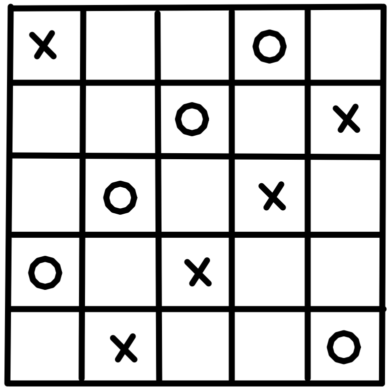
LEGENDRIAN AND TRANSVERSE INVARIANTS IN HEEGAARD FLOER HOMOLOGY

Vera Vértesi (MIT-IAS)

10/4/2011

Grid Diagrams for Knots



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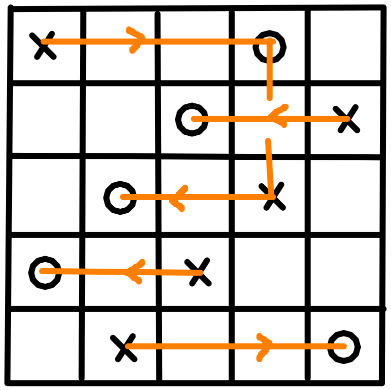

- every row/column contains one "X" and one "O"
- connect "X" to "O" horizontally

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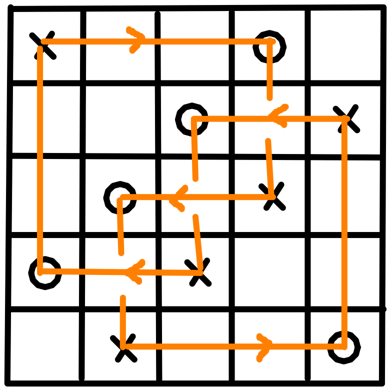

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- vertical strands are OVER the horizontal strands

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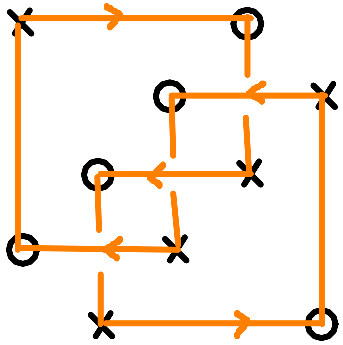

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$\rightsquigarrow$ knot (or link) in $\mathbb{R}^3$

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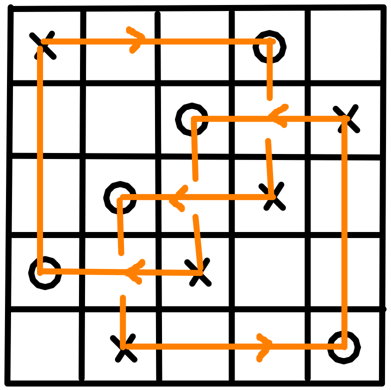

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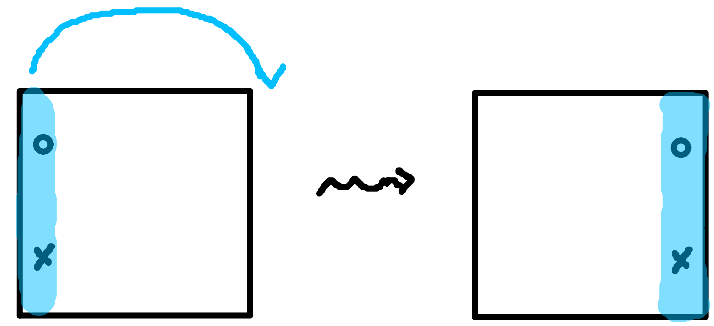
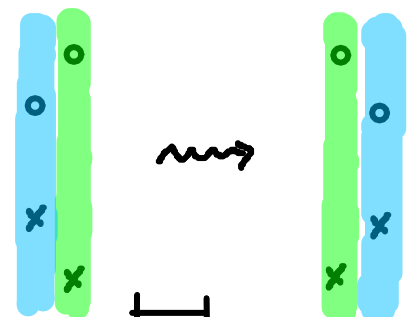
grid moves:

