Circle homeomorphisms and shears with $\ell^2$ structure

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(joint work with Dragomir Saric and Yilin Wang)
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Massachusetts Institute of Technology
Plan of the talk

Goal: compare circle homeomorphisms defined in terms of shears with ones from Teichmüller theory. Main result:

**Theorem** (Saric, Wang, W.)

\[
\mathcal{C}^{1,\alpha} \subset \mathcal{H} \subset WP(\mathbb{T}) \subset S
\]

if and only if \( \alpha > 1/2 \).
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- Look at circle homeomorphisms with finitely supported shears to motivate the definition of *diamond shear* coordinates
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See preprint at arXiv:2211.11497 for more details and references!
Shears and shear coordinates
Shear in terms of cross ratio

The cross ratio of four points $a, b, c, d$ along a circle or line is

$$\text{cr}(a, b, c, d) = \frac{(b - a)(d - c)}{(c - b)(d - a)} \in \mathbb{R}.$$ 

The cross ratio is invariant under Möbius transformations and has the symmetry $\text{cr}(a, b, c, d) = \text{cr}(c, d, a, b)$.

**Example.** $\text{cr}(\infty, -1, 0, \lambda) = \lambda$ for $\lambda \in (0, \infty)$. 
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Definition. The shear of a quadrilateral $Q$ with vertices $a, b, c, d \in \mathbb{T}$ along its diagonal $e = (a, c)$ is defined

$$
\text{s}(Q, e) = \log \text{cr}(a, b, c, d).
$$
Shear in terms of hyperbolic length

The shear $s(Q, e)$ can also be computed as (signed) hyperbolic length:

$$s(Q, e) = \pm d_{\text{hyp}}(m_b(e), m_d(e)).$$

Here $m_b(e)$ is the intersection of the geodesic through $b$ perpendicular to $e$ and $m_d(e)$ is the intersection of the geodesic through $d$ perpendicular to $e$.

The shear $s(Q, e)$ measures how the two triangles on either side of $e$ are glued together to construct $Q$. 
\( \mathcal{F} = (V, E) \) tessellation of \( \mathbb{D} \) starting from \( \tau_0 = \{-1, i, 1\} \) generated by hyperbolic reflections. Vertices \( V = \mathbb{T} \cap \mathbb{Q}^2 \). The dual tree \( \mathcal{F}^* \) (where each triangle corresponds to a vertex, etc) is a trivalent tree.
Farey tessellation

\( \mathcal{F} = (V, E) \) tessellation of \( \mathbb{D} \) starting from \( \tau_0 = \{-1, i, 1\} \) generated by hyperbolic reflections. Vertices \( V = \mathbb{T} \cap \mathbb{Q}^2 \). The dual tree \( \mathcal{F}^* \) (where each triangle corresponds to a vertex, etc) is a trivalent tree.

Conjugating to \( \mathbb{H} \) by a Möbius transformation sending \( \{-1, i, 1\} \mapsto \{0, 1, \infty\} \), \( V \) is sent to \( \mathbb{Q} \). There is an edge \((p/q, r/s) \in E \) if and only if \( pr - qs = 1 \), and tessellation is invariant under the action of \( \text{PSL}(2, \mathbb{Z}) \) action.
Farey tessellation in $\mathbb{D}$ and shears

The *Farey quad* $Q_e$ around $e \in E$ is the pair of triangles in $\mathcal{F}$ with diagonal $e$.

Since $\mathcal{F}$ is generated by reflection, in terms of shears

$$s(Q_e, e) = 0 \quad \forall e \in E.$$
Tessellation from homeomorphism

A Möbius transformations is determined by its action on three points, so

\[ \text{Homeo}^+(\mathbb{T})/\text{Möb} \cong \{ h \in \text{Homeo}^+(\mathbb{T}) : h \text{ fixes } \pm 1, i \}. \]

Given a homeomorphism \( h \) fixing \( \pm 1, i \), we can define a tessellation \( h(\mathcal{F}) \) which contains the triangle \( \tau_0 \) and has vertices \( h(V) \) and edges

\[ h(E) = \{(h(a), h(b)) : (a, b) \in E \}. \]
Shear coordinates

**Definition.** If $h$ is a homeomorphism, its shear coordinate $s_h : E \rightarrow \mathbb{R}$ is

$$s_h(e) = s(h(Q_e), h(e)) \quad \forall e \in E.$$ 

**Remark.** Not all functions $s : E \rightarrow \mathbb{R}$ encode homeomorphisms. (There exist shear functions where the image of $V$ is not dense.)
Circle homeomorphisms
Quasisymmetric homeomorphisms $QS(\mathbb{T})$

A map $f : \mathbb{D} \to \mathbb{D}$ is quasiconformal if $f$ solves the Beltrami equation

$$f_z = \mu f_z,$$

for some Beltrami coefficient $\mu$ with $||\mu||_{\infty} < 1$. 
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A homeomorphism \( h : \mathbb{T} \to \mathbb{T} \) is quasisymmetric if and only if it is the boundary value of a quasiconformal map of \( \mathbb{D} \).
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A homeomorphism $h : \mathbb{T} \rightarrow \mathbb{T}$ is quasisymmetric if and only if it is the boundary value of a quasiconformal map of $\mathbb{D}$.

