This problem set is due on 2016–10–04 at 12n and to be submitted to the box outside of 4-174.

**Problem 1** (20 points). Show that any manifold $M$ can be embedded into $\mathbb{R}^n$ some $n$ depending on $M$.

**Problem 2** (30 points). Prove that any rank 1 real vector bundle over $S^2$ is trivial.

**Problem 3** (30 points). Let $\Sigma \subset \mathbb{R}^3$ be a 2-dimensional submanifold and suppose $n: \Sigma \to \mathbb{R}^3$ is unit normal, i.e., a smooth map such that for each $x \in \Sigma$ we have

$$|n(x)| = 1 \quad \text{and} \quad n \perp T_x \Sigma.$$  

Here we think of $T_x \Sigma$ as a subspace of $T_x \mathbb{R}^3$ and identify $T_x \mathbb{R}^3 = \mathbb{R}^3$. Define $\Pi: T^* \Sigma \otimes T^* \Sigma \to \mathbb{R}$, the second fundamental form of $\Sigma$, by

$$\Pi(v, w) := -(dn(v), w)_{\mathbb{R}^3}.$$  

1. Show that $\Pi(v, w) = \Pi(w, v)$.

2. Using the metric on $\Sigma$ induced by the standard metric on $\mathbb{R}^3$, we can think of $\Pi_x$ as a symmetric endomorphism of $T_x \Sigma$; in particular, it has two eigenvalues $\lambda_1(x)$ and $\lambda_2(x)$. We call

$$\kappa := \lambda_1 \lambda_2 \in C^\infty(\Sigma)$$  

the **Gauss curvature** of $\Sigma$ and

$$H := \frac{1}{2}(\lambda_1 + \lambda_2) \in C^\infty(\Sigma)$$  

the **mean curvature** of $\Sigma$.

Compute $\kappa$ and $H$ for $\Sigma = S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 + 1\}$ and the outward-pointing unit normal.

**Problem 4** (20 points). Prove Cartan’s formula

$$\mathcal{L}_v \alpha = d(v)\alpha + i(v)d\alpha$$

for any $\alpha \in \Omega^k(M)$ and $v \in \text{Vect}(M)$. 

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