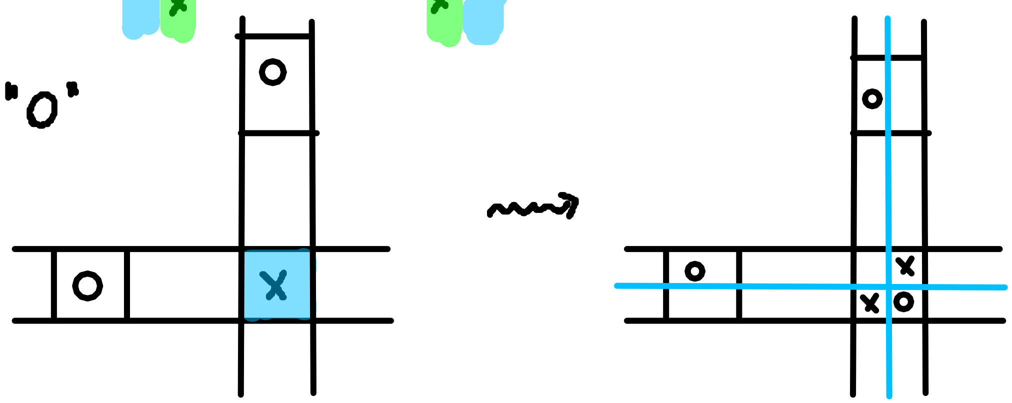
- cyclic permutation of columns and rows



- commutation of columns and rows whose "X"s and "O"s do not overlap



- stabilization of an "X" or an "O"



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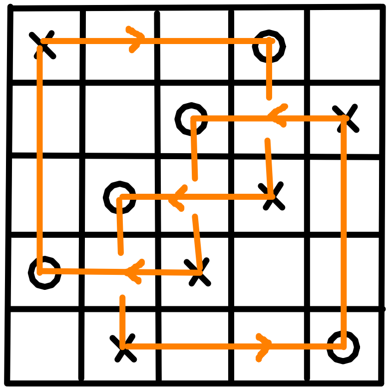

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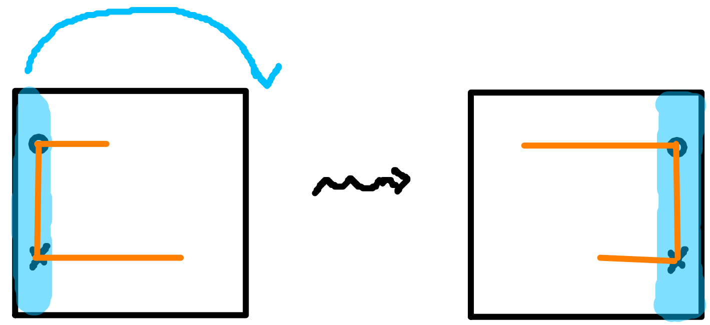

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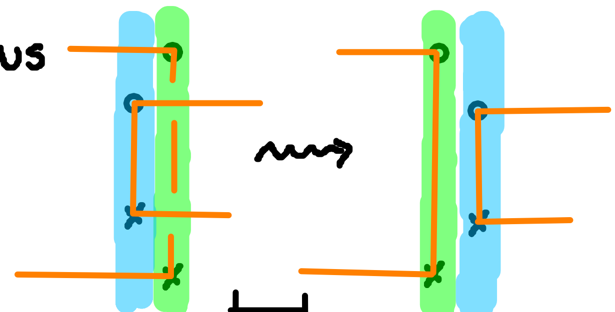

