circ \phi_{i_2i_3} \circ \phi_{i_3i_1} = \mathrm{Id}$ for any three embeddings i_1, i_2, i_3.

(ii) Show that if X is smooth then $\widetilde{\mathcal{D}}_{X,i}$ is canonically isomorphic to $\mathcal{D}_X$.

(iii) Let M be a $\mathcal{D}$-module on X, and i_*M be the corresponding $\mathcal{D}$-module on V. Set $\Gamma(M,i) := \{v \in i_*M : fv = 0 \text{ for all } f \in I_X\}$. Show that $\Gamma(M,i) = \mathrm{Hom}(\widetilde{\mathcal{D}}_{X,i}, M)$ and deduce that $\Gamma(M,i)$ is in fact independent on M.

In view of Exercise 1.7.1, we will denote $\mathcal{D}_{X,i}$ simply by $\mathcal{D}_X$, and $\Gamma(M,i)$ simply by $\Gamma(M)$. The functor $M \mapsto \Gamma(M)$ is called *the functor of global sections*. The object $\widetilde{\mathcal{D}}_X$ represents this functor. Note that in general this functor is not exact (it is only left exact); equivalently, the object $\widetilde{\mathcal{D}}_X$ is, in general, not projective.

We would now like to show that the algebra $\mathrm{End}(\widetilde{\mathcal{D}}_X)$ is isomorphic to the algebra $\mathcal{D}_X$ of Grothendieck differential operators on X.

As before, fix an embedding $i : X \to V$. By Kashiwara's equivalence, $\mathrm{End}(\widetilde{\mathcal{D}}_X) = \mathrm{End}(\mathcal{D}_V/I_X\mathcal{D}_V)$. Let us call this algebra A.

We have a natural isomorphism $A \cong (\mathcal{D}_V/I_X\mathcal{D}_V)^{I_X}$, given by $a \mapsto a(1)$, so A has a natural ascending filtration F by order (induced from that on $\mathcal{D}_V$). It is easy to check that $[F^nA, F^mA] \subset F^{n+m-1}A$, and $F^0A = \mathcal{O}(X)$.

THEOREM 1.7.1. There is a canonical filtered isomorphism $\phi : A \to \mathcal{D}_X$.

PROOF. We have a natural homomorphism

$$\phi : A \to \mathrm{End}_k(\mathcal{D}_V/I_X\mathcal{D}_V \otimes_{\mathcal{D}_V} \mathcal{O}_V) = \mathrm{End}_k(\mathcal{O}_X).$$

More explicitly, if $a \in A$ then consider $a(1) \in (\mathcal{D}_V/I_X\mathcal{D}_V)^{I_X}$, and pick a lift L_a of $a(1)$ to $\mathcal{D}_V$. Then L_a maps I_X to I_X, so it defines an operator on $\mathcal{O}_X = \mathcal{O}_V/I_X$ which does not depend on the choice of the lift L_a. This operator is $\phi(a)$.

It is clear that $\phi|_{F^0A}(f)$ is the operator of multiplication by $f \in \mathcal{O}_X$. This implies that ϕ lands in $\mathcal{D}_X$ and preserves filtrations.

Let us show that ϕ is injective. Suppose $\phi(a) = 0$. Then L_a maps $\mathcal{O}_V$ to I_X, so $L_a \in I_X\mathcal{D}_V$. Thus $a(1) = 0$ and hence $a = 0$.

Let us show that ϕ is surjective, and ϕ^{-1} preserves filtrations. Let $x_1, ..., x_n$ be linear coordinates on V, and for a multiindex $\mathbf{k}$ let $x^{\mathbf{k}} = \prod x_i^{k_i}$ be the corresponding monomial. Let $|\mathbf{k}| = \sum k_i$. Suppose $H \in \mathcal{D}_X$ is of order m. Let $f_{\mathbf{k}} := Hx^{\mathbf{k}} \in \mathcal{O}_X$ for all $\mathbf{k}$ with $|\mathbf{k}| \le m$. Let $\widetilde{f}_{\mathbf{k}}$ be lifts of $f_{\mathbf{k}}$

to $\mathcal{O}_V$, and let

$$\widetilde{H} = \sum_{|\mathbf{k}| \leq m} g_{\mathbf{k}} \partial^{\mathbf{k}} \in \mathcal{D}_V$$

be the unique operator such that $\widetilde{H} x^{\mathbf{k}} = \widetilde{f}_{\mathbf{k}}$.

We claim that $\widetilde{H}$ preserves the ideal I_X. Indeed, let $\pi : \mathcal{O}_V \to \mathcal{O}_V / I_X$ be the projection. Then by definition for all $\mathbf{k}$ with $|\mathbf{k}| \leq m$ we have $\pi(\widetilde{H} x^{\mathbf{k}}) = H x^{\mathbf{k}}$. Since H has order m (in the sense of Grothendieck), this implies that $\pi(\widetilde{H} x^{\mathbf{k}}) = H x^{\mathbf{k}}$ for all $\mathbf{k}$ (without any restriction on $|\mathbf{k}|$). Thus, for any $\psi \in k[x_1, ..., x_n]$, one has[2] $\pi(\widetilde{H}\psi) = H\psi$. Now take $\psi \in I_X$. Then we get that $\pi(\widetilde{H}\psi) = 0$ (as $\psi = 0$ in $\mathcal{O}_V/I_X$). Thus, $\widetilde{H}\psi \in I_X$, as claimed.

Let b_H be the image of $\widetilde{H}$ in $\mathcal{D}_V / I_X \mathcal{D}_V$. Then b_H is killed by right multiplication by I_X, so there exists $a_H \in A$ such that $b_H = a_H(1)$. Clearly, we have $\phi(a_H) = H$, and $\deg(a_H) = \deg(H)$. We are done. □

1.7.4. Exercises. In this collection of problems all varieties are assumed to be affine.

Exercise 1.7.2. Let X be an algebraic variety and let $\Delta : X \to X \times X$ be the diagonal embedding. Let J denote the ideal of $\Delta(X)$ in $\mathcal{O}_{X \times X}$ and let $X^{(n)}$ denote the closed subscheme of $X \times X$ corresponding to the ideal J^n.

FIX THIS EXERCISE

Exercise 1.7.3. Let $\eta : A \to B$ be a homomorphism of commutative algebras such that B is finite over A. Recall that in this case we have the functor $\eta^! : A - mod \to B - mod$ defined by

$$\eta^!(M) = \mathrm{Hom}_A(B, M).$$

Let X be a scheme over our base field k (the definitions below make sense (and are interesting) when k has arbitrary characteristic but we will consider only the case of characteristic 0). Recall that a *nilpotent extension* of X is a closed embedding $i : X \to Y$ where Y is another scheme and the ideal sheaf of X in Y is nilpotent. If Y and Z are two nilpotent extensions of X we say that $\eta : Y \to Z$ is a morphism of extensions if it is a morphism of schemes which restricts to the identity on X.

A *!-crystal* on X is a collection of the following data:

(1) An $\mathcal{O}_Y$-module M_Y for every nilpotent extension Y of X.
(2) An isomorphism $\alpha_\eta : M_Y \simeq \eta^! M_Z$ for every finite map of extensions $\eta : Y \to Z$.

This data is required to satisfy the following compatibility condition: for every chain $Y \xrightarrow{\eta} Z \xrightarrow{\rho} W$ of finite morphisms of nilpotent extensions of X we have $\alpha_{\eta\circ\rho} = \alpha_\eta \circ \alpha_\rho$. We denote by $Crys(X)$ the category of !-crystals on X.

Let X be a (not necessarily smooth) algebraic variety and take $M \in \mathcal{M}(\mathcal{D}_X)$ (defined via right modules). Let also $X \to Y$ be a nilpotent extension of X. We may imbed Y into some smooth variety Z. In this case M gives rise to a $\mathcal{D}_Z$-module M_Z on Z supported on X. Define M_Y to be the set of all elements of M_Z which are scheme-theoretically supported on Y.

a) Show that the collection $\{M_Y\}$ has a natural structure of a !-crystal.

b) Show that the resulting functor $\mathcal{M}(\mathcal{D}_X) \to Crys(X)$ is an equivalence of categories (hint: do this first for smooth X using problem 1).

1.8. Direct and inverse images preserve holonomicity

In the previous section we defined two "inverse image" functors $i^!$ and i^0. The first is left exact and the second is right exact. We start by elucidating the relationship between these two functors.

[2] Here and above, we abuse the notation, denoting the image of ψ in $\mathcal{O}_V/I_X$ by the same symbol ψ.

LEMMA 1.8.1. *For $k > \dim Y - \dim X$, $L^k i^0 = R^k i^! = 0$. Furthermore, we have*

$$i^! = L^{\dim Y - \dim X} i^0 \tag{$*$}$$

and

$$i^0 = R^{\dim Y - \dim X} i^! \tag{$**$}$$

(This is true even at the level of $\mathcal{O}$-modules.)

PROOF. As in the proof of Kashiwara's theorem, we can assume that X has codimension 1 in Y and is given by the equation $f = 0$. By definition, $i^0 M = \mathcal{O}_X \otimes_{\mathcal{O}_Y} M$. Note that the exact sequence

$$0 \to \mathcal{O}_Y \xrightarrow{f} \mathcal{O}_Y \to \mathcal{O}_X \to 0$$

gives a free resolution of $\mathcal{O}_X$. Tensoring with M, we get an exact sequence

$$M \xrightarrow{f} M \to \mathcal{O}_X \otimes_{\mathcal{O}_Y} M \to 0.$$

By definition $L^{-1} i^0 M = \mathrm{Ker} f = i^! M$ and $i^0 M = \mathrm{Coker} f = R^1 i^! M$. $\square$

LEMMA 1.8.2. *Let $i : X \hookrightarrow Y$ be an embedding of smooth varieties. Then $i^!$ and i_0 define inverse equivalences between the category of holonomic D_Y-modules and the category of holonomic D_X-modules.*

PROOF. As before, we can assume that X has codimension 1 in Y and is given by equation $f = 0$.

In the proof of Kashiwara's theorem, we showed that $i_0 M = \bigoplus_j M\partial^j$, where ∂ is a vector field on Y such that $\partial f = 1$. It's easy to see that in this case $d(i_0 M) = d(M) + 1$.

By Kashiwara's theorem $i_0 i^! N = N$ if $N \in \mathcal{M}_X(\mathcal{D}_Y)$. So $d(N) = d(i_0 i^! N) = d(i^! N) + 1$ and $d(i^! N) = d(N) - 1$. If N is holonomic, so is $i^! N$. $\square$

DEFINITION 1.8.3. Suppose $X \subset Y$ is singular. Then $M \in \mathcal{M}(\mathcal{D}_X) = \mathcal{M}_X(\mathcal{D}_Y)$ is said to be holonomic if it is holonomic as a $\mathcal{D}_Y$-module.

Assume Y and Y' are two different smooth varieties equipped with embeddings of X. Then we can embed both Y and Y' into some variety Z such that these two embeddings agree on X. So by Lemma 1.8.2, we see that a $\mathcal{D}_Y$-module supported on X is holonomic iff the corresponding D_Z-module is holonomic iff the corresponding $D_{Y'}$-module is holonomic. Thus this definition does not depend on the choice of Y.

THEOREM 1.8.4. *Let $\pi : \mathbb{A}^n \to \mathbb{A}^m$ be an affine map (i.e. a composition of a linear map and a translation). Then π_0, π^0 and their derived functors map holonomic modules to holonomic modules. Furthermore, for any holonomic $\mathcal{D}_{\mathbb{A}^n}$-module N and any holonomic $\mathcal{D}_{\mathbb{A}^m}$-module M,*

$$\sum_i c(L^i \pi_0 N) \le c(N)$$

and

$$\sum_i c(L^i \pi^0 M) \le c(M). \tag{1}$$

Assume we have a map of affine varieties $\pi : X \to Y$. If we have embeddings $X \to \mathbb{A}^p$ and $Y \to \mathbb{A}^q$, then we have a natural embedding of X into $\mathbb{A}^{p+q}$. Restricting the projection $\mathbb{A}^{p+q} \to A^q$ to X gives the map π, so combining Theorem 1.8.4 with Lemma 1.8.2, we see that π_0, π^0 and their derived functors preserve holonomicity. We shall see later that this statement is true even when π is an arbitrary map of algebraic varieties.

EXAMPLE 1.8.5. Let $\pi :$ be the map $X \to pt$. We claim that $L^i\pi_0 M = H^{n+i}_{dR}(M)$.

In our case, $\mathcal{D}_{X\to Y} = \mathcal{O}_X$. This means that $\pi_0 M = M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y} = M \otimes_{\mathcal{D}_X} \mathcal{O}_X$ for right modules and $\pi_0 M = \Omega^n(X) \otimes_{\mathcal{D}_X} M$ for left modules.

We earlier proved that $dR(\mathcal{D}_X)$ is a projective resolution of $\Omega^n(X)$ as a right $\mathcal{D}_X$-module. To compute $L^\bullet \pi_0 M$, we have to compute $H^\bullet(dR(\mathcal{D}_X)) = dR(\mathcal{D}_X) \otimes_{\mathcal{D}_X} M = dR(M)$. Thus $L^i\pi_0 M = H^{n+i}_{dR}(M)$. Theorem 1.8.4 now immediately gives the following corollary.

COROLLARY 1.8.6. *If M is holonomic,* $\dim H^i_{dR}(M) < \infty$.

Proof of Theorem 1.8.4. Any affine map is a composition of a standard projection and an affine embedding. It thus suffices to prove the theorem for embeddings and for standard projections.

If π is a projection, then π^0 is exact. By induction, we can assume that $m = n - 1$ and that π is the projection map $\mathbb{A}^n \cong \mathbb{A}^{n-1} \times \mathbb{A}^1 \to \mathbb{A}^{n-1}$. As we have $\mathcal{O}_{\mathbb{A}^n} = k[x_1, \dots, x_n]$ and $\mathcal{O}_{\mathbb{A}^{n-1}} = k[x_1, \dots, x_{n-1}]$, we can compute that

$$\pi^0 M = \mathcal{O}_{\mathbb{A}^n} \otimes_{\mathcal{O}_{\mathbb{A}^{n-1}}} M = \bigoplus_i M x_n^i.$$

Let $F_i M$ be a good filtration on M. It induces a good filtration on $\pi^0 M$, namely the one given by

$$F_k \pi^0 M = \sum_{i+j=k} F_j M \cdot x_n^i.$$

As M is holonomic, we have that $\dim F_j M = \frac{c(M)j^{n-1}}{(n-1)!} + \dots$. This implies that

$$\dim F_k \pi^0 M = \frac{c(M)k^n}{n!} + \dots,$$

and hence that, by Corollary 1.2.14, $\pi^0 M$ is holonomic and $c(\pi^0 M) \le c(M)$. As π^0 is exact, we have proven the theorem in this case.

Now suppose π is an embedding. By induction, we can assume that $m = n + 1$, and that $\mathbb{A}^n = \{x = 0\}$, where we define x to be x_{n+1}. Let M be a holonomic module on $\mathbb{A}^{n+1}$.

Choose a good filtration $F_i M$ on M (with respect to Bernstein's filtration on $\mathcal{D}_{\mathbb{A}^{n+1}}$). Let $N \subset M$ be the subset of elements on M annihilated by some power of x. Then $N = \pi_0 \pi^! M$ (in particular, N is a submodule.) In the proof of Kashiwara's theorem, we showed that x acts surjectively on N. This implies that $\mathrm{Coker}(x)$ on M is the same as $\mathrm{Coker}(x)$ on M/N. On M/N, multiplying by x has trivial kernel and M/N is holonomic as a quotient of holonomic module. Thus to prove that $\pi^0 M = \mathrm{Coker}(x)$ is holonomic, it is enough to assume that $\mathrm{Ker}(x)$ on M is 0.

The module $\pi^0 M = M/xM$ inherits filtration from M which might be not good. Since M is holonomic we know that

$$\dim F_i M = \frac{c(M)i^{n+1}}{(n+1)!} + \dots$$

By definition the map $F_i M/(x \cdot F_{i-1} M) \twoheadrightarrow F_i(M/xM)$ is surjective. Since x has no kernel it follows that $\dim x \cdot F_{i-1} M = \dim F_{i-1} M$ and hence

$$\dim F_i(M/xM) = \frac{c(M)i^n}{n!} + \dots$$

In lecture 2 we' discussed that such an equality for an arbitrary filtration implies that M/xM is holonomic and $c(M/xM) \le c(M)$. Hence $\pi^0 M$ is holonomic.

Now let's prove inequality (1). Let M_1 be maximal submodule of M supported on $\mathbb{A}^{n-1}$. From the following exact sequence

$$0 \to M_1 \to M \to M/M_1 \to 0$$

we know, that $c(M) = c(M_1)+c(M/M_1)$. By construction $\pi^0 M_1 = 0 = \pi^!(M/M_1)$. Thus $c(\pi^! M_1) = c(M_1)$ and $c(\pi^0 M/M_1) \le c(M/M_1)$. Since $\pi^! = L^{-1}\pi^0$, we have $c(L^{-1}\pi^0 M) + c(\pi^0 M) \le c(M)$. Our theorem is proved for π^0.

In order to prove this statement for π_0 we shall introduce the *Fourier transform* $F : \mathcal{M}(\mathcal{D}_V) \to \mathcal{M}(\mathcal{D}_{V^*})$, where V is vector space and V^* – its dual.

Since $\mathcal{D}_V$ is generated by $V^* \subset \mathcal{O}(V)$ and $V \subset Vec_V$, $\mathcal{D}_{V^*}$ is generated by $V \subset \mathcal{O}(V^*)$ and $V^* \subset Vec_{V^*}$, $\mathcal{D}_V$ and $\mathcal{D}_{V^*}$ are isomorphic via $V \leftrightarrow V$ and $V^* \leftrightarrow V^*$. Obviously, this isomorfhism preserves Bernstein's filtration. So F maps holonomic modules to holonomic.

Example 1.8.7. In case $V = \mathbb{A}^1$ isomorphism between $\mathcal{D}_V$ and $\mathcal{D}_V^*$ is given by $x \mapsto \frac{d}{dx}$ and $\frac{d}{dx} \mapsto -x$.

Let $\pi : V \to W$ be a linear map and $\widetilde{\pi} : W^* \to V^*$ be its dual.

$$F_W(\pi_0 M) = \widetilde{\pi}^0(F_V M)$$

Thus, if the theorem is true for π^0, it's true for π_0 also.

□

Corollary 1.8.8. *If M is holonomic on $\mathbb{A}^n$ then*

$$\sum \dim H^i_{dR}(M) \le c(M).$$

1.8.1. Application: Lie algebra coinvariants and Poisson homology. Let X be an affine variety over a field k of characteristic zero, and $\mathfrak{g}$ be a Lie algebra. Suppose that $\mathfrak{g}$ acts on X, i.e., we have a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathrm{Vect}(X)$. We will say that this action is *transitive* if X is irreducible and the linear map $\rho_x : \mathfrak{g} \to T_x X$ is surjective for all $x \in X$. It is easy to see that if the action is transitive then X is smooth, since $\dim \mathrm{Im}\rho_x$ is upper semicontinuous, while $\dim T_x X$ is lower semicontinuous.

Exercise 1.8.1. (a) Show that $\mathrm{Vect}(X)$ acts transitively on X if and only if X is irreducible and smooth.

(b) Show that every vector field v on X is tangent to the singular locus of X.

Hint. Consider the formal flow defined by v.

(c) Give a counterexample to (b) in characteristic p.

Hint. Consider the curve $y^2 = x^p$.

Let X_i be the set of $x \in X$ such that $\dim \mathrm{Im}\rho_x = i$. By semicontinuity of $\dim \mathrm{Im}\rho_x$, X_i is a locally closed subvariety of X, and X is stratified as a disjoint union of the X_i for $i \ge 0$. Clearly, every irreducible component of X_i has dimension $\ge i$.

Definition 1.8.9. We say that X has finitely many $\mathfrak{g}$-orbits if $\dim X_i = i$ (whenever $X_i \neq \emptyset$).

Exercise 1.8.2. Show that if X has finitely many $\mathfrak{g}$-orbits, then all X_i are smooth, and each connected component of X_i carries a transitive action of $\mathfrak{g}$.

Hint. Show that if x is a singular point of X_i then $\dim \mathrm{Im}\rho_x < i$, a contradiction.

In this situation, the connected components of X_i are called $\mathfrak{g}$-*orbits* of the action. Clearly, there are finitely many orbits, which justifies the terminology.

Example 1.8.10. Let X be a Poisson variety, and $\mathfrak{g} = \mathrm{Ham}(X)$, the Lie algebra of Hamiltonian vector fields on X, i.e.,vector fields of the form $\{f, ?\}$, where $f \in \mathcal{O}(X)$. In this case, $\mathfrak{g}$-orbits are called *symplectic leaves*.

EXERCISE 1.8.3. Let Y be an affine symplectic variety, and G a finite group acting faithfully on Y preserving the symplectic form. Show that the Poisson variety Y/G has finitely many symplectic leaves, and describe these leaves. In particular, the symmetric power S^nY has finitely many symplectic leaves. Generalize these results to the case when Y is itself a Poisson variety with finitely many symplectic leaves.

Our goal in this subsection is to use the theory of $\mathcal{D}$-modules to prove the following theorem, which appeared in [**ES**].

THEOREM 1.8.1. If X is an affine variety carrying an action of a Lie algebra $\mathfrak{g}$ with finitely many $\mathfrak{g}$-orbits, then the space of coinvariants $O(X)/\mathfrak{g}O(X)$ is finite dimensional. In particular, if X is a Poisson variety with finitely many symplectic leaves then the *zeroth Poisson homology* $HP_0(X) := \mathcal{O}(X)/\{\mathcal{O}(X), \mathcal{O}(X)\}$ is finite dimensional.

EXAMPLE 1.8.2. By Exercise 1.8.3, if Y is an affine symplectic variety with an action of a finite group G and $X = Y/G$, then $\mathcal{O}(X)/\{\mathcal{O}(X), \mathcal{O}(X)\}$ is finite dimensional. In the case when Y is a vector space with a bilinear symplectic form and G acts linearly, this was conjectured in [**AF**] and proved in [**BEG**].

PROOF. Embed X into an affine space V. Recall that the category of $\mathcal{D}$-modules on X can be realized as the category of $\mathcal{D}$-modules on V which are set-theoretically supported on X. We will abuse notation and denote $\mathcal{D}$-modules on X and the corresponding $\mathcal{D}$-modules on V supported on X in the same way.

Let $I_X \subset O(V)$ be the ideal of X. Consider the right $\mathcal{D}$-module $\widetilde{\mathcal{D}}_X := \mathcal{D}(V)/I_X\mathcal{D}(V)$ on V. We don't mention V in the notation since this $\mathcal{D}$-module, viewed as a $\mathcal{D}$-module on X, is actually independent of V (see Subsection 1.7.3); it represents the functor of global sections on the category of right $\mathcal{D}$-modules on X.

Let $\mathrm{Vect}_X(V)$ be the Lie algebra of vector fields on V tangent to X (i.e., preserving I_X). We have a natural homomorphism $\phi : \mathrm{Vect}_X(V) \to \mathrm{Vect}(X)$. We claim that this homomorphism is surjective: Indeed, let x_i be linear coordinates on V, and for any $v \in \mathrm{Vect}(X)$, let $v(x_i) = f_i \in O(X)$. Let $\widetilde{f}_i$ be any lifts of f_i to $\mathcal{O}(V)$, and define $\widetilde{v} \in \mathrm{Vect}(V)$ by $\widetilde{v}(x_i) := \widetilde{f}_i$. Then $\widetilde{v} \in \mathrm{Vect}_X(V)$, and $\phi(\widetilde{v}) = v$.

