Modular curves / $\mathbb{Z}$.

x

Last time: $Y(N)$ / $\mathbb{Z}[\frac{1}{N}]$: parametrized $\{(E/S, \ldots, E[N]\}$
$6, 7, -$

How to extend this to a moduli problem / $\mathbb{Z}$?

in char. p, E/k, $E[p] \simeq 0$ on $\mathbb{Z}/p$.
k-field, alg. closed

A: Drinfeld level structure. $E[p] \subset E$ subgroup-scheme, Cartier divisor.

<u>Defn:</u> a Drinfeld $\Gamma(N)$-structure on E/S is

a map $\phi: (\mathbb{Z}/N)^2 \to E(S)$, s.t. $E[N] = \sum_{(a,b \in \mathbb{Z}/N^2)} \phi(a,b)$

as Cartier divisors.

Ex: 1) E/k ~~ss~~ supersingular, k - alg. closed.

$\Gamma(p^n)$-structure: $(\mathbb{Z}/p^n)^2 \xrightarrow{\phi} E$ | $E[p^n] = \sum \phi(a,b)$.
$\qquad \shortparallel$
$\qquad p^{2n} \cdot [0]$.

2) E/k ordinary; $(\mathbb{Z}/p^n)^2 \twoheadrightarrow E[p^n]$.

Defn: $\Gamma_1(p^n)$: $E \xrightarrow{f} E'$ p^n-isog., w/ a generator of $\ker(f)$. i.e. an $x \in E(S)$ s.t.

$$\ker(f) = \sum_{i=\cancel{0}1}^{N} [i \cdot x].$$

Defn: $\Gamma_0(p^n)$: $E \xrightarrow[N\text{-isog.}]{f} E'$ s.t. $\ker(f)$ is cyclic, i.e. admits a generator (fppf-loc).

$Y_1(p^n) \longrightarrow Ell$

Wednesday, June 22, 2022 7:11 PM

Thm (Katz-Mazur): $p^n \geq 4$. $Y(p^n)$, $\cancel{Y_0(p^n)}$, $Y_1(p^n)$ are represented by regular schemes; $Y_?(p^n) \to Ell$ is finite, flat.
2-dim.

$\mathbb{Z}[x,y]/(xy=p)$

$Y(N)$ $N \geq 3$.

How to prove this:

Thm ("axiomatic regularity"): $\mathcal{P}$ – (rigid.) moduli problem on elliptic curves

$$\mathcal{P} \xrightarrow{f} (Ell)$$

Then if: 1) f is rel. representable, finite;

2) étale over $\mathbb{Z}[\frac{1}{p}]$

3) only depends on underlying p-div. group.

4) k-alg. closed of char. p, E_0/k supersingular, then:

$W := W(k)$.

a) $\mathcal{P}(E_0/k) = *$. $\mathcal{P}_{\tilde{E}/W[[t]]} \overset{\text{rep.}}{\simeq} (A, \mathfrak{m})$, $\mathfrak{m}$ is gen'd by 2 elts.

b) $\tilde{E} / \underline{W(k)[[T]]}$ univ. formal deformation.

$\Longrightarrow$ $\mathcal{P}$ is rep., regular.

We will check last condition.

Fix $k = \bar{k}$ char p. $F : Art_k \to Sets$. $(F(k) = *.)$

$Art_k =$ category of local Artin rings, w/ res. field $\simeq k$.

F is representable by (A, m)

$\underline{Q}$: given $f_1, f_2, \dots, f_n \in m$, when is $m = (f_1, \dots, f_n)$ true?

A: when $g : A \to R$ sends all $f_i \mapsto 0$ $\Leftarrow\Rightarrow$ the corresponding object O over R

$\mathrm{Spec}\, A/m \hookrightarrow \mathrm{Spec}\, A$.
$A/m \cong k$

$\hat{F}(R)$ is "constant", i.e. R is a k-alg., O is pulled back along $k \to R$.

In our case: E_0/k, $\tilde{E} / W[[T]]$ univ. form. def.,

$k = \bar{k}$ $R \mapsto P(R) = \{(E/R$ ell. curve, w/ P_level str$\}$ is represented by some (A, m).

$P = \Gamma(p^n)$.

$\Gamma = \Gamma(p^n)$.

$R \mapsto \{ (E/R,\ \phi : (\mathbb{Z}/p^n)^2 \to E(R)$ s.t. $\sum \phi(a,b) = E[p^n] \}$.

E_0/k. X-coordinate for $\hat{E}$, the formal group. $k[[T]]$.

$\phi \Leftrightarrow P, Q \in E(R)$. $f, g \in m_A$ are "$X(P), X(Q)$".

WTS: given E/R, $(0,0)$ is a Dr. p^n-level structure $\Leftrightarrow$

$R \in \mathrm{Art}_k$. $\Leftrightarrow p = 0$ on R, & $E \simeq R \otimes_k E_0$.

Pf: $(0,0)$ is a Dr. lev. str $\Rightarrow E[p^n] = p^{2n} \cdot [0]$ as subschemes

$F : E \to E^{(p)}$. $\ker F^{2n} = p^{2n} \cdot [0]$. $\Rightarrow E / E[p^n] \simeq$

$E \simeq E^{(p^{2n})} \simeq E^{(p^{4n})} \simeq \cdots \simeq E / \ker F^{2n}$.

$\Rightarrow E$ is constant. R local Artin k-alg., $\Rightarrow E$ constant. $\square$

E/R, (also find $p = 0$ on R.)

Bad reduction of modular curves.

Goal: describe $\Gamma_0(p^n)$, $\Gamma(p^n)$ $\otimes_{\mathbb{Z}} \mathbb{F}_p$.