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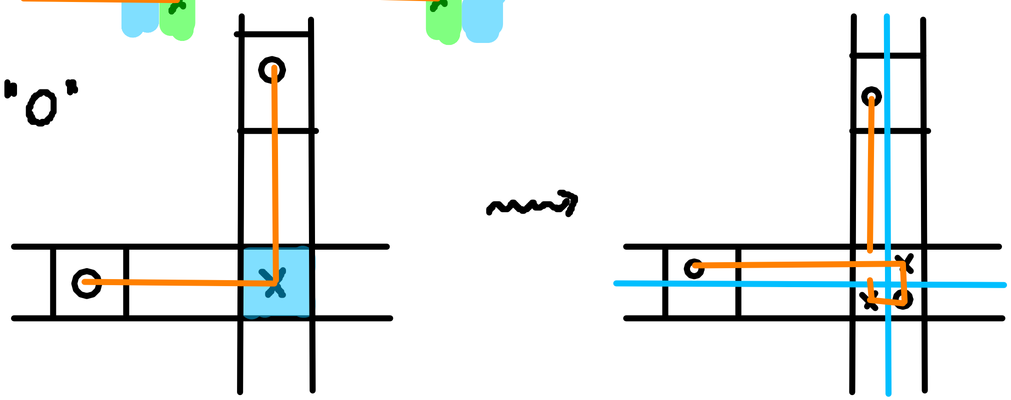
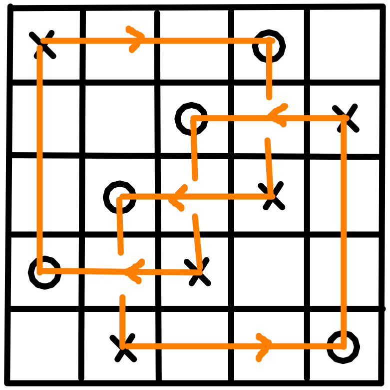


these moves give isotopic knots

GRID DIAGRAMS FOR KNOTS



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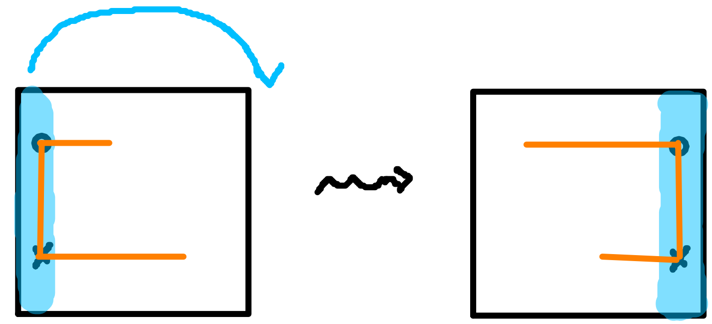

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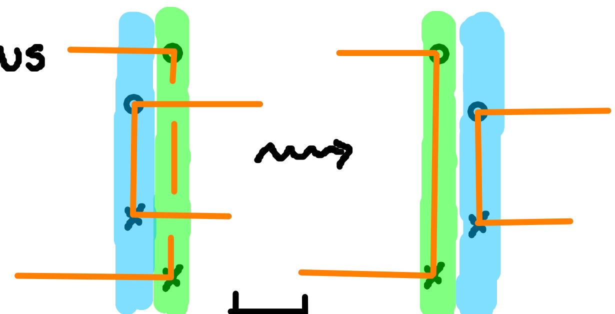