Let $\widetilde{\rho} : \mathfrak{g} \to \mathrm{Vect}_X(V)$ be any linear map which lifts the action $\rho : \mathfrak{g} \to \mathrm{Vect}(X)$; it exists since ϕ is surjective. Define

$$M_\rho := \mathcal{D}_X/\rho(\mathfrak{g})\mathcal{D}_X = \mathcal{D}(V)/(I_X\mathcal{D}(V) + \widetilde{\rho}(\mathfrak{g})\mathcal{D}(V)).$$

Again, absence of V from the notation is justified by the fact that M_ρ is independent on V (and the lift $\widetilde{\rho}$) as a $\mathcal{D}$-module on X.

LEMMA 1.8.11. *The singular support $SS(M_\rho)$ is contained in the union of the conormal bundles of the $\mathfrak{g}$-orbits.*

PROOF. Denote the $\mathfrak{g}$-orbits by Z_j.

Equip M_ρ by the filtration induced by the filtration on $\mathcal{D}(V)$. Then the associated graded module $\mathsf{gr}\, M_\rho$ is a quotient of the algebra $R := \mathcal{O}(T^*V)/(I_X\mathcal{O}(T^*V) + \widetilde{\rho}(\mathfrak{g})\mathcal{O}(T^*V))$. Closed points of $\mathrm{Spec}(R)$ are points $(x, p) \in V \times V^*$ such that $x \in X$ and $(p, \rho_x(a)) = 0$ for all $a \in \mathfrak{g}$. Hence $(p, T_xZ_j) = 0$ if $x \in Z_j$. Thus, if $x \in Z_j$ then (p, x) belongs to the conormal bundle of Z_j. $\square$

Lemma 1.8.11 implies that M_ρ is holonomic (since conormal bundles are Lagrangian).

Now,

$$\mathcal{O}(X)/\mathfrak{g}\mathcal{O}(X) = \mathcal{O}(V)/(I_X\mathcal{O}(V) + \widetilde{\rho}(\mathfrak{g})\mathcal{O}(V)) = M_\rho \otimes_{\mathcal{D}(V)} \mathcal{O}(V) = \pi_{*0}(M_\rho)$$

(the underived direct image of M_ρ), where $\pi : X \to \mathrm{pt}$ is the projection of X to a point. Since direct image preserves holonomicity, and since M_ρ is holonomic, we obtain that $\pi_{*0}(M_\rho)$ is a holonomic $\mathcal{D}$-module on a point, i.e., a finite dimensional vector space, as desired. $\square$

CHAPTER 2

D-modules on general algebraic varieties

2.1. $\mathcal{D}$-modules on arbitrary varieties

Let X be any algebraic variety (for simplicity we will only consider quasi-projective varieties; this will not affect the statements of our results). Then there exists a unique sheaf $\mathcal{D}_X$ of differential operators on X such that for any affine subset $U \subset X$, we have $\Gamma(U, \mathcal{D}_X) = \mathcal{D}_U$. This sheaf is naturally equipped with both a right and a left $\mathcal{O}_X$-module structure. The following lemma immediately follows from Lemma 1.6.2.

Lemma 2.1.1. *$\mathcal{D}_X$ is quasi-coherent with respect to either $\mathcal{O}_X$-module structure.*

A left (resp. right) $\mathcal{D}_X$-module is defined to be a quasi-coherent sheaf on X equipped with a left (resp. right) $\mathcal{D}_X$-module structure such that the two natural $\mathcal{O}_X$ actions coincide. As in the case of affine varieties, the category $\mathcal{M}^l(\mathcal{D}_X)$ of left $\mathcal{D}_X$-modules and the category $\mathcal{M}^r(\mathcal{D}_X)$ of right $\mathcal{D}_X$-modules are naturally equivalent via the functor $M \mapsto M \otimes \Omega^n(X)$, where $n = \dim X$.

Some of the statements that we proved only for affine varieties remain true (and are proved in the same way). For example, as Kashiwara's theorem is a local statement, it remains true for general varieties.

Let X be smooth. Then $\mathcal{D}_X$ is a filtered sheaf of algebras and

$$\mathsf{gr}\, \mathcal{D}_X = p_* \mathcal{O}(T^* X),$$

where $p : T^*X \to X$ is the standard projection. Using this, we can define the singular support $s.s.(M) \subset T^*X$ of a finitely generated $\mathcal{D}_X$-module M as in the affine case. The corresponding cycle will be denoted by $s.c.(M)$.

Definition 2.1.2. A finitely generated $\mathcal{D}$-module M is called holonomic iff $d(s.s.(M)) = n$.

It is still true that holonomic modules have finite length.

2.1.1. Inverse and direct images for arbitrary varieties. Consider a map $\pi : X \to Y$ and $N \in \mathcal{M}^l(\mathcal{D}_Y)$. Then we can define the inverse image $\pi^0 N$ as $\mathcal{O}_X \otimes_{\pi^* \mathcal{O}_Y} \pi^*(N)$. Note that we did not need to do anything to generalize from the case of affine varieties.

Now let us try to generalize our definition of the direct image functor. Start by defining $\mathcal{D}_{X \to Y} = \pi^0 \mathcal{D}_Y$. Let $M \in \mathcal{M}^r(\mathcal{D}_X)$ be a right $\mathcal{D}_X$-module. If we set $\pi_0 M = \pi_*(M \otimes_{\mathcal{D}_X} \mathcal{D}_{X \to Y})$, then π_0 is neither left nor right exact, as the functor $\otimes_{\mathcal{D}_X} \mathcal{D}_{X \to Y}$ is right exact, while the functor π_* is left exact. Moreover, this definition of π_0 is not compatible with composition.

In fact, the direct image can be defined correctly only at the level of derived categories. We will discuss this definition later.

Remark 2.1.3. If π is a closed embedding, this definition is still good. In fact, this definition will work as long as π is affine (i.e., the inverse image of an affine open is affine.)

2.1.2. $\mathcal{D}$-affine varieties.

DEFINITION 2.1.4. An algebraic variety X is called $\mathcal{D}$-affine if the functor of global sections $M \mapsto \Gamma(X, M)$ is an equivalence between $\mathcal{M}^l(\mathcal{D}_X)$ and the category of modules over $\mathcal{D}_X^{glob} = \Gamma(X, \mathcal{D}_X)$).

REMARK 2.1.5. If we replace $\mathcal{D}$ by $\mathcal{O}$ in this definition, by a theorem of Serre we will get the class of affine varieties.

THEOREM 2.1.6. *$\mathbb{P}^n$ is $\mathcal{D}$-affine.*

PROOF. We claim that a variety X is $\mathcal{D}$-affine if and only if the functor of global sections Γ is exact on $\mathcal{D}_X$-modules and there are no nonzero $\mathcal{D}_X$-modules M with $\Gamma(X, M) \neq 0$.

It is easy to see that these two properties are implied by $\mathcal{D}$-affinity. The other implication follows from a more general fact, for which we need the following definition to formulate.

DEFINITION 2.1.7. Let $\mathcal{A}$ be an abelian category with infinite direct limits. An object $P \in \mathcal{A}$ is called a projective generator if P is projective and $\operatorname{Hom}(P, X) \neq 0$ for any $X \neq 0$. The following result is well-known.

LEMMA 2.1.8. *Let $\Lambda = \operatorname{End} P$ and $F : \mathcal{A} \to \{$right Λ-modules$\}$ be the functor defined by $F(X) = \operatorname{Hom}(P, X)$. Then F is an equivalence of categories.*

If Γ is exact on $\mathcal{D}_X$-modules and $\Gamma(X, M) \neq 0$ for any $M \neq 0$, then $\mathcal{D}_X$ is a projective generator of $\mathcal{M}(\mathcal{D}_X)$ (since $\Gamma(X, M) = \operatorname{Hom}(\mathcal{D}_X, M)$). In this case $\Lambda = \operatorname{End} \mathcal{D}_X = (\mathcal{D}_X^{glob})^{op}$. So X is $\mathcal{D}$-affine by the lemma above.

Now let us prove Theorem 2.1.6. In our case $X = \mathbb{P}^n = \mathbb{P}(V)$. Let $\widetilde{V} = V \backslash \{0\}$, with $j : \widetilde{V} \to V$ denoting the natural inclusion and $\pi : \widetilde{V} \to \mathbb{P}(V)$ the natural projection.

Let M be a $\mathcal{D}_{\mathbb{P}^n}$-module. By the projection formula, we have $\pi_*\pi^0 M = M \otimes \pi_*\mathcal{O} = \bigoplus_{k \geq 0}(M \otimes \mathcal{O}(k))$. (Note here that we are using the sheaf-theoretic pushforward π_*, instead of the direct image functor on $\mathcal{D}$-modules.) Thus

$$\Gamma(\widetilde{V}, \pi^0 M) = \bigoplus_{k \in \mathbb{Z}} \Gamma(\mathbb{P}^n, M \otimes \mathcal{O}(k)).$$

Denote by $\mathcal{E}$ the Euler vector field $\sum x_i \frac{\partial}{\partial x_i}$. Under the natural action of $\mathcal{E}$ on $\Gamma(\widetilde{V}, \pi^0 M)$, the subspace corresponding to $\Gamma(\mathbb{P}^n, M \otimes \mathcal{O}(k))$ can be identified with the eigenspace of eigenvalue k. Thus we have $\Gamma(\mathbb{P}^n, M) = \Gamma(\widetilde{V}, \pi^0 M)^{\mathcal{E}}$. As $\mathcal{E}$ acts semi-simply on $\Gamma(\widetilde{V}, \pi^0 M)$, taking $\mathcal{E}$ invariants is an exact functor, and it suffices to prove that the functor $M \mapsto \Gamma(\widetilde{V}, \pi^0 M)$ is exact.

Consider the following sequence of functors:

$$M \mapsto \pi^0 M \mapsto j_0 \pi^0 M \mapsto \Gamma(\widetilde{V}, \pi^0 M).$$

Here the first functor is exact (as π is flat) and the third functor is exact (as V is affine), but the second functor is not necessarily exact. For any open embedding $j : U \to X$ the functor of direct image j_0 is left exact and all higher derived functors are supported on $X \backslash U$. Consider the embedding $j : \widetilde{V} \hookrightarrow V$. For any short exact sequence of $\mathcal{D}_{\mathbb{P}^n}$-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

we have the long exact sequence

$$0 \to j_0\pi^0 M_1 \to j_0\pi^0 M_2 \to j_0\pi^0 M_3 \to R^1 j_0 \pi^0 M_1 \to \dots.$$

By Kashiwara's theorem, $R^1 j_0 \pi^0 M_1$ is a direct sum of δ_0s, where δ_0 is the $\mathcal{D}$-module of δ-functions at 0. Thus the eigenvalues of $\mathcal{E}$ on $\Gamma(V, \delta_0)$ are $-1, -2, -3, \dots$. As the eigenvalues of $\mathcal{E}$ on $\Gamma(V, j_0\pi^0 M_3)$ are nonnegative, $\alpha : \Gamma(V, j_0\pi^0 M_3) \to \Gamma(V, R^1 j_0 \pi^0 M_1)$ must be zero. Hence the functor $M \mapsto \Gamma(\widetilde{V}, \pi^0 M)$ is exact and hence the functor $M \mapsto \Gamma(\mathbb{P}(V), M) = \Gamma(\widetilde{V}, \pi^0 M)^{\mathcal{E}}$ is also exact.

Let us prove that $\Gamma(\mathbb{P}^n, M) \neq 0$ if $M \neq 0$. There exists an integer $k \geq 0$ such that $\Gamma(\mathbb{P}^n, M \otimes \mathcal{O}(k)) \neq 0$. We take a $0 \neq m \in \Gamma(\widetilde{V}, \pi^0 M)$ such that $\mathcal{E}m = km$. Now observe that $\mathcal{E}(\frac{\partial}{\partial x_i} m) = (k-1)\frac{\partial}{\partial x_i} m$, so if there exists i such that $\frac{\partial}{\partial x_i} m \neq 0$, then there exists $l \in \Gamma(\widetilde{V}, \pi^0 M)$ with eigenvalue $k-1$. But if $\frac{\partial}{\partial x_i} m = 0$ for all i, then $\mathcal{E}m = 0$, and $\Gamma(\mathbb{P}^n, M) \neq 0$. Now an induction on k shows that $\Gamma(\widetilde{V}, \pi^0 M)^{\mathcal{E}} \neq 0$. $\square$

2.1.3. Equivariant $\mathcal{D}$-modules. Consider a variety X with an action of an affine algebraic group G. Let us review the notion of a G-equivariant coherent sheaf. The group structure gives us a multiplication map $m : G \times G \to G$, and the action of G gives us a map $\rho : G \times X \to X$. We have a commutative diagram

$$\begin{tikzcd} & G \times G \times X \arrow[dl, "m \times \mathrm{id}"'] \arrow[dr, "\mathrm{id} \times \rho"] & \\ G \times X \arrow[dr, "\rho"] & & G \times X \arrow[dl, "\rho"'] \\ & X & \end{tikzcd}.$$

Definition 2.1.9. A G-equivariant quasi-coherent sheaf on X is a quasi-coherent sheaf $\mathcal{F}$ on X equipped with an isomorphism $\phi : \rho^*\mathcal{F} \to \mathcal{O}_G \boxtimes \mathcal{F}$ making the following diagram commutes:

$$\begin{tikzcd} (\mathrm{id} \times \rho)^* \rho^* \mathcal{F} \arrow[r, "(\mathrm{id} \times \rho)^* \phi"] \arrow[d, equal] & (\mathrm{id} \times \rho)^*(\mathcal{O}_G \boxtimes \mathcal{F}) \arrow[r] & \mathcal{O}_G \boxtimes \rho^* \mathcal{F} \arrow[d, "\mathcal{O}_G \boxtimes \phi"] \\ (m \times \mathrm{id})^* \rho^* \mathcal{F} \arrow[r, "(m \times \mathrm{id})^* \phi"] & (m \times \mathrm{id})^*(\mathcal{O}_G \boxtimes \mathcal{F}) \arrow[r] & \mathcal{O}_G \boxtimes \mathcal{O}_G \boxtimes \mathcal{F} \end{tikzcd}$$

We now wish to define a G-equivariant $\mathcal{D}_X$-module. We start with the following definition.

Definition 2.1.10. A weakly G-equivariant $\mathcal{D}$-module on X is a $\mathcal{D}_X$ module $\mathcal{F}$ with a G-equivariant coherent sheaf structure where ϕ is $\mathcal{D}_X$-linear.

We will have two different actions of $\mathfrak{g} = \mathrm{Lie}(G)$ on $\mathcal{F}$. First of all, the G action on X gives us maps $\mathfrak{g} \to \mathrm{Vect}(X) \to \Gamma(\mathcal{D}_X)$, and so the $\mathcal{D}$-module structure on $\mathcal{F}$ gives us a $\mathfrak{g}$ action on $\mathcal{F}$. Note that this action does not depend on the choice of equivariant structure ϕ.

On the other hand, we have a $\mathfrak{g}$ action on $\mathcal{O}_G \boxtimes \mathcal{F}$ coming from the G action on $G \times X$ given by $g \cdot (h, x) = (gh, x)$. Translating this along ϕ, we get a $\mathfrak{g}$ action on $\rho^*\mathcal{F}$. Restricting to $1 \times X$, this gives us another $\mathfrak{g}$ action on $\mathcal{F}$.

Definition 2.1.11. A G-equivariant $\mathcal{D}_X$-module is a weakly equivariant $\mathcal{D}_X$-module where these two $\mathfrak{g}$ actions agree (or, equivalently, where ϕ is $\mathcal{D}_{G \times X}$-linear.)

Remark 2.1.12. A given $\mathcal{D}_X$-module may have many weakly G-equivariant structures, but if G is connected, then it can only have one G-equivariant structure. This is because the $\mathfrak{g}$ action on $\mathcal{F}$ is determined by the embedding $\mathfrak{g} \to \Gamma(\mathcal{D}_X)$ and this action can be integrated to a G-equivariant structure in an unique way (recall that we always work in characteristic 0.)

Furthermore, any $\mathcal{D}_X$-linear map of $\mathcal{D}_X$-modules is automatically compatible with the G action. This is because such a map is necessarily $\mathfrak{g}$-linear, which implies that it is in fact G-linear. These two facts combined show that the category of G-equivariant $\mathcal{D}_X$ modules is a full subcategory of the category of $\mathcal{D}_X$ modules. Stated another way, G-equivariance of a $\mathcal{D}$-module is a property, not a structure.

EXAMPLE 2.1.13. Consider the case where X is a point. Then $\mathcal{D}_X \cong k$, and so a $\mathcal{D}_X$-module is a just a vector space. A weakly G-equivariant $\mathcal{D}_X$-module is then simply an algebraic representation of G. This representation gives a G-equivariant structure if and only if $\mathfrak{g}$ acts trivially, which (as we are in characteristic 0) implies that G_0 acts trivially, where G_0 is the connected component of the identity. Thus a G-equivariant $\mathcal{D}_X$ module is just a representation of G/G_0.

The notion of weakly equivariant $\mathcal{D}$-module often arises in the following setting. Let T be an algebraic torus and let $\tilde{X}$ be a T-torsor over X.

DEFINITION 2.1.14. A monodromic $\mathcal{D}_X$-module (with respect to the torsor $\tilde{X} \to X$) is a weakly T-equivariant $\mathcal{D}_{\tilde{X}}$-module.

2.1.4. Exercises.

EXERCISE 2.1.1. (a) Let X be a smooth irreducible variety over the complex numbers. Compute $\mathrm{Tor}_i^{D(X)}(\Omega(X), \mathcal{O}(X))$, where $\Omega(X)$ and $\mathcal{O}(X)$ are the right (resp. left) $D(X)$-modules of top forms and functions on X, respectively.

(b) Recall that the Hochschild homology of an algebra A is

$$HH_i(A, A) := \mathrm{Tor}_i^{A-\mathrm{bimod}}(A, A).$$

Compute $HH_i(D(X), D(X))$ for affine X (apply (a) and Kashiwara's theorem for the diagonal embedding).

EXERCISE 2.1.2. Let G be a finite group acting faithfully on a smooth irreducible affine complex algebraic variety X.

(a) Show that a G-equivariant D-module on X is the same thing as a module over the algebra $A := \mathbb{C}[G] \ltimes D(X)$.

(b) Prove that A is a simple algebra.

Hint: Assume that I is a nonzero ideal in A, and let $z = g_0b_0 + ... + g_mb_m$ be the shortest (i.e., with smallest m) nonzero element of I ($g_i \in G$, $b_i \in D(X)$). Show that one may assume that $g_0 = 1$ and b_0 is a nonzero function on X, and the possible functions b_0 together with 0 form an ideal in $\mathcal{O}(X)$ invariant under $\mathrm{Vect}(X)$. Deduce that one may assume that $b_0 = 1$. Then consider the commutator of z with a vector field to lower m if $m > 0$. Deduce that $m = 0$, and $I = A$.

(c) Let $e = \frac{1}{|G|}\sum_{g\in G} g$ be the symmetrizer. Prove that the functor $M \mapsto eM = M^G$ defines an equivalence from the category of G-equivariant D-modules on X to the category of $D(X)^G$-modules.

EXERCISE 2.1.3. Keep the notation of the previous problem. By Noether's theorem, the algebra $\mathcal{O}(X)^G$ is finitely generated, so it defines an algebraic variety X/G (which in general is singular). One can show that points of X/G correspond bijectively to G-orbits on X, which motivates the notation.

Let us say that $g \in G$ is a reflection if the fixed point set X^g has a component of codimension 1 in X.

(a) Show that if G does not contain reflections, then the natural homomorphism $\phi : D(X)^G \to D(X/G)$ is an isomorphism (where for a variety Y, $D(Y)$ denotes the algebra of Grothendieck differential operators on Y). Deduce that in this case $D(X/G)$ is Noetherian on both sides.

Hint: Use that if $U \subset X$ is an open set with $X \setminus U$ of codimension ≥ 2, then any section over U of any vector bundle on X uniquely extends to all of X).

(b) Is ϕ an isomorphism in general (i.e., if G may contain reflections)?

(c) Use (a) to explicitly describe $D(Y)$ when Y is the quadratic cone $xy = z^2$ in the 3-dimensional space.

(d) In (c), is the functor Γ of global sections an equivalence from the category of right D-modules on Y to the category of right $D(Y)$-modules?

Hint: Consider the modules concentrated at the vertex of the cone in both categories.

(e) Show that for any X, G, X/G is locally isomorphic to X'/G', where X' is smooth and G' does not contain reflections.

Hint. Use Chevalley's theorem that if G is a subgroup of $GL(V)$ generated by reflections, then V/G is an affine space (equivalently, is smooth).

(f) Show that for any X, G, the algebra $D(X/G)$ is Noetherian on both sides.

EXERCISE 2.1.4. For the rest of this problem set we will drop the condition that X is affine.

(a) Let $\omega \in \Omega^1(X)$ be a 1-form. Show that there is an automorphism $\mathcal{D}_X \to \mathcal{D}_X$ restricting to the identity on $\mathcal{O}_X$ and restricting to the map $v \to v + \langle \omega, v \rangle$ on Vect_X iff ω is closed.

(b) Let Ω^1_{cl} be the sheaf of closed 1-forms. For $h \in H^1(\Omega^1_{cl})$ define a sheaf of algebras D_h on X locally isomorphic to $\mathcal{D}_X$.

(c) Recall that the isomorphism classes of line bundles are in bijection with cohomology classes in $H^1(X, \mathcal{O}^*)$, where $\mathcal{O}^*$ is the sheaf of invertible functions. Consider the morphism of sheaves $\mathcal{O}^* \to \Omega^1_{cl}$ given by $f \mapsto \frac{df}{f}$. For a line bundle L let $c_1(L) \in H^1(\Omega^1_{cl})$ be the image of the corresponding class in $H^1(\mathcal{O}^*)$ under the induced map $H^1(\mathcal{O}^*) \to H^1(\Omega^1_{cl})$. Identify $\mathcal{D}_{c_1(L)}$ with the sheaf of differential operators acting on the sections of L.

EXERCISE 2.1.5. For which $i \in \mathbb{Z}$ is $\mathbb{P}^n$ "affine with respect to $\mathcal{D}_{\mathcal{O}(i)}$?" In other words, for which i is the functor of global sections an equivalence of categories between the category of quasicoherent $\mathcal{D}_{\mathcal{O}(i)}$-modules and the category of modules over the global sections of $\mathcal{D}_{\mathcal{O}(i)}$?

EXERCISE 2.1.6. Assume that X is a $\mathcal{D}$-affine variety and that G is an affine algebraic group acting on X. Let $\Gamma(\mathcal{D}_X)$ be the ring of global sections of $\mathcal{D}_X$. Show that the category of G-equivariant $\mathcal{D}_X$-modules is equivalent to the category of $\Gamma(\mathcal{D}_X)$-modules M endowed with a G-action whose differential coincides with the action of $\mathfrak{g}$ on M coming from the embedding $\mathfrak{g} \to \Gamma(\mathcal{D}_X)$.

EXERCISE 2.1.7. Let $X = \mathbb{P}^1$ and $G = SL(2)$. Show that $\Gamma(\mathcal{D}_X)$ is equal to the quotient of $U(\mathfrak{g})$ by the ideal generated by the Casimir element C.

EXERCISE 2.1.8. Let G be an algebraic group and K be a subgroup. A $(\mathfrak{g}, K)$-module is a vector space M endowed with compatible actions of $\mathfrak{g}$ and K. (Here, compatible means that the two natural actions of the lie algebra of K coincide.) Here, we will take $G = PGL(2)$ and K the subgroup of classes of diagonal matrices. Use the above exercises to show that there exist exactly 3 irreducible $(\mathfrak{g}, K)$-modules on which C acts by 0. Write out these 3 irreducible modules explicitly.

