Large deviations for the 3D dimer model

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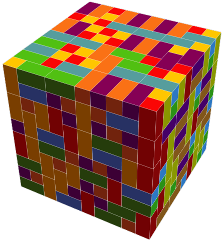
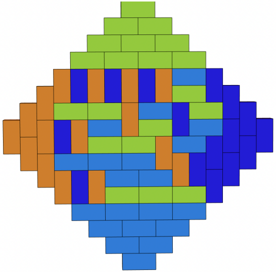
What is the dimer model?

- Dimers in 2D are *dominoes*, e.g. 1×2 or 2×1 blocks.
- Dimers in 3D are *bricks*, e.g. $2 \times 1 \times 1$ or $1 \times 2 \times 1$ or $1 \times 1 \times 2$ blocks.

A dimer tiling of a region $R \subset \mathbb{Z}^2$ or $\mathbb{Z}^3$ is a collection of dimer tiles such that every square/cube is covered by exactly one tile.

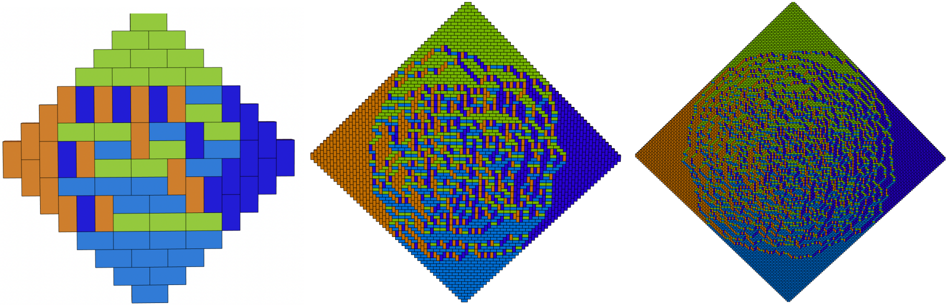




A lot is known about the dimer model in 2D (e.g. the study of domino tilings) and the goal of this work is to try to understand some things about the 3D dimer model (e.g. the study of brick tilings).

Large deviations in 2D

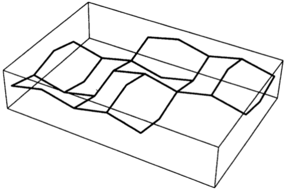
In 2000, Cohn, Kenyon and Propp [CKP] proved an LDP for the 2D dimer tilings and showed that there is a **limit shape**. The set up is: let $R \subset \mathbb{R}^2$ be a compact simply connected region, h_b is a Lipschitz function on ∂R. Let $R_n \subset \frac{1}{n}\mathbb{Z}^2$ be a sequence of grid regions approximating R from within. For a sequence of uniform random tilings of R_n, if their rescaled boundary "heights" on ∂R_n converge to h_b as $n \to \infty$, their heights functions converge to a deterministic "limit shape".



Example: finer and finer aztec diamonds. The interface between the frozen/rough regions is exactly a circle in the limit. The behavior depends on the boundary conditions on R_*.

A bit more about dimers in 2D

- In 2D, there is a correspondence between 2D dimer tilings and Lipschitz functions called *height functions* (up to an additive constant).



Example of a height function of a tiling. From [T].

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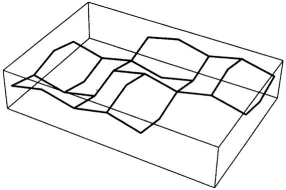
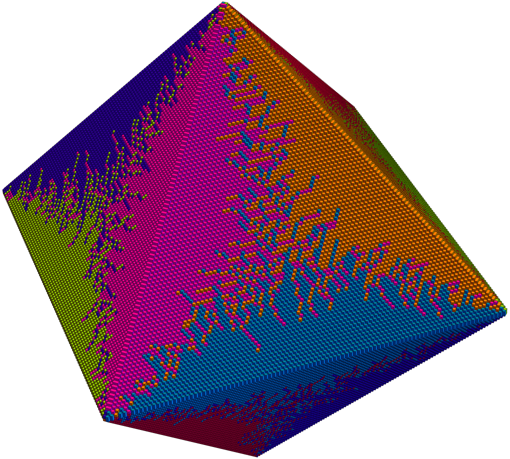


Example of a height function of a tiling. From [T].

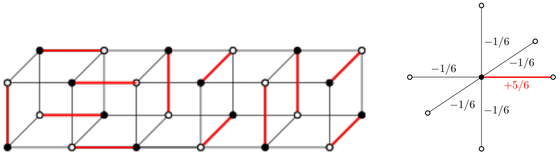
- The large deviations principle and limit shape theorem that CKP prove are in terms of height functions. (E.g. two tilings are close if their height functions are close in the sup norm)
- The rate function is $C - \text{Ent}_2(\cdot)$ where Ent_2 is an entropy functional.
- Linear algebra methods in 2D called *Kasteleyn theory* make it possible to compute Ent_2 totally explicitly.

What about in 3D?



- Is there a large deviations principle and limit shape in 3D?
- Yes! But the height function correspondence and Kasteleyn theory both break down in 3D.
- What can we use instead of height functions?

