Kashiwara-Vergne Problem

Problem Set

1. Let $F : g \to g$ be of the form $x \mapsto \phi(x)f(x)f(x)^{-1}$ and such that it defines a diffeomorphism locally around 0. Show that

$$F \circ \hat{J}(F) \alpha = \alpha,$$

for any distribution $\alpha$ supported near 0. Hint: Set $F_t(x) := f(tx)f(tx)^{-1}$ and show that $F_t^* \phi = J(F_t) \phi \in C^\infty(g)_g$ for a test function $\phi$. Here $C^\infty(g)_g$ denotes the vector space of $g$-coinvariants.

2. i) Show that $\text{TAut}.(x + y) := \{ F(x + y) | F \in \text{TAut} \} = x + y + \text{Lie}_{\geq 2}(x,y)$ where $\text{Lie}_{\geq 2}(x,y)$ denotes the space of Lie series in degrees 2 and higher.

ii) Show that $\text{TAut}.([x,y]) \neq [x,y] + \text{Lie}_{\geq 3}(x,y)$.

3. i) Let $g(x) \in k[[x]]$ be any power series in one variable. Find a formula for $\partial_x g \in k[[x]] \otimes k[[x]] \simeq k[[x_1,x_2]]$.

ii) Use i) to show that

$$\text{bch}(x,y) = \log(e^xe^y) = x + \frac{\text{ad}_x}{e^{\text{ad}_x} - 1} y + O(y^2).$$

4. Let $\mathcal{G} = k[S^1, \Sigma]$ be the Goldman-Turaev Lie bialgebra on a surface $\Sigma$.

i) Viewing elements in $\Lambda^\bullet \mathcal{G}$ as collections of loops in $\Sigma$ in general position use resolution of double points to define an operator $\Delta : \Lambda^\bullet \mathcal{G} \to \Lambda^\bullet \mathcal{G}$ and show that $\Delta^2 = 0$.

ii) Show that $\Delta^2 = 0$ is equivalent to $\mathcal{G}$ being an involutive Lie bialgebra.

5. Compute $\mathcal{G}$ (that is, give a description for $\mathcal{G}$ together with $[\cdot,\cdot]$ and $\delta$) for $\Sigma$ a torus.

6. Verify that $t\text{div}^A(\sigma_{\exp}(e^x)) = 0$. 

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