Legendrian Classification of Twist Knots

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joint work with: J. Etnyre
L. Ng

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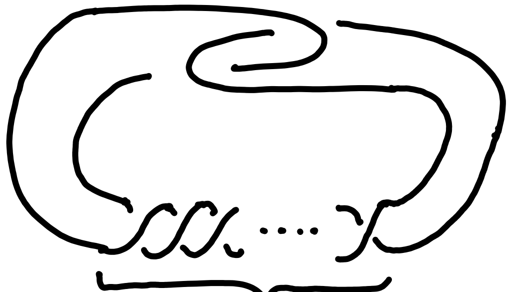
(joint work with J. Etnyre & L. Ng)

twistknot:



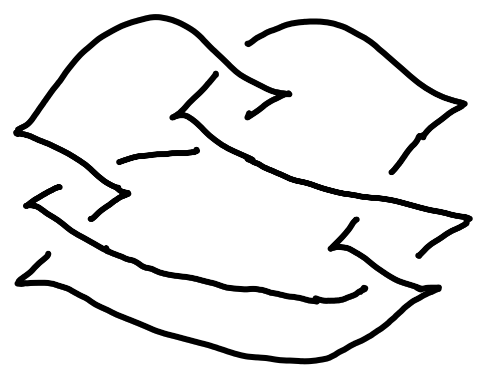
$=: K_n$

n half twists

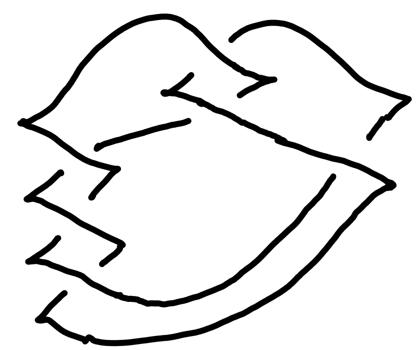
Thm (Chekanov, 1997)

The classical invariants are not sufficient to classify Legendrian isotopy classes of a given knot type.

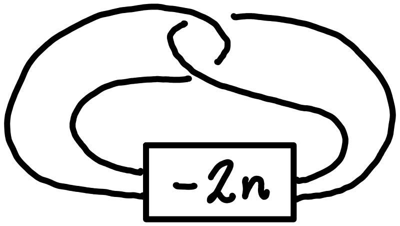
$m(5_2)$



$\overset{\text{Leg.}}{\not\simeq}$



Thm (Epstein – Fuchs – Meyer, 2001)



K_{-2n}

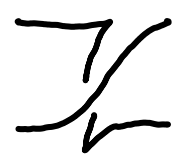
has (at least) n non-Legendrian isotopic Legendrian representations (with the same classical invariants):

$$\mathcal{E}(k,l) \overset{Leg}{\cong} \mathcal{E}(k',l') \iff \begin{array}{l} k+l = k'+l' = 2n-2 \\ \{k,l\} = \{k',l'\} \end{array}$$

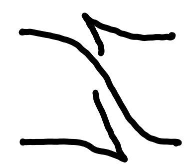
Thm (Etnyre – Ng – V, 2009)

K_{-2n} has exactly $\left\lfloor \frac{n^2}{2} \right\rfloor$ different Legendrian representations of the form:

in the box:



or



"Z" "S"

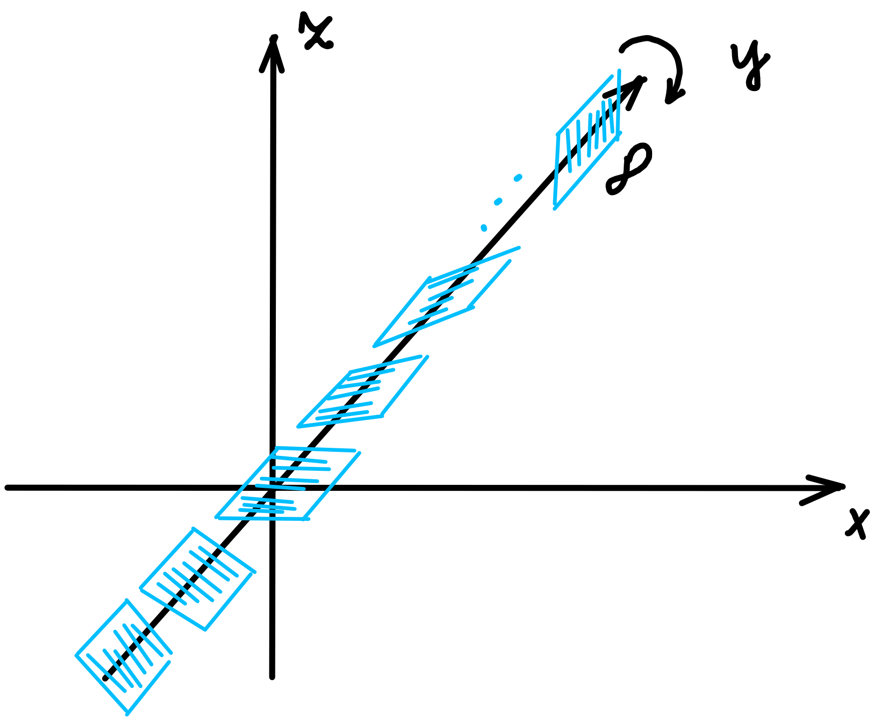
Rmk: for $m(5_2)$ $(n=2)$ $n = \left\lfloor \frac{n^2}{2} \right\rfloor = 2$

Contact Structures

a totally non-integrable planefield on a 3-manifold

standard contact structure on $\mathbb{R}^3$: $\xi_{st} = \ker(dz - y\,dx)$

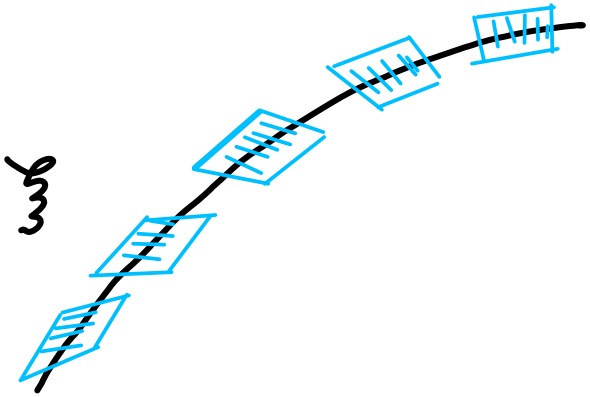
$$= \left\langle \frac{\partial}{\partial y},\ y\frac{\partial}{\partial z} + \frac{\partial}{\partial x} \right\rangle$$



Thm (Darboux) Every contact structure is locally isotopic to ξ_{st}.

Legendrian Knots

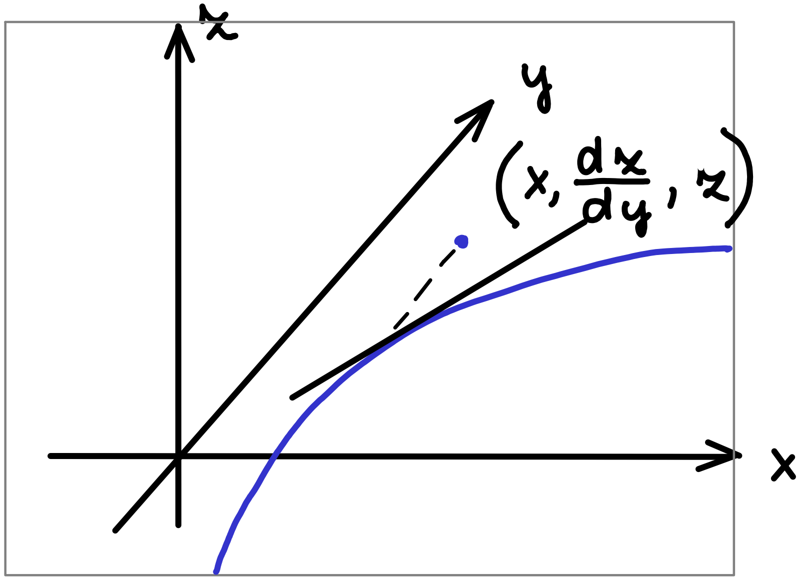
a knot K is Legendrian if $TK \in \xi$



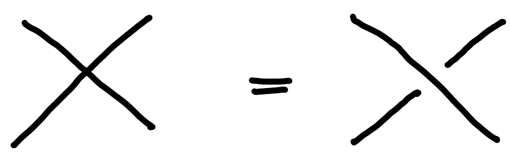
Fact: Every knot can be put in Legendrian position

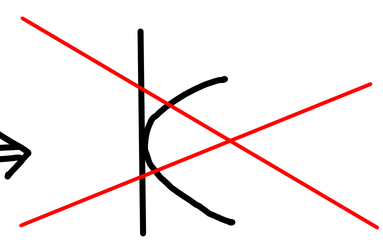
in $(\mathbb{R}^3, \xi_{st})$: $TK \in \xi_{st} = \ker(dz - y\,dx) \iff y = \frac{dz}{dx}$

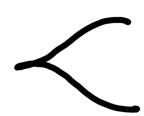
projection to the (x,z)-plane: front projection



the slope determines the y-coordinate:

$\Longrightarrow$ • 

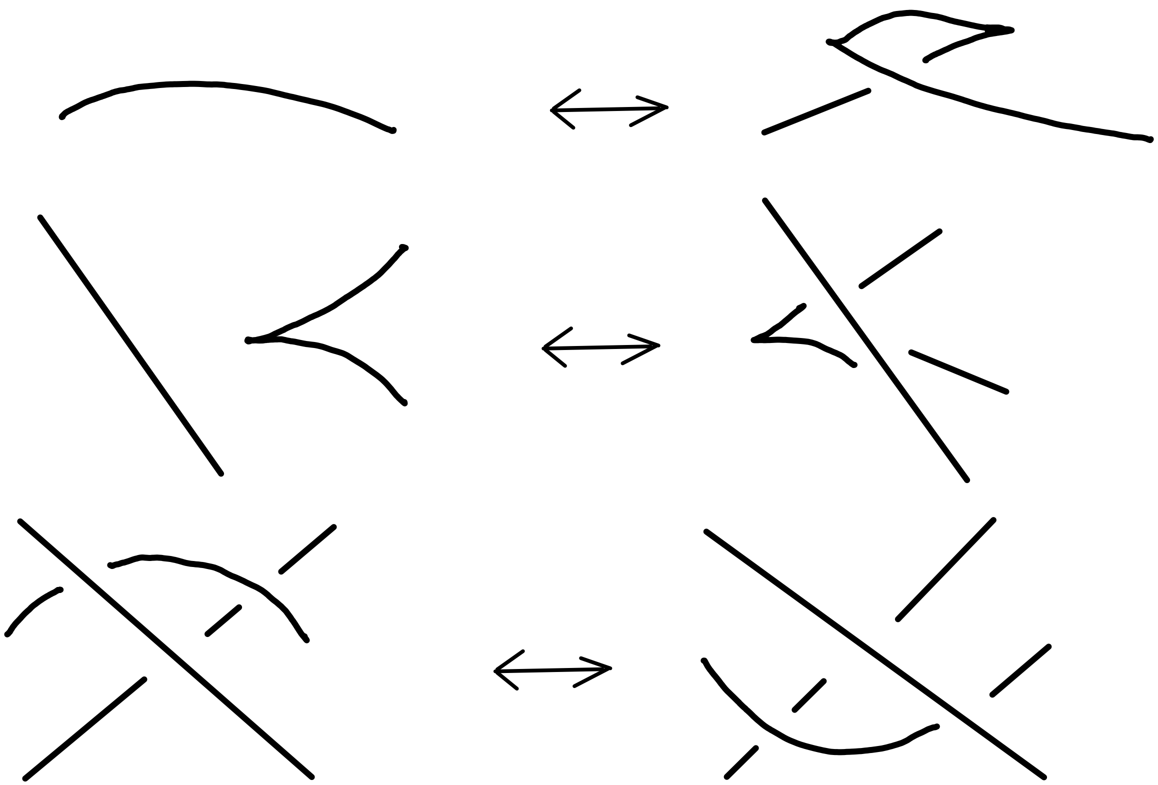
• $\infty \neq y = \frac{dz}{dx}$ $\Rightarrow$ 

instead: 

EQUIVALENCE OF LEGENDRIAN KNOTS

Legendrian isotopy: isotopy through Legendrian knots

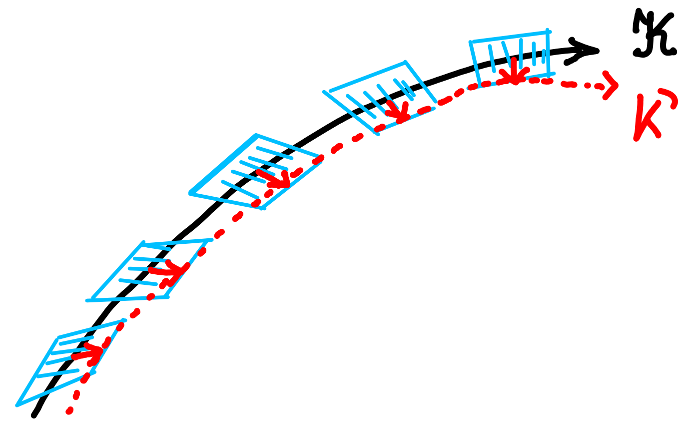
Legendrian Reidemeister moves



Thm two front projections correspond to Legendrian isotopic knots $\Leftrightarrow$ related by a sequence of Legendrian Reidemeister moves