Tiling flows



Dual picture where dimers are edges; f_τ values around one vertex.

Given a dimer tiling of τ of $\mathbb{Z}^3$ (or $\mathbb{Z}^d$ for any d), we can associate a divergence free discrete vector field. For each edge e oriented from black to white,

$$f_\tau(e) = \begin{cases} 1 - 1/6 = 5/6 & e \in \tau \\ -1/6 & e \notin \tau \end{cases}$$

- In 2D, a div-free flow f has a scalar potential h, e.g. $\nabla h = f$. The scalar potential of the tiling flow in 2D is the height function. So f_τ is a generalization of ∇h.
- The metric d_W on flows is a version of *Wasserstein distance*. Two flows are close if we don't have to move flow too much or too far to transform one into the other.
- The fine-mesh limits of tiling flows are measurable vector fields g that we call *asymptotic flows*.

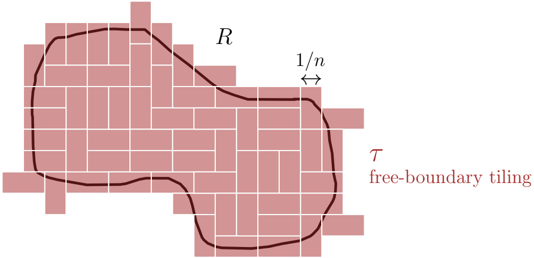
Large deviations in 3D

Theorem (Chandgotia, Sheffield, W.)

(Rough statement) For any asymptotic flow g on R,

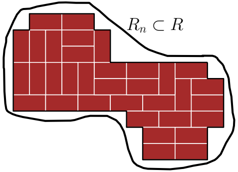
$$\lim_{\delta\to 0}\lim_{n\to\infty} n^{-3}\mathit{Vol}(R)^{-1}\log\#\{\text{free-boundary tilings } \tau \text{ of } R\cap \frac{1}{n}\mathbb{Z}^3 : d_W(f_\tau, g) < \delta\} = \mathit{Ent}_3(g).$$

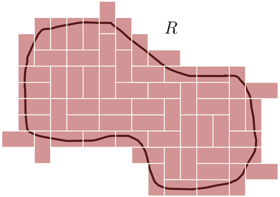
Like in 2D, the entropy functional Ent_3 is an average of a "local entropy" function ent_3:
$\mathit{Ent}_3(g) = \frac{1}{\mathit{Vol}(R_*)}\iiint_R \mathit{ent}_3(g(x))\,dx.$



Dictionary between 2D and 3D set ups

	2D	**3D**
object associated to tiling τ	height function h	tiling flow f_τ
topology (e.g. how to compare tilings)	sup norm on height functions	Wasserstein metric d_W on tiling flows
limits of discrete objects	asymptotic height functions $AH(R, h_b)$, Lipschitz functions	asymptotic flows $AF(R, b)$, divergence-free measurable vector fields
rate function	$\mathrm{Ent}_2(\nabla h)$	$\mathrm{Ent}_3(f_\tau)$
approximation method	all tilings of fixed region $R_n \subset \frac{1}{n}\mathbb{Z}^2$ contained in R	all free-boundary tilings of $R \cap \frac{1}{n}\mathbb{Z}^3$





Limit shape in 3D

$AF(R, b)$ is the space of asymptotic flows with boundary value b. We show that $\text{Ent}_3(\cdot)$ has a unique maximizer f in $AF(R, b)$, so we also get a limit shape theorem.

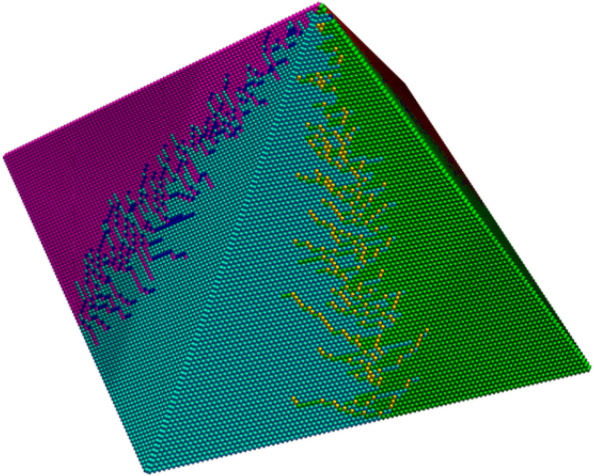
- Technical issue: boundary value convergence.
- There exists a sequence of "thresholds" $(\theta_n)_{n\geq 0}$ with $\theta_n \to 0$ such that for n large enough, any $g \in AF(R, b)$ can be approximated by a tiling with boundary values within θ_n of b.

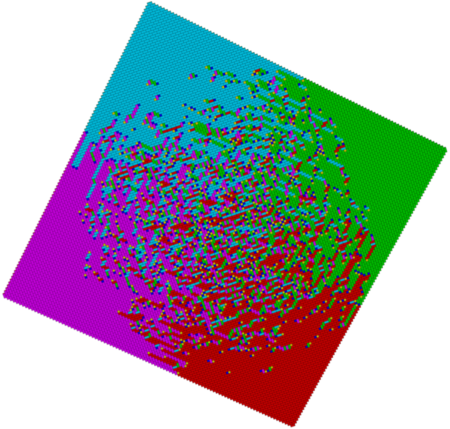
Theorem (Chandgotia, Sheffield, W.)