Shears for quasisymmetric maps are totally classified (Saric). One model of universal Teichmüller space is $\text{QS}(\mathbb{T})/\text{Möb}$, where $\text{Möb} = \text{PSU}(1, 1)$ is the Möbius transformations preserving the disk.
Weil-Petersson homeomorphisms $\text{WP}(\mathbb{T})$

*Weil-Petersson Teichmüller space* $\text{WP}(\mathbb{T})/\text{Möb}$ is a subspace of universal Teichmüller space that has received a lot of interest lately. The class now has at least 26 definitions, often with $L^2$ structure.
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**Definition 1.** A homeomorphism $h : \mathbb{T} \to \mathbb{T}$ is Weil-Petersson if there exists an extension $f : \mathbb{D} \to \mathbb{D}$ such that:

- $f$ is quasiconformal, i.e. $f$ solves $f_z = \mu f z$ for $\mu$ with $||\mu||_\infty < 1$.
- The Beltrami coefficient $\mu$ is in $L^2$ for the hyperbolic metric on the disk, i.e.

$$
\int_{\mathbb{D}} \frac{|\mu(z)|^2}{(1 - |z|^2)^2} \, dA(z) < \infty.
$$
Weil-Petersson homeomorphisms WP(\(\mathbb{T}\))

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**Definition 2 (Shen).** A homoemorphism \(h\) is Weil-Petersson if and only if it is absolutely continuous and \(\log h' \in H^{1/2}\), i.e.

\[
\iint_{\mathbb{T} \times \mathbb{T}} \left| \frac{\log h'(x) - \log h'(y)}{x - y} \right|^2 \, dx \, dy < \infty.
\]
We define Hölder classes

\[ C^{1,\alpha} = \{ h : \mathbb{T} \to \mathbb{T} : \log h' \text{ is } \alpha\text{-Hölder} \}. \]
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**Corollary** (of the \( H^{1/2} \) characterization of WP). The inclusion \( C^{1,\alpha} \subset WP(\mathbb{T}) \) holds if and only if \( \alpha > 1/2 \).
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**Corollary** (of the \( H^{1/2} \) characterization of WP). The inclusion \( C^{1,\alpha} \subset WP(\mathbb{T}) \) holds if and only if \( \alpha > 1/2 \).

If \( \log h' \) is \( \alpha \)-Hölder, then

\[
\iint_{\mathbb{T} \times \mathbb{T}} \left| \frac{\log h'(x) - \log h'(y)}{x - y} \right|^2 \, dx \, dy \leq \text{const.} \iint_{\mathbb{T} \times \mathbb{T}} |x - y|^{2\alpha - 2} \, dx \, dy,
\]

which is finite if and only if \( 2\alpha - 2 > -1 \) hence if and only if \( \alpha > 1/2 \).
Square summable shears

We define the set of \textit{square summable shear functions}

\[ S = \{ s : E \rightarrow \mathbb{R} : \sum_{e \in E} s(e)^2 < \infty \} \]
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**Question.** How close is $S$ to $WP(\mathbb{T})$?
Square summable shears

We define the set of square summable shear functions

$$\mathcal{S} = \{ s : E \rightarrow \mathbb{R} : \sum_{e \in E} s(e)^2 < \infty \}$$

**Question.** How close is $\mathcal{S}$ to $\text{WP}(\mathbb{T})$?

Turns out $\mathcal{S}$ is “far” from (much bigger than) $\text{WP}(\mathbb{T})$, and this can be seen by looking at circle homomorphisms with finitely supported shears.
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Turns out \( S \) is “far” from (much bigger than) \( WP(\mathbb{T}) \), and this can be seen by looking at circle homomorphisms with finitely supported shears.

This will motivate the definition of *diamond shear* coordinates, and the space \( \mathcal{H} \) (square summable diamond shears), which we show is much closer to \( WP(\mathbb{T}) \).
Circle homeomorphisms with finitely supported shears
Finitely supported is piecewise Möbius

A shear function $s : E \to \mathbb{R}$ is *finitely supported* if $s(e) \neq 0$ for only finitely many $e \in E$.

**Example**

A finitely supported shear function is always induced by a piecewise Möbius circle homeomorphism with “breakpoints” in the vertices $V$ of $\mathcal{F}$.

