σ -conjugacy classes and the Kottwitz map

Let F/\mathbf{Q}_p a finite extension, $\Gamma = \operatorname{Gal}(\overline{F}/F)$, $L \subseteq \overline{F}$ the maximal unramified extension of F. Let v denote the valuation on L, normalized so that $v(\pi_F) = 1$. To an algebraic group G/F we assign the abelian group B(G) of σ -conjugacy classes of elements in G(L). This is naturally a pointed set and can be interpreted as $H^1(\langle \sigma \rangle, G(L))$.

The goal of this note is to define a natural map

$$B(G) \longrightarrow X^*(Z(\widehat{G})^{\Gamma})$$

which will be an isomorphism when G is a torus. We will focus on the case of tori, and then generalize. This is an exposition of (a subset of) Kottwitz's paper "Isocrystals with additional structure."

1 The Kottwitz map

1.1 Tori

We begin with the case of a torus T/F. Then we are looking to define a map

$$B(T) \longrightarrow X^*(\widehat{T}^{\Gamma}) \cong X_*(T)_{\Gamma}.$$

In fact we will define the isomorphism in the opposite direction.

Proposition 1.1. There is a natural isomorphism of functors $A : X_*(-)_{\Gamma} \to B(-)$ from the category of tori over F to the category of abelian groups (unique up to negation).

We normalize so that the resulting map $\operatorname{End}(\mathbf{G}_m) \to B(\mathbf{G}_m)$ sends 1 to 1.

Let us, for the moment, suppose we know that such an F exists. We will be able to compute an explicit formula for what it should be, and then check that this indeed works.

Lemma 1.2. The functor $X_*(-)$ is pro-represented by the pro-torus

$$\mathbf{\Gamma}_F \coloneqq \varprojlim_{E/F} R_{E/F} \mathbf{G}_m$$

where the limit runs over finite Galois extensions E/F and the transition maps are norms.

Proof. For each E/F finite Galois let

$$\mu_E: \mathbf{G}_{m,E} \to (R_{E/F}\mathbf{G}_m)_E \cong \prod_{\Gamma_{E/F}} \mathbf{G}_{m,E}$$

be the map $a \mapsto (a, 1, 1, ...)$ where the first slot corresponds to $1 \in \Gamma_{E/F}$. Let $\mu \in X_*(T)$ with field of definition K, so $\mu : \mathbf{G}_{m,K} \to T_K$. The key claim is that there is a unique map $f_{\mu} : R_{K/F}\mathbf{G}_m \to T$ such that $\mu = f_{\mu} \circ \mu_K$. This is equivalent to saying that there is a unique Galois-equivariant map

$$f_{\mu,K}:\prod_{\Gamma_{K/F}}\mathbf{G}_{m,K}\to T_K$$

which is μ when restricted to the first component. But $\Gamma_{K/F}$ acts on $\prod \mathbf{G}_{m,K}$ by $\sigma(x_g)_g = (x_{g\sigma})_g$, so the desired map is given explicitly as

$$f_{\mu,K}((x_g)_g) = \prod_g g^{-1}\mu(x_g)$$

Compatibility with the transition maps follows from the above stated uniqueness and the compatibility of the maps μ_E with the transition maps.

Proposition 1.3. Let T/F a torus and E/F a Galois extension splitting T. Let $E_0 = E \cap L$. Then for any $\mu \in X_*(T)$ we have

$$A_T(\mu) = N_{E/E_0}\mu(\pi_E) \in T(E_0)$$

where π_E is any uniformizer of E.

Proof. By enlarging E, we can assume there is a map $f : R_{E/F}\mathbf{G}_m \to T$ such that $f \circ \mu_E = \mu$. By functoriality, we have a diagram

We have an equality

$$f(N_{E/E_0}\mu_E(\pi_E)) = N_{E/E_0}f(\mu_E(\pi_E)) = N_{E/E_0}\mu(\pi_E)$$

(since f is defined over F, it is Galois-equivariant on E-points). So we are reduced to the case $T = R_{E/F}\mathbf{G}_m$. In this case, we have a commuting diagram

$$\mathbf{Z}[\Gamma_{E/F}] \longrightarrow B(T)
\downarrow \qquad \qquad \downarrow^{N_{E/F}}
\mathbf{Z} \longrightarrow B(\mathbf{G}_m)$$

and the right-hand downward arrow is an isomorphism. We have another commutative diagram as follows:

(there are two different kinds of norm maps here, one $T \to \mathbf{G}_m$ denoted $N_{E/F}$ and one $E \to E_0$ denoted N_{E/E_0}). The key observation is that $N_{E/F}\mu_E(\pi_E) = \pi_E$, so

$$N_{E/F}(N_{E/E_0}\mu_E(\pi_E)) = N_{E/E_0}(\pi_E) = u\pi_F$$

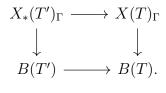
for a *p*-adic unit *u*. So $N_{E/E_0}\mu_E(\pi_E)$ is the element of B(T) that maps to the class of π_F in $B(\mathbf{G}_m)$, so we're done.

Corollary 1.4. Suppose E is unramified (so T is also unramified). Then $A_T(\mu) = \mu(\pi_F)$.