2.2. Derived categories

In this section, we will not always provide proofs for our claims. Our goal is just to sketch the theory of derived categories.

Consider a map $\pi : X \to Y$. There is no reasonable way to define the direct image functor π_* as a functor between the abelian categories $\mathcal{M}(\mathcal{D}_X)$ and $\mathcal{M}(\mathcal{D}_Y)$. In order to define such a functor, we need to work in the derived category.

2.2.1. Universal property of the derived category. Let $\mathcal{A}$ be an abelian category and $\mathcal{C}(\mathcal{A})$ be the category of all complexes with elements in $\mathcal{A}$. We define $\mathcal{C}^+(\mathcal{A})$ to be the full subcategory of complexes $K^\bullet$, such that $K^i = 0$ for $i \ll 0$. We similarly define $\mathcal{C}^-(\mathcal{A})$ and $\mathcal{C}^b(\mathcal{A})$ to be the category of complexes bounded from above and the category of bounded complexes.

Let $\mathcal{C}_0(\mathcal{A})$ be the category of (unbounded) complexes with zero differential. We have a functor $H : \mathcal{C}(\mathcal{A}) \to \mathcal{C}_0(\mathcal{A}) = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}$.

THEOREM 2.2.1. *There exist an unique (up to canonical equivalence) pair of a category $\mathcal{D}(\mathcal{A})$ (called the derived category of $\mathcal{A}$) and a functor $Q : \mathcal{C}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ such that*

1. *If $f : K^\bullet \to L^\bullet$ is a quasi isomorphism, then $Q(f)$ is an isomorphism.*
2. *This pair is universal with the property 1, i.e., for any functor $F : \mathcal{C}(\mathcal{A}) \to \mathcal{D}'$ satisfying 1, there exists an unique functor $G : \mathcal{D}(\mathcal{A}) \to \mathcal{D}'$ such that $F = G \circ Q$.*

Note that taking $\mathcal{D}'$ to be $\mathcal{A}$ and F to be the functor which sends a complex to its cohomology at i, we get a functor $H^i : \mathcal{D}(\mathcal{A}) \to \mathcal{A}$.

More generally, let $\mathcal{C}$ be any category and $\mathcal{S}$ any class of morphisms. Then there exist a category $\mathcal{C}[\mathcal{S}^{-1}]$ and a functor $\mathcal{C} \to \mathcal{C}[\mathcal{S}^{-1}]$ satisfying conditions *1* and *2* of the theorem above.

2.2.2. Structures on derived categories. The derived category of an abelian category will have some extra structure. First of all, there is a natural abelian group structure on the hom-sets of a derived category, which gives the derived category the structure of an additive category. Furthermore, we have shift functors and distinguished triangles, which give the derived category the structure of a *triangulated category*. We will not discuss the general theory of triangulated categories here, but we will consider the special case of the derived category.

The shift functors for the derived category are easy to describe: For any i there exists a functor $[i]$ given by $K^\bullet \mapsto K^\bullet[i]$, where $K^j[i] = K^{j+i}$ for any j. We have $[i] \circ [j] = [i+j]$. It will be harder to describe the distinguished triangles of the derived category.

Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. We want to formalize a notion of derived functor $RF : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$. Composing RF with H^i should give our old derived functors, and so we need a notion of exactness for RF that corresponds to our original long exact sequence. To do this, we replace the notion of a short exact sequence with the notion of a distinguished triangle.

We first define the cone of a morphism of complexes $K^\bullet \xrightarrow{F} L^\bullet$. Define a complex $C(f)^\bullet$, called the cone of f, by the following formulas: $C(f)^\bullet = K^\bullet[1] \oplus L^\bullet$, i.e., $C(f)^i = K^{i+1} \oplus L^i$ with differential $d(k^{i+1}, l^i) = (-d_K k^{i+1}, f(k^{i+1}) + d_L(l^i))$.

Exercise 2.2.1. Show that if f is an embedding, then $C(f)$ is quasi isomorphic to $L^\bullet/K^\bullet$.

Lemma 2.2.2. *The sequence*

$$\cdots \to H^i(K) \to H^i(L) \to H^i(C(f)) \to H^{i+1}(K) \to \dots$$

is exact.

Proof. If the sequence $K^\bullet \to L^\bullet \to C(f)$ were exact, this lemma would follow from the long exact sequence of cohomology associated to a short exact sequence of complexes. In our case it is not exact, but we can replace $L^\bullet$ by a quasi isomorphic complex such that the sequence will become exact.

Let's define a complex, called the cylinder of f, by $Cyl(f) = K^\bullet \oplus K^\bullet[1] \oplus L^\bullet$ with the following differential

$$d : (k^i, k^{i+1}, l^i) \mapsto (d_K k^i - k^{i+1}, -d_K k^{i+1}, f(k^{i+1}) - d_L l^i).$$

It can be checked that the natural inclusion $L^\bullet \to Cyl(f)$ is a quasi isomorphism and the sequence $K^\bullet \to Cyl(f) \to C(f)$ is exact. $\square$

Definition 2.2.3. A distinguished triangle in $\mathcal{D}(\mathcal{A})$ is a "triangle" $X \to Y \to Z \to X[1]$ which is the image under Q of $K^\bullet \to L^\bullet \to C(f) \to K^\bullet[1]$ for some complexes $K^\bullet, L^\bullet$.

This notion is not perfect; given a morphism $f : X \to Y$ in $\mathcal{D}(\mathcal{A})$, there is no canonical way to complete it to a distinguished triangle. There exists a distinguished triangle $X \to Y \to Z$, but it is only unique up to non-canonical isomorphism. This problem will not be relevant for us.

Definition 2.2.4. Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A functor $F : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ is called exact if it maps distinguished triangles to distinguished triangles.

Let X and Y be objects of $\mathcal{A}$. Then we can redefine the Ext functors by $\mathrm{Ext}^i(X,Y) = \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X^\bullet, Y^\bullet[i])$ (We will have $\mathrm{Hom}_{\mathcal{A}}(X,Y) = \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X^\bullet, Y^\bullet)$). If $\mathcal{A}$ has either enough projectives or injectives (see the end of the next section), this definition coincides with previous one.

2.2.3. An explicit construction of the derived category. Let $\mathcal{S}$ be any class of morphisms in a category $\mathcal{A}$.

DEFINITION 2.2.5. $\mathcal{S}$ is called a localizable class of morphisms if

1) For any two morphisms $s : B \to C, t : A \to B$ in $\mathcal{S}$, $s \circ t$ is also in $\mathcal{S}$.
2) For any two morphisms $s \in \mathcal{S}$ and f there exists an object W and morphisms $t \in \mathcal{S}$ and g such that the following diagrams are commutative

$$\begin{tikzcd} W \arrow[r, dashed, "g"] \arrow[d, dashed, "t\in\mathcal{S}"'] & Z \arrow[d, "s\in\mathcal{S}"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \qquad \begin{tikzcd} W & Z \arrow[l, dashed, "g"'] \\ X \arrow[u, dashed, "t\in\mathcal{S}"] & Y \arrow[l, "f"] \arrow[u, "s\in\mathcal{S}"'] \end{tikzcd}$$

3) Let $f, g : X \to Y$. Then there exist $s \in \mathcal{S}$ such that $sf = sg$ iff there exist $t \in \mathcal{S}$ such that $ft = gt$.

If $\mathcal{S}$ is a localizable class, then $\mathcal{C}[\mathcal{S}^{-1}]$ has a nice description. Morphisms in this category are given by diagrams

$$\begin{tikzcd} & X' \arrow[dl, "s\in\mathcal{S}"'] \arrow[dr, "f"] & \\ X & & Y. \end{tikzcd}$$

Two diagrams
$$\begin{tikzcd} & X' \arrow[dl, "s\in\mathcal{S}"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & X'' \arrow[dl, "t\in\mathcal{S}"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$
define the same morphism if there exist an object X''' and morphisms $\mathcal{S} \ni r : X''' \to X'$, $f : X''' \to X''$, such that the following diagram is commutative

$$\begin{tikzcd} & X''' \arrow[dl, "r"'] \arrow[dr, "h"] & \\ X' \arrow[d, "s"'] \arrow[drr] & & X'' \arrow[d, "h"] \arrow[dll] \\ X & & Y. \end{tikzcd}$$

LEMMA 2.2.6. *If $\mathcal{S}$ is a localizable class, then $\mathcal{C}[\mathcal{S}^{-1}]$ is the category with the same objects as $\mathcal{C}$ and with morphisms given by equivalence classes of the diagrams above.*

We want to define the derived category $\mathcal{D}(\mathcal{A})$ as localization of $\mathcal{C}(\mathcal{A})$ by quasi-isomorphisms. Unfortunately quasi-isomorphisms do not form a localizable class.

EXAMPLE 2.2.7. Let $\mathcal{A}$ be the category of abelian groups. Consider the complex

$K^\bullet = \begin{array}{ccccccc} \{0 & \to & \mathbb{Z} & \xrightarrow{*2} & \mathbb{Z} & \to & 0\} \\ & & {\scriptstyle -1} & & {\scriptstyle 0} & & \end{array}$ and the complex $L^\bullet 0 \to \mathbb{Z}/2\mathbb{Z} \to 0$. There is a natural quasi isomorphic $s : K^\bullet \to L^\bullet$. Let $f : K^\bullet \to K^\bullet$ be the multiplication by 2 map. We claim that the third condition for quasi-isomorphisms being a localizable class is not satisfied. Take $f : K^\bullet \to K^\bullet$ be the multiplication by 2 map and g to be the zero map. Then $sf = sg = 0$, but there is no quasi isomorphism $t : L^\bullet \to K^\bullet$ such that $ft = 0$. If there was such a quasi-isomorphism, then $t(L^0)$ would have to be nonzero (as there is nontrivial cohomology in the zeroth degree). But as $t(L^0)$ is a subgroup of Z, $2t(L^0)$ must also be nonzero.

To get around this problem, we need to first replace $\mathcal{C}(\mathcal{A})$ by a homotopy category $\mathcal{K}(\mathcal{A})$ where quasi-isomorphisms form a localizable class.

Let $f : K^\bullet \to L^\bullet$ be a map between complexes. Then we define f to be homotopic to 0 if there exists a collection of maps $h_i : K^i \to L^{i-1}$

$$\begin{tikzcd}
\dots \arrow[r, "d"] & K^{i-1} \arrow[r, "d"] \arrow[d, "f"'] & K^i \arrow[r, "d"] \arrow[d, "f"'] \arrow[dl, dashed, "h_i"'] & K^{i+1} \arrow[r, "d"] \arrow[d, "f"'] \arrow[dl, dashed, "h_{i+1}"'] & \dots \\
\dots \arrow[r, "d"] & L^{i-1} \arrow[r, "d"] & L^i \arrow[r, "d"] & L^{i+1} \arrow[r, "d"] & \dots
\end{tikzcd}$$

such that $f = dh + hd$. If $f, g : K^\bullet \to L^\bullet$ are two such morphisms, they are said to be homotopic if there exist a map h as above, with $f - g = dh + hd$.

LEMMA 2.2.8. *If f is homotopic to 0 then it is sent to 0 in the derived category.*

This is true because if f is homotopic to 0, it can be factorized through the natural inclusion of $K^\bullet$ into the cone of $id : K^\bullet \to K^\bullet$, which is sent to the zero object in the derived category (as it is quasi-isomorphic to zero).

DEFINITION 2.2.9. Let $\mathcal{A}$ be any abelian category. Then the homotopy category $\mathcal{K}(\mathcal{A})$ is the category with objects $Ob(\mathcal{K}(\mathcal{A})) = Ob(\mathcal{C}(\mathcal{A}))$ and morphisms $Mor(\mathcal{K}(\mathcal{A})) = Mor(\mathcal{C}(\mathcal{A}))/\{f \mid f \text{ is homotopic to } 0\}$.

Defined in this way, $\mathcal{K}(\mathcal{A})$ is an additive category. There is a well defined "cohomology at i" functor from $\mathcal{K}(\mathcal{A})$ to $\mathcal{A}$, so the notion of quasi-isomorphism is well defined.

THEOREM 2.2.10. *Quasi-isomorphisms form a localizable class in $\mathcal{K}(\mathcal{A})$.*

In the example above, the map f is homotopic to 0, so there is no contradiction to quasi-isomorphisms being a localizable class in $\mathcal{K}(\mathcal{A})$.

THEOREM 2.2.11. *The derived category $\mathcal{D}(\mathcal{A})$ is the localization of $\mathcal{K}(\mathcal{A})$ by quasi-isomorphisms.*

COROLLARY 2.2.12. *$\mathcal{D}(\mathcal{A})$ is additive.*

To define addition of morphisms, we use that the existence of a common denominator for two morphisms

$$\begin{tikzcd}
 & X' \arrow[dl, "s\in\mathcal{S}"'] \arrow[dr, "f"] & \\
X & & Y
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & X'' \arrow[dl, "t\in\mathcal{S}"'] \arrow[dr, "g"] & \\
X & & Y.
\end{tikzcd}$$

In other words, there exists a pair of diagrams

$$\begin{tikzcd}
 & X''' \arrow[dl, "p\in\mathcal{S}"'] \arrow[dr, "\tilde{f}"] & \\
X & & Y
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & X''' \arrow[dl, "p\in\mathcal{S}"'] \arrow[dr, "\tilde{g}"] & \\
X & & Y
\end{tikzcd}$$

that are equivalent to the given two. The sum of the last two diagrams is then given by $\tilde{f} + \tilde{g}$.

As before, we can define categories $\mathcal{D}^b(\mathcal{A}), \mathcal{D}^+(\mathcal{A}), \mathcal{D}^-(\mathcal{A})$. Assume $\mathcal{A}$ has enough projectives, i.e., that for any $X \in \mathcal{A}$, there exists a projective P which maps surjectively to X. Let $\mathcal{K}^-(\mathcal{P})$ denote the category of bounded from above complexes $\cdots \to P^i \to P^{i+1} \to \dots$, where all P^j are projective, with morphisms up to homotopy. Then the natural functor $\mathcal{K}^-(\mathcal{P}) \to \mathcal{D}^-(\mathcal{A})$ is an equivalence of categories. From this it easily follows that for any $X, Y \in \mathcal{A}, \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y[i]) = \mathrm{Ext}^i(X, Y)$. To see this, we can simply replace X by its projective resolution.

Similarly, we say that $\mathcal{A}$ has enough injectives if for any $X \in \mathcal{A}$ we have a monomorphism $X \to I$ for I injective. In this case, we get an equivalence of categories $\mathcal{K}^+(\mathcal{I}) \to \mathcal{D}^+(\mathcal{A})$.

2.2.4. The derived category of R-modules. We now consider the case where $\mathcal{A}$ is the category of R-modules for some ring R. Our main goal is to show that $\mathcal{A}$ has enough injectives. We will state this as a theorem:

THEOREM 2.2.13. *The category of R-modules has enough injectives.*

This will be used in the next section to show that the category of $\mathcal{D}$-modules has enough injectives. (In the category of R-modules, there are also enough projectives, as every module admits a surjection from a free module. This is in general not true for the category of $\mathcal{D}$-modules.) Although this is true for all rings R, we will restrict ourselves to rings over a field k.

We have a natural forgetful functor $f : \mathcal{A} \to \mathrm{Vect}(k)$, where $\mathrm{Vect}(k)$ is the category of k-vector spaces. We start by constructing a right adjoint g to f. For V a k-vector space, $g(A)$ will have underlying vector space $\mathrm{Hom}_{\mathrm{Vect}(k)}(R, A)$. The R-module structure on $g(A)$ will be given by setting $(r \cdot a)(x) = a(x \cdot r)$, where a is an element of $\mathrm{Hom}_{\mathrm{Vect}(k)}(R, A)$. It is straightforward to check that g indeed defines a functor and that g is right adjoint to f.

Now we claim that for any vector space I, $g(I)$ is also injective. By the adjunction between f and g, the functor $M \mapsto \mathrm{Hom}_{\mathcal{A}}(M, g(I))$ is isomorphic to the functor $M \mapsto \mathrm{Hom}_{\mathrm{Vect}(k)}(f(M), I)$. As both f and $\mathrm{Hom}_{\mathrm{Vect}(k)}(-, I)$ are exact, $\mathrm{Hom}_{\mathcal{A}}(-, g(I))$ must also be exact and so $g(I)$ is an injective object of $\mathcal{A}$.

For any R-module M, we can choose a monomorphism $f(M) \to I$ with I injective. This will correspond to a map $M \to g(I)$. We can factor our map of vector spaces $M \to I$ as $M \to g(I) \to I$, with the map $g(I) \to I$ given by evaluation at 1. As the map $M \to I$ was injective, our map $M \to g(I)$ must also be injective, as desired.

2.2.5. Derived functors. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive and left exact functor. We would like to give a universal property for the derived functor $RF : \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{A})$. We denote by $\mathcal{K}^+(F)$ the natural extension of F to a functor $\mathcal{K}^+(\mathcal{A}) \to \mathcal{K}^+(\mathcal{B})$.

DEFINITION 2.2.14. The derived functor of F is a pair (RF, ε_F) where $RF : \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ is an exact functor and $\varepsilon_F : \mathcal{Q}_{\mathcal{B}} \circ \mathcal{K}^+(F) \to RF \circ \mathcal{Q}_A$ is a morphism of functors satisfying the following universality condition:

For every exact functor $G : \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ and a morphism of functors $\varepsilon : \mathcal{Q}_{\mathcal{B}} \circ \mathcal{K}^+(F) \to G \circ \mathcal{Q}_A$, there exists an unique morphism $\eta : RF \to G$ for which the following diagram is commutative:

$$\begin{tikzcd}
& \mathcal{Q}_B \circ \mathcal{K}^+(F) \arrow[dl, "\varepsilon_F"'] \arrow[dr, "\varepsilon"] & \\
G \circ \mathcal{Q}_{\mathcal{A}} & & RF \circ \mathcal{Q}_{\mathcal{A}} \arrow[ll, "\eta\circ\mathcal{Q}_{\mathcal{A}}"']
\end{tikzcd}$$

It is easy to see that if (RF, ε_F) exists, then it is unique up to canonical isomorphism. The harder problems are to show existence and to compute RF. For this we have to introduce some additional definitions.

DEFINITION 2.2.15. Let $\mathcal{R}$ be some class of objects of $\mathcal{A}$. We say that $\mathcal{R}$ is admissible with respect to F if

(1) For every acyclic complex $\cdots \to K^i \to K^{i+1} \to \dots$ in $\mathcal{C}^+(\mathcal{A})$ with $K^i \in \mathcal{R}$, $F(K^\bullet)$ is also acyclic.
(2) For any $X \in \mathcal{A}$, there exists $Y \in \mathcal{R}$ such that there is a monomorphism from X to Y.

EXAMPLE 2.2.16. Consider the left exact functor $\mathrm{Hom}(X, \cdot)$. Injective objects form an admissible class with respect to it.

We can define all these notions for right exact functors as well.

If $\mathcal{R}$ is an admissible class with respect to F, then RF exists and can be defined as follows: any object of $\mathcal{D}(\mathcal{A})$ is isomorphic to the image of some object $K \in \mathcal{C}^+(\mathcal{R})$. We set $RF(Q_{\mathcal{A}}(K)) = Q_{\mathcal{B}}(F(K))$. This definition turns out to be independent of the choice of K.

PROPOSITION 2.2.17. 1. *$H^n(RF(K^\bullet))$ is a subquotient of*

$$\bigoplus_{p+q=n} R^pF(H^q(K^\bullet)).$$

2. *Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$. If there exist admissible classes $\mathcal{R}_{\mathcal{A}}$ and $\mathcal{R}_{\mathcal{B}}$ for F and G such that $F(\mathcal{R}_{\mathcal{A}}) \subset \mathcal{R}_{\mathcal{B}}$, then $R(G \circ F) = RG \circ RF$.*

An analogous theorem is true for right exact functors.

We now give two specific derived functors.

EXAMPLE 2.2.18. Let R be a commutative ring. Consider the right exact functor $N \mapsto M \otimes_R N$. The corresponding derived functor exists and will be denoted by $N \mapsto M \overset{L}{\otimes} N$, where $M \overset{L}{\otimes} N$ is an object in $\mathcal{D}^-(R-\text{modules})$.

EXAMPLE 2.2.19. Let $\mathcal{A}$ be an abelian category. If $\mathcal{A}$ has enough injectives, then the derived functor of $\mathrm{Hom}(X, -)$ exists and will be denoted by $Y \mapsto \mathrm{RHom}(X, Y) \in \mathcal{D}^+(\mathcal{A})$.

We can construct the functor $M \overset{L}{\otimes} N$ by deriving with respect to either M or N. These two functors will in fact be equivalent. If $\mathcal{A}$ has enough injectives and projectives, the same is true for RHom.

We now go back to $\mathcal{D}$-modules. Define $D(\mathcal{D}_X) = \mathcal{D}^b(\mathcal{M}(\mathcal{D}_X))$.

THEOREM 2.2.20. *If X is smooth, $\mathcal{D}^b(\mathcal{M}_{hol}(\mathcal{D}_X)) \simeq D_{hol}(\mathcal{D}_X)$, where the last category is the full subcategory of $D(\mathcal{D}_X)$ of complexes with holonomic cohomologies.*

This kind of statement is not true in general. For example $\mathcal{D}^b_{f.dim.}(\mathfrak{g}-\text{modules}) \not\simeq \mathcal{D}^b(\text{ finite dimensional } \mathfrak{g}-\text{modules})$, where $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$.

2.2.6. Exercises.

EXERCISE 2.2.2. Let $\mathcal{A}$ be an abelian category. Assume that $D(\mathcal{A})$ is equivalent to $\mathcal{C}_0(\mathcal{A})$. Prove that $\mathcal{A}$ is semi-simple.

EXERCISE 2.2.3. Let $\mathcal{A}$ be a full abelian subcategory of an abelian category $\mathcal{B}$. Denote by $D^b_{\mathcal{A}}(\mathcal{B})$ the full subcategory of all complexes in $D^b(\mathcal{B})$ whose cohomologies lie in $\mathcal{A}$. We have the obvious functor $D^b(\mathcal{A}) \to D^b_{\mathcal{A}}(\mathcal{B})$.

a) Is this functor always an equivalence of categories?

b) Prove that if the above functor is an equivalence of categories then $\mathcal{A}$ satisfies Serre's condition: for every short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{B}$ such that $X, Z \in \mathcal{A}$, we have $Y \in \mathcal{A}$.

c) Show that the converse of b) is still not true in general (hint: take $\mathcal{B}$ to be the category of $\mathfrak{g}$-modules where $\mathfrak{g}$ is a semi-simple Lie algebra over $\mathbb{C}$ and take $\mathcal{A}$ to be the category of finite-dimensional modules).

d) Let R be a ring. Take $\mathcal{B}$=the category of left R-modules, $\mathcal{A}$=the category of finitely generated R-modules. What can you say about this case?

Exercise 2.2.4. Let X be a scheme of finite type over a field k. Let $\mathcal{A}$ denote the category of quasi-coherent sheaves on X and let $\mathcal{B}$ denote the category of all sheaves of $\mathcal{O}_X$-modules. Show that in this case the functor $D^+(\mathcal{A}) \to D^+_{\mathcal{A}}(\mathcal{B})$ is an equivalence of categories. As a corollary we see that if $\mathcal{F}$ is a quasicoherent sheaf, then $H^i(X, \mathcal{F})$ is the same whether we compute it in the category of all sheaves or in the category of quasi-coherent sheaves.