Classical Invariants

Thurston-Bennequin invariant



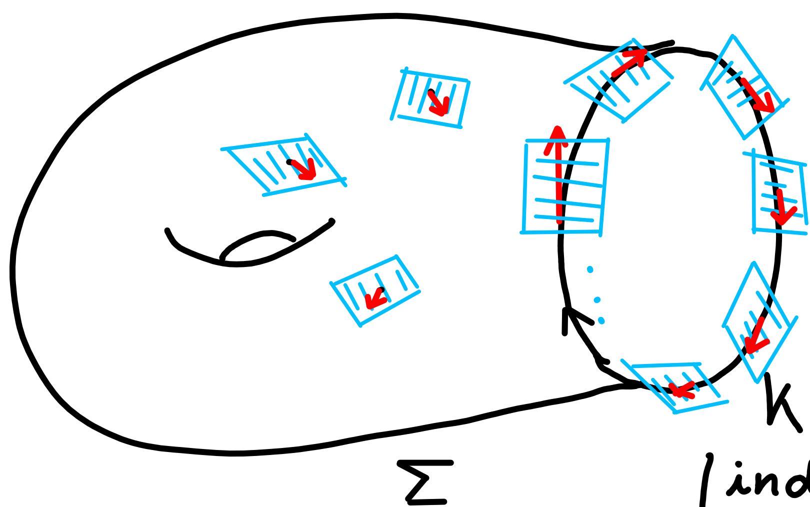
K': push off of $\mathcal{K}$ in ξ

$tb(\mathcal{K}) = lk(\mathcal{K}, K')$ ($\mathcal{K}$ is nullhomologous)

rotation number

Σ a Seifert surface of $\mathcal{K}$

$rot_\Sigma(\mathcal{K})$ is the relative euler number of ξ on Σ w.r.t. $T\mathcal{K}$

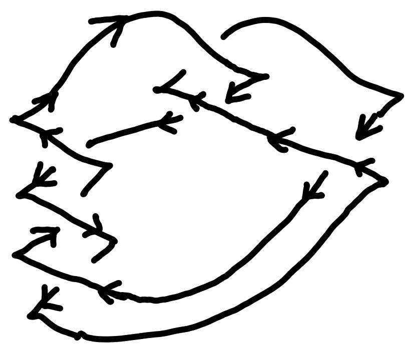


s - section of ξ over Σ

$rot_\Sigma(\mathcal{K}) = \langle PD[s^{-1}(0)], [\Sigma] \rangle$

(independent of Σ if $H_2 = 0$)

Classical Invariants of Front Projections in $(\mathbb{R}^3, \xi_3)$



$$tb(K) = w(K) - \tfrac{1}{2}\#\{\text{cusps}\}$$

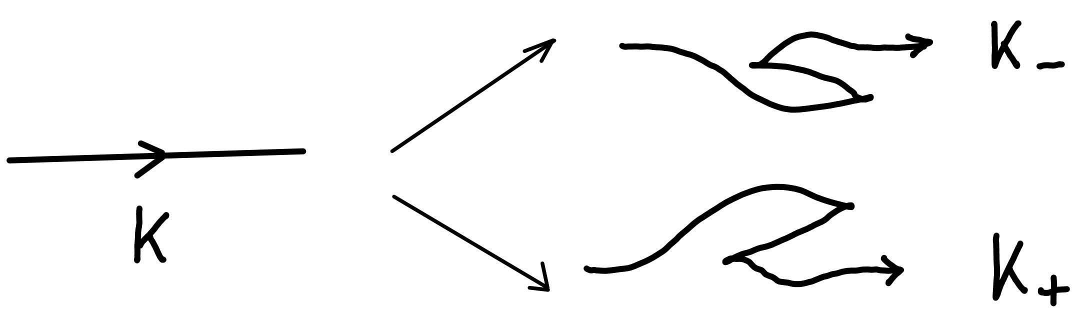
$$6 - \tfrac{1}{2}10 = 1$$

$$rot(K) = \tfrac{1}{2}\left(\#\left\{\begin{matrix}\text{downward}\\ \text{cusps}\end{matrix}\right\} - \#\left\{\begin{matrix}\text{upward}\\ \text{cusps}\end{matrix}\right\}\right)$$

$$\tfrac{1}{2}\left(\ 5 \quad - \quad 5\ \right) = 0$$

Rmk tb can be always decreased :

stabilization



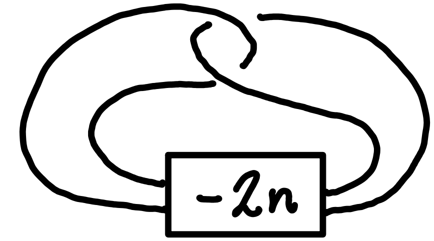
$$tb(K_\pm) = tb(K) - 1$$

$$rot(K_\pm) = rot(K) \pm 1$$

but ! can not always be increased

$\overline{tb}(K)$ = maximal tb amongst all Legendrian representations

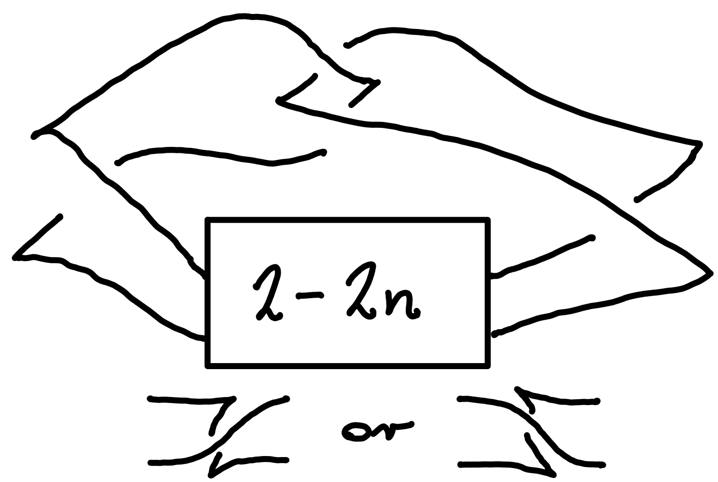
Statement of Result for K_{-2n}



The number of different Legendrian representations of K_{-2n} with given classical invariants

tb \ rot	-3	-2	-1	0	1	2	3
1				$\lfloor \frac{n^2}{2} \rfloor$			
2			$\lfloor \frac{n}{2} \rfloor$		$\lfloor \frac{n}{2} \rfloor$		
3		$\lfloor \frac{n}{2} \rfloor$		1		$\lfloor \frac{n}{2} \rfloor$	
4	$\lfloor \frac{n}{2} \rfloor$		1		1		$\lfloor \frac{n}{2} \rfloor$
$\vdots$							

- the max tb ones are of the form:
- and all other representations are stabilisations of the max tb representatives



Ingredients of the Proof

→ prove that Legendrian knots are the same

- combinatorics (Legendrian Reidemeister moves)
- convex surface theory

→ prove that Legendrian knots are different

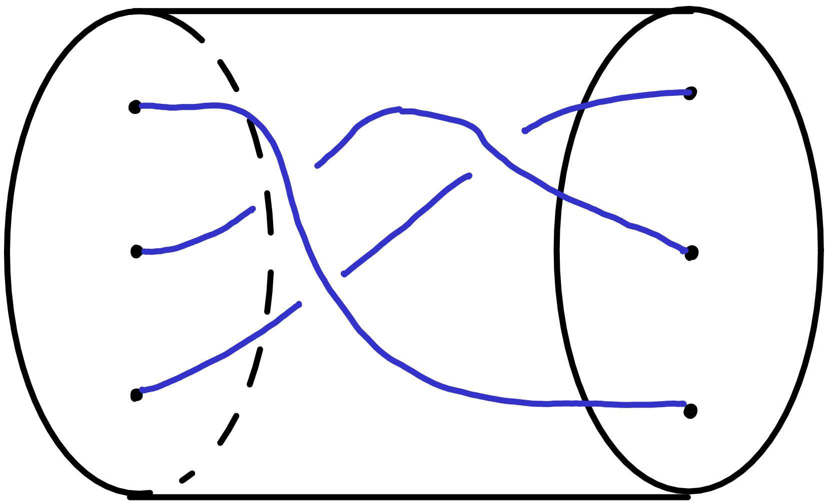
- Chekanov - Eliashberg DGA
- Legendrian invariant in HFK
  (Lisca - Ozsváth - Stipsicz - Szabó)

to understand Legendrian representations of

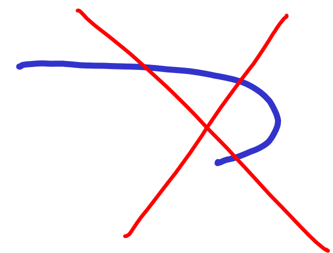
we need to understand

Legendrian representations of braids

BRAIDS

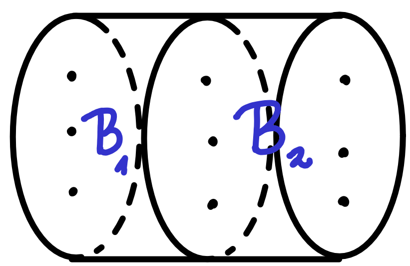


- n points on both sides
- connected by n monotone strands

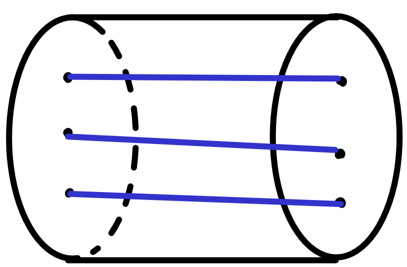


braids / isotopy form a group:

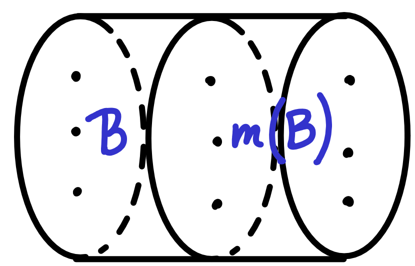
- multiplication:



- identity:

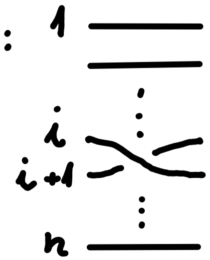


- inverse
  "
  mirror



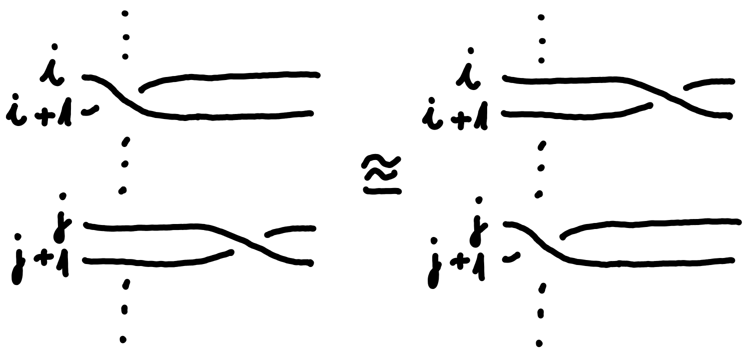
The Braid Group (B_n)

- generators :

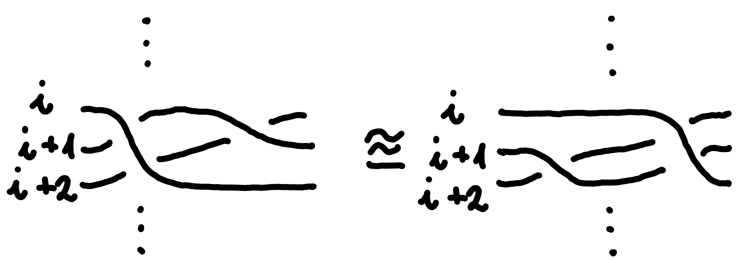


σ_i

- relations :