So we now understand the map $X_*(-)_{\Gamma} \to B(-)$ completely explicitly for tori. It is easy to check that this is a natural transformation of functors. We also note that if $\mu = \mathrm{id} \in$ $\mathrm{End}(\mathbf{G}_m)$ then we have $A_{\mathbf{G}_m}(\mathrm{id}) = 1 \in B(\mathbf{G}_m) \cong \mathbf{Z}$. It follows that $A_{\mathbf{G}_m}$ is bijective. It follows that A_T is bijective for any $T = R_{E/F}\mathbf{G}_m$ and consequently for any product of such tori.

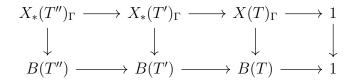
Lemma 1.5. For any torus T, the map $A_T : X_*(T)_{\Gamma} \to B(T)$ is surjective.

Proof. There is some torus T' of the form $R_{E/F}\mathbf{G}_m$ which surjects onto T. Since $A_{T'}$ is an isomorphism and $B(T') \to B(T)$ is surjective, we get the desired conclusion via commutativity of the diagram



Proposition 1.6. For any torus T, the map $A_T : X(T)_{\Gamma} \to B(T)$ is bijective.

Proof. Choose T' as in Lemma 1.5 and let T'' be the kernel of $T' \to T$. Then we have $A_{T''}: X_*(T'')_{\Gamma} \to B(T'')$ is surjective. Applying the four-lemma to the diagram



yields the desired result.

1.2 Groups with simply connected derived subgroup

We now want to extend this to reductive groups. For the sake of simplicity, we will assume that G_{der} is simply connected (which will suffice in many applications) but what follows can be done for a general connected reductive group. Define the torus $T \coloneqq G/G_{der}$. Then we have canonical maps

$$B(G) \longrightarrow B(T) \xrightarrow{\sim} X_*(T)_{\Gamma} \xrightarrow{\sim} X^*(\widehat{T}^{\Gamma})$$

so it will suffice to see that $\widehat{T} = Z(\widehat{G})$. This is a general fact about reductive G with simply connected G_{der} (since the dual of a simply connected group is centerless).

In fact, this map is surjective and we can identify a natural subset of B(G) which bijects onto $X^*(Z(\widehat{G})^{\Gamma})$.

2 σ -conjugacy classes and the slope morphism

2.1 The slope morphism

We again begin by working in the category of tori over F. We show that there is a functorial homomorphism $B(-) \to X_*(-)^{\Gamma}_{\mathbf{Q}}$ for tori. This is easier; indeed an element $b \in T(L)$ defines an element of $\operatorname{Hom}(X^*(T)^{\Gamma}_{\mathbf{Q}}, \mathbf{Q}) \cong X_*(T)^{\Gamma}_{\mathbf{Q}}$ via

$$\lambda \mapsto v(\lambda(b)).$$

Since we are only looking at Galois equivariant characters, changing b to a σ -conjugate will change $\lambda(b)$ to a σ -conjugate, which does not change the valuation. So this is well-defined as a map from B(T).

We want a generalization to connected reductive groups (at least with simply connected derived subgroup). Let **D** denote the pro-torus over F with $X^*(\mathbf{D}) = \mathbf{Q}$. Explicitly,

$$\mathbf{D} = \varprojlim_{n \in \mathbf{N}} \mathbf{G}_m$$

where the transition maps are the power maps. Then $X_*(T)^{\Gamma}_{\mathbf{Q}} = \text{Hom}(\mathbf{D}, T)$. The correct generalization is then to show that \mathbf{D} represents B(-) in the category of connected reductive groups over F.

Let G be a reductive group, and let ρ be a faithful representation on an L-vector space V. There is also a natural map from B(G) to isocrystal structures on V (a σ -semilinear endomorphism on V) where $g \in G(L)$ is assigned to the map $\Phi = g(1 \otimes \sigma) : V \otimes L \to V \otimes_{\sigma} L$. This gives rise to a slope decomposition on V, which is equivalent to a **Q**-grading and hence an action of **D**. So we get a corresponding map $\alpha_{\rho,g} \in \text{Hom}(\mathbf{D}, \text{GL}(V))$.

Thus, the choice of $g \in G(L)$ determines a map of Tannakian categories $\operatorname{Rep}(G) \to \operatorname{Rep}(\mathbf{D})$ and hence an element $\nu_g \in \operatorname{Hom}(\mathbf{D}, G)$. The assignment $g \mapsto \nu_g$ gives a functorial map $\nu : B(G) \to \operatorname{Hom}(\mathbf{D}, G)$.

2.2 Basic elements

Definition 2.1. We say that $g \in G(L)$ is *basic* if $\nu_g : \mathbf{D} \to G$ factors through Z(G). We write $B(G)_b$ for the basic σ -conjugacy classes.

Now suppose G_{der} is simply connected and let $T = G/G_{der}$ as before.

Proposition 2.2. The map $B(G)_b \to B(T)$ is bijective. In particular, the restriction

$$B(G)_b \to X_*(Z(\widehat{G})^{\Gamma})$$

is bijective.

This identifies a natural subset of B(G) which bijects onto the image of the Kottwitz map; we do not include the proof here.