2.3. The derived category of $\mathcal{D}$-modules

Let us now apply the machinery of derived categories to $\mathcal{D}$-modules. We will denote by $D(\mathcal{D}_X)$ the bounded derived category of $\mathcal{D}_X$-modules. When we want to stress that we work with left (resp. right) $\mathcal{D}_X$-modules we will write $D^l(\mathcal{D}_X)$ (resp. $D^r(\mathcal{D}_X)$).

2.3.1. Injective $\mathcal{D}$-modules. Our first goal will be to show that the category of $\mathcal{D}_X$-modules has enough injectives. We start by noting that the direct product of any collection of injective objects is injective. This follows immediately from the fact that $\mathrm{Hom}(-, \prod N_i) \cong \prod \mathrm{Hom}(-, N_i)$.

Take a point $x \in X$ with corresponding local ring $\mathcal{O}_X$. We set $\mathcal{D}_{X,x}$ to be the stalk $\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{O}_x$ of $\mathcal{D}_X$ at x. For any $\mathcal{D}_{X,x}$-module M, we have a $\mathcal{D}_X$-module $M(x)$ scheme-theoretically supported at x. If M is injective, then $M(x)$ will be as well. To see this, note that the functor $f : \mathcal{M}(\mathcal{D}_{X,x}) \to \mathcal{M}(\mathcal{D}_X)$ given by $M \mapsto M(x)$ is right adjoint to the functor $g : \mathcal{M}(\mathcal{D}_X) \to \mathcal{M}(\mathcal{D}_{X,x})$ given by $\mathcal{F} \mapsto \mathcal{F}_x = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_x$. Thus, we have an isomorphism of functors $\mathrm{Hom}_{\mathcal{M}(\mathcal{D}_X)}(-, M(x)) \cong \mathrm{Hom}_{\mathcal{M}(\mathcal{D}_{X,x})}(g(-), M)$. If M is injective, the right hand side is the composition of the two exact functors g and $\mathrm{Hom}_{\mathcal{M}(\mathcal{D}_{X,x})}(-, M)$, so $\mathrm{Hom}_{\mathcal{M}(\mathcal{D}_X)}(-, M(x))$ is exact and $M(x)$ is injective.

Now take a $\mathcal{D}_X$-module $\mathcal{F}$. We want to find a monomorphism from $\mathcal{F}$ into an injective $\mathcal{D}_X$-module. For every $x \in X$, by Theorem 2.2.13, we can find an $\mathcal{D}_{X,x}$ module M_x with an inclusion $\mathcal{F}_x \to M_x$. This gives in turn a map $\mathcal{F} \to M_x(x)$. Picking such a M_x for all $x \in X$, we get a map from $\mathcal{F}$ to the product of the $M_x(x)$. As the product of a collection of injective objects is injective, it only remains to show that this map is a monomorphism. But for any x, the corresponding map on stalks at x is a monomorphism, and thus our original map must be a monomorphism. This shows that the category of $\mathcal{D}_X$-modules has enough injectives.

2.3.2. Duality. We can define a duality functor $\mathbb{D} : D^l(\mathcal{D}_X) \to D^r(\mathcal{D}_X)$ by $M \mapsto \underline{\mathrm{RHom}}(M, \mathcal{D}_X[n])$ for any left $\mathcal{D}_X$-module. Here $\underline{\mathrm{Hom}}(M, N)$ is the quasi coherent sheaf of $\mathcal{D}_X$-modules whose sections on an affine open subset $U \subset X$ are given by $\mathrm{Hom}_{D_U}(\Gamma(U, M), \Gamma(U, N))$ and $\underline{\mathrm{RHom}}$ is the derived functor of $\underline{\mathrm{Hom}}(M, N)$ with respect to N. Finitely generated locally free $\mathcal{D}_X$-modules form an admissible class with respect to $\underline{\mathcal{H}om}(-, N)$, and so we can compute $\underline{\mathrm{RHom}}$ (and in particular $\mathbb{D}$) using them.

We can also define a map $\mathbb{D} : D^l(\mathcal{D}_X) \to D^l(\mathcal{D}_X)$ by $\mathbb{D}(M) = \underline{\mathrm{RHom}}(M, \mathcal{D}_X \otimes \Omega_X^{-1}[n])$, where $n = \dim X$. Our two duality maps are obviously identified under the canonical equivalence between $D^l(\mathcal{D}_X)$ and $D^r(\mathcal{D}_X)$.

Theorem 2.3.1. $\mathbb{D}^2 \simeq \mathrm{Id}$.

Proof. Let $\mathcal{R}$ be the class of locally free finitely generated $\mathcal{D}_X$ modules. As noted above, this is an admissible class for $\mathbb{D}$. We have a natural morphism $\mathrm{Id} \to \mathbb{D}^2$. It it is easy to check that this is an isomorphism for every object in $\mathcal{R}$, so as $\mathcal{R}$ is admissible, it is an isomorphism in general. □

THEOREM 2.3.2. $\mathrm{RHom}(M,N) \simeq \mathrm{RHom}(\mathbb{D}N, \mathbb{D}M)$ *for any left* $\mathcal{D}_X$*-modules* M *and* N*.*

PROOF. There exist a natural morphism $\mathrm{RHom}(M,N) \to \mathrm{RHom}(\mathbb{D}N, \mathbb{D}M)$. For locally free modules this is an isomorphism, so as in the proof of the previous theorem, we can conclude that it is an isomorphism in general. □

2.3.3. Inverse image. Let $\pi : X \to Y$ be a morphism and $\pi^\bullet$ be the sheaf theoretic inverse image. Define the inverse image functor $\pi^! : D^l(\mathcal{D}_Y) \to D^l(\mathcal{D}_X)$ by $M \mapsto \mathcal{O}_X \overset{L}{\otimes}_{\pi^\bullet \mathcal{O}_Y} \pi^\bullet M[\dim X - \dim Y]$. Here $\overset{L}{\otimes}$ means the derived functor.

If π is a closed embedding, then Lemma 1.8.1 tells us that $\pi^! : D^b(\mathcal{D}_Y) \to D^b(\mathcal{D}_X)$ is the left derived functor of the functor $\pi^! : \mathcal{M}(\mathcal{D}_Y) \to \mathcal{M}(\mathcal{D}_X)$ between abelian categories.

As before, we define $\mathcal{D}_{X\to Y} = \pi^! \mathcal{D}_Y[\dim Y - \dim X] = \mathcal{O}_X \overset{L}{\otimes}_{\pi^\bullet \mathcal{O}_Y} \pi^\bullet \mathcal{D}_Y$.

2.3.4. Direct image. Let $\pi : X \to Y$ be any morphism. Define the direct image of a right $\mathcal{D}_X$-module $M \in D^b(\mathcal{D}_X)$ by

$$\pi_*(M) = R\pi_\bullet(M \overset{L}{\otimes}_{\mathcal{D}_X} \mathcal{D}_{X\to Y}).$$

Here $\pi_\bullet$ is the usual sheaf theoretic direct image. By definition $\pi_*(M)$ is a complex of sheaves of $\mathcal{D}_Y$-modules. A priori it is not clear why these sheaves are quasi coherent. One way of showing this is to use the following theorem:

THEOREM 2.3.3. *[Bernstein] Let* $\mathcal{A}$ *be a quasi coherent sheaf of associative algebras on* X*. Then*

$$\mathcal{D}^b_{q.coh.}(\mathcal{M}(\mathcal{A})) \simeq \mathcal{D}^b(\mathcal{M}_{q.coh}(\mathcal{A})),$$

where the first category is the full subcategory of $\mathcal{D}^b(\mathcal{M}(\mathcal{A}))$ *consisting of complexes with quasi-coherent cohomology.*

Taking a decomposition of π into a locally closed embedding and a projection and using the Grothendieck spectral sequence, it can be shown that $\pi_*(M)$ has quasi-coherent cohomology.

We will take a different approach, based on explicit resolutions. The main idea is the following: suppose there exists a complex of quasi coherent sheaves $K^\bullet = K^\bullet(M)$, quasi isomorphic to $M \overset{L}{\otimes}_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$. Choose a cover of X by affine open subsets U_i. Consider the Cech complex $\check{C}(K^\bullet)$ with respect to this cover, i.e., the total complex of the bicomplex

$$\bigoplus_\alpha (j_\alpha)_* K^\bullet|_{U_\alpha} \to \bigoplus_{\alpha_1,\alpha_2} (j_{\alpha_1,\alpha_2})_* K^\bullet|_{U_{\alpha_1}\cap U_{\alpha_2} \to \dots},$$

where $j_{\alpha_1,\dots,\alpha_k} : U_{\alpha_1} \cap \cdots \cap U_{\alpha_k} \to X$. $K^\bullet$ and $\check{C}(K^\bullet)$ are quasi isomorphic and moreover

$$\pi_\bullet \check{C}(K^\bullet) = R\pi_\bullet(K^\bullet).$$

This reduces us to showing that $\pi_\bullet (j_{\alpha_1,\dots,\alpha_k})_* K^\bullet|_{U_{\alpha_1}\cap\dots\cap U_{\alpha_k}}$ is quasicoherent. But this is easy to see from the fact that any morphism from an affine variety is affine.

Now, let us find such a complex $K^\bullet(M)$ We will use the Koszul complex $Kos(M)$ of M, which is quasi-isomorphic to M. This complex is defined as follows: we know that $dR(\mathcal{D}_X)$ is a locally free resolution of the sheaf K_X of the top forms on X. Thus $dR(\mathcal{D}_X) \otimes K_X^{-1}$ is a locally free resolution of $\mathcal{O}_X$. Let $Kos(M) = M \otimes dR(\mathcal{D}_X) \otimes K_X^{-1}$, i.e., take $Kos(M)^i = M \otimes \Omega^i_X \otimes \mathcal{D}_X \otimes K_X^{-1}$. This complex is obviously a resolution of M. It carries a left $\mathcal{O}_X$ action and a right $\mathcal{D}_X$ action (as we took the $\mathcal{O}$-linear tensor product of a left and a right $\mathcal{D}$-module). As M may not be locally free over $\mathcal{D}_X$, $Kos(M)$ may also not be locally free. However, it is easy to see that $Kos(M)$ is still locally free in the ∂-direction. (This means that for any choice of coordinates $x_1,\dots,x_n,\partial_1,\dots,\partial_n, Kos(M)$ consists of free $k[\partial_1,\dots,\partial_n]$-modules.)

Modules, which are locally free in the ∂-direction form an admissible class with respect to $\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}$, as $\mathcal{D}_{X\to Y}$ is locally free over $\mathcal{O}_X$. This implies that $Kos(M)\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}$ is quasi-isomorphic to $M\overset{L}{\otimes}\mathcal{D}_{X\to Y}$. Since the $\mathcal{O}_X$-action and $\mathcal{D}_X$-action commute, $K^\bullet(M)=Kos(M)\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}$ is a complex of quasi-coherent $\mathcal{O}_X$-modules. Using the arguments above, we see that $\pi_\bullet(K^\bullet(M))=\pi_*(M)$ is a complex of quasi coherent modules.

2.3.5. An exact triple.

THEOREM 2.3.4. *Let $i:Z\to X$ be a closed immersion and let $j:U\to X$ be the complementary open immersion. Then for every $\mathcal{F}\in D(\mathcal{D}_X)$, we have an exact triple $i_*i^!\mathcal{F}\to\mathcal{F}\to j_*\mathcal{F}|_U$.*

PROOF. By Theorem 1.7.3 we know that i_* is left adjoint to $i^!$, which gives us a natural map $i_*i^!\mathcal{F}\to\mathcal{F}$. Similarly, it is not hard to show that j_* is right adjoint to $j^!:\mathcal{F}\mapsto\mathcal{F}|_U$, which gives us our second map $\mathcal{F}\to j_*\mathcal{F}|_U$. Both maps are functorial in $\mathcal{F}$.

Now note that the functors i_* and $j^!$ are exact and the functors $i^!$ and j^* are right exact. This shows that it suffices to check that our maps give us an exact triangle when $\mathcal{F}$ is an injective $\mathcal{D}_X$-module, in which case we are working entirely in the abelian category. Here, $i_*i^!\mathcal{F}$ is the sub $\mathcal{D}_X$-module of sections of $\mathcal{F}$ supported on Z. This immediately shows the exactness of

$$0\to i_*i^!\mathcal{F}\to\mathcal{F}\to j_*\mathcal{F}|_U.$$

It remains to show that $\mathcal{F}\to j_*\mathcal{F}|_U$ is surjective. Actually, it suffices to show this for a set of injectives $\mathcal{F}$ into which every $\mathcal{D}_X$-module injects. But we constructed such a set in Section 2.3.1, and for those $\mathcal{F}$ it is clear that $\mathcal{F}\to j_*\mathcal{F}|_U$ is surjective.

□

2.3.6. Some computations of direct images.

EXAMPLE 2.3.5. Let $\pi:X\to pt$. We claim that $\pi_*M=R\pi_\bullet(dR(M^l))$ where $M^l=M\otimes K_X^{-1}$. Indeed, if $Y=pt$, then $\mathcal{D}_{X\to Y}=\mathcal{O}_X$. As before, $M\overset{L}{\otimes}_{\mathcal{D}_X}\mathcal{D}_{X\to Y}$ is quasi isomorphic to $Kos(M)\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}$, where $(Kos(M)\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y})^i=(M\otimes\Omega_X^i\otimes\mathcal{D}_X\otimes K_X^{-1})\otimes_{\mathcal{D}_X}\mathcal{O}_X=M\otimes\Omega_X^i\otimes K_X^{-1}=dR(M^l)^i$. We see that $\pi_*M=R\pi_\bullet(dR(M^l))$.

This also shows that for M a left $\mathcal{D}$-module $\pi_*(M)$ is isomorphic to the hypercohomology of $dR(M)$.

Let $\pi:X\to Y$ be any morphism. It can be decomposed as a product of a locally closed embedding and a smooth morphism.

(1) If $\pi:X\to Y$ is an open embedding, direct image for $\mathcal{D}_X$-modules coincides with the usual derived direct image for $\mathcal{O}_X$-modules.
(2) If $\pi:X\to Y$ is a closed embedding (or more generally affine), we can write $\pi_*(M)=\pi_\bullet(M\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y})$.
(3) Suppose $\pi:X\to Y$ is smooth and let M be a left $\mathcal{D}_X$-module. In this case $\pi_*(M)=R\pi_\bullet(dR_{X/Y}(M))$, where $dR_{X/Y}(M)$ is the relative de Rham complex

$$0\to M\to M\otimes\Omega^1_{X/Y}\to\cdots\to M\otimes\Omega^{\dim X-\dim Y}_{X/Y}\to 0.$$

The sheaf of relative 1-forms $\Omega^1_{X/Y}$ is defined by the following exact sequence

$$0\to\pi^*\Omega^1_Y\to\Omega^1_X\to\Omega^1_{X/Y}\to 0.$$

and $\Omega^i_{X/Y}=\Lambda^i\Omega^1_{X/Y}$. The proof of this statement is analogous to that in the case of $Y=pt$, considered in example above.

2.3.7. Base change.

THEOREM 2.3.6. *For any diagram*

$$\begin{tikzcd} X \times_Y S \arrow[r, "\tilde{\tau}"] \arrow[d, "\tilde{\pi}"'] & X \arrow[d, "\pi"] \\ S \arrow[r, "\tau"'] & Y \end{tikzcd}$$

we have a natural isomorphism of functors $\tau^!\pi_*\mathcal{F} = \tilde{\pi}_*\tilde{\tau}^!\mathcal{F}$, *where* $\mathcal{F} \in D^b(\mathcal{D}_X)$.

PROOF. Let us decompose τ as the composition of a locally closed embedding and a projection.

(1) If τ is an open embedding, the statement is obvious, as direct image is compatible with restriction to open subsets.

(2) Assume $\tau : S \to Y$ is a projection. Let $S = Y \times Z$, so $X \times_Y S = X \times Z$. We have the following diagram

$$\begin{tikzcd} X \times Z \arrow[r, "\tilde{\tau}"] \arrow[d, "\tilde{\pi}"'] & X \arrow[d, "\pi"] \\ Y \times Z \arrow[r, "\tau"'] & Y \end{tikzcd}$$

For any $\mathcal{D}_X$-module $\mathcal{F}$, $\tilde{\tau}^!\mathcal{F} = \mathcal{F} \boxtimes \mathcal{O}_Z[\dim Z]$ and $\tilde{\pi}_*\tilde{\tau}^!(\mathcal{F}) = \pi_*\mathcal{F} \boxtimes \mathcal{O}_Z[\dim Z] = \tau^!\pi_*\mathcal{F}$.

(3) Suppose that τ is a closed embedding. In this case we will denote it by i (this will be our standard notation for a closed embedding). Consider the following diagram

$$\begin{tikzcd} W \arrow[r, hook, "\tilde{i}"] \arrow[d] & X \arrow[d, "\pi"] & U \arrow[l, hook', "\tilde{j}"] \arrow[d] \\ S \arrow[r, hook, "i"] & Y & V \arrow[l, hook', "j"] \end{tikzcd}$$

Here S is a closed subscheme in Y and V is its complement. Applying Theorem 2.3.4 to the top and bottom row, we get a diagram

$$\begin{tikzcd} \pi_*\tilde{i}_*\tilde{i}^! * \mathcal{F} \arrow[r] & \pi_*\mathcal{F} \arrow[r] \arrow[d, equal] & \pi_*\tilde{j}_*\mathcal{F}|_U \arrow[d, equal] \\ i_*i^!\pi_*\mathcal{F} \arrow[r] & \pi_*\mathcal{F} \arrow[r] & j_*\pi_*\mathcal{F}|_V \end{tikzcd}$$

Here the last terms are isomorphic by (1).

We need to construct an isomorphism between the two left terms in this diagram. Note that if we were living in an abelian category and the rows of the above diagram were short exact sequences we would get such an isomorphism automatically. However, here we are dealing with derived categories and we cannot derive the existence of such an isomorphism from general reasons because there does not exist a canonical cone in the derived category. However, it follows from the 5-lemma that if we construct a morphism between the two left terms of the above diagram making the whole diagram commutative then this morphism will necessarily be an isomorphism. This can be done in the following way.

Let $R \in D_S(\mathcal{D}_Y)$ (the category of $\mathcal{D}_Y$-modules supported on S) and $T \in D(\mathcal{D}_V)$. Then

$$\mathrm{Hom}(R, j_*T) = \mathrm{Hom}(R|_V, T) = 0.$$

Let $R = \pi_*\tilde{i}_*\tilde{i}^!\mathcal{F}$. The bottom exact triple in our diagram gives a long exact sequence

$$0 = \mathrm{Hom}(R, j_*\pi^*\mathcal{F}|_U[-1]) \to \mathrm{Hom}(R, i_*i^!\pi_*\mathcal{F}) \to$$

$$\to \mathrm{Hom}(R, \pi_*\mathcal{F}) \to \mathrm{Hom}(R, j_*\pi^*\mathcal{F}) = 0.$$

So $\mathrm{Hom}(R, i_*i^!\pi_*\mathcal{F}) \simeq \mathrm{Hom}(R, \pi_*\mathcal{F})$. Since we have a canonical element in $\mathrm{Hom}(R, \pi_*\mathcal{F})$ we also get an element in $\mathrm{Hom}(R, i_*i^!\pi_*\mathcal{F})$. The fact it makes the whole diagram commutative is clear.

Let us now explain why the isomorphism of functors constructed above doesn't depend on the decomposition of of τ as a product of a closed embedding and a smooth morphism. □

EXAMPLE 2.3.7. If $\pi : X \to Y$ is a smooth map, then $\pi^!$ maps coherent modules to coherent modules (in this case $\pi^!$ is exact up to a shift). If $\pi : X \to Y$ is projective, then π_* maps coherent modules to coherent modules. This last statement is clear when π is a closed embedding, so it suffices to show it in the case that π is a projection $X = \mathbb{P}^N \times Y \to Y$. As pushforward is local, we can further reduce to the case of Y affine. By Theorem 2.1.6, $\mathcal{D}_X$ is a projective generator in the category of coherent modules, so it's enough to prove the statement for $\mathcal{D}_X$. We have $\mathcal{D}_{X\to Y} = \mathcal{O}_{\mathbb{P}^N} \boxtimes \mathcal{D}_Y$. Thus,

$$(K_X \otimes \mathcal{D}_X) \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y} = ((K_{\mathbb{P}^N} \boxtimes K_Y) \otimes_{\mathcal{O}_X} (\mathcal{D}_{\mathbb{P}^N} \boxtimes \mathcal{D}_Y)) \otimes_{\mathcal{D}_X} (\mathcal{O}_{\mathbb{P}^N} \boxtimes \mathcal{D}_Y) = K_{\mathbb{P}^N} \boxtimes (K_Y \otimes \mathcal{D}_Y).$$

So $\pi_*\mathcal{D}_X = \mathcal{D}_Y[-N]$.

THEOREM 2.3.8. *Let $\pi : X \to Y$ be projective. Then*

(1) π_* *is left adjoint to* $\pi^!$.
(2) $\mathbb{D}\pi_* = \pi_*\mathbb{D}$.

PROOF. Theorem 1.7.3 takes care of the case where π is a closed embedding. Thus, as in the above example, we can reduce to the case where $X = \mathbb{P}^N \times Y$ and π is the natural projection, Y affine.

Let us prove 1. Let $\mathcal{F} \in D(\mathcal{D}_X)$ and $\mathcal{G} \in D(\mathcal{D}_Y)$. We want to construct a natural isomorphism $\mathrm{RHom}(\pi_*\mathcal{F}, \mathcal{G}) \simeq \mathrm{RHom}(\mathcal{F}, \pi^!\mathcal{G})$. It is enough to do this when $\mathcal{F} = \mathcal{D}_X$, since by Theorem 2.1.6 $\mathcal{D}_X$ is a projective generator and thus $D(\mathcal{D}_X)$ is equivalent to the homotopy category of free complexes.

As we have computed, $\pi_*\mathcal{D}_X = \mathcal{D}_Y[-N]$. So

$$\mathrm{RHom}(\pi_*\mathcal{D}_X, \mathcal{G}) = \mathrm{RHom}(\mathcal{D}_Y[-N], \mathcal{G}) = R\Gamma(Y, \mathcal{G})[N]$$

On the other hand $\pi^!\mathcal{G} = \mathcal{O}_{\mathbb{P}^N} \boxtimes \mathcal{G}[N]$, and hence

$$\mathrm{RHom}(\mathcal{D}_X, \pi^!\mathcal{G}) = \mathrm{R}\Gamma(\pi^!\mathcal{G}) = \mathrm{R}\Gamma(Y, \mathcal{G})[N].$$

To prove 2, it is again enough to construct the above isomorphism for $\mathcal{D}_X$, which can done by means of a similar calculation. □

THEOREM 2.3.9. *Let $\pi : X \to Y$ be smooth. Then*

(1) $\mathbb{D}\pi^![\dim Y - \dim X] = \pi^!\mathbb{D}[\dim X - \dim Y]$.
(2) $\pi^![2(\dim Y - \dim X)]$ *is left adjoint to* π_*.

CHAPTER 3

The derived category of holonomic $\mathcal{D}$-modules

3.1.

So far we have discussed the general properties of the derived category of holonomic D-modules. However, it turns out that many interesting things can only be done for holonomic D-modules. So, let us turn to a more detailed study of the derived category of holonomic D-modules.

Let us first summarize (and reformulate a little) what we already know about the direct and inverse image functors. Let $\pi : X \to Y$. There are two functors associated with π: π_* and $\pi_!$ (the latter is not always defined; recall that $\pi_! M$ exists if $\pi_* \mathbb{D} M$ is coherent). If $\pi_!$ is defined, we can construct a canonical morphism $\pi_! \to \pi_*$ as follows.

