Problem 1 (30 points). Suppose \( f \in L^2([0, 1]) \) has sumable Fourier coefficients, i.e.,
\[
\sum_{n=1}^{\infty} |\langle f_n, f \rangle| < \infty,
\]
then the function \( u: (0, \infty) \times [0, 1] \to \mathbb{R} \) constructed in Theorem 5.19 can be extended to a continuous function \([0, \infty) \times [0, 1] \to \mathbb{R}\).

Problem 2 (20 points). Suppose \( \sigma \in C^\infty([0, \infty) \times [0, 1]) \). Give a formula for the solution of
\[
\partial_t u + \Delta u = \sigma
\]
satisfying Dirichlet boundary conditions and \( u(0, \cdot) = 0 \).

Problem 3 (30 points). Define \( f \in L^2([0, 1]) \) by
\[
f(x) = \begin{cases} 
  x & x \leq 1/2 \\
  1/2 - x & x > 1/2.
\end{cases}
\]
Find a smooth function \( u: (0, \infty) \times [0, 1] \to \mathbb{R} \) which solves
\[
\partial_t u + \Delta u = 0,
\]
satisfies the Dirichlet boundary conditions
\[
u(t, 0) = u(t, 1) = 0 \quad \text{for all } t > 0,
\]
and with
\[
\lim_{t \to 0} u(t, \cdot) = f \text{ in } L^2([0, 1]).
\]
Problem 4 (10 points). Is there a $f \in L^2([0, 1])$ such that if $u: (0, \infty) \times [0, 1] \to \mathbb{R}$ is the function constructed in Theorem 5.19, then the Fourier series of $u(1, \cdot)$ is

$$\sum_{n=1}^{\infty} \frac{f_n}{n^2}?$$

Problem 5 (10 points). If $f: [0, T) \to \mathbb{R}$ is a $C^2$-function with $f'' \geq 0$, then

$$f(t) \geq f(0) + f'(0)t.$$