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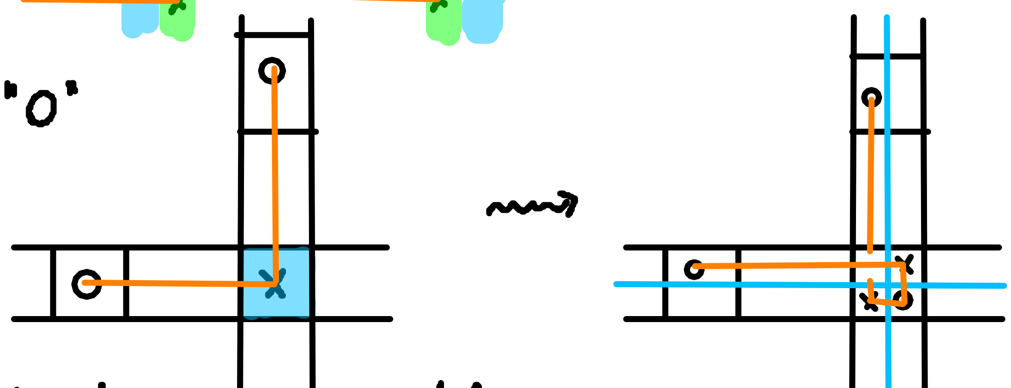
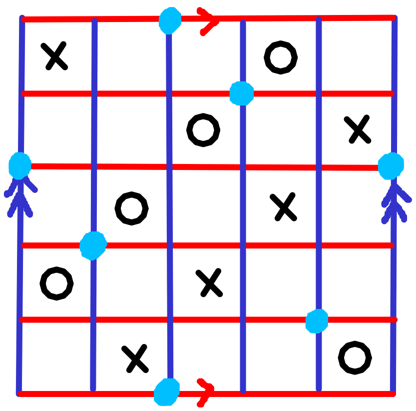


Thm (Cromwell, Dynnikov) two grid diagrams define isotopic knots if they are related by a sequence of grid moves.

KNOT FLOER HOMOLOGY

Strategy: define a homology for any grid diagram
then prove it is independent of grid moves



- identify the left and right, and top and bottom of the grid

generators: N-tuples of intersection points such that
- every horizontal line contains 1 point
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$$\widehat{CFK}(G) = \bigoplus \widehat{CFK}_M(G)$$

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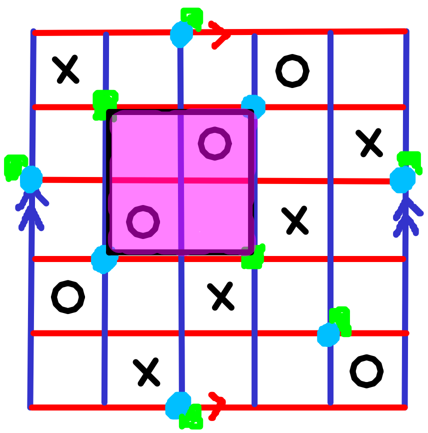

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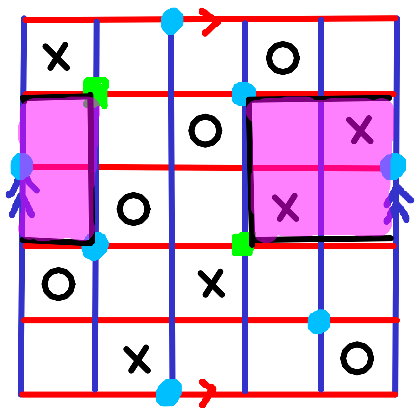

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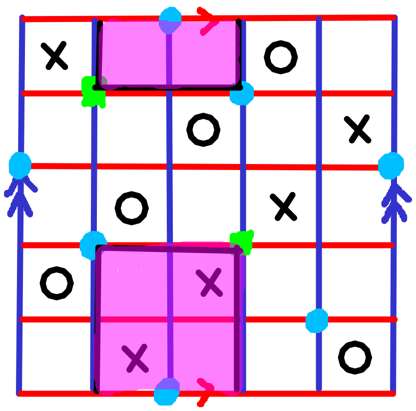

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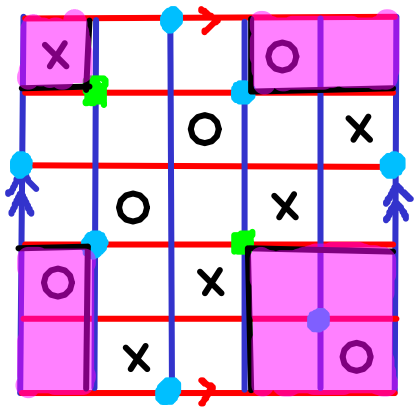