First decompose π as a product of an open embedding and projective morphism (as usual we shall leave the verification that the resulting morphism does not depend on the choice of this decomposition to the reader). Recall that if π is projective then $\pi_!$ and π_* coincide. Thus it is enough to construct our morphism for an open embedding. If we have an open embedding $j : U \hookrightarrow X$ then for any module M on U we have a natural adjunction $\mathrm{Hom}(N, j_* M) \cong \mathrm{Hom}(N|_U, M)$. For $N = j_! M$ we have $N|_U = M$ and hence we have a natural isomorphism $\mathrm{Hom}(j_! M, j_* M) \cong \mathrm{Hom}(M, M)$. Our morphism $j_! M \to j_* M$ will be the map corresponding to the identity map $M \to M$.

Similarly, we have the functors $\pi^!$ and π^* (where the latter is only sometimes defined). Using Theorem 2.3.9, it is easy to define and prove the key properties of π^* when π is smooth.

Theorem 3.1.1. *Let $\pi : X \to Y$ be a smooth morphism. Then*
(1) *$\pi^!$ maps $D_{coh}(\mathcal{D}_Y)$ to $D_{coh}(\mathcal{D}_X)$.*
(2) *$\pi^* = \mathbb{D}\pi^!\mathbb{D} = \pi^![2(\dim Y - \dim X)]$.*
(3) *π^* is left adjoint to π_*.*

When π is an open embedding, $\pi^! = \pi^*$.

Of course it would be very useful to have a situation in which we are guaranteed that π^* and $\pi_!$ exist. It turns out that this is always the case for holonomic modules. More precisely, we have the following theorem.

Theorem 3.1.2. *The functors π_*, $\pi^!$, $\mathbb{D}$ and $\boxtimes$ send complexes of holonomic modules to complexes of holonomic modules.*

Corollary 3.1.3. *For any morphism $\pi : X \to Y$ we can define $\pi_! = \mathbb{D}\pi_*\mathbb{D} : D_{hol}(\mathcal{D}_X) \to D_{hol}(\mathcal{D}_Y)$ and $\pi^* = \mathbb{D}\pi^!\mathbb{D} : D_{hol}(\mathcal{D}_Y) \to D_{hol}(\mathcal{D}_X)$.*

Defined in this way, $\pi_!$ is left adjoint to $\pi^!$ and $\pi_!$ is right adjoint to π^*.

Theorem 3.1.2 and Corollary 3.1.3 can be referred to as establishing Grothendieck's formalism of six functors in the setting of holonomic $\mathcal{D}$-modules. We do not give a precise definition, but note that we have 6 kinds of functors between the categories $D_{hol}(\mathcal{D}_X)$ - namely, the functors $\pi^*, \pi_*, \pi^!, \pi_!$ (for a morphism $\pi : X \to Y$), $\mathbb{D}$ and $\boxtimes$.

Before proving this theorem, let us give an application. We will use Theorem 3.1.2 to classify the irreducible holonomic $\mathcal{D}$-modules. As a first step, we prove the following lemma.

LEMMA 3.1.4. *Let M be a holonomic module on X with supp$M = X$. Then there exists a smooth open $U \subset X$ such that $j^!M$ is $\mathcal{O}_U$-coherent on U. (Here $j : U \hookrightarrow X$.)*

Recall that an $\mathcal{O}_U$-coherent $\mathcal{D}_U$-modules may be thought of as a vector bundle on U endowed with a flat connection.

PROOF. We know that $j^!M$ is $\mathcal{O}_U$-coherent iff $s.s.(M) = \{$ zero-section in $T^*U\}$.

Let us consider the following diagram

$$\begin{tikzcd} s.s.(M) \arrow[r, hook] \arrow[dr] & T^*X \arrow[d] \\ & X \end{tikzcd}$$

Since $s.s.(M)$ is conic it follows that any fiber of the map $s.s.(M) \to X$ is either 0 or of dimension greater or equal to 1. Since $\dim s.s.(M) = \dim X$ the fiber over generic point cannot be of dimension > 0. So, over the generic point the diagonal arrow is an isomorphism. Hence there exists U as above , such that $j^!M$ is $\mathcal{O}$-coherent. □

It is clear that we may choose the open subset U above to be affine.

Let us now look at the case when M is irreducible. In this case let us try to restore M from $j^!M$. Here $j : U \to X$ is any open embedding. First of all, if M is irreducible, so is $j^!M$. Indeed, suppose we have a short exact sequence $0 \to K \to j^!M \to N \to 0$ with $K, N \neq 0$. Then we have the map $M \to j_*N$ which in fact factorizes as $M \to H^0(j_*N) \to j_*N$ (since j_*N lives in degress ≥ 0. Let $\widetilde{K}$ denote the kernel of this map. Then $j^!\widetilde{K} = K$ which shows that $\widetilde{K}$ is non-zero but different from M. This contradicts irreducibility of M.

Let us now set $N = j^!M$ and pretend that we are only given N nut not M and we want to effectively construct M. In fact it turns out that we can do more: to every holonomic $\mathcal{D}_U$-module we are going to associate (canonically) a newholonomic $calD_X$-module $j_{!*}N$ called the *intermediate* or *minimal* (or Deligne-Goresky-MacPherson) extension of N to X which in particular will solve our problem in the case when N is irreducible. This extension will in fact be uniquely characterized in the following way.

THEOREM 3.1.5. *Let X be an irreducible variety and let $U \subset X$ be an open subset. For every holonomic $\mathcal{D}_U$-module N there exists unique $\mathcal{D}_X$-module M satisfying the following properties:*

(1) $j^!(j_{!*}(N)) = N$;
(2) $j_{!*}$ *has no submodules or quotients concentrated on $X \backslash U$.*

Moreover, we claim that if N is irreducible then $j_{!*}(N)$ (defined by (1) and (2) above) is also irreducible. Indeed, suppose there is a submodule $M \hookrightarrow j_{!*}(N)$. Applying the functor $j^!$ we get $j^!M \to N$. Since N is irreducible, this map is either an isomorphism or it is equal to 0. Assume first that this map is 0; in this case M is concentrated on $X\backslash U$ which contradicts property (2). In the case when this map is an isomorphism it follows $j_{!*}(N)/M$ is concentrated on $X\backslash U$ which again contradicts property (2).

In particular, it follows that if N is an irreducible $\mathcal{D}_U$-module then N has unique irreducible extension to X - namely the intermediate extension $j_{!*}N$.

3.1.1. Construction of intermediate extension. Let us now construct the extension $j_{!*}N$ satisfying properties (1) and (2) of Theorem 3.1.5. As is suggested by both the name and the notation it should somehow be constructed out of $j_!N$ and j_*N.

REMARK 3.1.6. Let $\mathcal{A}$ and $\mathcal{B}$ two be abelian categories. Let us introduce the following notations: $D^{\geq 0} = \{K^\bullet \mid H^i(K^\bullet) = 0 \text{ for } i < 0\}$ and $D^{\leq 0} = \{K^\bullet \mid H^i(K^\bullet) = 0 \text{ for } i > 0\}$. Then for any left exact functor $F : \mathcal{A} \to \mathcal{B}$ we have $RF : D^{\geq 0}(\mathcal{A}) \to D^{\geq 0}(\mathcal{B})$ and for any right exact functor $G : \mathcal{A} \to \mathcal{B}$ we have $LG : D^{\leq 0}(\mathcal{A}) \to D^{\leq 0}(\mathcal{B})$.

In particular, $j_* : D^{\geq 0}_{hol}(\mathcal{D}_U) \to D^{\geq 0}_{hol}(\mathcal{D}_X)$ and $j_! : D^{\leq 0}_{hol}(\mathcal{D}_U) \to D^{\leq 0}_{hol}(\mathcal{D}_X)$.

Consider the map $j_!N \to j_*N$. By the remark above $j_!N \in D^{\leq 0}_{hol}$ and $j_*N \in D^{\geq 0}_{hol}$. Hence this map factorizes through the chain

$$j_!N \to H^0(j_!N) \to H^0(j_*N) \to j_*N. \tag{3.1}$$

Let $j_{!*}$ be the image of this map. Note that is j is an affine.embedding (we can in fact always reduce the situation to this case) then $j_{!*}N$ is simply the image of $j_!N$ in j_*N.

We claim that $j_{!*}N$ defined in this way satisfies properties (1) and (2) of Theorem 3.1.5.

First, all the maps in (3.1) become isomorphisms after restricting to U. Thus the restriction of $j_{!*}N$ to U is equal to N. Hence property 1 is satisfied.

Let us now show property 2. We have the maps $j_!N \to j_{!*}N \to j_*N$. It is easy to see from the definition that for any non-zero $\mathcal{D}_X$-module L the induced maps $\mathrm{Hom}(j_{!*}N, L) \to \mathrm{Hom}(j_!N, L)$ and $\mathrm{Hom}(L, j_{!*}N) \to \mathrm{Hom}(L, j_*N)$ are injective.

Let L be any $\mathcal{D}_X$-module concentrated on $X\backslash U$. Then $\mathrm{Hom}(j_!N, L) = \mathrm{Hom}(N, j^!L) = 0$ since $j^!L = 0$. Similarly, $\mathrm{Hom}(L, j_*N) = \mathrm{Hom}(j^*L, N) = \mathrm{Hom}(j^!L, N) = 0$. This means that j_*N has no submodules concentrated on $X\backslash U$ and $j_!N$ has no quotients concentrated on $X\backslash U$. Hence by the above observation it follows that $\mathrm{Hom}(j_{!*}N, L) = \mathrm{Hom}(L, j_{!*}N) = 0$.

Warning. We have shown above that $j_{!*}N$ has neither quotients nor submodules concentrated on $X\backslash U$. However, we do not claim (and in general it is not true) that $j_{!*}N$ has no *subquotients* concentrated on $X\backslash U$.

Let us now show that the properties (1) and (2) of Theorem 3.1.5 define $j_{!*}N$ uniquely.

Let M be $\mathcal{D}_X$ module satisfying these properties. In particular, $j^!(M) = j^*M = N$. Thus by adjointness the map $j_!N \to j_*N$ factorizes through the sequence

$$j_!N \to M \to j_*N.$$

Since M is concentrated in cohomological degree 0 it follows that this sequence in fact factorizes through

$$j_!N \to H^0(j_!N) \xrightarrow{\alpha} M \xrightarrow{\beta} H^0(j_*N) \to j_*N.$$

Also all of these maps become isomorphisms when restricted to U. Thus the cokernel of α is a quotient module of M which is concentrated on $X\backslash U$ and therefore it is 0. Similarly the kernel of β is a submodule of M concentrated on $X\backslash U$ and hence it is 0. In other words, α is surjective and β is injective. This means that M is the image of the map $H^0(j_!N) \to H^0(j_*N)$ which finishes the proof.

Example. Let $X = \mathbb{A}^1$, $U = \mathbb{A}^1\backslash\{0\}$ and $N = \mathcal{O}_U$. In this case $j_!\mathcal{O}_U = \mathbb{D}j_*\mathcal{O}_U$ and there are two exact sequences $0 \to \mathcal{O}_X \to j_*\mathcal{O}_U \to \delta_0 \to 0$ and $0 \to \delta_0 \to j_!\mathcal{O}_U \to \mathcal{O}_X \to 0$, dual to each other. So we have a sequence $j_!\mathcal{O}_U \twoheadrightarrow \mathcal{O}_X \hookrightarrow j_*\mathcal{O}_U$, which means that $j_{!*}\mathcal{O}_U = \mathcal{O}_X$.

If N is $\mathcal{D}_U$-module generated by x^λ, where $\lambda \neq \mathbb{Z}$, then $j_!N \to j_*N$ is an isomorphism, since both modules are irreducible.

Note that in fact $j_{!*}$ is defined not only for an open embedding j but for any locally closed embedding.

Let us point out several properties of $j_{!*}$.

LEMMA 3.1.7. (1) *$j_{!*}N$ is functorial in N.*

(2) *If N is irreducible then so is $j_{!*}N$.*
(3) $j_{!*}(\mathbb{D}N) = \mathbb{D}(j_{!*}N)$.

PROOF. (1) follows immediately from the fact that $j_!$ and j_* are functors. Note, however, that $j_{!*}$ is neither left exact nor right exact. As a result the functor $j_{!*}$ makes sense only for modules and it doesn't make sense for general objects of the derived category.

(2) we have already discussed above.

To prove (3) let us check that $\mathbb{D}(j_{!*}N)$ satisfies properties (1) and (2) of Theorem 3.1.5. Property (1) is in fact obvious. To prove (2) let us remark that $\mathbb{D}$ is a contravariant equivalence of categories which does not change the support of a module. Hence it transforms submodules concentrated on $X\backslash U$ to quotient modules concentrated on $X\backslash U$ and vice versa. □

Here is some interesting application of the notion of intermediate extension (probably one of the most important ones). Let X be singular variety. Let $U \subset X$ be smooth dense open subset. Set

$$\mathrm{IC}_X = j_{!*}O_U \in \mathcal{M}_{hol}(\mathcal{D}_X)$$

We shall call IC_X the *intersection cohomology $\mathcal{D}$-module of X.*

If $\pi : X \to pt$, then $\pi_*(IC_X)$ is called the *intersection cohomology of X.* In the case when X is smooth one clearly has $\mathrm{IC}_X = \mathcal{O}_X$ since $\mathcal{O}_X$ is irreducible. Thus by Theorem ?? in the case when X is smooth its intersection cohomology is naturally isomorphic to the ordinary cohomology (with complex coefficients) of the underlying analytic space.

The intersection cohomology $\mathcal{D}$-module of X has many nice properties. For example since $\mathcal{O}_U$ is self-dual then so is IC_X. Assume for example X is proper, then $\mathbb{D}\pi_* = \pi_*\mathbb{D}$. Hence

$$\mathbb{D}\pi_*(IC_X) = \pi_*\mathbb{D}(IC_X) = \pi_*(IC_X)$$

i.e. the intersection cohomology of an proper variety satisfies Poincare duality.

3.1.2. Description of irreducible holonomic modules. It is clear from the above discussion are essentially classified by pairs (Z, N), where $Z \subset X$ is an irreducible smooth locally closed and N is an irreducible $\mathcal{O}$-coherent module on Z. The corresponding module is then $j_{!*}N$ where $j : Z \to X$ is the natural embedding. Indeed, if M is an irreducible $\mathcal{D}_X$-module, let $Y \subset X$ be the support of M. This is an irreducible closed subset of X and by Kashiwara's theorem M corresponds to some irreducible holonomic $\mathcal{D}_Y$-module M_Y. We now choose Z to be any smooth open subset of Y such that $M_Y|_Z$ is $\mathcal{O}$=coherent.

Moreover, if $Z' \subset Z$ is open, then $(Z', M\mid_{Z'})$ and (Z, M) correspond to the same holonomic module on X. In this case we say that (Z, M) and (Z', M') are equivalent. Let us generate by this an equivalence relation on the pairs (Z, M) as above. Then irreducible holonomic $\mathcal{D}_X$-modules are in one-to-one correspondence with equivalence classes of pairs (Z, M).

3.1.3. Exercises.

EXERCISE 3.1.1. Assume that an algebraic group G acts on an algebraic variety X with finitely many orbits. Let $x_1, \ldots x_k$ be representatives of orbits and let Z_i be the stabilizer of x_i in G.

1. Define the notion of a G-equivariant module.
2. Prove that irreducible G-equivariant modules are in one to one correspondence with set of pairs $\{i$, irreducible representation of $Z_i/Z_i^0\}$.

EXERCISE 3.1.2. Let $j : \mathbb{A}^1 \to \mathbb{P}^1$ be the natural embedding. Let $p(x)$ be any non-constant polynomial and let $M(e^p)$ denote the $\mathcal{D}_{\mathbb{A}^1}$-module generated by the function e^p.

a) Prove that $\mathbb{D}(M(e^p)) \simeq M(e^{-p})$.

b) Show that the natural map $j_!(M(e^p)) \to j_*(M(e^p))$ is an isomorphism (hint: prove that $j_*(M(e^p))$ is irreducible and then use (a)).

EXERCISE 3.1.3. Let $j : \mathbb{G}_m \to \mathbb{A}^1$. Show that the direct image of $j_!(\mathcal{O})$ along the map $\mathbb{A}^1 \to \operatorname{Spec} k$ is zero.

3.2. Proof of Theorem 3.1.2

We now turn to the proof of Theorem 3.1.2. It is easy to see that the theorem is true for the functor $\boxtimes$. Also, we already know that it is true for the functor $\mathbb{D}$. It remains to prove it for inverse and direct images.

Let us first prove this Theorem 3.1.2 for $\pi^!$. Any morphism can be decomposed as a product of closed embedding and projection. Let $\pi : X = Y \times Z \to Y$ be projection. For any $\mathcal{F} \in D_{hol}(\mathcal{D}_Y)$ $\quad \pi^!\mathcal{F} = \mathcal{F} \boxtimes \mathcal{O}_Z[\dim Z]$ and hence is holonomic. The case when π is closed embedding, we've already proved.

Now let us consider the case of π_*. Any arbitrary morphism $\pi : X \to Y$ can be decomposed as a product of a closed embedding and projective morphism. We have already proved the theorem for closed embedding. The case of projective morphism follows from the **main step**: the case of an open embedding.

First way of proving this statement for projective morphisms (without using the main step).
Let $\mathcal{F}$ be any holonomic $\mathcal{D}_Y$-module. Then $\pi_*\mathcal{F}$ is coherent and has finite dimensional fibers. This means that for any $i : y \hookrightarrow Y$ and $\tilde{i} : \pi^{-1}(y) \hookrightarrow Y$ $\quad i^!\pi_*\mathcal{F} = \pi_*\tilde{i}^!\mathcal{F}$ is finite dimensional (this is coherent module on a point and hence a vector space). Theorem follows from the following proposition.

PROPOSITION 3.2.1. *[Bernstein] Let $\mathcal{F} \in D_{coh}(\mathcal{D}_X)$. Then $\mathcal{F}$ is holonomic iff it has finite dimensional fibers.*

Second way of proving theorem in case of projective morphism.
We have to prove that for $\pi : Y \times \mathbb{P}^N \to Y$ $\quad \pi_* : D_{hol}(\mathcal{D}_X) \to D_{hol}(\mathcal{D}_Y)$. We have already proved the analogous claim for $\pi : \mathbb{A}^n \to \mathbb{A}^m$. Using the same method one can prove it for $\pi : Y \times \mathbb{A}^N \to Y$ (namely reducing to the case of affine Y and $N = 1$, where $\pi_*M = \{M \overset{\frac{\partial}{\partial x}}{\to} M\}$).

Now let us consider $\pi : Y \times \mathbb{P}^N \to Y$. We'll do induction on N. Let $j : U = Y \times \mathbb{A}^N \hookrightarrow X = Y \times \mathbb{P}^N$. For any sheaf $\mathcal{F}$ on X, there is the following exact triangle $\mathcal{F} \to j_*\mathcal{F}|_U \to K$. Here K is supported on $Y \times \mathbb{P}^{N-1}$.

General remark: Let $\mathcal{F} \to \mathcal{G} \to \mathcal{H} \to \mathcal{F}[1]$ be an exact triangle. Then if two of the terms are holonomic, so is the third one. To see this one should write the corresponding long exact sequence of cohomologies.

In our case $\mathcal{F}$ is holonomic by assumption and $j_*\mathcal{F}|_U$ is holonomic by main step (which we have not proved yet). Hence K is also holonomic. Applying π_* to that triangle we get

$$\pi_*\mathcal{F} \to \pi_*j_*\mathcal{F}|_U \to \pi_*K.$$

Let us denote $\tilde{\pi} : Y \times \mathbb{A}^N \to Y$. Then $\pi_*j_* = \tilde{\pi}_*$ and hence $\pi_*j_*\mathcal{F}|_U$ is holonomic by the previous step. By induction hypothesis π_*K is also holonomic and hence so is $\pi_*\mathcal{F}$.

Proof of the main step:

Let $j : U \hookrightarrow X$ be open embedding. Since holonomicity is a local property, we can assume that X is affine. We can also assume that U is affine and moreover U is a zero locus of $p \in \mathcal{O}_X$. (If not, we can cover U with finitely many affine open subsets $\{U_\alpha\}$ and consider the Check complex $\check{C}(M)$ with respect to this cover. This is a complex $\mathcal{D}_X$-modules $j_{\alpha_1,\ldots,\alpha_k *}(M|_{U_{\alpha_1}\cap\cdots\cap U_{\alpha_k}})$ where

$j_{\alpha_1,\dots,\alpha_k} : U_{\alpha_1} \cap \cdots \cap U_{\alpha_k} \hookrightarrow X$ and $\check{C}(M) = j_*(M) \in D(\mathcal{D}_X)$. Hence j_*M is holonomic when $j_{\alpha_1,\dots,\alpha_k *}(M\mid_{U_{\alpha_1}\cap\cdots\cap U_{\alpha_k}})$ are holonomic.)

Example. Let $X = \mathbb{A}^n$, $\quad p \in \mathcal{O}_X$ and $\lambda \in \Bbbk$. Consider $\mathcal{D}_U$-module $M(p^\lambda)$ (here U is the zero locus of p). At the begining of the course we've proved that j_*M is holonomic.

One way of proving the main step is to reduce it to the case $X = \mathbb{A}^n$, embedding X into an affine space. Consider U – zero locus of p and generalize the proof for $M(p^\lambda)$ to any module M.

We'll give another proof of this fact. Here is the sketch of it.

Step 1. We can assume that M is generated by some element $u \in M$. Let $N = j_*M$. Consider a submodule $N_k \subset N$ generated by up^k. Then for $k \gg 0 \quad N_k$ is holonomic and $N_k = N_{k+1}$.

Step 2. Lemma on b-functions.
Let ${}^\lambda X = X \otimes \Bbbk(\lambda)$ be the same variety as X considered over a field $\Bbbk(\lambda)$. Let ${}^\lambda M = M \otimes p^\lambda$. Then there exist $d \in \mathcal{D}_{{}^\lambda X}$ such that $d(up^\lambda) = up^{\lambda-1}$. (This is equivalent to say, that there exist $d \in \mathcal{D}_X[\lambda]$ and $b(\lambda) \in \Bbbk[\lambda]$ such that $d(up^\lambda) = b(\lambda)up^{\lambda-1}$.)

If lemma on b-functions holds, then ${}^\lambda M$ is generated by $up^{\lambda+k}$. And for $k \gg 0$ it is holonomic by step 1.

Step 3. Let $\widetilde{M}$ be $\mathcal{O}$-$\mathcal{D}$-module on $\mathbb{A}^1 \times X$ (i.e. an $\mathcal{O}$-module on $\mathbb{A}^1$ and $\mathcal{D}$-module on X), defined as follows:

$$\widetilde{M} = span\{q(\lambda)mp^{\lambda+i} \mid i \in \mathbb{Z}, m \in M, q(\lambda) \in \Bbbk[\lambda]\}.$$

For any $\alpha \in \mathbb{A}^1$ let $M_\alpha = \widetilde{M}/(\lambda - \alpha)\widetilde{M}$. We claim that for generic α this module is holonomic. Indeed, let Z be singular support in $\mathcal{D}_X$-direction of $\widetilde{M}$ in $T^*X \times \mathbb{A}^1$. After passing to a generic point of $\mathbb{A}^1$ we'll get the singular support of ${}^\lambda M$, which is holonomic by step 2. This means that $\dim(Z \otimes \Bbbk(\lambda)) = \dim X$.

So, we have a map $Z \to \mathbb{A}^1$ with fibers $Z_\alpha = s.s.M_\alpha$, $M_0 = M$ (and in fact $M_i = M$ for any $i \in \mathbb{Z}$) . The general fiber of this map has dimension $\dim X$. Hence there exist an integer i, such that $\dim Z_i = \dim X$ and this means that M is holonomic.

