Archimedean local heights

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Notes for Summer 2022 Gross-Zagier learning seminar

First: arithmetic intersection theory
Later: specialize to modular curves.

Review Gross-Zagier

$$L'(\cdots)\text{ "=" }\underbrace{\langle\ \ ,\ \ \rangle_{NT}}_{}$$

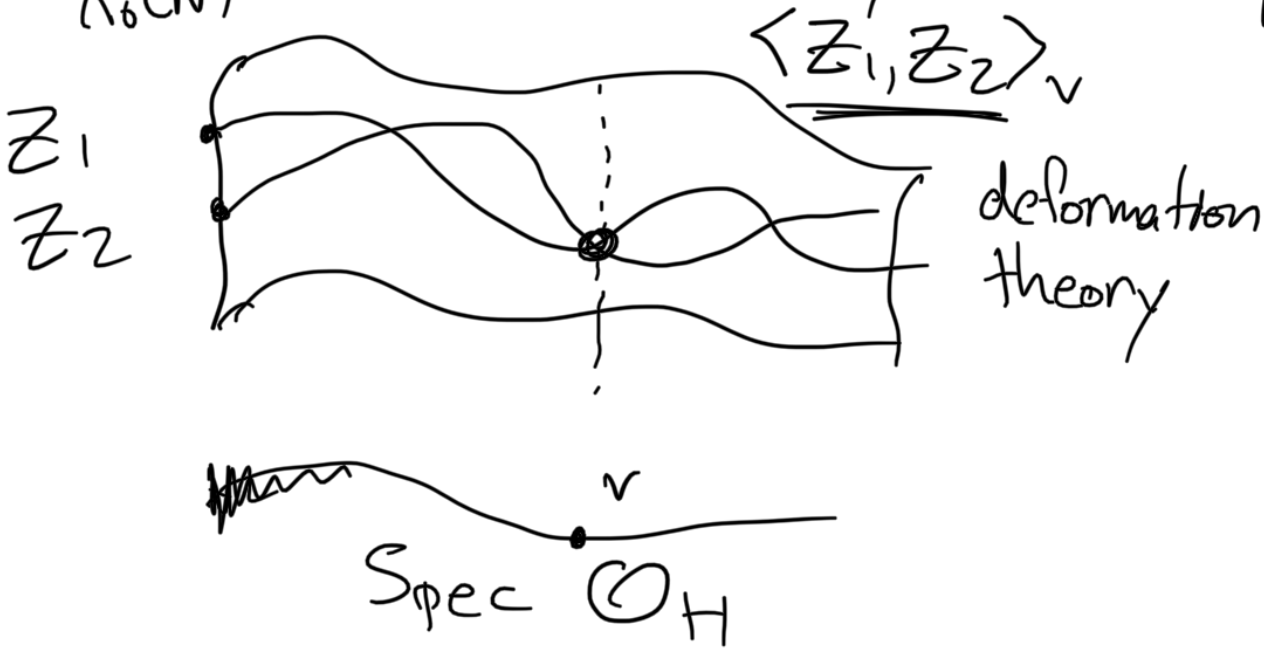
Heegner divisors

Néron-Tate height $J(X_0(N))$

$$= \sum_{v<\infty} \underbrace{\langle\ \ ,\ \ \rangle_v}_{} + \boxed{\sum_{v|\infty} \langle\ \ ,\ \ \rangle_v}$$

Intersection theory   Today

$X_0(N)$



Today X is smooth proj. connected variety / $\mathbb{C}$, e.g. $X = X_0(N)_{\mathbb{C}}$

If X is a curve:

$$\mathrm{Div}^0(X) \times \mathrm{Div}^0(X) \xrightarrow{\langle\ ,\ \rangle_\infty} \mathbb{R}$$

- defined when disjointly supp.
- symm. bilinear
- $y \longmapsto \langle a, y\rangle_v$ is continuous ($y = \sum m_i y_i$)

* for f mero.

$$\langle \mathrm{div}(f), \sum m_i y_i \rangle_\infty = \sum m_i \log|f(y_i)|^2.$$