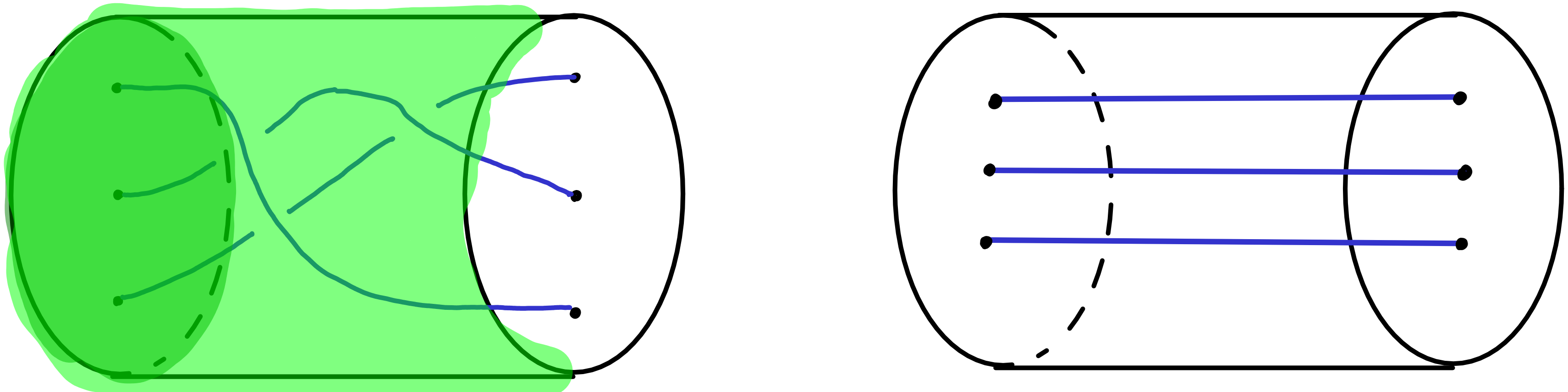


$\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| > 1$



$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

The Mapping Class Group of a Punctured Disc

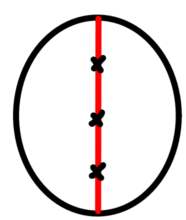


straighten the strands by a diffeomorphism fixing $\{0\} \times D^2 \cup I \times \partial D^2$

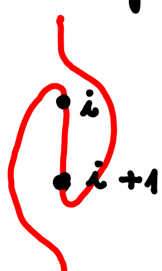
$\rightsquigarrow$ induces a diffeomorphism $\varphi : (D^2, n \text{ pts}) \longrightarrow (D^2, n \text{ pts})$

$$B_n \cong \left(\text{braids} \Big/ \substack{\text{isotopy} \\ \text{rel endpoints}}, \ \cdot \right) \xleftrightarrow{1 \text{ to } 1} \left(\{ (D^2, n \text{ pts}) \circlearrowleft \varphi \} \Big/ \substack{\text{isotopy} \\ \text{rel } \partial D^2}, \ \circ \right)$$

Rmk. all info about the diffeomorphism can be encoded in the image of the red curve

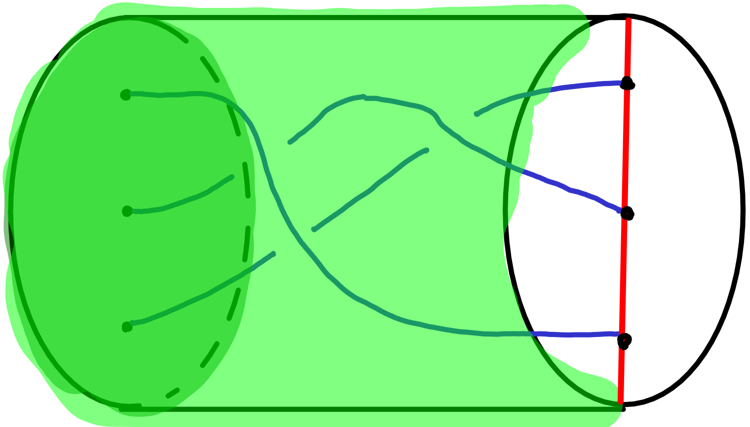
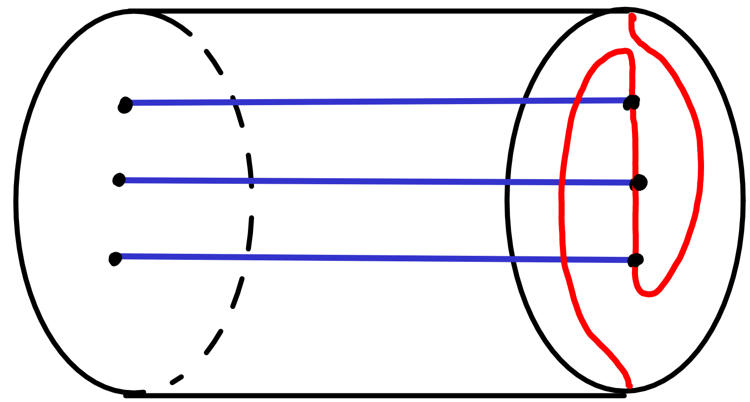


generator :



$\longleftrightarrow \sigma_i$

The Mapping Class Group of a Punctured Disc

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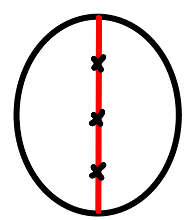
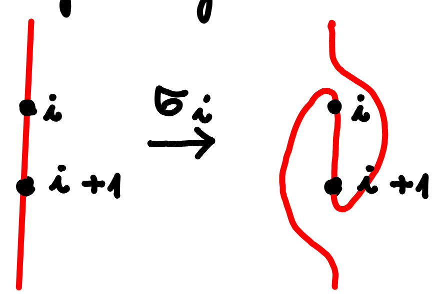
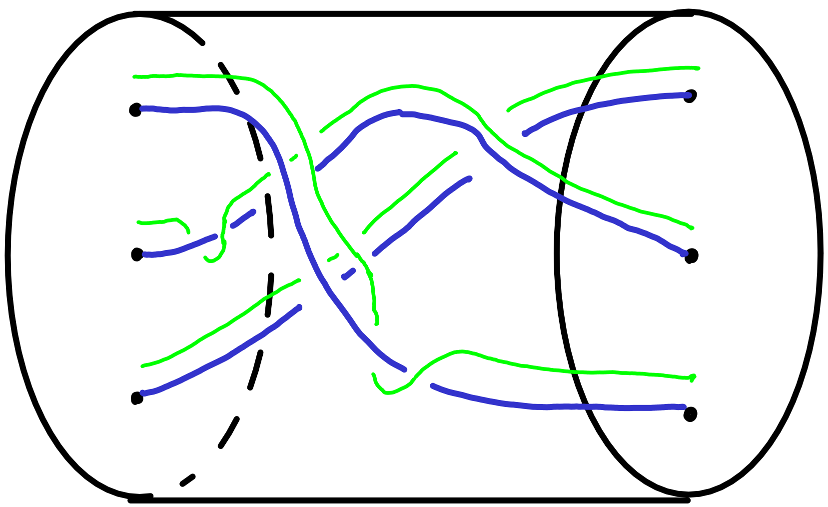


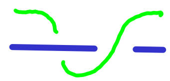
generator :

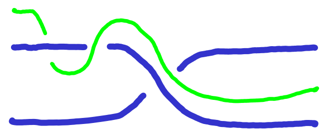
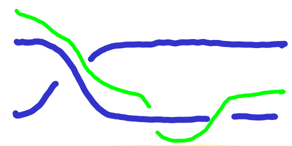


Framed Braids



braids with a framing on each strand

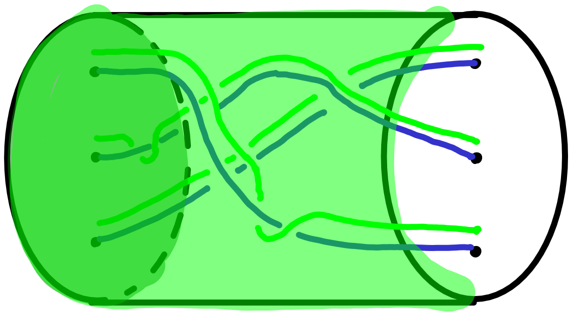
new generators :  τ_i $\langle \tau_i \rangle = \mathbb{Z}^n$

$B_n \circlearrowleft \mathbb{Z}^n$:  $\cong$ 

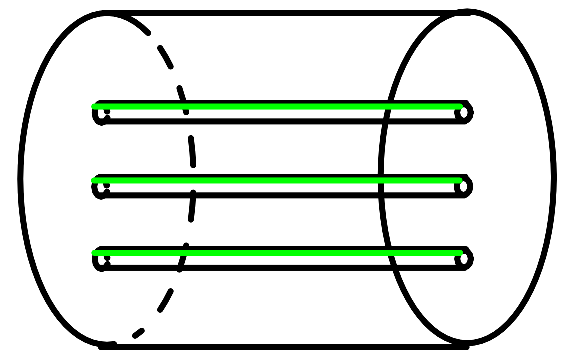
$$\sigma_i \tau_i \sigma_i^{-1} = \begin{cases} \tau_{i-1} & \text{if } \quad j = i-1 \; ; \\ \tau_{i+1} & \text{if } \quad j = i \; ; \\ \tau_i & \text{otherwise} \; . \end{cases}$$

So the framed braid group is : $F_n = B_n \ltimes \mathbb{Z}^n$

The Mapping Class Group of a Punctured Disc



- leave out a small neighborhood of each strand, s.t. the framing is on the boundary



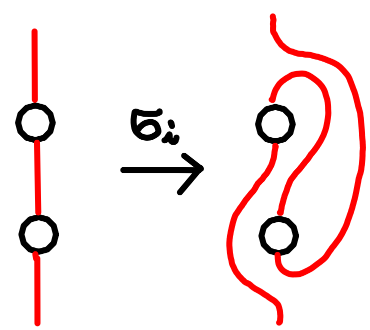
- by fixing the green part straighten everything

$\rightsquigarrow$ a diffeomorphism $\psi : \left(D^2 - \nu(n \text{ pts}), \partial(\nu(n \text{ pts}))\right) \circlearrowleft$

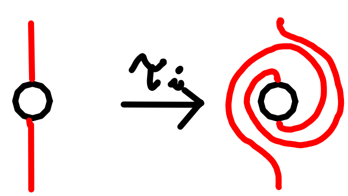
$\updownarrow$

F_n

the generators:

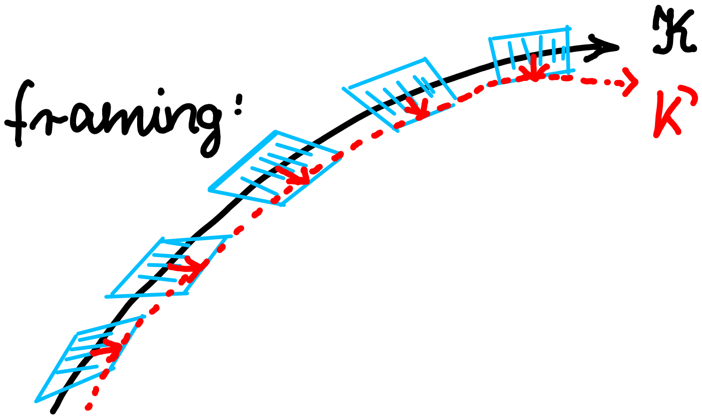


&



Legendrian Representations of Braids

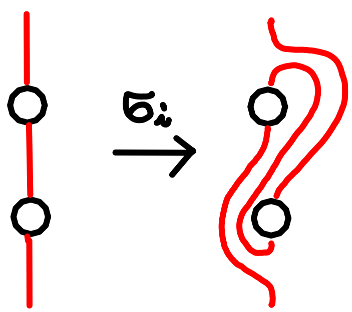
Remember: A Legendrian arc has a natural framing: the Thurston-Bennequin framing



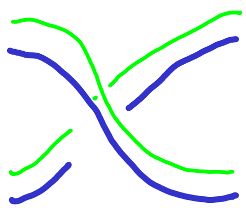
→ the set of Legendrian braids

$$L_n \subseteq F_n$$

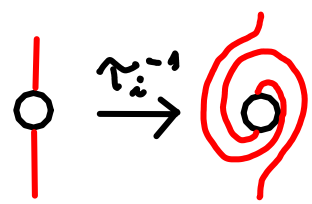
Question: Which framed braids can be represented by Legendrian braids?