Now let us deduce step 1 from the following lemma

LEMMA 3.2.2. *Consider an inclusion $I : \mathcal{M}_{hol}(\mathcal{D}_X) \to \mathcal{M}_{coh}(\mathcal{D}_X)$. It has a right adjoint functor $G : \mathcal{M}_{coh}(\mathcal{D}_X) \to \mathcal{M}_{hol}(\mathcal{D}_X)$ defined by $G(N) = \{$ the maximal holonomic submodule of $N\}$. This functor commutes with restriction to open subsets.*

Let $N = j_*M$. By the lemma $G(N)\mid_U = M$. For $k \gg 0$ we have $up^k \in G(N)$ and hence $N_k \subset G(N)$. In particular N_k is holonomic. Since $G(N)$ is holonomic, it has finite length and this means taht $N_k = N_{k+1}$ for k large enough.

Now let us deduce step 2. Consider ${}^\lambda N = j_*({}^\lambda M)$ and apply step 1 to it. We'll get ${}^\lambda N_k = {}^\lambda N_{k+1}$ for $k \gg 0$. And this gives us the lemma on b-functions.

For any $k \quad {}^\lambda N$ is generated by $up^{\lambda+k}$, so ${}^\lambda N = {}^\lambda N_k$ and hence is holonomic.

Proof of Lemma 3.2.2: Let us give an another construction of G. Given N we can constuct $\mathbb{D}(N)$, which is a complex, living in negative degrees. Let $G(N) = \mathbb{D}H^0(\mathbb{D}(N))$. Defined in such a way, G obviously commutes with restriction to an open sets. To prove, that it coincides with G, let's prove that it's right adjoint to I. Let $K \in \mathcal{M}_{hol}(\mathcal{D}_X)$ be holonomic. Then

$$\mathrm{Hom}(K, \mathbb{D}H^0(\mathbb{D}N)) = \mathrm{Hom}(H^0(\mathbb{D}N), \mathbb{D}K) = \mathrm{Hom}(\mathbb{D}N, \mathbb{D}K)$$

since $\mathbb{D}N \in D^{\leq 0}(\mathcal{D}_X)$, and $\mathbb{D}K$ is a module, since K is holonomic. So, $\mathrm{Hom}(K, \mathbb{D}H^0(\mathbb{D}N)) = \mathrm{Hom}(N, K)$ and our functor coincedes with G. $\square$

3.3.

Let $\pi : X \to Y$ be any morphism. By the theorem we proved last time, the functor $\pi^* = \mathbb{D}\pi^!\mathbb{D} : D_{hol}(\mathcal{D}_Y) \to D_{hol}(\mathcal{D}_X)$ is well defined.

Proposition 3.3.1. (1) *π^* is left adjoint to π_*.*
(2) *If π is smooth $\pi^! = \pi^*[2(\dim Y - \dim X)]$.*

Proof. (1) Let's decompose π as a product of an open embedding and projective morphism.

If $\pi : U \hookrightarrow X$ is an open embedding, then $\pi^* = \pi^!$ and is simply the restriction to U. We've already discussed that in this case π^* is left adjoint to π_*.

If π is projective, then $\pi_! = \pi_*$ and

$$\mathrm{Hom}(\pi^* M, N) = \mathrm{Hom}(\mathbb{D}\pi^!\mathbb{D}M, N) = \mathrm{Hom}(\mathbb{D}N, \pi^!\mathbb{D}M) =$$

$$= \mathrm{Hom}(\pi_!\mathbb{D}N, \mathbb{D}M) = \mathrm{Hom}(\pi_*\mathbb{D}N, \mathbb{D}M) = \mathrm{Hom}(M, \pi_* N).$$

(2) Let $\pi : Z \times Y \to Y$, where Z is smooth. Then $\pi^! M = \mathcal{O}_Z \boxtimes M[\dim Z]$ and $\pi^* M = \mathbb{D}\pi^!\mathbb{D}M = \mathbb{D}(\mathcal{O}_Z \boxtimes \mathbb{D}M[\dim Z]) = \mathcal{O}_Z \boxtimes M[-\dim Z] = \pi^! M[-2\dim Z]$.

Any smooth morphism $\pi : X \to Y$ is formally a projection, i.e. for any $x \in X$ and $y = \pi(x)$ there exist a formal neighborhood of x $\hat{X}_x = \hat{Y}_y \times Z$ (here $\hat{Y}_y$ is a formal neighborhood of y and Z is smooth) such that $\pi |_{\hat{X}_x} : \hat{Y}_y \times Z \to \hat{Y}_y$. This proves proposition. □

3.3.1. Elementary comlexes.

Let Z be a smooth variety and $j : Z \hookrightarrow X$ be locally closed embedding.

Definition. If M is an $\mathcal{O}$-coherent $\mathcal{D}_Z$-module, then $j_* M$ is called an elementary complex.

Let $\mathcal{A}$ be an abelian category.

Definition. An object $K \in D(\mathcal{A})$ is glued from L_1 and L_2, if there exist an exact triangle $L_1 \to L_2 \to K \to L_1[1]$.

Definition. Let $\mathcal{L}$ be a class of objects of $D(\mathcal{A})$ invariant under shift functor. Then $\mathcal{K}$ is called class of objects glued from $\mathcal{L}$, if it satisfies the following properties

(1) $\mathcal{L} \subset \mathcal{K}$
(2) If $L_1, L_2 \in \mathcal{K}$, then K, glued from L_1 and L_2 is also in $\mathcal{K}$.

Lemma 3.3.2. *Every object of $D_{hol}(\mathcal{D}_X)$ is glued from elementary objects.*

Proof. Let us assume, that $M \in D_{hol}(\mathcal{D}_X)$ is a module.

We'll prove this lemma by induction on $\dim(suppM)$. Any module on a point is by definition an elementary complex. By induction hypothesis we know this lemma for any M, such that $suppM = Z \subset X$ is a proper subvariety. Suppose $suppM = X$. Then there exist an open subset $U \subset X$, such that $M |_U$ is $\mathcal{O}$-coherent. Consider the following exact sequence

$$M \to j_*(M |_U) \to K,$$

where $j : U \hookrightarrow X$. Here K is supported on $Z = X \backslash U$ and hence by induction hypothesis is glued from elementary complexes. Since $j_*(M |_U)$ is an elementary complex by definition, M is also glued from elementary complexes. □

Example. (from problem set)
Let $X = \mathbb{A}^1$ and $U = \mathbb{A}^1 \backslash 0 \subset U$. For any $n \in \mathbb{Z}$ let us introduce $\mathcal{D}_U$-module $\mathcal{E}_n = M(x^\lambda)/\lambda^n$.

Topological interpretation. For every n $\mathcal{E}_n$ has regular singularities and hence corresponds to some representation of $\pi_1(U) = \mathbb{Z}$. Namely, $\mathcal{E}_n$ corresponds to the representation $\mathbb{Z} \ni 1 \mapsto$ Jordan block of size n.

Consider $\mathcal{D}_U$-module $\mathcal{E}_2$. Let us describe $j_{!*}\mathcal{E}_2$ as a module, glued from elementary complexes. (Here $j : U \hookrightarrow X$.) For $\mathcal{E}_2$ we have the following exact sequence $0 \to \mathcal{O}_U \to \mathcal{E}_2 \to \mathcal{O}_U \to 0$. This gives us two other exact sequences

$$0 \to j_*\mathcal{O}_U \to j_*\mathcal{E}_2 \to j_*\mathcal{O}_U \to 0$$

and

$$0 \to j_!\mathcal{O}_U \to j_!\mathcal{E}_2 \to j_!\mathcal{O}_U \to 0.$$

Hence the following two sequences are exact

$$0 \to j_*\mathcal{O}_U \to j_{!*}\mathcal{E}_2 \to \mathcal{O}_X \to 0;$$

$$0 \to \mathcal{O}_X \to j_{!*}\mathcal{E}_2 \to j_!\mathcal{O}_U \to 0.$$

So $j_{!*}\mathcal{E}_2$ is glued from $\mathcal{O}_X$ and $j_*\mathcal{O}_U$ or from $j_!\mathcal{O}_U$ and $\mathcal{O}_X$.

Because of the following exact sequences

$$0 \to \mathcal{O}_X \to j_*\mathcal{O}_U \to \delta \to 0,$$

$$0 \to \delta \to j_!\mathcal{O}_U \to \mathcal{O}_X \to 0,$$

we can say, that $j_{!*}\mathcal{E}_2$ has form $\mathcal{O}_X)\delta)\mathcal{O}_X$ (i.e. $j_{!*}\mathcal{E}_2$ is glued from δ and two copies of $\mathcal{O}_X$ in the indicated order).

More generally, $j_{!*}\mathcal{E}_n$ has a form $\mathcal{O}_X)\delta)\mathcal{O}_X)\dots)\delta)\mathcal{O}_X$.

3.3.2. A little bit more about intersection cohomology module. In the examples below we'll use the following notations

$$H^i(M) = H^i(\pi_*M); \quad H^i_c(M) = H^i(\pi_!M).$$

Examples (Poincare duality). Let X be any variety, $M \in D_{hol}(\mathcal{D}_X)$ and $\pi : X \to pt$.

(1) If X is smooth then $\pi_*M = H^*_{dR}(M)$. On the other hand

$$\pi_*M = \mathbb{D}\pi_!\mathbb{D}M = (\pi_!\mathbb{D}M)\check{}.$$

So, $H^i(\pi_*M) = H^{-i}(\pi_!\mathbb{D}M)\check{}$.

(2) Suppose X is smooth and proper and $M = \mathcal{O}_X$. Then $\mathbb{D}\mathcal{O}_X = \mathcal{O}_X$ and $\pi_* = \pi_!$. Hence $H^i(\mathcal{O}_X)\check{} = H^{-i}(\mathcal{O}_X)$.

(3) Let X be projective, but not smooth. Let $j : U \hookrightarrow X$ be a smooth open subset of X. We've defined the intersection cohomology module $IC_X = j_{!*}\mathcal{O}_U = \mathrm{Image}(H^0 j_!\mathcal{O}_U \to H^0 j_*\mathcal{O}_U)$.

For any module M on U the intermediate extension $j_{!*}M$ has the following properties:

(1) $j_{!*}M \mid_U = M$,
(2) $j_{!*}M$ has no submodules or quotients, supported on $X \backslash U$,
(3) $\mathbb{D}(j_{!*}M) = j_{!*}(\mathbb{D}M)$. In our case $\mathbb{D}(IC_X) = IC_X$, since $\mathbb{D}\mathcal{O}_U = \mathcal{O}_U$.
So, $H^i(IC_X)\check{} = H^{-i}(IC_X)$.
(4) If X is any variety, then $H^i(IC_X) = H_c^{-i}(IC_X)\check{}$.

3.3.3. Examples.

(1) Let $X \subset \mathbb{A}^2$ be given by equation $xy = 0$. Then $X = X_1 \sqcup X_2$, where $X_1 = \{x = 0\}$ and $X_2 = \{y = 0\}$. It's easy to see, that $IC_X = i_{1_*}\mathcal{O}_{X_1} \oplus i_{2_*}\mathcal{O}_{X_2}$ (conditions (a) and (b) are satisfied, and those two determine the intermediate extension uniquely). Here $i_1 : X_1 \to X$ and $i_2 : X_2 \to X$. For cohomology we have

$$\dim H^i(IC_X) = \begin{cases} 2 & \text{if } i = -1; \\ 0 & \text{otherwise.} \end{cases}$$

(2) Let X be a smooth variety and Γ – a finite group, acting on X freely at the general point. Let $Y = X/\Gamma$ and $\pi : X \to Y$ be a natural morphism. Then Γ acts on $\pi_*\mathcal{O}_X$ as follows: let $\Gamma \ni \gamma : X \to X$. By definition $\pi\gamma = \pi$ and $\gamma_*\mathcal{O}_X \cong \mathcal{O}_X$. Consider the following composition

$$\pi_*\mathcal{O}_X \to (\pi \circ \gamma)_*\mathcal{O}_X = \pi_*(\gamma_*\mathcal{O}_X) \cong \pi_*\mathcal{O}_X$$

It gives us a map $\pi_*\mathcal{O}_X \xrightarrow{\sim} \pi_*\mathcal{O}_X$, which depends on γ.

LEMMA 3.3.3. *In the situation above, let $j : V \hookrightarrow Y$, where V is smooth. Then*

$$j_{!*}((\pi_*\mathcal{O}_X) \mid_V) = \pi_*\mathcal{O}_X.$$

PROOF. Let $\tilde{j} : U \hookrightarrow X$, where $U = \pi^{-1}V$. Since Γ acts freely at the general point of X, π is proper. Hence

$$j_!((\pi_*\mathcal{O}_X) \mid_V) = \pi_*(\tilde{j}_!\mathcal{O} \mid_U) \text{ and } j_*((\pi_*\mathcal{O}_X) \mid_V) = \pi_*(\tilde{j}_*\mathcal{O} \mid_U).$$

So

$$\mathrm{Im}(j_!(\pi_*\mathcal{O}_X \mid_V) \to j_*(\pi_*\mathcal{O}_X \mid_V)) = \pi_*(\mathrm{Im}(\tilde{j}_!\mathcal{O}_U \to \tilde{j}_*\mathcal{O}_U)) = \pi_*\mathcal{O}_X,$$

since X is smooth. And this means that $j_{!*}((\pi_*\mathcal{O}_X) \mid_V) = \pi_*\mathcal{O}_X$. □

COROLLARY 3.3.4. $IC_Y = (\pi_*\mathcal{O}_X)^\Gamma$.

COROLLARY 3.3.5. $H^i(IC_Y) = H^{n+i}(Y^{an}, \mathbb{C}) = H^{n+i}(X^{an}, \mathbb{C})^\Gamma$, *where* $n = \dim X$.

3.3.4. Example. Let Y be a quadratic cone in $\mathbb{C}^3$. Then $H^*(IC_Y) = H^*(Y^{an}, \mathbb{C})[2]$. This follows from Corollary 3.3.5 together with the fact that $Y \simeq \mathbb{C}^2/ZZ_2$.

3.3.5. Exercises.

EXERCISE 3.3.1. Let M denote the $\mathcal{D}$-module on $\mathbb{G}_m$ consisting of expressions $p(x, \lambda)x^{\lambda+i}$ where $p \in k[x, x^{-1}, \lambda]$ (note that we do not allow division by λ). Define $\mathcal{E}_n = M/\lambda^n M$ (again considered as a $\mathcal{D}$-module on $\mathbb{G}_m$). Note that multiplication by λ induces a nilpotent endomorphism of $\mathcal{E}_n$.

a) Show that $\mathcal{E}_n$ is $\mathcal{O}$-coherent of rank n and that every irreducible subquotient of $\mathcal{E}_n$ is isomorphic to $\mathcal{O}$ (i.e., that $\mathcal{E}_n$ is a successive extension of n copies on $\mathcal{O}$).

b) Show that $\mathcal{E}_n$ is indecomposable.

c) Show that $\mathcal{E}_n$ is uniquely determined by the conditions a and b (up to an isomorphism).

d) Explain the existence and uniqueness of $\mathcal{E}_n$ "topologically" (using the notion of monodromy).

EXERCISE 3.3.2. In this problem we want to compute $j_{!*}(\mathcal{E}_n)$. Here j is the embedding of $\mathbb{G}_m$ into $\mathbb{A}^1$.

a) Show that there exists an indecomposable $\mathcal{D}_{\mathbb{A}^1}$-module N satisfying the following conditions: there exist short exact sequences

$$0 \to j_*\mathcal{O}_{\mathbb{G}_m} \to N \to \mathcal{O}_{\mathbb{A}^1} \to 0$$

and

$$0 \to \mathcal{O}_{\mathbb{A}^1} \to N \to j_!\mathcal{O}_{\mathbb{G}_m} \to 0.$$

In particular, N has $\mathcal{O}_{\mathbb{A}^1}$ as both submodule and a quotient module and δ_0 as a subquotient (sitting "between" the two $\mathcal{O}$'s). Construct N both explicitly and by computing the corresponding Ext-groups.

b) Prove that δ is neither a submodule, nor a quotient of N.

c) Show that a) and b) imply that $N = j_{!*}(\mathcal{E}_2)$. This example shows that in general when $j : U \to X$ is an open embedding the module $j_{!*}(M)$ may have subquotients concentrated on $X \backslash U$ (we only know that it has neither quotients nor submodules concentrated on the complement).

d) Explain what $j_{!*}(\mathcal{E}_n)$ looks like.

CHAPTER 4

$\mathcal{D}$-modules with regular singularities

4.1. Lectures 14 and 15 (by Pavel Etingof): Regular singularities and the Riemann-Hilbert correspondence for curves

Let X be a C^∞-manifold. Recall that a local system on X consists of the following data:

1) A vector space V_x for every point $x \in X$

2) An isomorphism $\alpha_\gamma : V_{x_1} \to V_{x_2}$ for every C^∞-path γ starting at x_1 and ending at x_2.

This data should depend only on the homotopy class of γ and should be compatible with composition of paths.

If X is connected, then by choosing a point $x \in X$ we may identify the category of local systems on X with the category of representations of the fundamental group $\pi_1(X, x)$ (we shall often omit the point x in the notations).

Let X be a complex manifold. Then we have equivalance of categories "holomorphic vector bundles on X with connection=representations of $\pi_1(X)$ (=local systems on X)".

This is not true in the algebraic setting. For example let $X = \mathbb{A}^1$ and and consider the connection on the trivial vector bundle on X given by the formula:

$$\nabla(f(x)) = (f'(x) - f(x))dx.$$

Then this connection has a nowhere vanishing flat section given by the function e^x. Hence the corresponding local system is trivial. On the other hand it is clear (for example, bacause e^x is not a polynomial function) that over $\mathbb{C}[x]$ the above connection is not isomorphic to the trivial one. Thus we cannot hope to have an equivalence between the category of algebraic vector bundles with connection with the category of local systems.

It turns out that we can single out some nice subcategory of $\mathcal{O}$-coherent D-modules on smooth algebraic variety X (called D-modules with regular singularities) for which the above equivalence is still valid. Today we are going to do it for curves.

4.1.1. Regular connection on a disc.

First of all let us develop some analytic theory and then we'll apply it to the algebraic setting. Let D denote the complex disc $\{x \in \mathbb{C} | |x| < r\}$ and let D^* denote the punctured disc. Let $\mathcal{O}_D$ denote the algebra of holomorphic functions on D and let $\mathcal{O}_D[x^{-1}]$ be the algebra of meromorphic functions. Also we denote by $\Omega_D[x^{-1}]$ the space of meromorphic one-forms on D which are holomorphic outside 0. Let also Ω_{log} denote the space of 1-forms with pole of order at most 1 at 0.

By a meromorphic connection on D we mean a vector bundle M with a connection $\nabla : M \to M \otimes \Omega_D[x^{-1}]$.

By a morphism of meromorphic connections we mean a *meromorphic* map $\alpha : M_1 \to M_2$ which is holomorphic outside 0 and which is compatible with the connections. Thus meromorphic connections form a category (note that there is no functor from this category to the category of vector bundles on D).

If we choose a meromorphic trivialization of M then ∇ is given by a matrix A of meromorphic one forms. A is defined uniquely up to gauge trasformations $A \mapsto gAg^{-1} + g^{-1}dg$ where g is a holomorphic function on D^* taking values in the vactor space $Mat(n)$ of $n \times n$-matrices which is meromorphic at 0.

DEFINITION 4.1.1. We say that ∇ has regular singularities if there exists a trivialization a as above such that all the entries of A have a pole of order at most one 1. More invariantly (M, ∇) is regular if it is isomorphic to some (M', ∇') where $\nabla' : M' \to M' \otimes \Omega_{log}$.

In other words, M' above stable under the algebra generated by $\mathcal{O}_D$ and the vector field $x\frac{d}{dx}$. In a coordinate-free way it means that $\nabla(M') \subset M' \otimes \Omega_{log}$.

Examples. Let us give some examples of regular and non-regular connections. First of all any connection with connection matrix having poles of order ≤ 1 is regular. We claim that the converse is true if $\text{rank}(M) = 1$. Indeed, in this case the connection matrix A is just a differential one-form which is meromorphic at 0. The RS-condition says that there exists a meromorphic function g such that $A + g^{-1}dg$ has pole of order ≤ 1 at 0. But $g^{-1}dg$ also has a pole of order ≤ 1 at 0. Hence the same is true for A.

Here is an example of a connection matrix of rank 2 which has poles of order > 1 but still defines a maromorphic connection with regular singularities. Namely let β an arbitrary meromorphic function on D and consider the connection ∇_β whose connection matrix is

$$A_\beta = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$$

We claim that for every *beta* the connection ∇_β has regular singularities at 0. Indeed, it is easy to see that there exists a meromorphic function u on D such that $-u' + \beta$ has pole of order ≤ 1. Define

$$g = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}.$$

Then it is easy to see that the matrix $gA_\beta g^{-1} + g^{-1}dg$ has poles of order ≤ 1 at 0. Hence ∇_β has regular singularities.

Here are some first properties of RS.

Given two meromorphic connections (M_1, ∇_1) and (M_2, ∇_2) w may consider their tensor product $\nabla_1 \otimes \nabla_2$ (this is a connection on $M_1 \underset{\mathcal{O}_D 2}{M}$). It is easy to see that if both ∇_1 and ∇_2 have RS then their tensor product has that property too.

Also given a meromorphic connection (M, ∇) we may define the dual meromorphic coneection $\nabla^\vee$ on $M^\vee = \underline{\text{Hom}}(M, \mathcal{O}_D)$. If ∇ has RS then $\nabla^\vee$ has RS too.

Given two meromorphic connections as above we can define their inner Hom by

$$\underline{\text{Hom}}(M_1, M_2) = M_1^\vee \otimes M_2.$$

It follows from the above that if ∇_1 and ∇_2 have RS then the same is true for the corresponding connection on $\underline{\text{Hom}}(M_1, M_2)$.

Here is an analytic characterisation of regular singularities (RS) on D.

DEFINITION 4.1.2. Let f be a vector-valued function defined in some sector

$$\{z = \rho e^{i\theta} |\ 0 < |\rho| < r,\ \alpha < \theta < \beta\}.$$

We say that f has *moderate growth* if there exist some constants C and γ such that

$$||f(\rho e^{i\theta})|| \leq C\rho^{-\gamma}.$$

THEOREM 4.1.3. *A meromorphic connection ∇ regular if and only if for every sector $\alpha < arg(x) < \beta$ the horizontal sections for ∇ on this sector have moderate growth.*

PROOF. Suppose first that ∇ is regular. Thus we are looking for the asymptotics of solutions of the equation

$$\frac{dF}{dz} = A(z)F$$

where A is an $n \times n$-matrix of meromorphic functions having poles of order ≤ 1 at 0 and F is a function of z with values in $\mathbb{C}^n$. Let $\widetilde{A}(z) = zA(z)$. Then $\widetilde{A}$ is regular at 0. So, we have the equation

$$\rho \frac{d}{d\rho} = \widetilde{A}(\rho e^{i\theta})F.$$

LEMMA 4.1.4. *Let $f, B : [0, L] \to Mat_n(\mathbb{C})$ be two C^1-functions such that*

$$f'(t) = B(t)f(t).$$

Then we have

$$||f(L)|| \leq ||f(0)||e^{L \max ||B||}. \tag{4.1}$$

PROOF. We have

$$f(L) = \lim_{n\to\infty} \prod_{j=0}^{n-1} e^{\frac{L}{n}B(\frac{j}{n})} f(0).$$

Let us set $M = \max ||B||$. Then for $\varepsilon > 0$ and n sufficiently large we have

$$||f(L)|| \leq \limsup_{n\to\infty} \prod_{j=0}^{n-1} ||e^{\frac{L}{n}B(\frac{j}{n})}|| \cdot ||f(0)|| \leq \prod_{j=0}^{n-1}(1 + \frac{L}{n}(1+\varepsilon)M)||f(0)|| \leq e^{L(1+\varepsilon)M}||f(0)||.$$

As $n \to \infty$ we see that the above inequality becomes true for all $\varepsilon > 0$. □

□

THEOREM 4.1.5. *Restriction to D^* is an equivalence of categories "meromorphic connections with regular singularities= connections on D^*".*

PROOF. The restriction functor is clearly exact and faithful. Also we claim that it is surjective on objects. Indeed, since $\pi_1(D^*) = \mathbb{Z}$ it follows that every connection on D^* of rank n is isomorphic to a connection on the trivial rank n vector bundle on D^* with connection form equal to $A\frac{dz}{z}$ where A is some constant $n \times n$-matrix. Extending this bundle in the trivial way to D we get a connection with regular singularities.