(Rough statement) Let τ_n be a uniform random free-boundary tiling of $R \cap \frac{1}{n}\mathbb{Z}^3$, conditioned so that boundary values of its tiling flow f_{τ_n} are within θ_n of b.
Then for any $\epsilon > 0$, the probability that $d_W(f_{\tau_n}, f) > \epsilon$ goes to 0 exponentially fast in n^3 as $n \to \infty$.

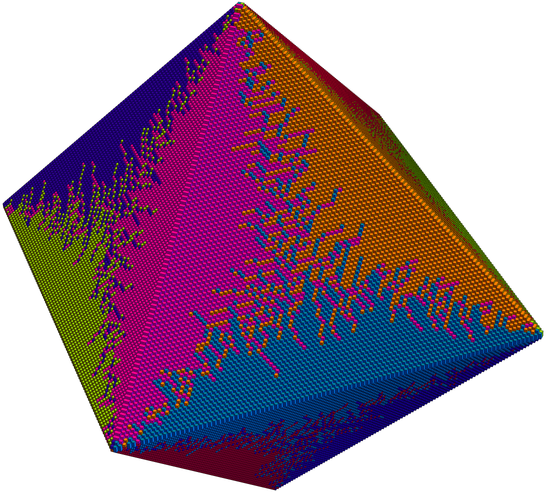
Remark: **we don't know how to compute $\text{Ent}_3(\cdot)$ or $\text{ent}_3(\cdot)$!** Different work goes in to controlling this. While we know that a limit shape exists, all we know about what they look like is from simulations.

Simulations: pyramid

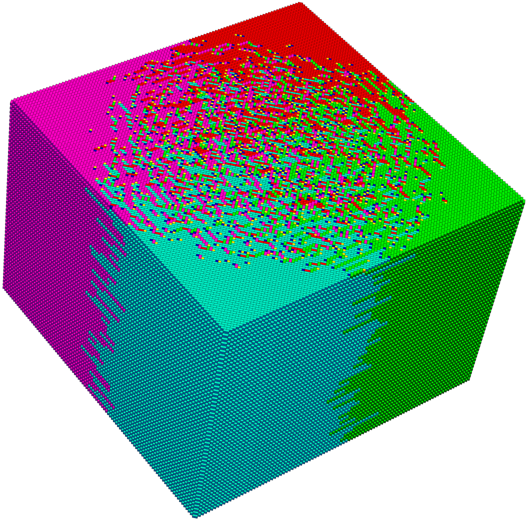




Simulations: aztechedron (again)

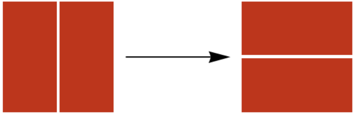


Simulations: alternating stack



Open problem: local move connectedness in 3D?

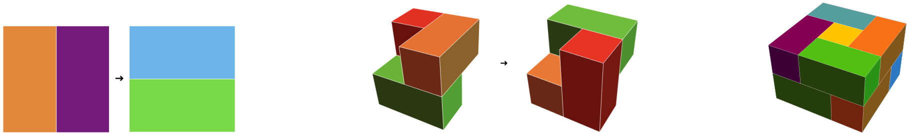
In 2D, pictures like this are made using *Glauber dynamics* with one local move called a flip:



Any two tilings τ_1, τ_2 of a simply connected region in 2D are connected by a finite sequence of flips, so the state space of dimer tilings of any region $R \subset \mathbb{Z}^2$ is connected under flips. (This is proven in [T] using height functions.)

Open problem: local move connectedness in 3D?

In 3D, connectedness under flips fails, even for small boxes:



A flip, a trit, and a flip-rigid configuration called a *hopfion*.

A natural but subtle question is whether flips and trits are enough. If $L, M, N > 2$, this is an open question:

Problem

Are all 3D dimer tilings of an $L \times M \times N$ box connected under flips and trits?

See [FKMS] and [FHNQ] (and more) for the status of this problem. Result we know: for homologically equivalent tilings of the 3-dimensional torus, there is no finite collection of local moves that is sufficient. (In 2D, all homologous tilings are connected under flips.)

Thank you for listening!

[CKP] Henry Cohn, Richard Kenyon, James Propp (2001)
A variational principle for domino tilings.
Journal of the American Mathematical Society 14, 297-346.

[FHNQ] Michael Freedman, Matthew B. Hastings, Chetan Nayak, Xiao-Liang Qi (2011)
Weakly coupled non-Abelian anyons in three dimensions
Physical Review B 84, 245119.

[FKMS] Juliana Freire, Caroline J. Klivans, Pedro H. Milet, Nicolau C. Saldanha (2022)
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Transactions of the American Mathematical Society 375, 1579-1605.

[T] William P. Thurston (1990)
Conway's Tiling Groups
The American Mathematical Monthly Vol. 97, No. 8, 757-773.