Consider $[p^n\text{-isog.}]_{/k}$: $R \mapsto \{(E/R, E \to E' \ p^n\text{-isog.})\}$

$\cong \{(E/R, \Gamma \subset E[p^n] \mid \mathrm{rk}\, \Gamma = p^n\}$

$\Gamma_0(p^n) \subset [p^n\text{-isog}]$:

$\| $

$\{(E/R, \Gamma \subset E[p^n] \mid \Gamma \text{ is cyclic of rank} = p^n\}$

Crossings theorem:

$$\begin{tikzcd} X \arrow[d, "\pi"] & \bigsqcup_{i \in I} Z_i \arrow[l, hook', "i"'] \arrow[dl] \\ Y \arrow[d] & \\ \mathrm{Spec}(k) & \end{tikzcd}$$

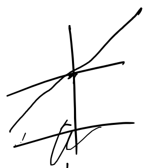
$Y^{ss} \subset Y$, $\forall y_0 \in Y^{ss}$, $\exists!\, x_0 \in X$, $x_0 \mapsto y_0$

$\forall i$, $\exists!\, z_0 \in Z_i$ s.t.

$$\begin{tikzcd} x_0 \arrow[d, mapsto] & z_0 \arrow[l, mapsto] \arrow[dl, mapsto] \\ y_0 & \end{tikzcd}$$

Y - smooth curve,

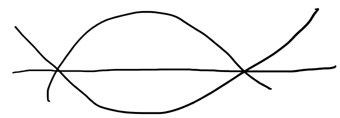
$X \to Y$ finite flat, Z_i,



$Z_i \hookrightarrow X$ closed, Z_i^{red} smooth.

Y - connected over $Y \setminus Y^{ss}$, $X \simeq \bigsqcup Z_i \big|_{Y^{ord}}$.

$\hat{\mathcal{O}}_{X,x_0} \simeq k[[x,y]]/(f)$.



Thm: in this case, the Z_i are the irr. components of X,

$\forall$ s.s. x_0, $\hat{\mathcal{O}}_{X,x_0} \simeq k[[x,y]]/\prod_{i \in I} f_i^{e_i}$, where $\hat{\mathcal{O}}_{Z_i} \simeq k[[x,y]]/f_i^{e_i}$

Let us explain these conditions for [p^n-isog.].

<u>Lemma?</u>: E/S ordinary ell. curve, $\Gamma \subset E$ p^n-subgroup. then Zariski-locally in S, $\exists!$ (a,b) s.t. $a, b \geq 0$, $a+b=n$, s.t.

1) $\Gamma \cap \ker F^n = \ker F^a$; 2) $\Gamma / \ker(F^a)$ is a fin. etale cyclic subgroup, of rk p^b.

$S = \mathrm{Spec}(k)$, $k = \bar{k}$. $E[p^\infty] = \underline{\mu_{p^\infty} \times \mathbb{Q}_p/\mathbb{Z}_p}$.

<u>Cor</u>: $E \xrightarrow{f} E'$ p^n-isogeny. then $\exists!$ (a,b) s.t.

$$E \xrightarrow{F^a} E^{(p^a)} \simeq E'^{(p^b)} \xrightarrow{V^b} E'. \quad (E\text{-ordinary.})$$

Observe: E/k - supersingular, $\Gamma \subset E$ p^n-subgroup. then

$$\Gamma = \ker(F^n).$$

E - ordinary, $E \xrightarrow{f} E'$ p^n-isogeny. Then $\exists!$ (a,b) $a+b=n$ s.t.

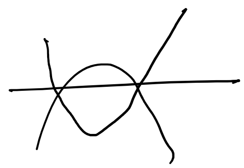
$$E \xrightarrow{F^a} E^{(p^a)} \xrightarrow{\sim} E'^{(p^b)} \xrightarrow{V^b} E'.$$

$P_{a,b,n} = [p^n\text{-isog}, (a,b)]$ - moduli scheme parametrizing such.

$$Ell \times Ell \xrightarrow{(F^a, F^b)} Ell \times Ell$$

$$(E, E') \mapsto (E^{(p^a)}, E'^{(p^b)}).$$

$$P_{a,b,n} \simeq (F^a \times F^b)^{-1}(\Delta).$$



Thm: $[p^n\text{-isog}] = \bigcup_{a+b=n} P_{a,b,n}$.

$$\begin{array}{c} [p^n\text{-isog}] \\ \downarrow \\ Ell \end{array}$$: fiber over any s.s. E is $*$. and ord E is $\bigsqcup_{a+b=n} (-)$.

<u>Thm:</u> $[\Gamma_0(p^n)] \otimes \mathbb{F}_p = \bigtimes_{a+b=n} [\Gamma_{0,a,b}(p^n)]$.

$n=1$: $[\Gamma_0(p)] = [p\text{-isog.}]$.

$\Gamma_0(p) \otimes \mathbb{F}_p = \underset{\substack{a=1\\b=0}}{Ell} \times \underbrace{Ell}_{mult = p}$ $\quad \begin{matrix} a=0 \\ b=1 \end{matrix}$ $\quad (E, E')$ $\quad (E')^{(p)} \simeq E$.

$\boxed{\Gamma_0(p)^{\text{coarse}} = }$ $\quad Ell \xrightarrow{F} Ell$.

Final example: $\Gamma(p^n)$.

$\{E/k, \ \phi : (\mathbb{Z}/p^n)^2 \twoheadrightarrow E[p^n]\}$.

E s.s.: $\exists !\ \phi$.

E ord.: $\phi : (\mathbb{Z}/p^n)^2 \to \mathbb{Z}/p^n$, i.e. $\phi \in \mathbb{P}^1(\mathbb{Z}/(p^n))$. $\square$