Let us denote the restriction functor by R. To prove Theorem 4.1.5 it is enough now to show that for two meromorphic connections (M_1, ∇_1) and (M_2, ∇_2) with regular singularities we have

$$\mathrm{Hom}(M_1, M_2) = \mathrm{Hom}(R(M_1), R(M_2))$$

(here in the left hand side we look at the morphisms in the category of meromorphic connections on D and in the right hand side we deal with morphisms in the category of connections (local systems) on D^*).

Let $\phi \in \mathrm{Hom}(R(M_1), R(M_2))$. We can regard ϕ as a flat section of $\underline{\mathrm{Hom}}(M_1, M_2)$ on D^*. By Theorem 4.1.3 ϕ has moderate growth. Also ϕ is holomorphic on D^*. Such a section is automatically holomorphic on D. Hence ϕ comes from an element of $\mathrm{Hom}(M_1, M_2)$. □

4.1.2. Regular connections on an arbitrary curve. Let X be a smooth projective curve, $j: Y \hookrightarrow X$ an open subset, S – the complement of Y (finite set). Let $\mathcal{D}_X^S$ denote the subsheaf of algebras of the sheaf $\mathcal{D}_X$ generated (locally) by $\mathcal{O}_X$ and vector field which vanish on S. We say that an $\mathcal{O}_Y$-coherent $\mathcal{D}_Y$-module N has regular singularities if there exists an $\mathcal{O}_X$-coherent (=vector bundle on X) submodule M of $j_*(N)$ which is stable under $\mathcal{D}_X^S$.

It is clear that the category of $\mathcal{O}_Y$-coherent RS $\mathcal{D}_Y$-modules is closed under subquotients.

This definition of regular singularities is connected with what we studied before in the following way.

Let M be as above. Let $s \in S$ and let D be a small disc around s (not containing any other point from S). Then the restriction of M to D acquires a meromorphic connection with regular singularities.

THEOREM 4.1.6. *The natural functor "$\mathcal{O}$-coherent $\mathcal{D}$-modules on Y with regular singularities" $\to$ "connection on Y^{an} (i.e. Y considered as a complex analytic manifold)" is an equivalence of categories. In particular, we have an equivalence "$\mathcal{O}$-coherent $\mathcal{D}$-modules on Y with regular singularities" $\simeq$ "representations of $\pi_1(Y)$.*

PROOF. Let us denote the functor in question by $N \mapsto N^{\mathrm{an}}$ (we shall call the analytification functor). This functor is clearly exact and faithful. Hence to show that it is an equivalence of categories we must show the following two statements:

1. The analytification functor is surjective on objects.
2. For every two $\mathcal{O}_Y$-cohere $\mathcal{D}_Y$-modueles N_1, N_2 we have

$$\mathrm{Hom}(N_1, N_2) = \mathrm{Hom}(N_1^{\mathrm{an}}, N_2^{\mathrm{an}}).$$

Let us prove 1. Let $(N^{\mathrm{an}}, \nabla^{\mathrm{an}})$ be a holomorphic vector bundle with a connection on Y. We must show that there exists an algebraic vector bundle M on X with a connection ∇ with poles of first order along S such that $(M, \nabla)|_Y^{\mathrm{an}} = (N^{\mathrm{an}}, \nabla^{\mathrm{an}})$. By Theorem 4.1.5 such an M exists locally in the analytic topology. Thus globally we get a holomorphic vector bundle M^{an} on X^{an} with a meromorphic connection ∇^{an} having poles of first order along S. By GAGA M^{an} ha unique algebraic structure and thus gives rise to an algebraic vector bundle M on X. Thus ∇^{an} becomes a holomorphic section of some algebraic vector bundle on X and therefore is also algebraic by GAGA. $\square$

THEOREM 4.1.7. *The notion of RS-modules is stable under extensions and subquotients.*

Enough to prove this for the disc – there we have to make an explicit calculation. Alternatively, we can use therem Theorem 4.1.5 to prove it.

In general we say that a holonomic $\mathcal{D}$-module on a (not necessarily projective) curve X is RS if at the generic point it is an $\mathcal{O}$-coherent module which is RS.

As an example let us consider $X = \mathbb{C}$. Consider $\mathcal{D}$-modules on $\mathbb{C}$ which have RS and which are $\mathcal{O}$-coherent on $\mathbb{C}^*$. We can completely describe this category. Namely let $\tau : \mathbb{C}/\mathbb{Z} \to \mathbb{C}$ be any lift of the natural projection (we shall assume that $\tau(0) = 0$. We claim that this category is equivalent to the following one:

"pairs of finite-dim. vector spaces E, F with morphisms $u : E \to F$ and $v : F \to E$ such that the eigenvalues of vu lie in the image of τ (it is then automatically true also for uv)". The functor in one direction is described in the following way. If M is a module as above then let M^α (for $\alpha \in \mathbb{C}$) denote the generalized α-eigenspace of $x\frac{d}{dx}$. Then we define

$$F = \oplus_{\alpha \in Im(\tau)} M^{\alpha-1}, \quad E = \oplus_{\alpha \in Im(\tau)} M^{\alpha}$$

We also let v be multiplication by x and let u be $\frac{d}{dx}$.

The functor in the opposite direction is: given (E, F, u, v) define

$$M = \mathbb{C}[x] \otimes E \oplus \mathbb{C}[\frac{d}{dx}] \otimes F$$

Also define the action of x and $\frac{d}{dx}$ by

$$\frac{d}{dx}(1 \otimes e) = 1 \otimes u(e) \quad x(1 \otimes f) = 1 \otimes v(f)$$

(the action of x on $\mathbb{C}[x] \otimes E$ and the action of $\frac{d}{dx}$ on $\mathbb{C}[\frac{d}{dx}] \otimes F$ are assumed to be the natural ones).

It is easy to see that these two functors are mutually inverse.

4.2. Regular singularities in higher dimensions

We begin with the definition.

Definition 4.2.1. a) Let $\mathcal{E}$ be an $\mathcal{O}$-coherent $\mathcal{D}_X$-module. Then $\mathcal{F}$ is called regular singular (or with regular singularities - we shall denote this by RS) if its restriction to any curve is RS.
b) Let $\mathcal{F}$ be an irreducible holonomic $\mathcal{D}_X$-module. Let $U \subset X$ be an open dense subset such that $\mathcal{F}|_U$ is $\mathcal{O}$-coherent. We say that $\mathcal{F}$ is RS if $\mathcal{F}|_U$ is RS.
c) A holonomic $\mathcal{D}_X$-module $\mathcal{F}$ is called RS if all its irreducible subquotients are RS.
d) A holonomic $\mathcal{D}_X$-complex $\mathcal{F}^{\cdot}$ is RS if all its cohomology sheaves are RS.

We denote by $\mathcal{M}_{rs}(\mathcal{D}_X)$ the full subcategory of $\mathcal{M}_{hol}(\mathcal{D}_X)$, consisting of RS-modules, and by $D_{rs}(\mathcal{D}_X)$ the full subcategory of $D(\mathcal{D}_X)$ consisting of regular singular $\mathcal{D}_X$-complexes.

The definition of RS modules may seems a little awkward. We have chosen this definition so that the following lemma will be obvious in our setting.

Lemma 4.2.2. *The category $\mathcal{M}_{rs}(\mathcal{D}_X)$ is closed with respect to subquotients and extensions.*

Now we come to the main result about RS modules.

Theorem 4.2.3. *Let $\pi : X \to Y$ be a morphism of algebraic varieties. Then*
a) The functors $\mathbb{D}$, π_*, $\pi^!$, $\pi_!$ and π^* preserve the category

$$D_{rs}(\mathcal{D}) \subset D_{hol}(\mathcal{D}).$$

b) <u>RS − criterion</u>
A holonomic $\mathcal{D}_X$-complex $\mathcal{F}^\bullet$ is RS if and only its restriction $i_C^!\mathcal{F}^\bullet$ to any curve $C \subset X$ is RS.

Remark. It would be more natural to take b) as a definition of RS complexes. But then it would be difficult to prove "subquotient" properties. So we prefer the definition, which makes these properties trivial, and transfers all the difficulties into the “cohomological part", where we have an appropriate machinery to work with.

The proof of Theorem 4.2.3 contains two technical results both due to P. Deligne. The first describes the RS property of $\mathcal{O}$-coherent $\mathcal{D}$-modules without referring to curves. The second proves that π_* preserves RS in some simple case.

4.2.1. $\mathcal{D}$-modules with regular singularities along a divisor. Let X be a non-singular algebraic variety. By a regular extension of X we shall mean a nonsingular variety X^+, containing X as an open subset, such that $X^\nu = X^+ \setminus X$ is a divisor with normal crossings. We denote by $J \subset \mathcal{O}_{X^+}$ the ideal of X^ν and by T^ν the subsheaf of vector fields preserving J. Also we denote by $\mathcal{D}_X^\nu$ the subalgebra of $\mathcal{D}_{X^+}$, generated by T^ν and $\mathcal{O}_{X^+}$.

Let $\mathcal{E}$ be an $\mathcal{O}$-coherent $\mathcal{D}_X$-module. Set $\mathcal{E}^+ = (i_{X\to X^+})_*\mathcal{E}$.

Proposition 4.2.4. *(P. Deligne). The following conditions are equivalent.*

(i) $\mathcal{E}^+$ is a union of $\mathcal{O}$-coherent D_X^ν submodules
(ii) For any extended curve
$\sigma : (C^+, C) \longrightarrow (X^+, X)$ *(i.e., $\sigma : C^+ \to X^+$, such that $\sigma(C) \subset X$, $\sigma(c) \in X^+ \setminus X$)* $\mathcal{E}^+|_C$ *has RS at c.*
(iii) For each irreducible component W of X^ν there is an extended curve $\sigma : (C^+, c) \longrightarrow (X^+, X)$ which intersects W transversally at c such that $\mathcal{E}^+|_C$ has RS at c.

Corollary 4.2.5. *Suppose X^+ is a complete regular extension of X, $\mathcal{E}$ is an $\mathcal{O}$-coherent D_X-module. Then $\mathcal{E}$ is RS iff $\mathcal{E}^+$ is a union of $\mathcal{O}$-coherent D_X^ν-modules.*

Let us not turn to the proof of of Theorem 4.2.3 assuming these results.

Lemma 4.2.6. *Let $\pi : Y \to X$ be a morphism, where Y is a surface, X is a curve, X, Y are irreducible. Let $\mathcal{H}$ be an $\mathcal{O}$-coherent RS D_Y-module. Then for some open subset*

$$X_0 \subset X \qquad \pi_*(H)|_{X_0} \quad \text{is RS.}$$

We will prove this lemma in 6.

We also will use the following version of Hironaka's desingularisation theorem.

Proposition 4.2.7. *Let $\pi : Y \to X$ be a morphism. Then there exists a regular extension $i : Y \to Y^+$ and a morphism $\pi^+ : Y^+ \to X$ such that π is the restriction of π^+ to Y and π^+ is a proper morphism.*

We will call the triple (π^+, Y^+, i) the resolution of the morphism π.

Now let us start the proof of Theorem 4.2.3. By definition RS is closed with respect to the duality $\mathbb{D}$, and hence D_{RS} is closed with respect to $\mathbb{D}$.

4.2.2. Proof of Theorem 4.2.3 for π_*. We have a morphism $\pi : Y \to X$ and an RS D_Y-complex $\mathcal{H}$. We want to prove that $\pi_*(\mathcal{H})$ is RS. The proof is by induction on the dimension of $S = \mathit{Supp}\ \mathcal{H}$. So we assume that the statement is true for $\dim S < n$. Also we assume that RS-criterion of Theorem 4.2.3 is true for $\dim S < n$.

Step 1. Let $\pi = i : Y \to Y^+$ be an inclusion into a regular extension of Y, $\mathcal{H}$ be an RS $\mathcal{O}$-coherent D_Y-module. Then $i_*(\mathcal{H})$ is an RS D_{Y^+}-module.

Since i is an affine morphism $i_*(H) = i_+(H)$. Without loss of generality we can assume Y^+ to be complete. By Deligne's proposition $i_+(H)$ is a union of $\mathcal{O}$-coherent D_Y-modules. Hence arbitrary irreducible subquotient F of $i_+(H)$ has this property.

Let $Z^+ = \operatorname{Supp} F$. Then it is easy to check that Z^+ is an irreducible component of an intersection of some components of the divisor X^ν and $F = L(Z, E)$, where Z is an open subset of Z^+. It is clear that $E^+ = i_{Z \to Z^+}(E)$ is a union of $\mathcal{O}$-coherent D_Z^ν-modules, since D_Z^ν is a quotient of the algebra D_Y^ν and E^+ is a subquotient of H^+. Hence E is RS, i.e., F is RS.

4.2.3. Sketch of the proof of the key lemma. We have a smooth morphism $\pi : Y \to X$ with dim $Y = 2$, dim $X = 1$. Then, after deleting several points from X, we can find a regular complete extension Y^+ of Y and a morphism $\pi^+ : Y^+ \to X^+$, where X^+ is the regular completion of X, such that

(i): $\pi^{-1}(X^\nu) \subset Y^\nu$, where $X^\nu = X^+ \setminus X$, $Y^\nu = Y^+ \setminus Y$
(ii): $\pi^{-1}(X^\nu)$ contains all singularities of Y^ν.

Denote by T_Y^ν and T_X^ν sheaves of vector fields on Y^+ and X^+, which preserve Y^ν and X^ν. Conditions (i), (ii) imply that each local vector field $\xi \in T_X^\nu$ can be lifted locally to a vector field

$\xi' \in T_Y^\nu$. This means that the natural morphism of sheaves on Y^+

$$\alpha : T_Y^\nu \longrightarrow (\pi^+)^* T_X^\nu = O_{Y^+} \bigotimes_{\pi^+ \cdot O_{X^+}} \pi^+ \cdot (T_X^\nu)$$

is epimorphic.

We denote by $T_{Y/X}^\nu$ the kernel of α. Consider sheaves of algebras D_Y^ν and D_X^ν on Y^+ and X^+, generated by T_Y^ν and by T_X^ν and denote by $M^R(D_Y^\nu)$, $M^R(D_X^\nu)$ corresponding categories of right D^ν-modules, and by $D^R(D_Y^\nu), D^R(D_X^\nu)$ derived categories (here I prefer to work with right D-modules as all formulae are simple).

Let us put $D_{Y\to X}^\nu = \mathcal{O}_{Y^+}\, bigotimes_{\pi^+ \cdot O_{X^+}} \pi^+ \cdot (D_X^\nu)$. This module is $D_Y^\nu - \pi^+ \cdot (D_X^\nu)$-bimodule. Using $D_{Y\to X}$ let us define the functor

$$\pi_*^\nu : D^R(D_Y^\nu) \longrightarrow D^R(D_X^\nu) \qquad \text{by}$$

$$\pi_*^\nu(E) = R(\pi^+)_\bullet (E \overset{L}{\bigotimes_{D_Y^\nu}} D_{Y\to X}^\nu).$$

Statement (i) Let $\mathcal{H}$ be a right D_Y-module, $H^+ = (i_Y)_+ H \in M^R(D_{Y^+})$. Then, if we consider $\mathcal{H}^+$ as D_Y^ν-module, we have

$$\pi_*^\nu(H^+) = \pi_*(H^+) \qquad \text{as} \quad D_X^\nu\text{-module.}$$

(ii) if E is an $\mathcal{O}$-coherent D_Y^ν-module, then

$$\pi_*^\nu(E) \quad \text{is} \quad O\text{-coherent} \quad D_X^\nu\text{-module.}$$

This statement implies the key lemma. Indeed, if H is an RS $\mathcal{O}$-coherent (right) D_Y-module, then H^+ is an inductive limit of $\mathcal{O}_{Y^+}$-coherent D_Y^ν-modules and hence $\pi_*(\mathcal{H}^+) = \pi_*^\nu(\mathcal{H}^+)$ is an inductive limit of $\mathcal{O}_{X^+}$-coherent D_Y^ν-modules, i.e., it is RS.

Proof of statement (i) is an immediate consequence of the projection formula and the fact that $D_Y^\nu|_Y = D_Y$, $D_{Y\to X}^\nu|_Y = D_{Y\to X}$.

(ii) Consider "De Rham" resolution of $D_{Y\to X}$

$$0 \longrightarrow D_Y^\nu \bigotimes_{\mathcal{O}_Y} T_{Y/X}^\nu \longrightarrow D_Y^\nu \longrightarrow D_{Y\to X}^\nu \longrightarrow 0.$$

Using it we see that as $\mathcal{O}_{X^+}$-module

$$\pi_*^\nu(E) = R(\pi^+)_\bullet (E \otimes T_{Y/X}^\nu \longrightarrow E).$$

Since π^+ is a proper morphism, $R\pi^+$ maps coherent $\mathcal{O}_{Y^+}$-modules into coherent $\mathcal{O}_{X^+}$-modules, i.e., $\pi_*^\nu(E)$ is $\mathcal{O}$-coherent for $\mathcal{O}$-coherent $\mathcal{E}$.

4.2.4. The following statement, due to P. Deligne, is a very useful criterion of RS.

Criterion Let X^+ be an irreducible complete normal (maybe singular) variety, $X \subset X^+$ an open nonsingular subset, E an $\mathcal{O}$-coherent D_X-module. Assume that for any component W of $X^\nu = X^+ \setminus X$ of codimension 1 in X^+, S is RS along W (i.e., E satisfies conditions (i), (ii), (iii) in 4 along W). Then E is RS.

Unfortunately, the only proof of this criterion I know is analytic. I would like to have an algebraic proof.

4.2.5. RS-modules with given exponents. Let us fix some $\mathbb{Q}$-linear subspace $\Lambda \subset kK$, containing 1. Let C be a curve, C^+ its regular extension $c \in C^+ \setminus C$, F an RS $\mathcal{O}$-coherent D_C-module, $F^+ = (i_C)_+ F$. For any $\mathcal{O}$-coherent D^ν submodule $E \subset F^+$ we denote by $\Lambda_c(E)$ the set of eigenvalues of the operator $d = t\partial$ in the finite-dimensional space E/tE (t is a local parameter at c, see 1). Now we define

$$\Lambda(F) = \bigcup_{c,E} \Lambda(E) \quad \text{for all } O\text{-coherent}$$

D^ν-submodules of F^+ and all points $c \in C^+ \setminus C$.

The set $\Lambda(F)$ is called the set of exponents of F. We say that F is $RS\Lambda$ if $\Lambda(F) \subset A$. We say that D_X-complex $\dot{F}$ is $RS\Lambda$ if for any curve $C \subset X$ all cohomology sheaves of $i_C^!(\dot{F})$ are $RS\Lambda$.

It is not difficult to prove that all functors $D, ¶_*, ¶^!, ¶_!, ¶^*$ preserve $D_{RS\Lambda(D_X)}$ – one should repeat proofs in 1-5 with minor modifications. Apparently criterion 6 is also true for $RS\Lambda$ (for $\Lambda = \mathbb{Q}$ it is proved by Kashiwara). I would like to have an algebraic proof of it.

CHAPTER 5

The Riemann-Hilbert correspondence and perverse sheaves

5.1. Riemann-Hilbert correspondence

5.1.1. Constructible sheaves and complexes. Let X be a complex algebraic variety. We denote by X^{an} the correspondent analytic variety, considered in classical topology.

Let $\mathbb{C}_X$ be the constant sheaf of complex numbers on X^{an}. We denote by $\mathrm{Sh}(X^{\mathrm{an}})$ the category of sheaves of $\mathbb{C}_X$-modules, i.e., the category of sheaves of $\mathbb{C}$-vector spaces. Derived category of bounded complexes of sheaves will be denoted by $D(X^{an})$. We shall call sheaves $\mathcal{F} \in Sh(X^{an})$ $\mathbb{C}_X$-*modules* and complexes $\mathcal{F}^{\cdot} \in D(X^{an})$ $\mathbb{C}_X$-*complexes*. We shall usually omit the $^{\cdot}$ superscript when it does not lead to a confusion.

We shall call a $\mathbb{C}_X$-module $\mathcal{F}$ *constructible* if there exists a stratification $X = \cup X_i$ of X by locally closed *algebraic* subvarieties X_i, such that $F|_{X_i^{\mathrm{an}}}$ is a locally constant (in classical topology) sheaf of finite-dimensional vector spaces. We shall call a $\mathbb{C}_X$-complex $\mathcal{F}^{\cdot}$ constructible if all its cohomology sheaves are constructible $\mathbb{C}_X$-modules. The full subcategory of $D(X^{\mathrm{an}})$ consisting of constructible complexes will be denoted by $D_{\mathrm{con}}(X^{\mathrm{an}})$.

Any morphism $\pi : Y \to X$ of algebraic varieties induces the continuous map $\pi^{\mathrm{an}} : Y^{\mathrm{an}} \to X^{\mathrm{an}}$ and we can consider functors

$$\begin{aligned} \pi_!, \pi_* : D(Y^{\mathrm{an}}) &\longrightarrow D(X^{\mathrm{an}}) \\ \pi^*, \pi^! : D(X^{\mathrm{an}}) &\longrightarrow D(Y^{\mathrm{an}}). \end{aligned}$$

Also we shall consider the Verdier duality functor

$$\mathbb{D} : D(X^{\mathrm{an}}) \longrightarrow D(X^{\mathrm{an}}).$$

THEOREM 5.1.1. *The functors $\pi_*, \pi_!, \pi^*, \pi^!$ and $\mathbb{D}$ preserve subcategories $D_{con}(\quad)$. On this categories $\mathbb{D} \circ \mathbb{D} \simeq Id$ and*

$$\mathbb{D}\pi^* D = \pi^!, \qquad \mathbb{D}\pi_*\mathbb{D} = \pi_!.$$

5.1.2. De Rham functor. Denote by $\mathcal{O}_X^{\mathrm{an}}$ the structure sheaf of the analytic variety X^{an}. We will assign to each $\mathcal{O}_X$-module $\mathcal{F}$ corresponding "analytic" sheaf of $\mathcal{O}_X^{an}$-modules M^{an}, which is locally given by

$$M^{an} = \mathcal{O}_{X^{\mathrm{an}}} \underset{\mathcal{O}_X}{\otimes} M.$$

This defines an exact functor

$$\mathrm{an} : \mathcal{M}(\mathcal{O}_X) \longrightarrow \mathcal{M}(\mathcal{O}_{X^{an}}).$$

In particular, the sheaf $\mathcal{D}_X^{an}$ is the sheaf of analytic (holomorphic) differential operators on X^{an} and we have an exact functor

$$\mathrm{an} : \mathcal{M}(\mathcal{D}_X) \longrightarrow \mathcal{M}(\mathcal{D}_X^{\mathrm{an}}).$$

Since this functor is exact it induces a functor

$$\mathrm{an} : D(\mathcal{D}_X) \longrightarrow D(\mathcal{D}_X^{\mathrm{an}}).$$

DEFINITION 5.1.2. Define the De Rham functor $\mathrm{DR} : D(\mathcal{D}_X) \to D(X^{\mathrm{an}}) = D(\mathsf{Sh}(X^{\mathrm{an}}))$ by

$$\mathrm{DR}(M^\cdot) = \Omega_X^{\mathrm{an}} \underset{\mathcal{D}_X^{\mathrm{an}}}{\otimes} (M^\cdot)^{\mathrm{an}}.$$

Remarks. 1. We know that the complex $dR(\mathcal{D}_X)$ is a locally projective resolution of the right $\mathcal{D}_X$-module Ω_X. Hence

$$DR(M^\cdot) = dR(\mathcal{D}_X^{\mathrm{an}}) \underset{\mathcal{D}_X^{\mathrm{an}}}{\otimes} (\mathcal{F}^\cdot)^{\mathrm{an}}[n] = dR((M^\cdot)^{an})[n],$$

where $n = \dim X$.