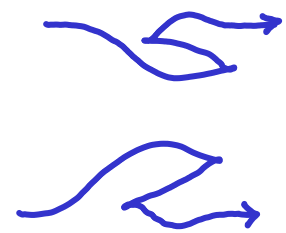
is



so ✓ &

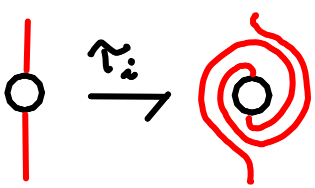


is

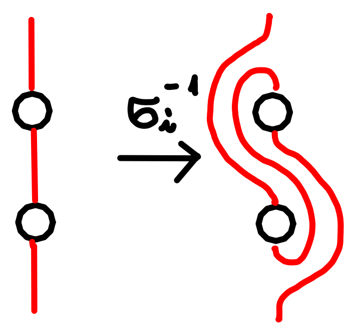


✓

but!



or

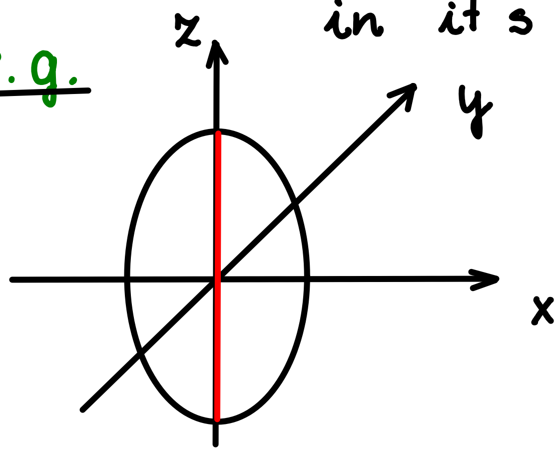


can not be represented

Convex Surfaces

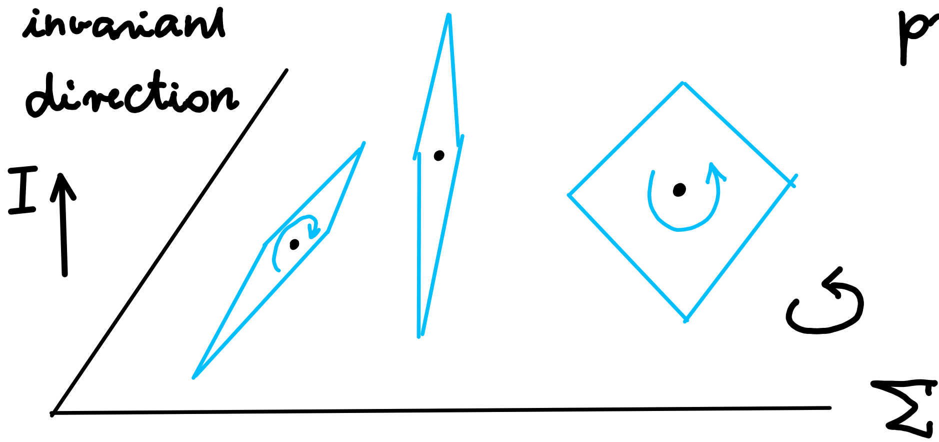
$\Sigma \hookrightarrow (Y, \xi)$ is convex if the contact structure is I-invariant in it's neighborhood

e.g.



$D = \{ y^2 + z^2 \leqslant 1 \} \times \{ x_0 \}$

is invariant in the $\xrightarrow{\frac{\partial}{\partial x}}$ direction



project ξ_x to $T\Sigma$

- at some pts is not onto
  $\shortparallel$
  1- dimensional submanifold of Σ
  Γ_z = dividing curve

- if it is onto
  - the orientation of ξ_x and of Σ agree Σ_+
  - the orientation of ξ_x and of Σ disagree Σ_-

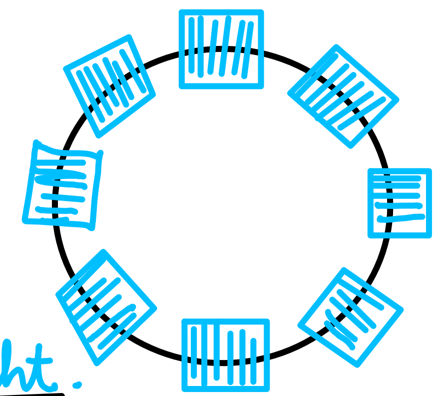
$\rightsquigarrow \quad \Sigma - \Gamma_z = \Sigma_+ \cup^* \Sigma_-$

More about Convex Surfaces – Overtwistedness

Thm (Giroux) • convex surfaces are generic

• The isotopy class of the dividing curve Γ_Σ only depends on the isotopy class of the contact structure near Σ

• Γ_Σ determines the isotopy class of contact structure near Σ.



Def. A contact structure is overtwisted if it contains a trivial knot with $tb = 0$.
Otherwise the contact structure is tight.

Thm (Eliashberg) The standard contact structure is tight

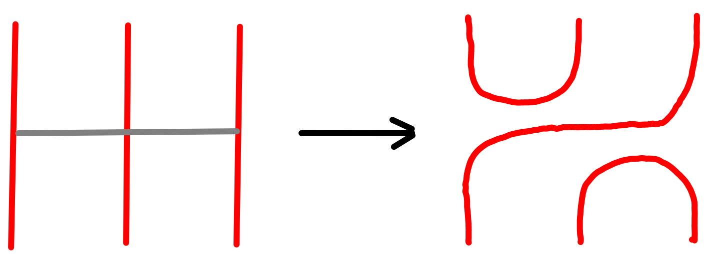
Thm (Giroux) In the neighborhood of Σ $(\neq S^2)$ the contact structure is tight $\Longleftrightarrow$ Γ_Σ contains no trivial curve

Convex Surface Theory

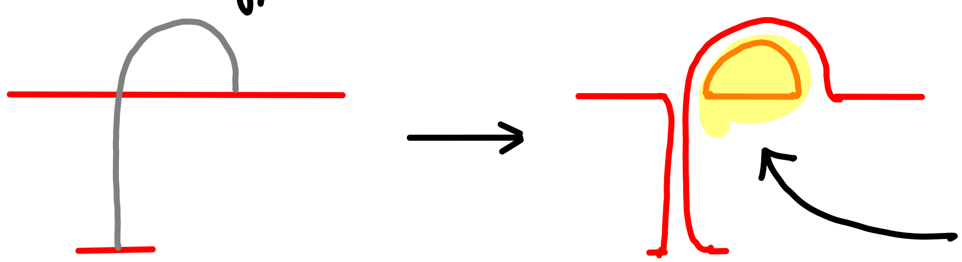
while isotoping Σ in a contact manifold

- if Σ remains convex at all times $\to \Gamma_\Sigma$ does not changes
- if Σ fails to be convex then Γ_Σ changes by a bypass attachment:

this is a local operation, Γ_Σ is unchanged on other parts



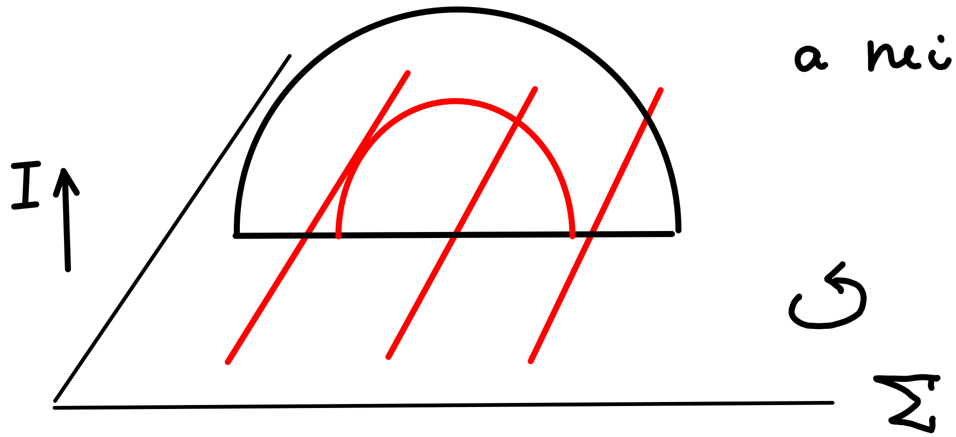
Rmk some bypasses cannot occur in tight contact structures



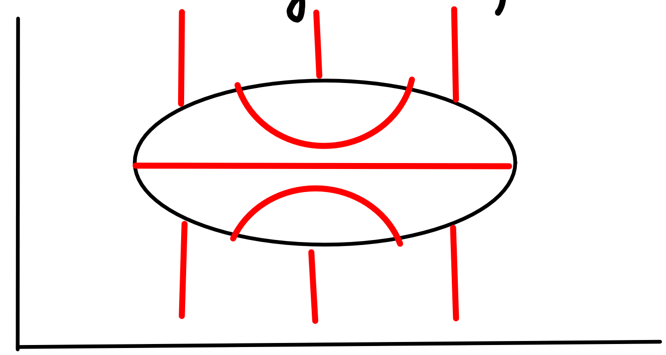
a trivial curve

Tight Contact Structures on $\Sigma \times I$

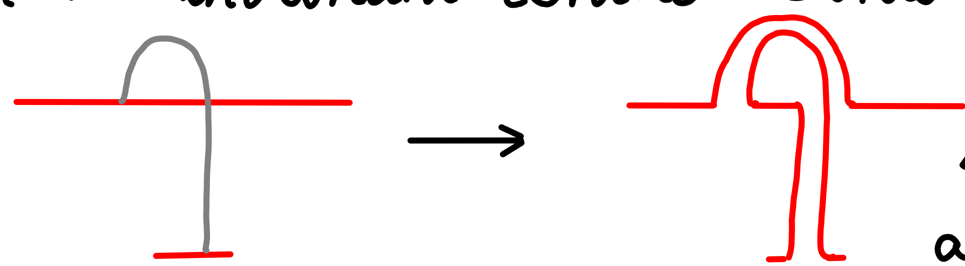
- a bypass defines a contact structure on $\Sigma \times I$:



a neighborhood of this from above:



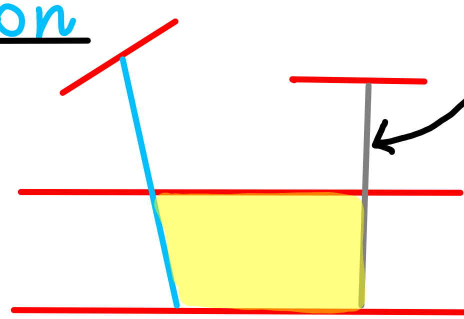
- a contact structure on $\Sigma \times I$ is built up from bypasses
- an I-invariant contact structure contains trivial bypasses:

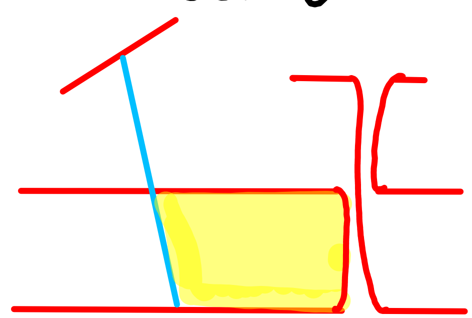


isotopic dividing curve

- bypass rotation

after attaching this the other become trivial:



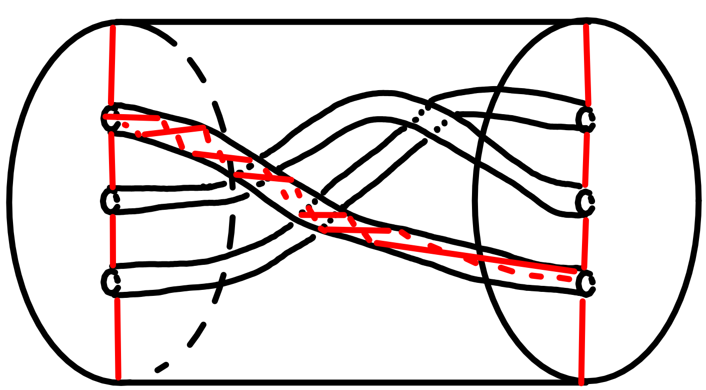


Back to Legendrian Braids

Fact • A Legendrian arc has a standard neighborhood with convex boundary having a 2 component dividing curve each of which representing the Thurston-Bennequin framing

$$\left\{\begin{array}{l}\text{Legendrian representations}\\ \text{of } A \text{ with given T-B framing}\\ \text{in } (D^3, \xi_{st})\end{array}\right\} \Big/ \text{Legendrian isotopy} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{l}\text{tight contact structures on}\\ D^3 - \nu(A) \text{ with the convex bdry}\\ \text{given by the T-B framing}\end{array}\right\} \Big/ \text{isotopy}$$

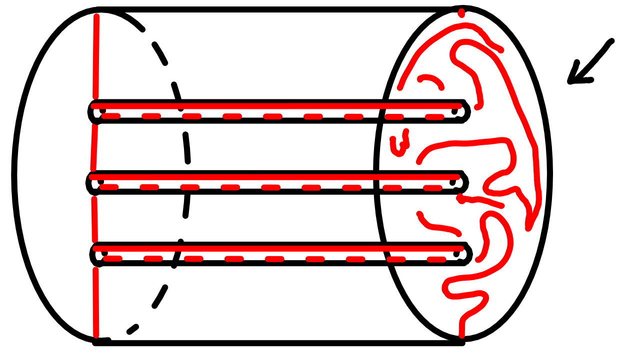
Given a Legendrian braid, take out its standard nbhd:



need to understand isotopy classes of tight contact structures with this boundary

