Homework Assignment 2 for 18.965

1. (a) Let A be an $n \times n$ matrix. Show that the infinite series
   (*) $\exp tA = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$

   converges uniformly on compact subintervals of the t-axis.

   (b) Show that $\exp tA$ is differentiable as a function of t and that

$$\frac{d}{dt}\exp tA = (\exp tA)A = A(\exp tA)\,.$$

   Hint: First show that if one differentiates the series above term by term one gets a series which is uniformly convergent on compact intervals.

   (c) Conclude from b) that $\exp tA$ is smooth in t.

2. Let $A = (a_{ij})$ be an $n \times n$ matrix and let v_A be the vector field on $\mathbb{R}^n$:

$$v_A = \Sigma(a_{ij}x_j)\frac{\partial}{\partial x_i}\,.$$

   Show that v_A generates a global one-parameter group of diffeomorphisms of $\mathbb{R}^n$.

   Hint: Let x_0 be an arbitrary point of $\mathbb{R}^n$.

   Show that the curve

$$t \to (\exp tA)(x_0),\ \ -\infty < t < \infty\,,$$

   is the (unique) integral curve of v_A passing through the point x_0.

3. From exercise 2 deduce that $(\exp sA)(\exp tA) = \exp(s+t)A$.

4. Let $GL(n)$ be the group of invertible $n \times n$ matrices and let $\phi : \mathbb{R} \to GL(n)$ be a homomorphism of the additive group of real numbers into $GL(n)$. Assuming ϕ is smooth, prove that there exists a $n \times n$ matrix, A, such that $\phi(t) = \exp tA$ for all t.

5. Let A and B be $n \times n$ matrices. Prove that the following properties are equivalent:

   (i) A and B commute (as matrices).

   (ii) $\exp tA$ and $\exp sB$ commute for all s and t.

   (iii) The Lie bracket of v_A and v_B is zero.

6. Let A be an $n \times n$ matrix. Prove that the following properties are equivalent:

   (i) The transpose of A is $-A$.

   (ii) $\exp tA$ is in $O(n)$ for all $t \in \mathbb{R}$.

   Hint: Let x_0 be an arbitrary point of $\mathbb{R}^n$.

   To prove i. implies ii. Compute

   $$\frac{d}{dt}||(\exp tA)(x_0)||^2$$

   and show that it's identically zero.