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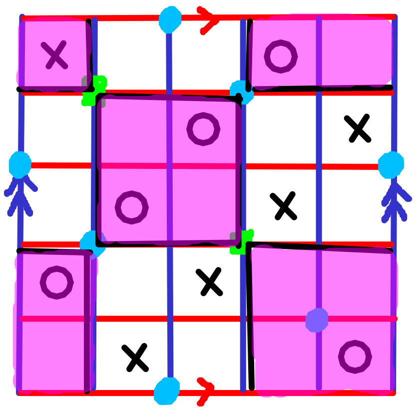

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- $\bullet$ and $\blacksquare$ differs in exactly 2 coordintes
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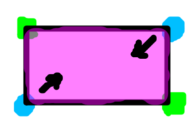
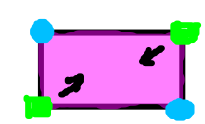
- 2 of which goes from $\bullet$ to $\blacksquare$:  the other 2 from $\blacksquare$ to $\bullet$ 

- a rectangle is emty if there is no "X", "O", other point $\bullet - \blacksquare$ in its interior

$$\hat{\partial}\underline{x} = \sum_{\exists \text{ emty rectangle } \underline{x} \to y} y$$

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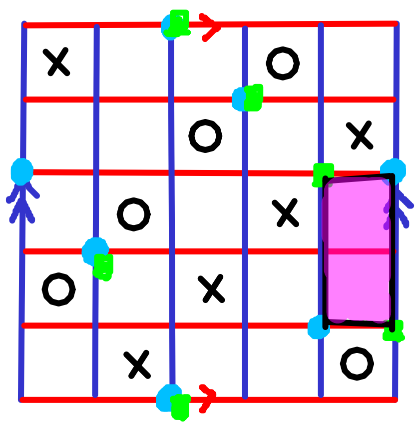

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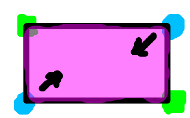
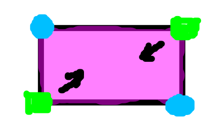
- 2 of which goes from $\bullet$ to $\blacksquare$:  the other 2 from $\blacksquare$ to $\bullet$ 

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e.g. $\hat{\partial} \bullet = \blacksquare$

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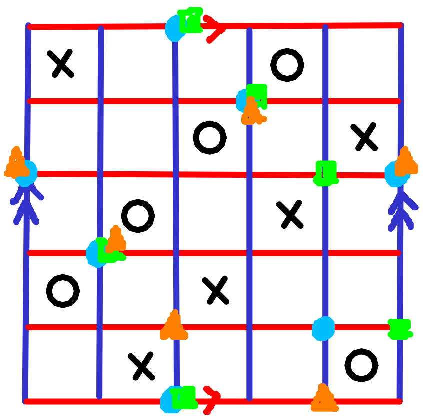

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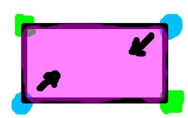
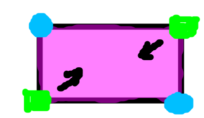
- 2 of which goes from $\bullet$ to $\blacksquare$:  the other 2 from $\blacksquare$ to $\bullet$ 
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e.g. $\hat{\partial}\bullet = \blacksquare + \blacktriangle$

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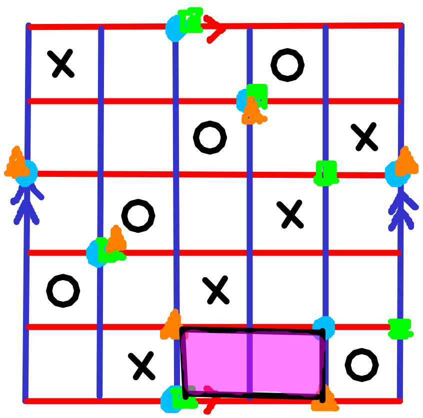