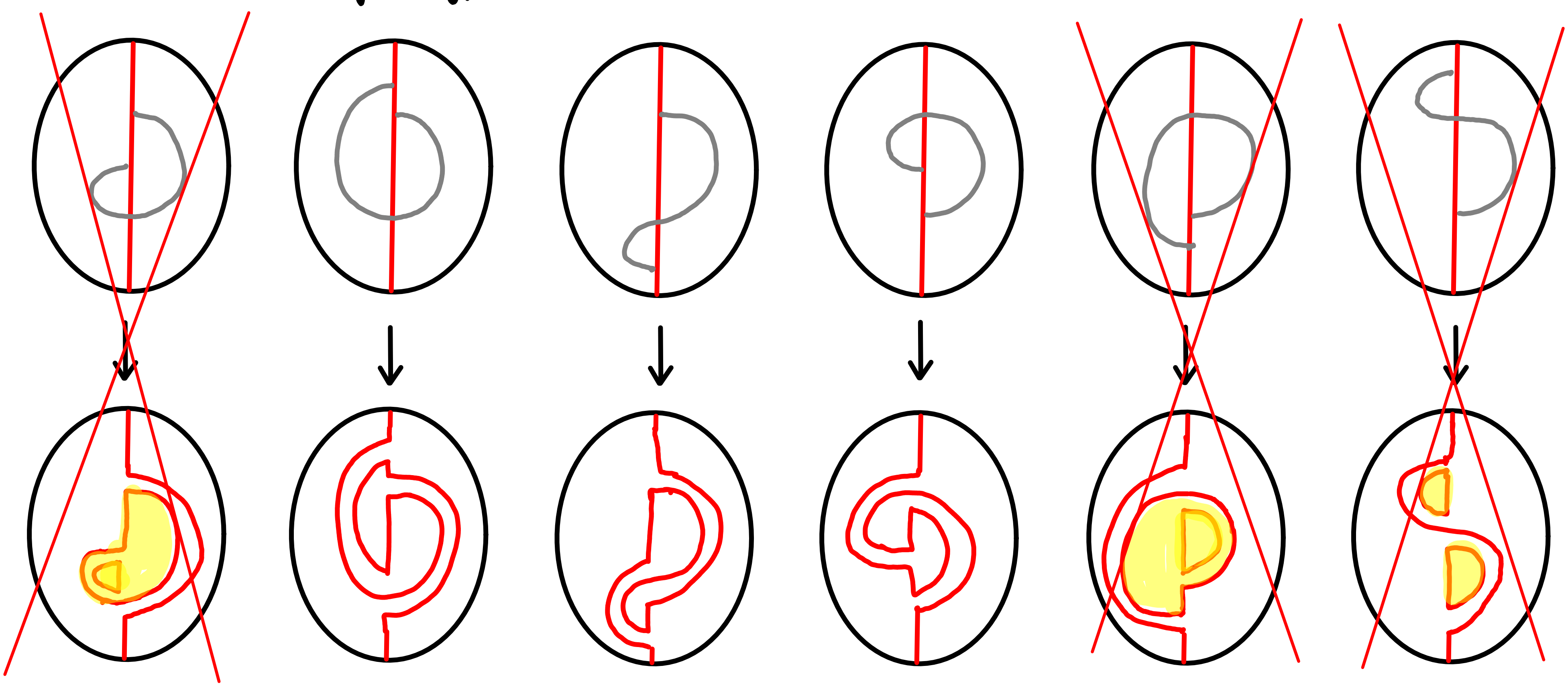
GENERATORS OF THE MONOID OF LEGENDRIAN BRAIDS

Straighten everything out !

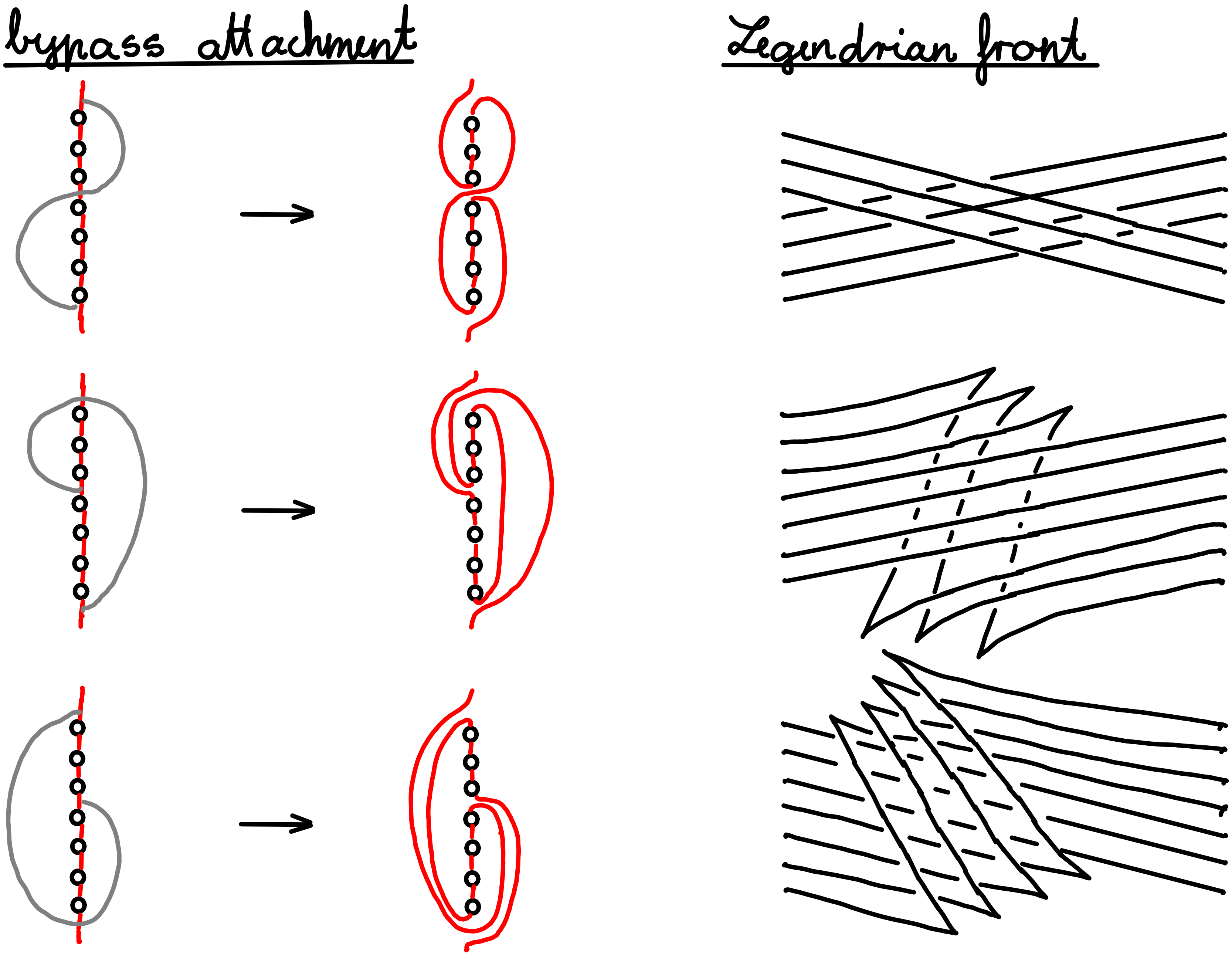


all framing-
-info
is encoded in
the dividing
curve

- the contact structure is built up from bypasses.
- What kind of bypasses can occur?

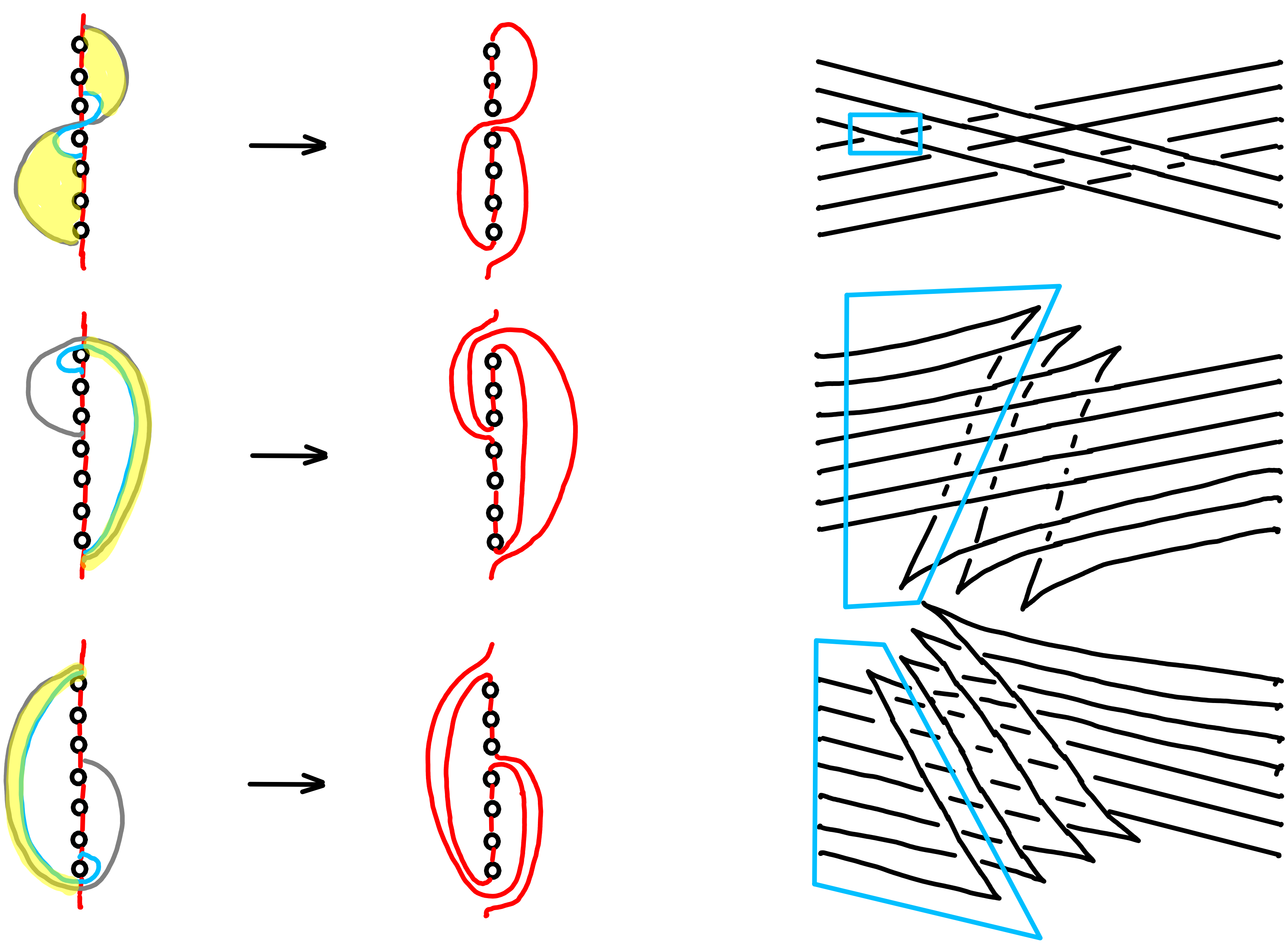


GENERATORS OF THE MONOID OF LEGENDRIAN BRAIDS – CTD

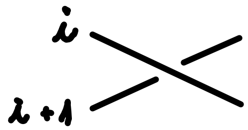
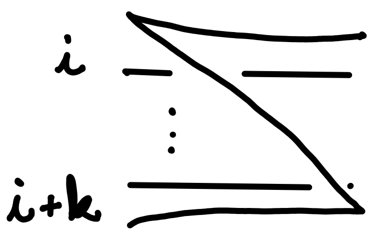
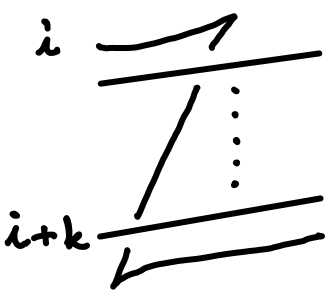


Implications by Bypass Rotation

by bypass rotation (| ⇒ |) on the Legendrian front



The Generators

Legendrian front	framed braid word
	$\sigma_i := X_i$
	$\tau_{i+k}^{-1}\,\sigma_{i+k-1}^{-1}\cdots\sigma_i^{-1} := S_{i,i+k}$
	$\tau_i^{-1}\,\sigma_i^{-1}\cdots\,\sigma_{i+k}^{-1} := Z_{i,i+k}$

Question: What are the relations?

