Problem 1 (30 points). Suppose \( u, v : U \rightarrow \mathbb{R} \) are both harmonic and set \( f := u|_{\partial U} \) and \( g := v|_{\partial U} \).

Prove that \( f \geq g \), but \( f \neq g \implies u > v \) in \( U \).

Also, show that \( \|u - v\|_{L^\infty(U)} \leq \|f - g\|_{L^\infty(\partial U)}. \)

Problem 2 (30 points). Let \( u : \mathbb{R}^n \rightarrow \mathbb{R} \) be a harmonic function. Suppose there are constants \( A, B > 0 \) and \( \alpha \in (0, 1) \) such that

\[
|u|(x) \leq A + B|x|^\alpha.
\]

Prove that \( u \) is constant.

Problem 3 (30 points). Prove that if \( u \) is slowly growing and weakly harmonic on \( \mathbb{R}^n \), and if

\[
K_t(x) := \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}
\]

denotes the heat kernel, then

\[
\int_{\mathbb{R}^n} (\Delta_x K_t(x - y))u(y) \, dy = 0.
\]

Problem 4 (10 points). Consider the annulus

\[
A := \{(x, y) \in \mathbb{R}^2 \cong \mathbb{C} : \sqrt{|x|^2 + |y|^2} \in [1, 2]\} = \bar{B}_2(0) \setminus B_1(0).
\]
The function $f : A \to \mathbb{R}$ defined by

$$f(x, y) = \frac{x}{x^2 + y^2} = \text{Re}(1/z).$$

is harmonic.

Is there a harmonic function $\tilde{f} : \bar{B}_2(0) \to \mathbb{R}$ such that

$$\tilde{f}|_A = f?$$