**edges $e$ in the support of $S$.**
Finitely supported shears and Weil-Petersson

For \( v \in V \), \( \text{fan}(v) \) is the edges \( e \in E \) incident to \( v \).

**Lemma.** If \( h : \mathbb{T} \rightarrow \mathbb{T} \) has \( s_h : E \rightarrow \mathbb{R} \) finitely supported, TFAE:

1. \( h \) is Weil Petersson;
2. \( h \) is \( C^{1,1} \) with breakpoints in \( V \);
3. The shear function \( s_h \) satisfies the *finite balance condition*, i.e. for all \( v \in V \), \( \sum_{e \in \text{fan}(v)} s(e) = 0 \).

**Note:** The shear function \( s : E \rightarrow \mathbb{R} \) supported on one edge is *not* WP.
**Definition of diamond shear**

**Definition.** Fix $e \in E$ (with dual edge $e^*$), and let $e_1, e_2, e_3, e_4$ be the edges around $Q_e$. The *diamond shear* basis element $\Delta_e$ corresponds to the shear function with $s(e_1) = s(e_3) = +1$, $s(e_2) = s(e_4) = -1$, and all other shears 0.

$\Delta_e$ is the shear coordinate of a piecewise Möbius map with 4 pieces.

\[
\text{Ex} \quad \Delta_e \\
\text{for} \\
e = (-1, 1)
\]

\[
\begin{align*}
\theta(e^*) &= 1 \\
s(e_1) &= s(e_3) = +1 \\
s(e_2) &= s(e_4) = -1
\end{align*}
\]

**Definition.** If a homeomorphism $h$ has a shear coordinate $s : E \to \mathbb{R}$ such that $s = \sum_{e^* \in E^*} \vartheta(e^*) \Delta_e$ then $h$ has *diamond shear coordinate* $\vartheta : E^* \to \mathbb{R}$.

**Note:** not all shear functions can be written as diamond shears.
Lemma. If $h$ has $s_h$ finitely supported, then $h$ is Weil-Petersson if and only if $h$ has a diamond shear coordinate.

Proof sketch. “Pruning the tree.” By the previous Lemma, $h$ is Weil-Petersson if and only if $s_h$ satisfies the finite balanced condition.
Definition of $\mathcal{H}$

Not all shear functions can be written as diamond shears. We let $\mathcal{P} \subset \mathbb{R}^E$ be the subset of shear functions $s$ such that $s$ can be written in terms of diamond shear coordinates $\vartheta$, $\Psi : \mathcal{P} \to \mathbb{R}^{E^*}$ sends $s \mapsto \vartheta$.

**Definition.** The set of *square summable diamond shears* is

$$\mathcal{H} = \{ s \in \mathcal{P} : \vartheta = \Psi(s), \sum_{e^* \in E^*} \vartheta(e^*)^2 < \infty \}.$$
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**Remark.** From the condition for a shear function to encode a quasisymmetric homeomorphism, it follows that

$$\mathcal{H} \subset QS(\mathbb{T}).$$

In particular, all $s \in \mathcal{H}$ induce homeomorphisms.
Main theorems

Theorem (SWW).

\[ \mathcal{C}^{1,\alpha} \subset \mathcal{H} \subset WP(\mathbb{T}) \subset S \]

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The space \( \text{WP}(\mathbb{T}) \) has a metric (the Weil-Petersson metric), and \( \mathcal{H} \) has a natural topology coming from its \( \ell^2 \) structure.
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**Theorem (SWW).** Suppose that \( h, (h_n)_{n \geq 1} \in \mathcal{H} \) with diamond shear coordinates \( \vartheta, \vartheta_n \) respectively. If

\[
\lim_{n \to \infty} \sum_{e^* \in E^*} (\vartheta_n(e^*) - \vartheta(e^*))^2 = 0
\]

then \( h_n \) converges to \( h \) in the Weil-Petersson metric.
Main theorems

**Theorem (SWW).**

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**Corollary.** Piecewise-Möbius, \( C^1 \) maps with rational breakpoints are dense in \( \mathcal{H} \) and \( \text{WP}(\mathbb{T}) \).
Proof ideas
Given $h \in \mathcal{H}$, we explicitly construct an extension $f : \mathbb{D} \to \mathbb{D}$ and show it is quasiconformal and has Beltrami coefficient $\mu \in L^2(\mathbb{D}, d_{hyp})$. 
Proof ideas: $\mathcal{H} \subset WP$

Given $h \in \mathcal{H}$, we explicitly construct an extension $f : \mathbb{D} \to \mathbb{D}$ and show it is quasiconformal and has Beltrami coefficient $\mu \in L^2(\mathbb{D}, d_{\text{hyp}})$.

The dual tree $\mathcal{F}^*$ subdivides $\mathbb{D}$ into cells $\{C_v : v \in V\}$. Based on a construction by Kahn and Markovic, we construct $f$ extending $h$ that sends cells to cells.