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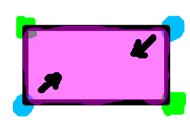
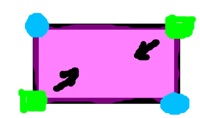
- 2 of which goes from to :  the other 2 from to 
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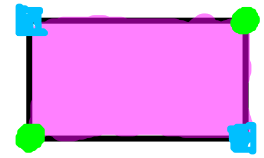
e.g. $\hat{\partial}$ = +

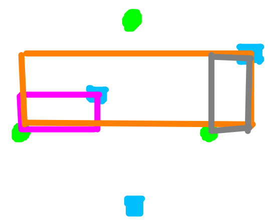
Gradings

Remember: $\hat{\partial}$ removed an inversion



Notation $\mathcal{J}(A,B) := \#\left\{ {}_{a}\square^{b} : a \in A,\ b \in B \right\}$

e.g.:



$\mathcal{J}({\color{green}A},{\color{cyan}B}) = 3$

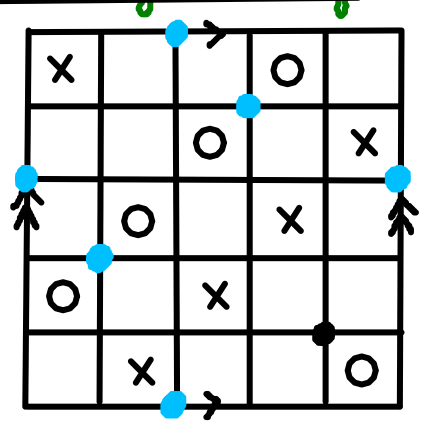
$$\mathcal{J}(A-B,C) := \mathcal{J}(A,B) - \mathcal{J}(B,C)$$

$\underline{x}$ generator, $\underline{X}$ set of "X"s, $\underline{O}$ set of "O"s on the grid

Maslov grading: $M(\underline{x}) = \mathcal{J}(\underline{x}-\underline{O}, \underline{x}-\underline{O}) + 1$

Alexander grading: $A(\underline{x}) = \frac{1}{2}\left(\mathcal{J}(\underline{x}-\underline{O}, \underline{x}-\underline{O}) - \mathcal{J}(\underline{x}-\underline{X}, \underline{x}-\underline{X}) - (N-1)\right)$

e.g.:



$M(\underline{x}) =$

$A(\underline{x}) =$

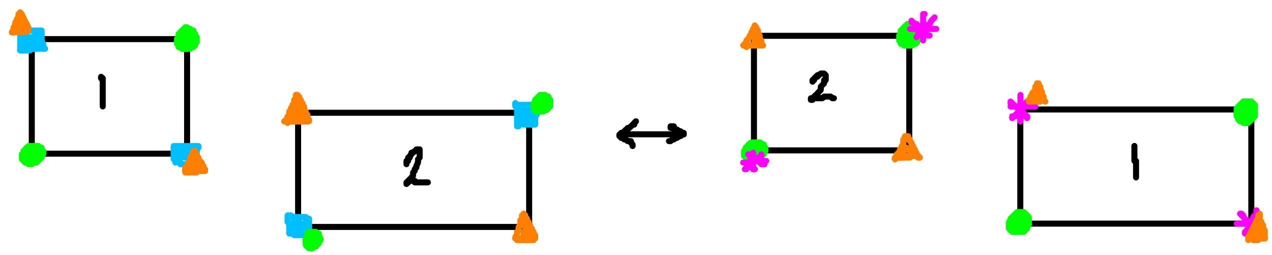
$(\widehat{CFK}, \hat{\partial})$ is a Chain Complex

