Problem 1 (20 points). Show that the heat kernel
\[ \Phi(t, x) := \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} \]
satisfies
\[ \partial_t \Phi + \Delta \Phi = 0. \]

Problem 2 (20 points). Suppose \( f \in C^0(\mathbb{R}^n) \) and there are constants \( a, b > 0 \) such that
\[ |f(x)| \leq ae^{b|x|^2}. \]
Prove that
\[ \lim_{t \to 0} \int_{\mathbb{R}^n} \Phi(t, x - y) f(y) \, dy = f(x). \]

Problem 3 (20 points). Suppose \( f : \mathbb{R}^n \to \mathbb{R} \) is uniformly continuous and bounded. Define \( f_t \in C^0(\mathbb{R}^n) \) by
\[ f_t(x) := \int_{\mathbb{R}^n} \Phi(t, x - y) f(y) \, dy. \]
Show that
\[ \|f_t - f\|_{C^0(\mathbb{R}^n)} \to 0. \]
(Together with an earlier exercise, this proves that \( f \in C^k(\mathbb{R}^n) \iff \{a_n\} \in \ell^1_k. \)

Problem 4 (20 points). Prove that if \( f \) is \( C^k([0, 1]) \), then there is a constant \( c > 0 \) such that the Fourier coefficients \( \{a_n\} \) of \( f \) satisfy
\[ n^k |a_n| \leq c. \]

Problem 5 (20 points). Sketch a proof of Theorem 8.4.