Step 1. Extend over $\mathcal{F}^*$ by hyperbolic stretching.

Step 2. Extend over a single cell $C_v, v \in V$.

Step 3. Stitch cells together again.
Proof ideas: extending over a cell $C_v$

For any $v \in V$, conjugating by appropriate Möbius transformations $\mathbb{H} \to \mathbb{D}$, we can send $h : \mathbb{T} \to \mathbb{T}$ to $\varphi : \mathbb{R} \to \mathbb{R}$ fixing $\infty$ and $C_v$ to $C_{\infty}$. Suffices to explain how to extend over $C_{\infty}$.
Proof ideas: extending over a cell $C_\nu$

For any $\nu \in V$, conjugating by appropriate Möbius transformations $\mathbb{H} \to \mathbb{D}$, we can send $h : \mathbb{T} \to \mathbb{T}$ to $\varphi : \mathbb{R} \to \mathbb{R}$ fixing $\infty$ and $C_\nu$ to $C_\infty$. Suffices to explain how to extend over $C_\infty$.

![Diagram showing extension of $\varphi$](image)

Extension $\psi$ (conjugate to $f$) of $\varphi$ sends $x + i u(x)$ (boundary of $C_\infty$) by hyperbolic stretching to the curve $\alpha(x) + i \beta(x)$. We extend over the rest of the cell on vertical lines:

$$\psi(x + iy) = \alpha(x) + i(\beta(x) - u(x) + y) \quad x + iy \in C_\infty.$$
Proof ideas: $C^{1,\alpha} \subset \mathcal{H}$

Analytic definition of diamond shears. If $h$ has a diamond shear coordinate $\vartheta_h$ and $e = (a, b) \in E$, then

$$
\vartheta_h(e) = \frac{1}{2} \log h'(a)h'(b) - \log \frac{h(a) - h(b)}{a - b}.
$$

Summability of Farey lengths. Let $\ell(a, b)$ be the length of the shorter circular arc from $a$ to $b$.

$$
\sum_{(a,b)\in E} \ell(a, b)^r < \infty
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if and only if $r > 1$. 
Proof ideas: $C^{1,\alpha} \subset \mathcal{H}$

**Analytic definition of diamond shears.** If $h$ has a diamond shear coordinate $\vartheta_h$ and $e = (a, b) \in E$, then

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**Proof sketch.** Suppose $h \in C^{1,\alpha}$. By the mean value theorem*, there is $c \in (a, b)$ so that

$$
\vartheta_h(e) = \frac{1}{2} (\log h'(a) - \log h'(c)) + \frac{1}{2} (\log h'(b) - \log h'(c)).
$$

Since $\log h'$ is $\alpha$-Hölder, $|\vartheta_h(e)|^2 \leq \text{const.} \ell(a, b)^{2\alpha}$, and the right hand side is summable if and only if $\alpha > 1/2$.  

\*
Comments on WP ∉ ℋ and WP ⊂ S

It turns out that for \( h : \mathbb{T} \rightarrow \mathbb{T} \) to even have diamond shear coordinate, \( h \) must have left and right derivatives at all \( v \in V \). But Weil-Petersson maps are allowed to have points of non-differentiability.
Comments on WP $\not\in \mathcal{H}$ and WP $\subset S$

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**Example.** $\varphi : \mathbb{R} \to \mathbb{R}$ defined by $\varphi(x) = x \log |x| - x$ outside $(-2, 2)$, and smoothed out in-between. The function $\log \varphi'(x) = \log \log |x|$ outside $(-2, 2)$ is in $H^{1/2}(\mathbb{R})$ so $\varphi \in WP(\mathbb{R})$. However since $\varphi$ does not have derivative at $\infty$, it does not have a diamond shear coordinate.
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**Example.** \( \varphi : \mathbb{R} \to \mathbb{R} \) defined by \( \varphi(x) = x \log |x| - x \) outside \((-2, 2)\), and smoothed out in-between. The function \( \log \varphi'(x) = \log \log |x| \) outside \((-2, 2)\) is in \( H^{1/2}(\mathbb{R}) \) so \( \varphi \in \text{WP}(\mathbb{R}) \). However since \( \varphi \) does not have derivative at \( \infty \), it does not have a diamond shear coordinate.

**Remark.** However one can compute that for \( n > 1 \),

\[
s_\varphi((n, \infty)) = \frac{1}{n \log n} + O\left(\frac{1}{n^2}\right).
\]

This is square summable, corresponding to the fact that \( \text{WP}(\mathbb{T}) \subset S \). The proof that \( \text{WP}(\mathbb{T}) \subset S \) uses a necessary condition for WP due to C. Wu.
Thank you for listening!