Need to prove $\partial^2 = 0$

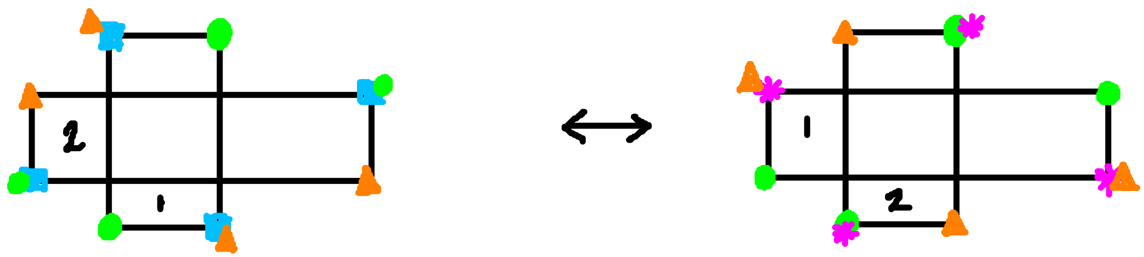
$$\partial^2 \mathbf{x} = \partial\Big(\sum_{\exists \text{ emty rectangle } \mathbf{x} \to \mathbf{y}} \mathbf{y}\Big) = \sum_{\exists \text{ emty rectangle } \mathbf{x} \to \mathbf{y}} \Big(\sum_{\exists \text{ emty rectangle } \mathbf{y} \to \mathbf{z}} \mathbf{z}\Big)$$

the coefficient of $\mathbf{z}$ is given by # 2 empty rectangles $\mathbf{x} \to \mathbf{y} \to \mathbf{z}$

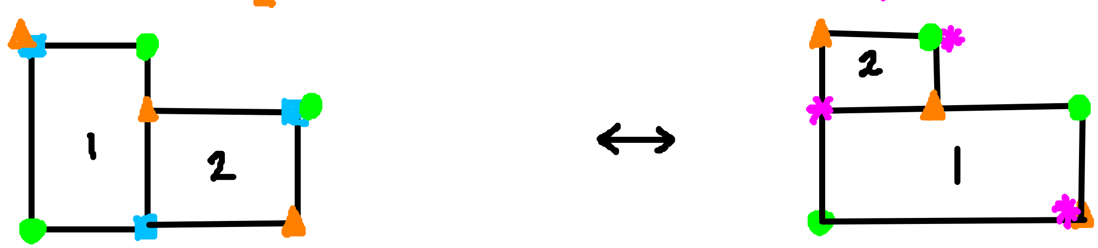
→ disjoint rectangles



→ the interiors intersect



→ common corner



$\Rightarrow \partial^2 = 0$ over $\mathbb{F}_2$

Thm (Manolescu-Ozsváth-Sarkar) $H_*(\widehat{CFK}, \hat{\partial}) \cong \widehat{HFK}(K) \otimes V^{\otimes n-1}$

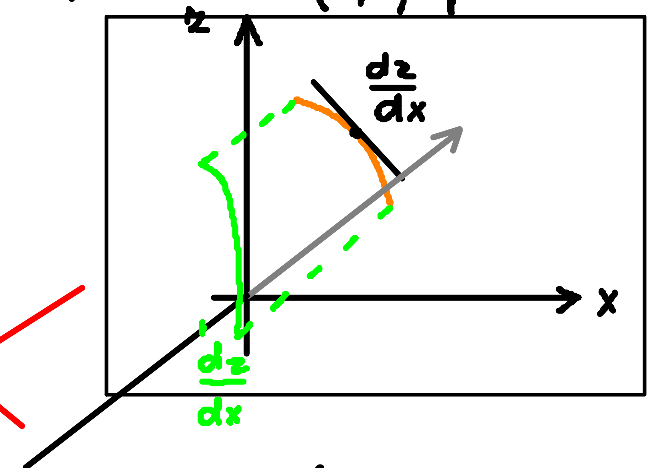
where $V = (\mathbb{F}_2)_{0,0} \oplus (\mathbb{F}_2)_{(-1,-1)}$