can interpret as archimedean part of arithmetic intersection theory

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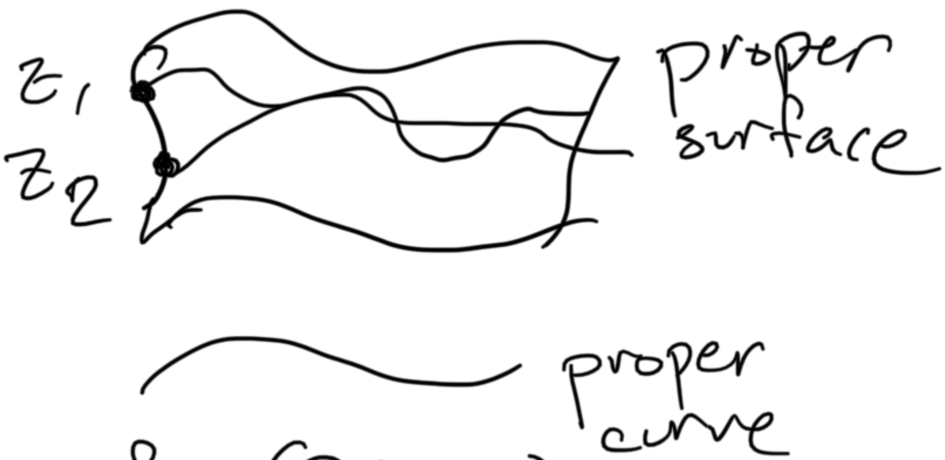
usual intersection



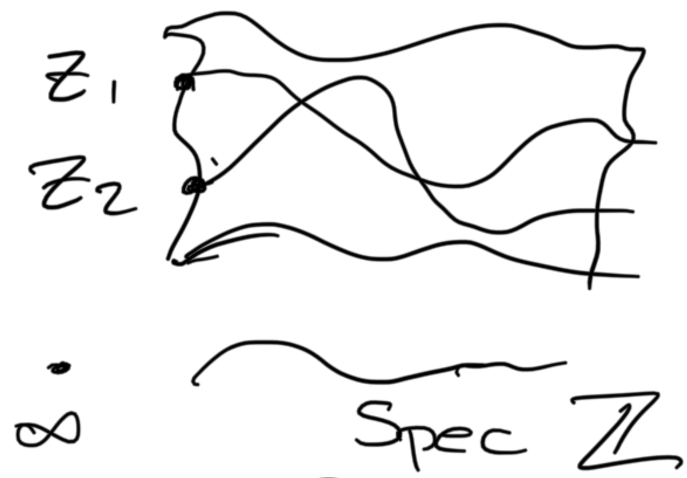
$\mathcal{L} = \mathcal{O}(z_2)$

$z_1 \cdot z_2 = \deg \mathcal{L}|_{z_1}$

$= \deg(s)$

any rational section of $\mathcal{L}|z_1$

arithmetic



$z_1 \cong \mathrm{Spec}\,\mathbb{Z}$.

$\mathcal{L} = \mathcal{O}(z_2)$.

"$\deg \mathcal{L}|_{\mathrm{spec}\,\mathbb{Z}}$" bad!

s rat. section.

$\|s\|_p = p^{-v_p(s/u_p)}$,

$u_p \cdot \mathbb{Z}_p = \mathcal{L} \otimes \mathbb{Z}_p$.

$\deg(s) = \sum_{p<\infty} \log \|s\|_p$

$\hat{\mathcal{L}} = (\mathcal{L}, \| \; \|_\infty)$ metric on $\mathcal{L} \otimes \mathbb{R}$.

$\widehat{\deg}\, \hat{\mathcal{L}} = \deg(s) = \sum_{p<\infty} \|s\|_p + \log \|s\|_\infty$.

indep. of choice $\left(\prod_{v \le \infty} \|a\|_v = 1 \right.$ for all $a \in \mathbb{Q}^\times$)

$\deg(s) = \deg(as)$.

Let X be sm. proj. conn. variety over $\mathbb{C}$. (if instead Spec $\mathcal{O}_H$)

Def A Hermitian line bundle on X is

$$\hat{\mathcal{L}} = (\mathcal{L}, \| \ \|)$$

$\mathcal{L}$: line bundle, $\| \ \|$: smooth Hermitian metric.

$$x \in X, \quad \mathcal{L}_x \xrightarrow{\| \ \|_x} \mathbb{R}.$$

Def $\widehat{\mathrm{Pic}}(X) = \{$ iso. classes of Hermitian line bundles. $\}$

$\hat{\mathcal{L}} \in \widehatiPic(X)$. mero section $s: X \dashrightarrow \mathcal{L}$ $\quad x \mapsto \|s\|_x$, $X \dashrightarrow \mathbb{R}$

$$(\hat{\mathcal{L}}, s) \longmapsto (\mathrm{div}(s), -\log \|s\|^2)$$

$$((\mathcal{O}(Z), \| \ \|), 1) \longleftarrow (Z, g)$$

h a local holo. section $\quad \mathcal{O}(Z) \hookrightarrow K(X)$

$$\|h\|^2 = |h|^2 e^{-g} \longleftarrow \boxed{\hat{Z}^1(X)}$$

$h: X \dashrightarrow \mathbb{C} \qquad \mathcal{O}(Z) \hookrightarrow K(X)$

Def Let Z be a divisor on X.
A Green function for Z on X is a smooth function $g: X \setminus |Z| \to \mathbb{R}$, s.t. $g \in L^1_{loc}(X)$ s.t. $\boxed{\text{Green current equation}}$

$$\frac{-1}{2\pi i} \partial\bar{\partial} g + \delta_Z = \omega_Z$$

for some smooth $(1,1)$-form ω_Z.

