This problem set is due on 2016–09–20 at 12n and to be submitted to the box outside of 4-174.

**Problem 1** (20 points). Show that the Hopf map $h: S^3 \to CP^1$ defined by

$$h(x_1, x_2, x_3, x_4) := [x_1 + ix_2 : x_3 + ix_4]$$

is smooth.

**Problem 2** (20 points). Show that $CP^1$ is diffeomorphic to $S^2$.

**Problem 3** (20 points). Let $Z_2 = \mathbb{Z}/2\mathbb{Z}$ act on $\mathbb{R}^3$ via multiplication by $-1$. Show that the quotient $\mathbb{R}^3/Z_2$ does not have an atlas.

**Problem 4** (20 points). Let $M$ be a manifold and $V$ be a finite dimensional vector space. Suppose \{U\alpha\} is an open cover of $M$ and $g^\alpha_\beta \in C^\infty(U_{\alpha\beta}, GL(V))$ satisfy the cocycle conditions. Let $\pi: E \to M$ denote the resulting vector bundle.

Let $\rho_{\wedge^k}: GL(V) \to GL(\wedge^2 V)$ denote the representation defined by

$$\rho_{\wedge^k}(g)(v_1 \wedge v_2 \wedge \ldots \wedge v_k) = gv_1 \wedge gv_2 \wedge \ldots \wedge gv_k,$$

for $v_i \in V$, and denote by $\rho_*$ denote the representation defined by

$$\rho_*(g)v^* = v^* \circ g^{-1},$$

for $v^* \in V^* = Hom(V, \mathbb{R})$.

Show that the vector bundle associated with the cocycle $\{\rho_{\wedge^k} \circ g^\alpha_\beta\}$ is isomorphic to $\wedge^k E$ and show that the vector bundle associated with the cocycle $\{\rho_* \circ g^\alpha_\beta\}$ is isomorphic to $E^*$.

**Problem 5** (20 points). Let $a_i, b_j \in C^\infty(\mathbb{R}^n)$ and set $v = \sum_{i=1}^n a_i \partial_{x_i}$ and $w = \sum_{j=1}^n b_j \partial_{x_j}$. Compute the Lie bracket $[v, w]$. Show that one can choose $n, a_i, b_j$ such that the value of the vector field $v$ at the point $0 \in \mathbb{R}^n$ vanishes, i.e., $v(0) = 0$, but nevertheless $(Z_v w)(0) \neq 0$.

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1In a former version of this question I wrote “$a_i, b_j \in C^\infty(M)$”, which could lead to a misunderstanding as to what $v(0)$ means. Thanks to Robert Jones for pointing this out.