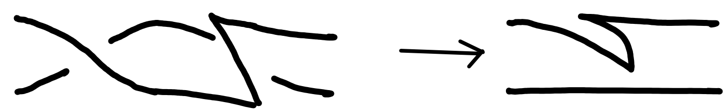
For 2-Braids

the generators :

$x \qquad s \qquad z \qquad St_i^- \qquad St_i^+$

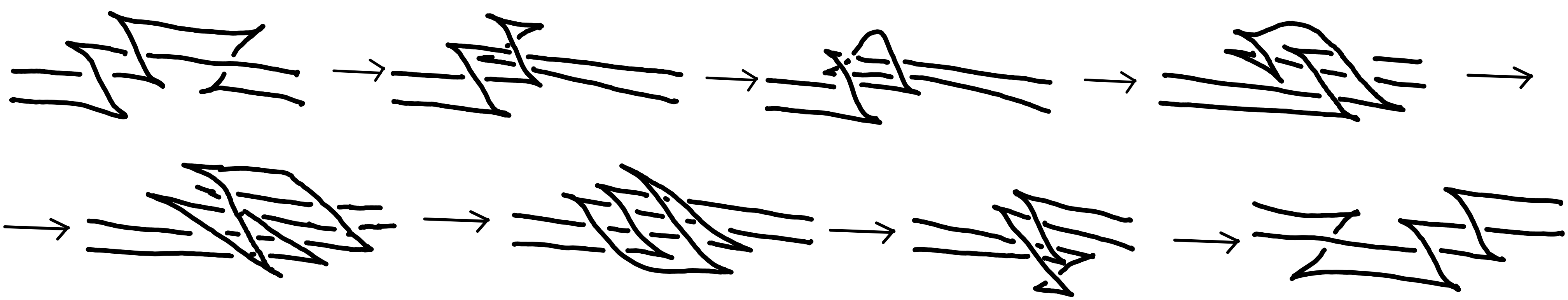
relations : • $St_i^{\pm} \begin{matrix} x \\ s \\ z \end{matrix} = \begin{matrix} x \\ s \\ z \end{matrix} St_{i+1}^{\pm}$

• $xs = St_1^-$



similarly : $xz = St_2^+$, $sx = St_2^-$, $zx = St_1^+$

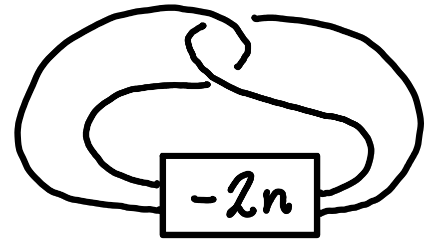
• $ssz = zss$



& similarly $zzs = szz$

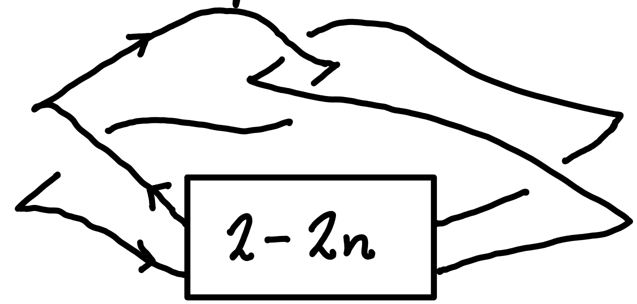
these are all the relations of L_2

Back to Twist Knots



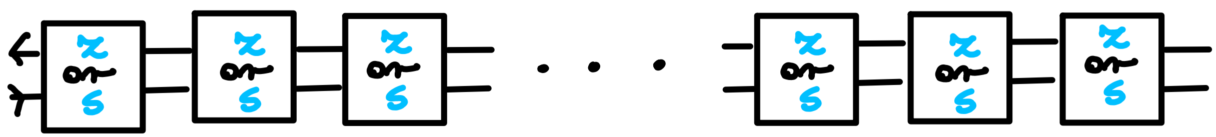
K_{-2n}

Thm (Etnyre - Ng - V) Any Legendrian representation of K_{-2n} is Legendrian isotopic to



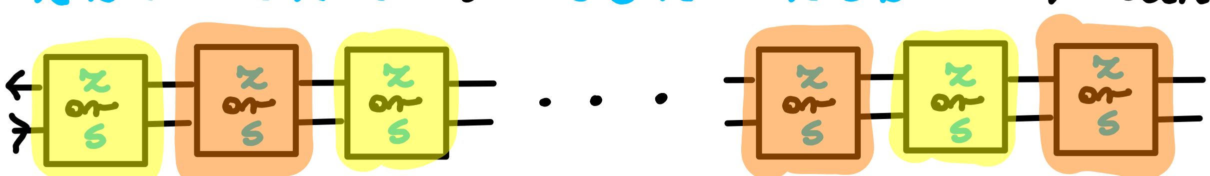
where  contains a Legendrian 2-braid.

 is a (stabilization of a) sequence of $2n-2$ x's & s's.

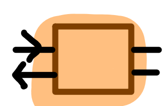
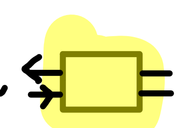


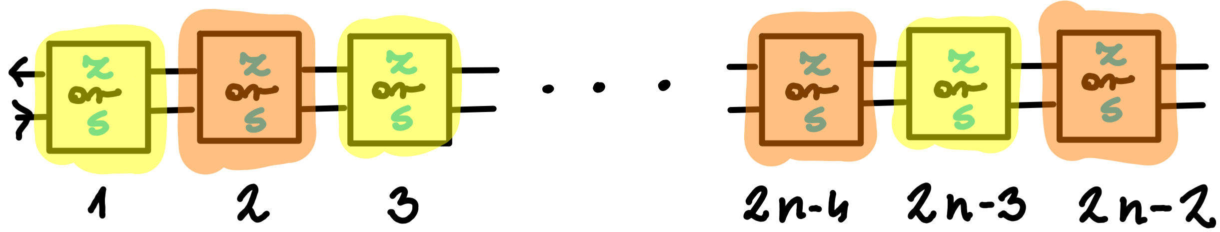
1 2 3 . . . $2n-4$ $2n-3$ $2n-2$

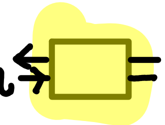
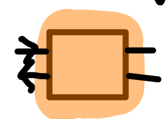
- $xxs = sxx$ & $ssx = xss$ $\Rightarrow$ can move a x or s by 2

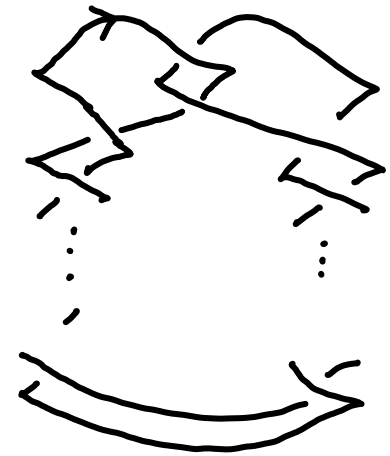


1 2 3 . . . $2n-4$ $2n-3$ $2n-2$

the Legendrian isotopy class only depends on the # of x's in  and in 



denote by $\mathcal{K}_{a,b}$ the Legendrian isotopy class with 'a' z's in  and 'b' z's in 



<u>Rmk.</u> the example of Epstein-Fuch-Meyer corresponds to the word:

$zz\cdots z\,ss\cdots s$ (thus $a=b$ or $b+1$)

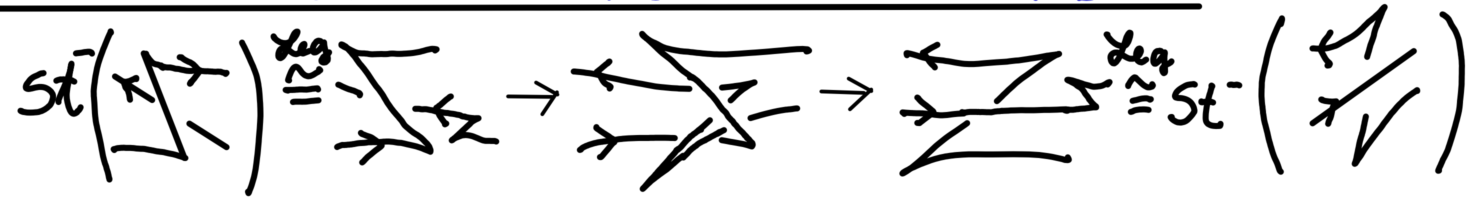
- by symmetry $\mathcal{K}_{a,b} \overset{\text{Leg}}{\approxeq} \mathcal{K}_{n-1-a,\,n-1-b}$

<u>Thm</u> (Etnyre-Ng-V) $\mathcal{K}_{a,b} \overset{\text{Leg}}{\approxeq} \mathcal{K}_{a',b'} \iff$ $a=a'$ & $b=b'$ or $a=n-1-a'$ & $b=n-1-b'$

thus $a,b \in \{0,1,\dots,n-1\} \Rightarrow \left\lceil \frac{n^2}{2} \right\rceil$ different Legendrian representations

THE LEGENDRIAN REPRESENTATIONS ARE DIFFERENT

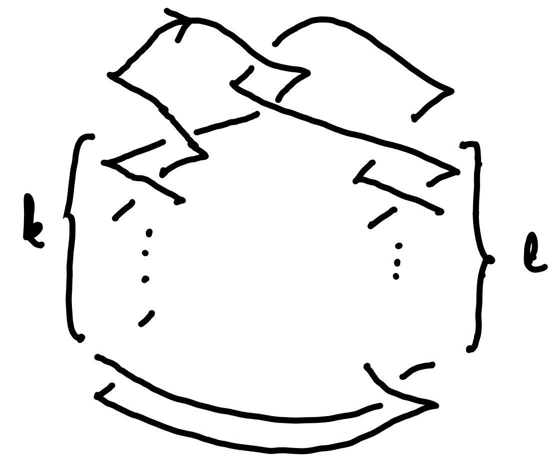
stabilisation :

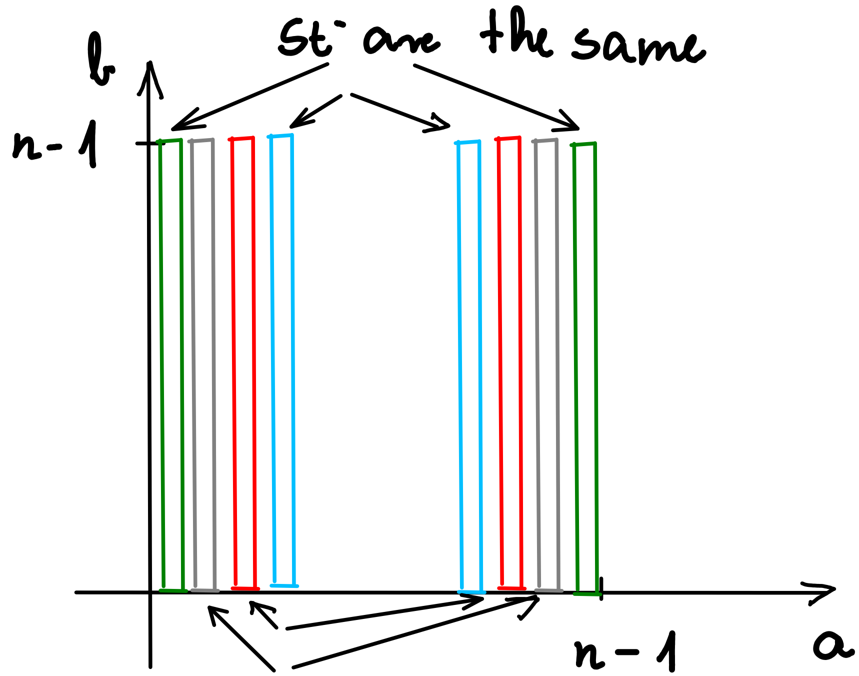


$\Rightarrow \quad St^-(\boxed{x}) \overset{Leg}{\cong} St^-(\boxed{s})$ similarly $St^+(\boxed{x}) \overset{Leg}{\cong} St^+(\boxed{s})$

$\Rightarrow \quad St^-(K_{a,b}) \overset{Leg}{\cong} St^-(K_{a',b}) \overset{Leg}{\cong} St^-(K_{a',n-1-b})$

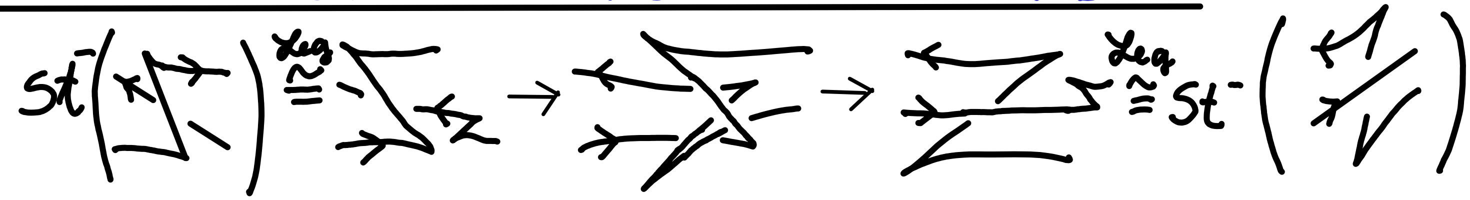
$\& \; St^+(K_{a,b}) \overset{Leg}{\cong} St^+(K_{a,b'}) \overset{Leg}{\cong} St^+(K_{n-1-a,b'})$