$$\bar{\partial} g = \left(\sum_{z_i} \frac{\partial g}{\partial \bar{z}_i} d\bar{z}_i \right) \quad \partial g \ldots$$

— as currents, (i.e., as functionals on $(n-1, n-1)$ forms.),

$$\delta_Z(\alpha) = \int_Z \alpha .$$

<u>Ex</u> $\hat{\mathcal{L}} = (\mathcal{L}, \|\ \|)$, $Z = \mathrm{div}(s)$ mero.

$$\underline{g = -\log\|s\|^2}$$

$$\frac{-1}{2\pi i}\partial\bar{\partial}\underbrace{g} + \delta_Z = \underbrace{c_1(\hat{\mathcal{L}})}_{\substack{\text{locally,}\\ \frac{1}{2\pi i}\partial\bar{\partial}\log\|s'\|^2}} .$$

(Poincaré–Lelong)

holo.

$$\hat{Z}^1(X) = \{(Z, g_Z)\}$$

divisor, Green function

$$(Z_1, g_{Z_1}) + (Z_2, g_{Z_2}) = (Z_1 + Z_2, g_{Z_1} + g_{Z_2}).$$

$$\to \hat{Z}^p = \{(Z, \underline{g_Z})\}$$

Green <u>current</u>.

<u>Ex</u> $X = \mathbb{P}^1$ $\quad((\infty), \log(1+|z|^2)) \in \hat{Z}^1(X)$

<u>Ex</u> $\hat{\mathcal{L}} = (\mathcal{O}_X, \|\ \|)$, $\underline{\|1\|} = 1$.

$$(\mathrm{div}(f), \underbrace{-\log|f|^2}) \in \hat{Z}^1(X)$$

principal arithmetic divisor $\hat{R}^1(X)$.

$$\left[\ \widehat{\mathrm{Pic}}(X) \xrightarrow{\ \sim\ } \widehat{\mathrm{CH}}^1(X) := \hat{Z}^1(X)/\hat{R}^1(X)\right.$$

$$(\mathcal{L}, \|\ \|) \longmapsto [(\mathrm{div}\, s), -\log\|s\|^2]$$

$\ldots$, $\ldots$ $\eta \ldots$].

define $\widehat{CH}^p$ similarly

Intersections commutative bilinear.

$$\widehat{CH}^1 \times \widehat{CH}^1 \xrightarrow{\cap} \widehat{CH}^2$$

$$(Z_1, g_{Z_1}), (Z_2, g_{Z_2}) \longmapsto (Z_1 \cap Z_2, g_{Z_1} * g_{Z_2}).$$

$*$-product.

$$g_{Z_1} * g_{Z_2} = g_{Z_1} \wedge \underbrace{\delta_{Z_2}}_{\text{Dirac } \delta} + \omega_{Z_1} \wedge g_{Z_2}$$

$$\underbrace{\omega_{Z_1}}_{\text{smooth }(1,1)\text{-form}} = \frac{-1}{2\pi i} \partial\bar\partial g_{Z_1} + \delta_{Z_1}$$

Specialize to X a curve (on $\mathbb{C}$).

$Z \in \mathrm{Div}(X)$, $\sum_x m_x x$

$$g_Z = \underbrace{-m_x \log|z_x|^2}_{\text{near } x \in X} + \text{smooth},$$

z_x is uniformizer at x.

$$\frac{-1}{2\pi i} \partial\bar\partial g_Z + \delta_Z = \omega_Z.$$

Fact If $Z \in \mathrm{Div}^\circ(X)$, can pick g_Z s.t. $\omega_Z = 0$. (secretly ω_Z is Chern form).

$$\int_X \omega_Z = \deg(Z).$$

(can arrange $\omega_Z = \deg Z \cdot \alpha$. fixed α, $\int \alpha = 1$.)

Lemma Such g_Z is unique up to additive constant.

to additive constants.

<u>Pf</u> Suppose g_Z, h_Z smooth function on X

$$\partial\bar\partial(g_Z - h_Z) = 0.$$

$g_Z - h_Z$ is harmonic, everywhere on X.

maximum principle $\Rightarrow$ $g_Z - h_Z$ is constant. $\square$.

