This problem set is due at the beginning of the lecture on 2016–02–18.

Problem 1 (10 points). Fix a constant $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$ and a smooth function $g = g(t, x) : \mathbb{R}^{n+1} \to \mathbb{R}$. What is the general solution of the inhomogeneous transport equation

$$\partial_t u(t, x) + \sum_{i=1}^n w_i \partial_i u(t, x) = g(t, x)?$$

Problem 2 (15 pts). Suppose $u : [0, T) \times \mathbb{R} \to \mathbb{R}$ is a solution the inviscid Burgers’ equation

$$\partial_t u + u \partial_x u = 0$$

and

$$\int_{\mathbb{R}} |u(0, x)|^2 \, dx < \infty.$$

Prove that the spatial $L^2$–norm is conserved, that is, for all for all $t \in [0, T)$ we have

$$\int_{\mathbb{R}} |u(t, x)|^2 \, dx = \int_{\mathbb{R}} |u(0, x)|^2 \, dx.$$

Problem 3 (25 points). Define the function $u : [0, 1) \times \mathbb{R} \to \mathbb{R}$ by

$$u(t, x) := \begin{cases} 
1 & x \leq t \\
\frac{1-x}{1-t} & x \in [t, 1] \\
0 & x \geq 1.
\end{cases}$$

Show that $u$ satisfies the inviscid Burgers’ equation in the weak sense. That is, for every $\phi \in C_c^\infty((0, 1) \times \mathbb{R})$, show that

$$\int_{(0,1) \times \mathbb{R}} (\partial_t \phi) u + (\partial_x \phi) \frac{u^2}{2} \, dt \, dx = 0.$$
Problem 4 (25 points). For $k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, set

$$
\ell^1_k := \left\{ (a_n) \in \mathbb{R}^N : \sum_{n=1}^{\infty} n^k |a_n| < \infty \right\}.
$$

Suppose $f \in L^2([0, 1])$ has the Fourier series

$$(4.1) \quad f = \sum_{n=1}^{\infty} a_n f_n$$

with $f_n := \sqrt{2} \sin(n \pi x)$ and $a_n := \langle f, f_n \rangle_{L^2} = \int_0^1 f(x) f_n(x) \, dx$.

Prove that if $(a_n) \in \ell^1_k$, then $f \in C^k([0, 1])$ and

$$
\lim_{N \to \infty} \left\| f - \sum_{n=1}^{N} a_n f_n \right\|_{C^k} = 0.
$$

Here

$$
\|f\|_{C^k} := \sum_{i=0}^{k} \sup_{x \in [0,1]} |\nabla^i f(x)|.
$$

Problem 5 (25 points). Define $f \in L^2([0, 1])$ by

$$
f(x) = \begin{cases} 1 & x \leq 1/2 \\ -1 & x \geq 1/2. \end{cases}
$$

Show that the coefficients of $f$ in (4.1) are

$$
a_{4k+2} = \frac{4\sqrt{2}}{(4k+2)\pi}
$$

and $a_n = 0$ if $n \not\equiv 2 \mod 4$. Show that

$$
\lim_{N \to \infty} \sum_{n=1}^{4N+2} a_n f_n \left( \frac{1}{2} - \frac{1}{4N+2} \right) = \frac{2}{\pi} \int_0^1 \frac{\sin(\pi x)}{x}.
$$

Remark. The right-hand side is approximately 1.18. Thus the partial sums of the Fourier expansion overshoot by about 9% times the height of the discontinuity at 1/2. This is called the Gibbs phenomenon.