The Legendrian Representations are Different

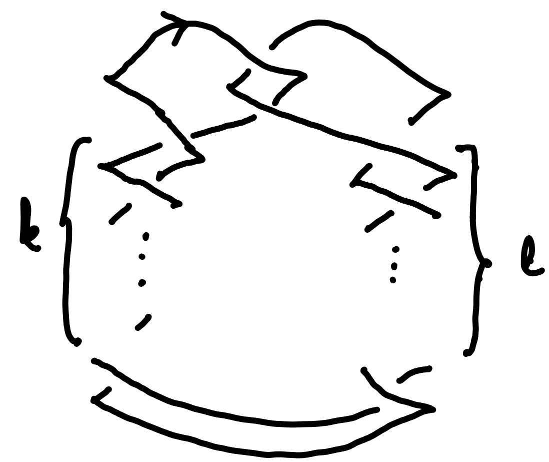
stabilisation :

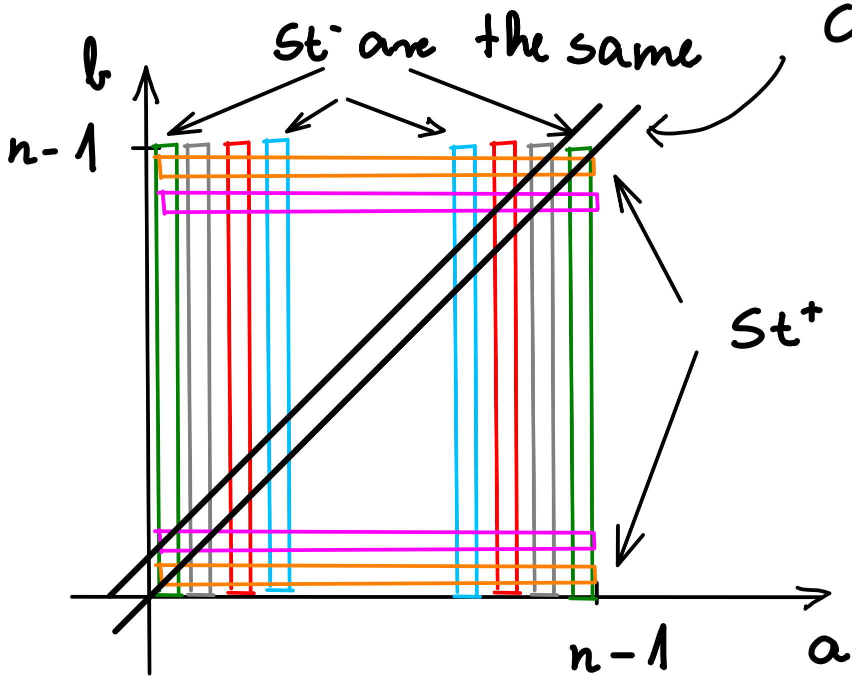


$\Rightarrow$ $St^-(\boxed{x}) \overset{Leg}{\cong} St^-(\boxed{s})$ similarly $St^+(\boxed{x}) \overset{Leg}{\cong} St^+(\boxed{s})$

$\Rightarrow$ $St^-(K_{a,b}) \overset{Leg}{\cong} St^-(K_{a',b}) \overset{Leg}{\cong} St^-(K_{a',n-1-b})$

& $St^+(K_{a,b}) \overset{Leg}{\cong} St^+(K_{a,b'}) \overset{Leg}{\cong} St^+(K_{n-1-a,b'})$





Chekanov knots

St^+ are the same

The Legendrian Representations are Different

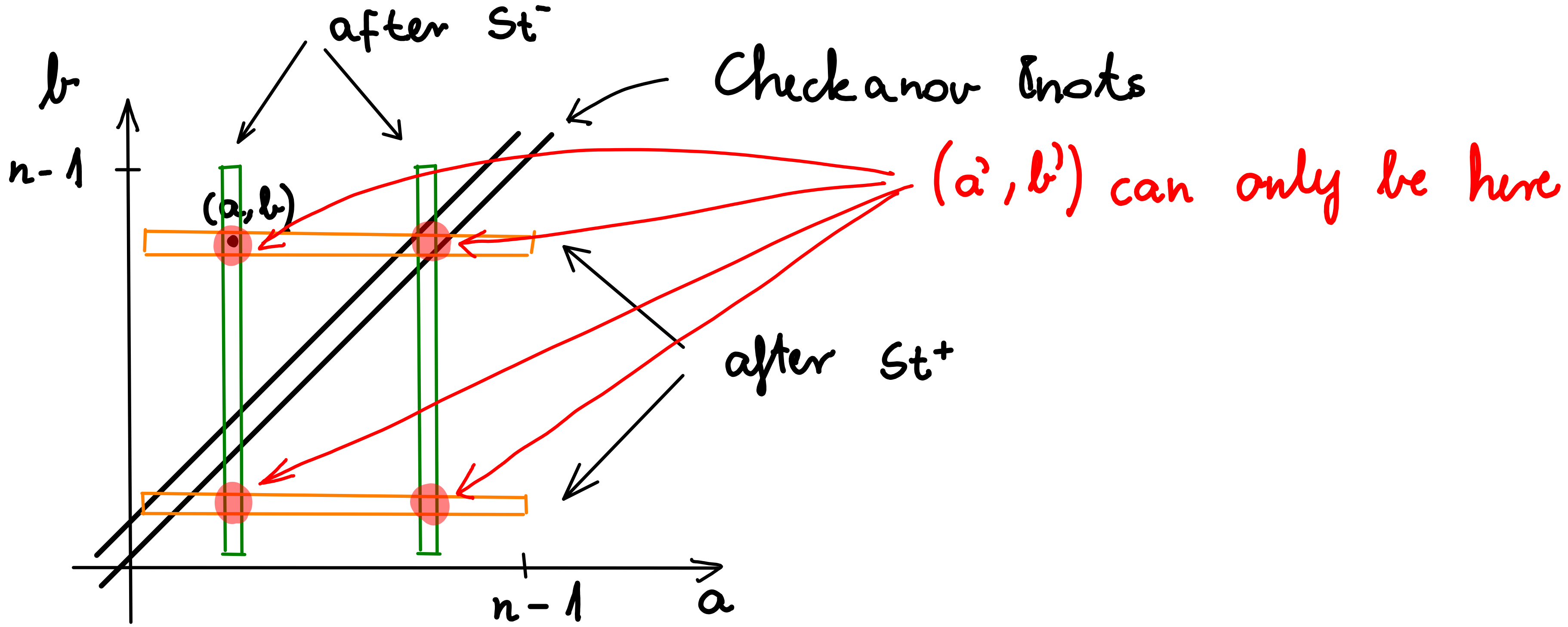
Thm (Ozsváth – Stipsicz)

$k, l > 0$, $k + l = 2n-2$, $2 \mid k$, $2 \mid l$

$(St^-)^m(\mathcal{E}(k,l)) \overset{Leg}{\cong} (St^-)^m(\mathcal{E}(k',l')) \Rightarrow k = k'$ or $2n-2-k'$

$(St^+)^m(\mathcal{E}(k,l)) \overset{Leg}{\cong} (St^+)^m(\mathcal{E}(k',l')) \Rightarrow l = l'$ or $2n-2-l'$

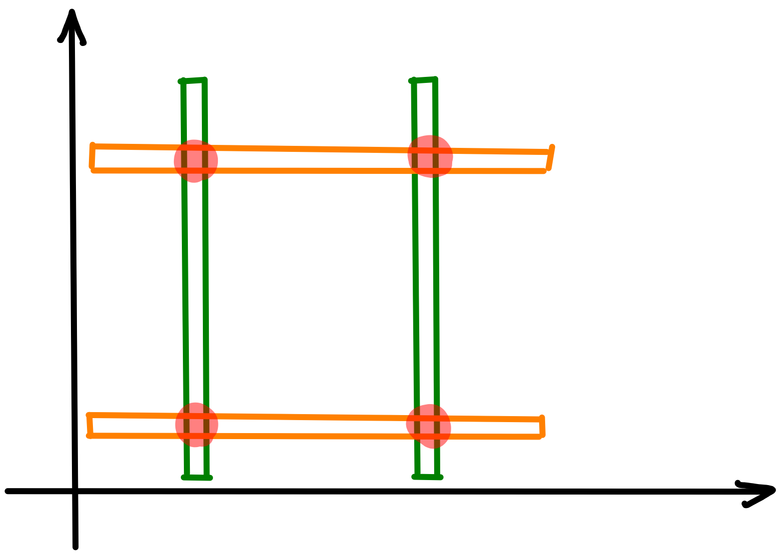
Which $K_{a',b'}$ can be Legendrian isotopic to a given $K_{a,b}$?



So far : $\mathcal{K}_{a,b} \overset{\mathrm{Leg}}{\cong} \mathcal{K}_{a',b'} \Rightarrow a = a'$ or $n-1-a'$

&

$b = b'$ or $n-1-b'$



<u>Cor</u> of <u>Thm</u> (Pushkar - Chekanov) about Legendrian ruling invariants :

$$\mathcal{K}_{a,b} \overset{\mathrm{Leg}}{\cong} \mathcal{K}_{a',b'} \Rightarrow |a+b+1-n| = |a'+b'+1-n|$$

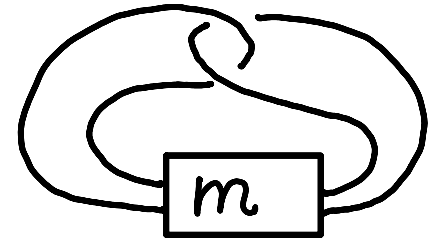
Combining these :

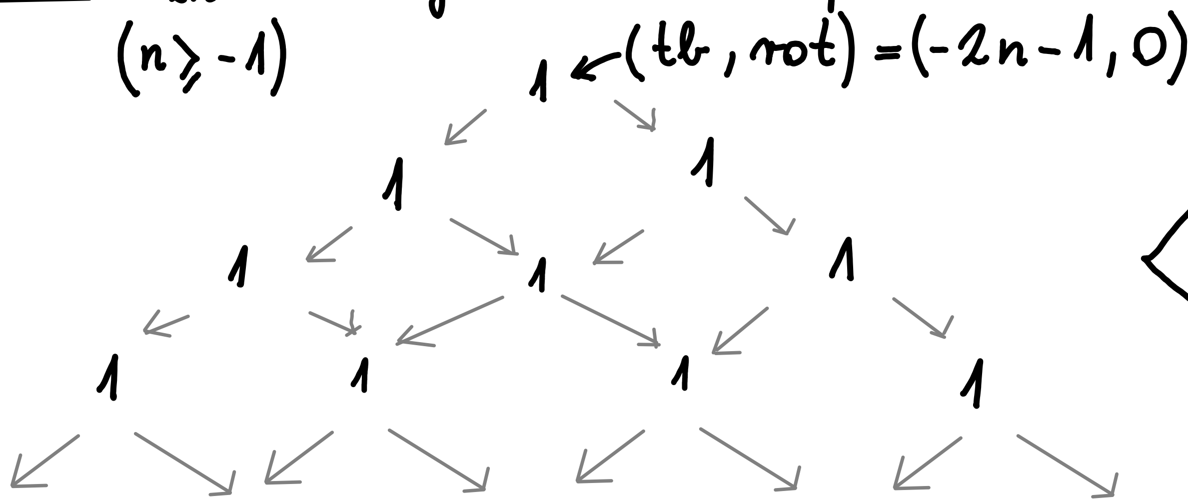
$$\left.\begin{array}{c} a = a' \text{ or } a = n-1-a' \\ \& \\ b = b' \text{ or } b = n-1-b' \\ \& \\ a+b = a'+b' \end{array}\right\} \Rightarrow \begin{array}{c} a = a' \ \& \ b = b' \\ \text{or} \\ a = n-1-a' \ \& \ b = n-1-b' \end{array}$$