<u>Rmk</u>. For such g_Z,

$\Updownarrow$
- g_Z is harmonic away from Z.
- g_Z has correct log. asymptotics at Z.

$Z_i \in \mathrm{Div}^0(X)$. (disjoint supp.)

$$\widehat{Z}^1 \times \widehat{Z}^1 \xrightarrow{\ \frown\ } \widehat{Z}^2$$

$$(Z_1, g_{Z_1}) \cdot (Z_1, g_{Z_1}) \longrightarrow (0, g_{Z_1} * g_{Z_2}).$$

assume, $\omega_{Z_1} = \omega_{Z_2} = 0$.

$$\widehat{Z}^2 \xrightarrow{\deg_\infty} \mathbb{R}$$

$$(0, \omega) \longmapsto \int_X \omega.$$

e.g. (1,1)-form real form.

Archimedean local height.

$$\underline{\langle Z_1, Z_2\rangle_\infty} = -\deg_\infty\left(\underline{(Z_1, g_{Z_1})\cdot(Z_2, g_{Z_2})}\right)$$

$$= -\deg_\infty(0, g_{Z_1} * g_{Z_2}).$$

$$\left[\ g_{Z_1} * g_{Z_2} = g_{Z_1} \wedge \delta_{Z_2} + \underbrace{\omega_{Z_1}} \wedge g_{Z_2}\right.$$

$\langle z_1, z_2 \rangle_\infty = -\int_X g_{z_1} \wedge \delta_{z_2}$

$z_2 = \sum m_x x$ $\quad = \boxed{-\sum m_x g_{z_1}(x)}$.

since $\deg z_2 = 0$, $g_{z_1} \leftrightarrow g_{z_1} + C$

Next, specialize to $X = X_0(N)$.

For $\langle \, , \, \rangle_\infty$, compute g_z for $z \in \mathrm{Div}^0(X)$ w/ $\omega_z = 0$.

$g_{z_1} + g_{z_1} = g_{z_1 + z_2} + C$.

linearity, enough to do

$z = (x) - (\infty)$. Green fn.

$G(x,y) := g_{(x)-(\infty)}(y)$

- harmonic in y (away from x, ∞).
- log singularities at $y = x$ $\quad y = \infty$.

$X_0(N) = \Gamma_0(N) \backslash \mathcal{H}$

$G : (\mathcal{H} \times \mathcal{H}) \setminus (\text{diag.}) \longrightarrow \mathbb{R}$

$G(x,y)$ $\Gamma_0(N)$-inv't in both variables.

Harmonic in second var.

appropriate log asymp.

$y \longrightarrow x$

$y \to \infty$

$y \to$ cusps.

Try $\tilde{G}(x,y) = \dfrac{\log|x-y|^2}{\log|\bar{x}-y|^2}$

Fact: (1) $\tilde{G}(\gamma x, \gamma y)$ for $\gamma \in \underline{PSL_2(\mathbb{R})}$.
$\qquad = \tilde{G}(x,y)$

(2) Harmonic in y

(3) $y \to x$ asymp. ✓.

$\tilde{G}(\gamma x, \gamma' y) \neq \tilde{G}(x,y)$

$\gamma, \gamma' \in \Gamma_0(N)$

Try $G(x,y) = \sum_{\gamma \in \Gamma_0(N)} \tilde{G}(x, \gamma y)$

diverges.

Try again: $\tilde{G}$ w/ (1), (3).

(2') $\Delta \tilde{G}(x,-) = \varepsilon \tilde{G}(x,-)$.

(then send $\varepsilon \to 0$)

(1) $\Rightarrow$ $\tilde{G}(x,y)$ is a function of hyperbolic distance $d(x,y)$.

$t = \cosh(d(x,y))$.

$\tilde{G}(x,y) = Q(t)$.

(2') $\Rightarrow$ (if $\varepsilon = s(s-1)$).

$$(1-t^2)\frac{d^2Q}{dt^2} - 2t\frac{dQ}{dt} + s(s-1)Q = 0.$$

(3) + ODE theory.

$$Q(t) = -2\int_0^\infty \left(t+\sqrt{t^2-1}\cosh t\right)^{-s} dt$$

$$\widetilde{G}(x,y,s) = Q(s,t).$$

$$\sum_{\gamma\in\Gamma_0(N)} \widetilde{G}(x,\gamma y,s) =: G(x,y,s).$$

converges for $\mathrm{Re}(s) > 1$,

log singularity on diagonal,

mero. cont. to $\mathrm{Re}(s) \geq 1/2$

$s \to 1$, pole at $s=1$,

subtract off Eisenstein series......

References

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§ II

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