In particular, if M is an $\mathcal{O}$-coherent $\mathcal{D}_X$-module corresponding to some vector bundle with a flat connection and $\mathcal{L} = M^{\mathrm{flat}}$ is the local system of flat sections of $\mathcal{F}$ (in classical topology), then by Poincaré lemma

$$\mathrm{DR}(M) = \mathcal{L}[n].$$

2. Kashiwara usually uses a slightly different functor $\mathrm{Sol} : D_{\mathrm{coh}}(\mathcal{D}_X)^o \to D(X^{\mathrm{an}})$ defined by

$$\mathrm{Sol}(M^\cdot) = \underline{\mathrm{RHom}}_{\mathcal{D}_X^{\mathrm{an}}}(M^{\mathrm{an}}, \mathcal{O}_X^{\mathrm{an}}).$$

We claim that $\mathrm{Sol}(\mathcal{M}^\cdot) = DR(\mathbb{D}M^\cdot)[-n]$. This follows from the following formula. Let P be any locally projective $\mathcal{D}_X$-module and let $P^\vee = \mathrm{Hom}_{\mathcal{D}_X}(P, \mathcal{D}_X^\Omega)$. Then

$$\mathrm{Hom}_{\mathcal{D}_X}(P, \mathcal{O}_X) = \Omega_X \underset{\mathcal{D}_X}{\otimes} (P^\vee),$$

Here is the main result about the relation between $\mathcal{D}$-modules and constructible sheaves.

THEOREM 5.1.3. *a) $DR(D_{hol}(\mathcal{D}_X)) \subset D_{con}(X^{an})$. Also on $D_{hol}(\mathcal{D}_X)$ we have*

$$\mathbb{D} \circ DR = DR \circ \mathbb{D}.$$

(note that in the laft hand side $\mathbb{D}$ means the Verdier duality and in the right hand side $\mathbb{D}$ stands for the duality of $\mathcal{D}$-modules).

For $M^\cdot \in D_{hol}(\mathcal{D}_X)$ and $N^\cdot \in D(\mathcal{D}_Y)$ we have

$$DR(M^\cdot \boxtimes N^\cdot) \approx DR(M^\cdot) \boxtimes DR(N^\cdot).$$

b) On the subcategory D_{rs} the functor DR commutes with $\mathbb{D}, \pi_*, \pi^!, \pi_!, \pi^*$ and $\boxtimes$.

c) $DR : D_{rs}(\mathcal{D}_X) \to D_{con}(X^{an})$ is an equivalence of categories.

5.1.3. Simple statements. First let us consider some basic properties of the functor DR.

(i) DR commutes with restriction to an open subset. For an étale covering $\pi : Y \to X$ the functor DR commutes with π_* and $\pi^!$.

(ii) For a morphism $\pi : Y \to X$ there exists a natural morphism of functors $\alpha : \mathrm{DR}\pi_* \to \pi_* \circ \mathrm{DR}$ which is an isomorphism for proper π.

In order to prove this let us consider the functor
$\pi_*^{an} : D(\mathcal{D}_Y^{\mathrm{an}}) \to D(\mathcal{D}_X^{\mathrm{an}})$ on the categories of $\mathcal{D}^{\mathrm{an}}$-complexes, which is given by

$$\pi_*^{an}(M^\cdot) = R\pi_\bullet^{\mathrm{an}}(\mathcal{D}_{X\leftarrow Y}^{an} \underset{\mathcal{D}_Y^{an}}{\otimes} M^\cdot).$$

We claim that $DR \circ \pi_*^{\mathrm{an}} = \pi_* \circ \mathrm{DR}$. Indeed,

$$\mathrm{DR}(\pi_*^{\mathrm{an}}(M^\cdot)) = \Omega_X^{\mathrm{an}} \overset{L}{\underset{\mathcal{D}_X^{an}}{\otimes}} R\pi_\bullet^{\mathrm{an}}(\mathcal{D}_{X\leftarrow Y}^{\mathrm{an}} \overset{L}{\underset{\mathcal{D}_Y^{an}}{\otimes}} M^\cdot) =$$

$$R\pi_\bullet^{\mathrm{an}}(\pi^\bullet(\Omega_X^{\mathrm{an}}) \overset{L}{\underset{\pi^\bullet D_X^{an}}{\otimes}} \mathcal{D}_{X\leftarrow Y}^{\mathrm{an}} \overset{L}{\underset{\mathcal{D}_Y^{an}}{\otimes}} M^\cdot) = R\pi_\bullet^{an}(\Omega_Y^{an} \overset{L}{\underset{D_Y^{an}}{\otimes}} M^\cdot),$$

since $\pi^\bullet\Omega_X \underset{\pi^\cdot\mathcal{D}_X\otimes}{\mathcal{D}}_{X\leftarrow Y} \approx \Omega_Y$ as a $\mathcal{D}_Y$-module.

Now there exists in general the natural morphism of functors

$$\mathrm{an}\circ R\pi_\bullet(M^\cdot) \longrightarrow R\pi_\bullet^{\mathrm{an}}(M^\cdot)^{\mathrm{an}}.$$

This functor is not an isomorphism in general, since direct image on the left and on the right are taken in different topologies. But according to Serre's "GAGA" theorem it is an isomorphism for proper π. Combining these 2 observations we obtain (ii).

(iii) On the category of coherent $\mathcal{D}_X$-complexes there exists a natural morphism of functors

$$\beta : \mathrm{DR}\circ\mathbb{D}(M^\cdot) \to \mathbb{D}\circ\mathrm{DR}(M^\cdot)$$

which is an isomorphism for $\mathcal{O}$-coherent $M^\cdot$ and which is compatible with the isomorphism $\pi_*\circ\mathrm{DR} = \mathrm{DR}\circ\pi_*$ for proper π, described in (ii).

By definition of the duality functor $\mathbb{D}$ in the category $D(X^{\mathrm{an}})$ we have

$$\mathbb{D}(\mathcal{F}^\cdot) = \underline{\mathrm{RHom}}_{\mathbb{C}_X}(\mathcal{F}^\cdot, \mathbb{C}_X[2\dim X]).$$

(Note that $\mathbb{C}_X[2\dim X]$ is the dualizing sheaf of X^{an}). Hence in order to construct β it is sufficient to construct a morphism

$$\beta' : \mathrm{DR}\circ\mathbb{D}(M)\otimes_{\mathbb{C}_X}\mathrm{DR}(M) \longrightarrow \mathbb{C}_X[2\dim X].$$

As we have seen above $\mathrm{DR}\circ\mathbb{D}(M)$ is naturally isomorphic to

$$\mathrm{Sol}(M)[\dim X] = \underline{\mathrm{RHom}}_{\mathcal{D}_X^{\mathrm{an}}}(M^{\mathrm{an}}, \mathcal{O}_X^{\mathrm{an}})[\dim X].$$

Let us realize $\mathrm{DR}(M)$ as $dR(M^{\mathrm{an}})$ and $\mathrm{DR}\circ\mathbb{D}(M)$ as $\underline{\mathrm{RHom}}_{\mathcal{D}_X^{\mathrm{an}}}(M^{\mathrm{an}}, \mathcal{O}_X^{\mathrm{an}}[\dim X])$. Hence we get a morphism

$$\beta'' : M^{\mathrm{an}}\otimes_{\mathbb{C}_X}\mathrm{DR}(M) \longrightarrow \mathcal{O}_X^{\mathrm{an}}[\dim X]).$$

Since $dR(\mathcal{O}_X^{\mathrm{an}}) = \mathbb{C}_X[2\dim X]$ by applying dR to both sides we get β'.

It is easy to check that the corresponding β is an isomorphism for $\mathcal{O}$-coherent M.

It remains to check that β is compatible with (ii). By it is enough to do it for projections $Z\times X\to X$ where Z is smooth for which the statement is straightforward.

(iv) There is a natural morphism of functors

$$\gamma : \mathrm{DR}(M\boxtimes N) \longrightarrow \mathrm{DR}(M)\boxtimes\mathrm{DR}(N)$$

which is an isomorphism for $\mathcal{O}$-coherent M.

The morphism γ is defined by the natural imbedding $\Omega_X^{\mathrm{an}}\boxtimes_{\mathbb{C}}\Omega_Y^{\mathrm{an}} \longrightarrow \Omega_{X\times Y}^{\mathrm{an}}$. If M is O-coherent and N is locally projectively then γ is an isomorphism by partial Poincaré lemma. This implies the general statement.

(v) There is a natural morphism of functors

$$\delta : \mathrm{DR}\circ\pi^!(M) \to \pi^! DR(M)$$

which is an isomorphism for smooth π.

It is enough to construct δ in the following two cases:

1. π is an open embedding.
2. π is a smooth projection.
3. π is a closed embedding.

In the first case the construction of δ is obvious (it is also clear that in this case δ is an isomorphism). Consider the second case. In this case the isomorphism δ can be constructed on generators − locally projective modules. Indeed, let $\pi : Y = T\times X\to X$ be the projection, then $\pi^!(M) = \mathcal{O}_T\boxtimes M[\dim T]$ and $\pi^!\mathrm{DR}(M) = \mathbb{C}_T\boxtimes\mathrm{DR}(M)[2\dim T] = \mathrm{DR}(\mathcal{O}_T)\boxtimes\mathrm{DR}(M)[\dim T]$.

Consider now the case of a closed embedding $i : Y \to X$. Using i_*, which commutes with DR, we will identify sheaves on Y with sheaves on X, supported on Y. Then $i_* i^! M = R\Gamma_Y M$ in both categories, which gives the natural morphism

$$\delta : \mathrm{DR} \circ i_* i^!(M) = \mathrm{DR}(R\Gamma_Y M) \longrightarrow R\Gamma_Y \mathrm{DR}(M) = i_* i^! \mathrm{DR}(M).$$

5.1.4. Proof of Theorem 5.1.3 a). Let M be a holonomic $\mathcal{D}_X$-complex. Consider the maximal Zariski open subset $U \subset X$ such that $DR(M)|_U$ is constructible. Since M is $\mathcal{O}$-coherent almost everywhere it follows that U is dense in X.

Let W be an irreducible component of $X \setminus U$. We want to show that $\mathrm{DR}(M)$ is locally constant on some Zariski dense open subset $W_0 \subset W$.

PROPOSITION 5.1.4. *We can assume that*

$$X = \mathbb{P} \times W, \qquad W = p \times W, \qquad \text{where } p \in \mathbb{P}$$

and that $V = U \cup W$ is open in X. Here $\mathbb{P}$ denotes some projective space.

Indeed, consider an étale morphism of some open subset of W onto an open subset of an affine space $\mathbb{A}^k$ and extend it to an étale morphism of a neighbourhood of W onto an open subset of $\mathbb{A}^n \supset \mathbb{A}^k$. By changing base from $\mathbb{A}^k$ to W we can assume that $V = U \cup W$ is an open subset of $X' = \mathbb{P}^{n-k} \times W$. Then we can extend M to some holonomic $\mathcal{D}$-module on X'.

Now consider the projection $\mathrm{pr} : X = \mathbb{P} \times W \to W$. Since it is a proper morphism it follows that $\mathrm{DR}(\mathrm{pr}_*(M)) = \mathrm{pr}_* \mathrm{DR}(M)$. Since $\mathrm{pr}_*(M)$ is a holonomic $\mathcal{D}_W$-complex, it is $\mathcal{O}$-coherent almost everywhere. Hence $\mathrm{DR}(\mathrm{pr}_*(M))$ is locally constant almost everywhere.

Put $S = \mathrm{DR}(M) \subset D(X^{\mathrm{an}})$. Replacing W by an open subset, we can assume that $\mathrm{pr}_*(S) = \mathrm{DR}(\mathrm{pr}_*(M))$ is locally constant. We have an exact triangle

$$S_U \to S \to S_{X \setminus U}$$

where $S_U = (i_U)_! S|_U)$ and $i_U : U \to X$ is the natural embedding.

By the choice of U the complex $S|_U$ is constructible. Hence the complex S_U is constructible. Thus the complex $pr_*(S_{X\setminus U})$ is constructible. Replacing W once again by an open subset we can assume that it is locally constant.

Now $S_{X\setminus U}$ is a direct sum of 2 sheaves $(i_W)_! S|_W$ and something concentrated on $\{X \setminus U\} \setminus W$ (here we use the fact that V is open in X). This implies that $S|_W$ is a direct summand of the locally constant sheaf $\mathrm{pr}_*(S_{X\setminus U})$ and hence itself is locally constant. $\square$

5.1.5. Proof of Theorem 5.1.3 b) for $\mathbb{D}$ and $\boxtimes$. Let us now show that DR commutes with $\mathbb{D}$ for holonomic complexes. Let $M \in D_{\mathrm{hol}}(\mathcal{D}_X)$. Put

$$\mathrm{Err}(M) = \mathrm{Cone}(\mathrm{DR} \circ \mathbb{D}(M) \to \mathbb{D} \circ \mathrm{DR}(M)).$$

This sheaf vanishes on a dense open subset where M is $\mathcal{O}$-coherent. Also we know that the functor Err commutes with direct image for proper morphisms. Repeating the above arguments we can show that $\mathrm{Err} = 0$, i.e., DR commutes with $\mathbb{D}$ on $D_{\mathrm{hol}}(\mathcal{D}_X)$.

Also the same arguments show that $\mathrm{DR}(M \boxtimes N) = DR(M) \boxtimes DR(N)$ for holonomic M.

Remark. Of course this proof is simply a variation of Deligne's proof of "Théorèmes de finitude" in SGA 4 1/2. Note the crucial role in the proof of both statements is played by the fact that we have a well-defined morphism between the two corresponding functors for *all* $\mathcal{D}$-modules. Then we use Deligne's trick to show that it is an isomorphism for holonomic ones.

5.1.6. Proof of Theorem 5.1.3 b) for direct image. Let us prove that the morphism

$$\mathrm{DR} \circ \pi_*(M) \to \pi_* \circ \mathrm{DR}(M)$$

is an isomorphism for $M \in D_{rs}(\mathcal{D}_Y)$.

Case 1. $\pi = j : Y \to X$ is a regular extension [1] and M is an $\mathcal{O}$-coherent D_Y-module with regular singularities. Blowing up Y and making use of (ii) above we see that it is enough to consider the case when Y has codimension 1 in X.

In this case the proof is straightforward, using the definition of *RS* (it was done by P. Deligne). Since M has RS there exists an $\mathcal{O}$-coherent submodule $M' \subset j_*M$ with respect to which our connection has a pole of order ≤ 1. It is clear that both $j_*(\mathrm{DR}(M)$ and $\mathrm{DR}(j_*(M)$ depend only on $(M')^{\mathrm{an}}$ (which is a meromorphic connection).

Now, locally in the neighbourhood of a point $x \in X \setminus Y$ we can choose coordinates $x_1, \ldots, x_n$ such that $X \setminus Y$ is given by the equation

$$x_1 = 0.$$

We may replace x by an *analytic* neighbourhood of x such that $\pi_1(X \setminus Y, x) = \mathbb{Z}$. Since the above fundamenta group is commutative, we can decompose M' into 1-dimensional subquotients. Using commutativity with $\boxtimes$ we can reduce to the case $\dim X = 1$. Hence as $\mathcal{O}_X$-module M' is generated by one element e, which satisfies the equation $x\partial(e) = \lambda e$. In this case our statement can be proved by a direct calculation. *Case 2.* M is an $\mathcal{O}$-coherent D_Y-module with regular singularities.

In this case we decompose $\pi = \pi^+ \circ j$, where $j : Y \to Y^+$ is a regular extension and $\pi^+ : Y^+ \to X$ is a proper morphism. Then we know that DR commutes with j_* by Case 1 and with π^+_* by Section 5.1.3 (ii).

General Case. It is sufficient to check the statement on generators. Hence we can assume that $M = i_*(N)$, where $i : Z \to Y$ is a locally closed imbedding and N an $\mathcal{O}$-coherent D_Z-module with regular singularities. Then

$$\mathrm{DR}(\pi_*(M)) = \mathrm{DR}(\pi \circ i)_*(N) \overset{\text{Case 2}}{=} (\pi \circ i)_* \,\mathrm{DR}(N) =$$
$$\pi_*(i_*\mathrm{DR}(N)) \overset{\text{Case 2}}{=} \pi_*\mathrm{DR}(i_*(N)) = \pi_*\mathrm{DR}(N).$$

It follows that on $D_{rs}(\mathcal{D}_X)$ the functor DR also commutes with $\pi_!$ since $\pi_! = \mathbb{D} \circ \pi_* \circ \mathbb{D}$ and we have already checked that DR commutes with $\mathbb{D}$.

5.1.7. Proof of Theorem 5.1.3 b) for inverse image. It is enough to prove Theorem 5.1.3 for the functor $\pi^!$ (since $\pi^* = \mathbb{D} \circ \pi^! \circ \mathbb{D}$).

In Section 5.1.3 (v) we have constructed the morphism $\delta : \mathrm{DR} \circ \pi^! \to \pi^! \circ \mathrm{DR}$ which is an isomorphism for smooth π. Hence it is sufficient to check that for RS $\mathcal{D}_Y$-complexes the morphism δ is an isomorphism for the case of a closed embedding $\pi = i : Y \hookrightarrow X$. Denote by $j : U = X \setminus Y \to X$ the embedding of the complementary open set. Then we have the morphism of exact triangles

$$\begin{tikzcd}
DR(i_*i^!M) \arrow[r] \arrow[d, "\delta"] & DR(M) \arrow[r] \arrow[d, "id"] & DR(j_*(M|_V)) \arrow[d, "\alpha"] \\
i_*i^!DR(M) \arrow[r] & DR(M) \arrow[r] & j_*(DR(M)|_V).
\end{tikzcd}$$

Since we already know that α is an isomorphism it follows that δ is an isomorphism.

---

[1]by a regular extension we mean an open embedding $j : Y \to X$ such that the corresponding embedding $X \setminus Y \to X$ is regular

5.1.8. Proof of Theorem 5.1.3 c). First of all, let us prove that DR gives an equivalence of $D_{rs}(\mathcal{D}_X)$ with a full subcategory of $D_{\text{con}}(X^{an})$. We should prove that for $M, N \in D_{rs}(\mathcal{D}_X)$ the map

$$\mathrm{DR} : \mathrm{Hom}_{D_{rs}}(M, N) \longrightarrow \mathrm{Hom}_{D_{\text{con}}}(\mathrm{DR}(M), \mathrm{DR}(N))$$

is an isomorphism.

It turns out that it is simpler to prove the isomorphism of RHom. Let π denote the morphism from X to pt. We know that

$$\mathrm{RHom}(M, N) = \pi_* \underline{\mathrm{RHom}}(M, N) = \pi_* \mathbb{D}M \otimes N.$$

Note that $\otimes$ in the sense of $\mathcal{D}$-modules is transformed to $\overset{!}{\otimes}$ in $D_{\text{con}}(X^{\text{an}})$. [2]

This implies that

$$\mathrm{DR}(\underline{\mathrm{RHom}}(M, N)) = \underline{\mathrm{RHom}}(\mathrm{DR}(M), \mathrm{DR}(N)).$$

This proves that DR gives an equivalence of the category $D_{rs}(\mathcal{D}_X)$ with a full subcategory of $D_{\text{con}}(X^{\text{an}})$.

Now let us prove that this subcategory contains all isomorphism classes of $D_{\text{con}}(X^{\text{an}})$. Since it is a full triangulated subcategory, it is sufficient to check that it contains some generators. As generators we can choose $\mathbb{C}_X$-complexes of the form $i_*(\mathcal{L})$ where $i : Y \to X$ is an imbedding and $\mathcal{L}$ is a local system on Y. We ca also assume that Y is smooth. Since DR commutes with direct images it is sufficient to check that there exists an $\mathcal{O}$-coherent $\mathcal{D}_Y$-module M such that $\mathrm{DR}(M) \simeq \mathcal{L}[\dim Y]$. This is a result by P. Deligne.

5.1.9. Perverse sheaves. Theorem 5.1.3 gives us a dictionary which allows to translate problems, statements and notions from $\mathcal{D}$-modules to constructible sheaves and back.

Consider one particular example. The category $D_{rs}(\mathcal{D}_X)$ of *RS*-complexes contains the natural full abelian subcategory *RS*-category of *RS*-modules.

How to translate it in the language of constructible sheaves?

From the description of the functor $i^!$ for locally closed imbedding one can immediately get the following

Criterion. Let M be a holonomic $\mathcal{D}_X$-complex. Then M is concentrated in nonnegative degrees (i.e., $H^i(M) = 0$ for $i < 0$) if and only if it satisfies the following condition.
$(*)_{rs}$ For any locally closed embedding $i : Y \to X$ there exists an open dense subset $Y_0 \subset Y$ such that $i^! M|_{Y_0}$ is an $\mathcal{O}$-coherent $\mathcal{D}_{Y_0}$-complex, concentrated in degrees ≥ 0.
In terms of constructible complexes this condition can be written as
$(*)_{\text{con}}$ For any locally closed embedding $i : Y \to X$ there exists an open dense subset $Y_0 \subset Y$ such that $i^! S|_{Y_0}$ is locally constant and concentrated in degrees $\geq$ - dim Y.

Thus we have proved the following.

Criterion. A constructible complex S lies in the abelian subcategory

$$\mathrm{DR}(D_{rs}(\mathcal{D}_X)) \quad \text{iff} \quad S \quad \text{and} \quad \mathbb{D}S \quad \text{satisfy} \quad (*)_{con}.$$

Such a complex is called a *perverse sheaf* on X^{an}.

---

[2]By the definition $\mathcal{F} \overset{!}{\otimes} \mathcal{H} = \Delta^! \mathcal{F} \boxtimes \mathcal{H}$ where $\Delta : X \to X \times X$ is the diagonal embedding.

5.1.10. Analytic criterion of regularity. For any point $x \in X$ let us denote by $\mathcal{O}_x^{\mathrm{an}}$ (resp. $\mathcal{O}_x^{\mathrm{form}}$) the algebra of convergent (resp. formal) power series on X at the point x. For any $\mathcal{D}_X$-complex M the natural inclusion $\mathcal{O}_x^{\mathrm{an}} \to \mathcal{O}_x^{\mathrm{form}}$ induces a morphism

$$\nu_x : \mathrm{RHom}_{\mathcal{D}_X}(M, \mathcal{O}_x^{\mathrm{an}}) \longrightarrow \mathrm{RHom}_{\mathcal{D}_X}(M, \mathcal{O}_x^{\mathrm{form}}).$$

We say that M is good at x if ν_x is an isomorphism.

THEOREM 5.1.5. (1) *Let M be an RS $\mathcal{D}_X$-complex. Then M is good at all points.*
(2) *Assume that X is proper. Then M is good at all points of X if and only if M is RS.*

PROOF. FILL IN LATER $\square$

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