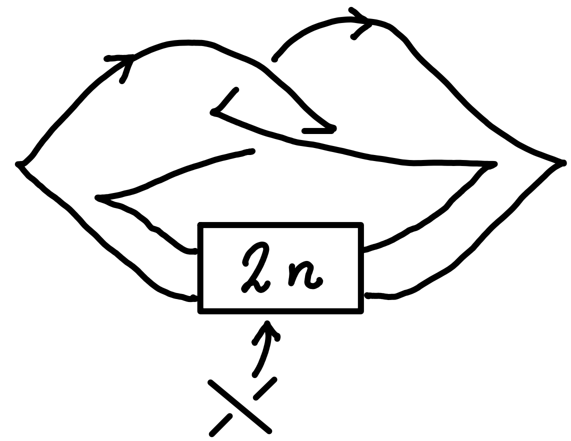
QED

So... the Results

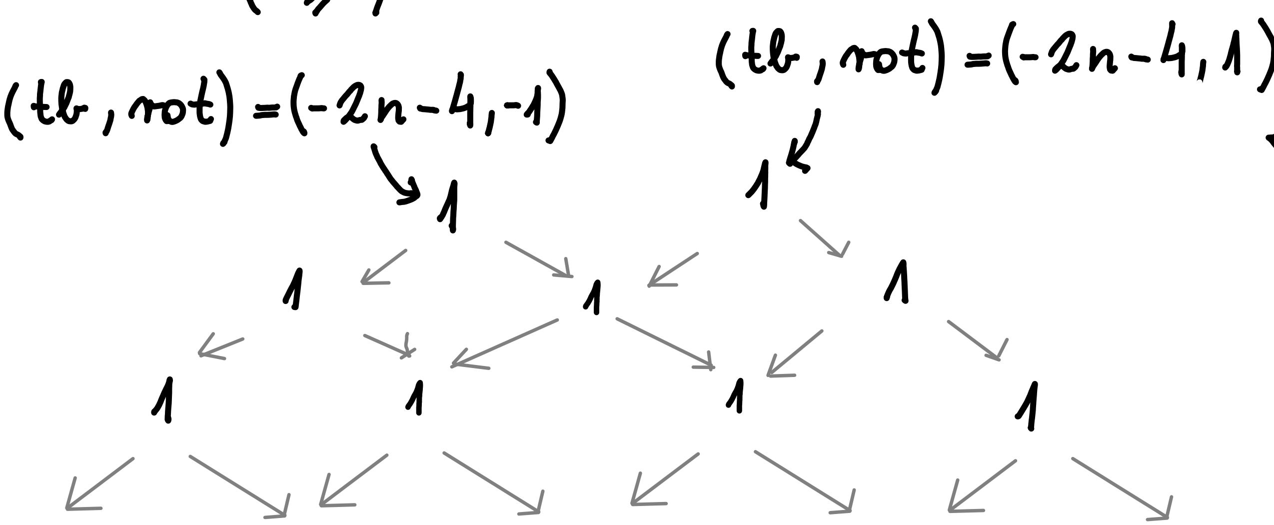
Thm: K_{2n} is Legendrian simple $(n \geq -1)$

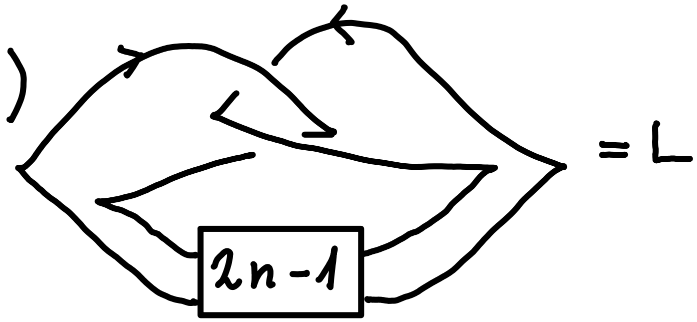


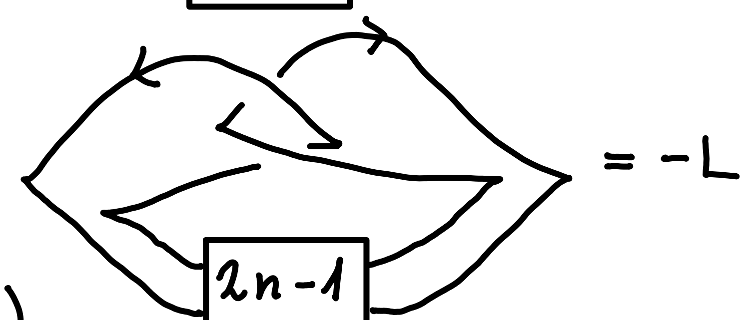




Thm: K_{2n-1} is Legendrian simple $(n \geq 1)$



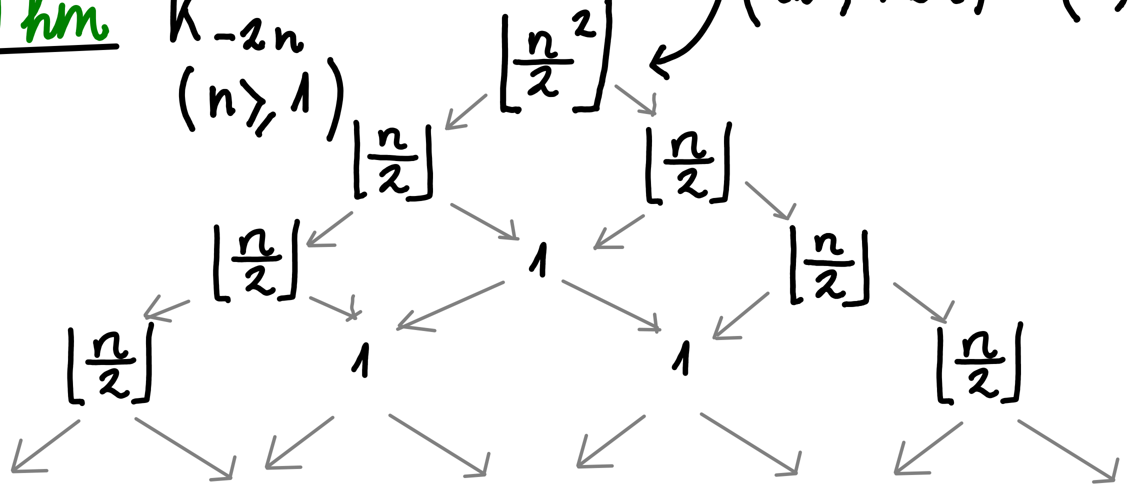


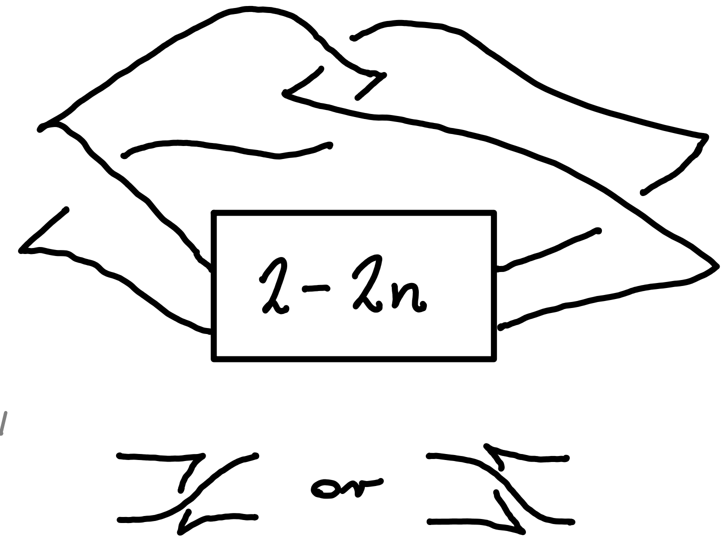


$L_- = (-L)_+$

Thm K_{-2n} $(n \geq 1)$

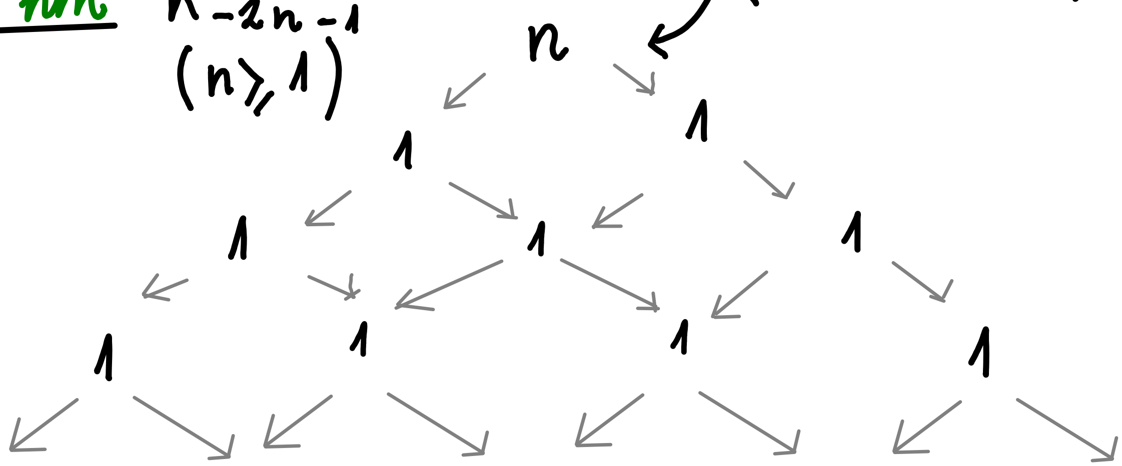
$(tb, rot) = (1, 0)$

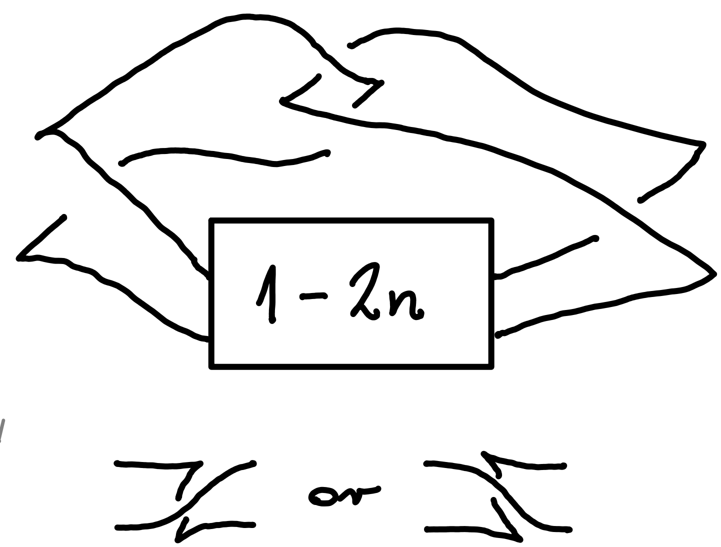




Thm K_{-2n-1} $(n \geq 1)$

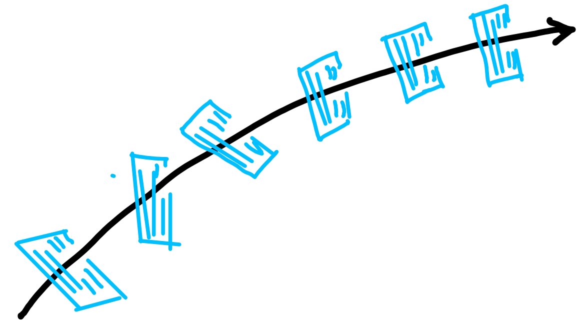
$(tb, rot) = (-3, 0)$





Transverse Knots

Def a knot T is *transverse* if $TT \pitchfork \xi$

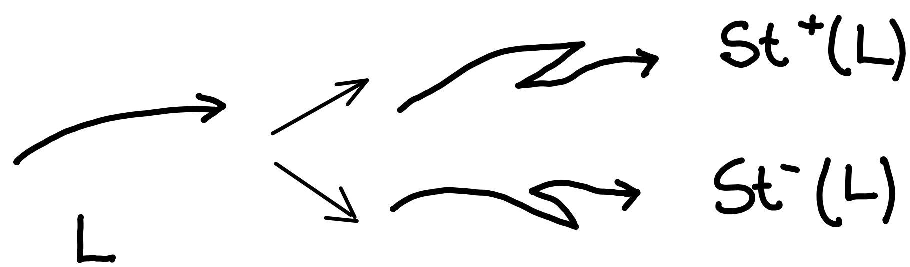


Fact transverse knots have Legendrian approximations

Def T transverse, L its Legendrian approximation

the *self linking #* of T: $\quad sl(T) = tb(L) - rot(L)$

Rem *stabilisation* of a Legendrian knot L is a local operation:



Fact T_0 & T_1 are transverse isotopic $\Leftrightarrow$ their Legendrian approximations have Legendrian isotopic negativ stabilisations.

Cor the self linking # is well defined.

Results for Transverse Knots

Thm K_{2n} $(n \geq -1)$ is transversely simple

$$\begin{array}{c} 1 \\ 1 \\ \vdots \end{array} \quad sl = -2n-1$$

Thm K_{2n-1} is transversely simple

$$n \geq 1 \quad \begin{array}{c} 1 \\ 1 \\ \vdots \end{array} \quad sl = -2n-3 \qquad\qquad n = -1 \quad \begin{array}{c} 1 \\ 1 \\ \vdots \end{array} \quad sl = -1$$

$$n \leq -1 \quad \begin{array}{c} 1 \\ 1 \\ \vdots \end{array} \quad sl = -3$$

Thm K_{-2n} $(n \geq 2)$

$$\begin{array}{c} \lfloor \frac{n}{2} \rfloor \\ 1 \\ 1 \\ \vdots \end{array} \quad sl = 1$$

Thanks
for your
Attention!