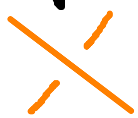
invariant of the knot

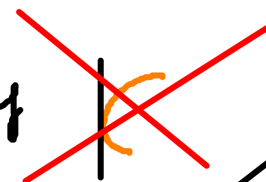
LEGENDRIAN KNOTS

Def: A knot is *Legendrian* if its projection to the (x,z)-plane determines the knot by

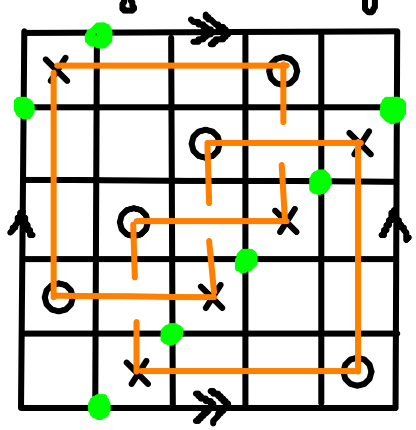
$$y = -\frac{dz}{dx}$$



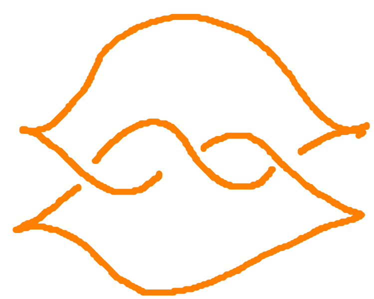
Note: • every crossing in the projection looks like 

• there is no vertical tangency 

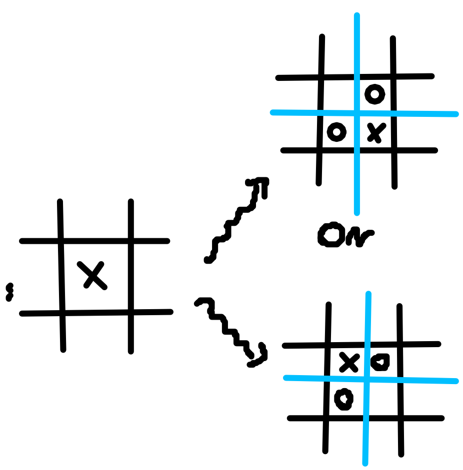
A grid diagram naturaly determines a Legendrian knot:



rotate by 45° & smooth some corners



a restricted set of gridmoves give Legendrian isotopies:



Thm: (Ozsváth-Szabó-Thurston) $\lambda(L) = [\bullet] \in \widehat{HFK}(K)$ is an invariant for Legendrian knots.

Applications of $\widehat{HFK}$

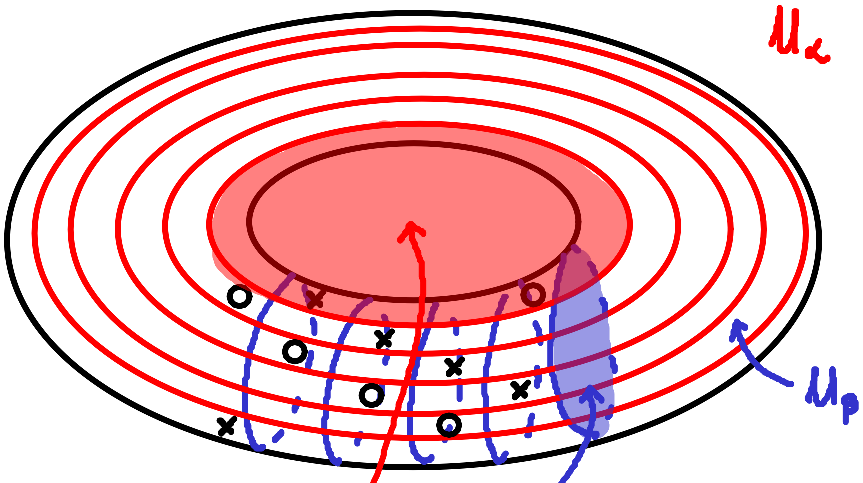
Thm (Ozsváth - Szabó) Euler characteristic is the Alexander poly: Δ

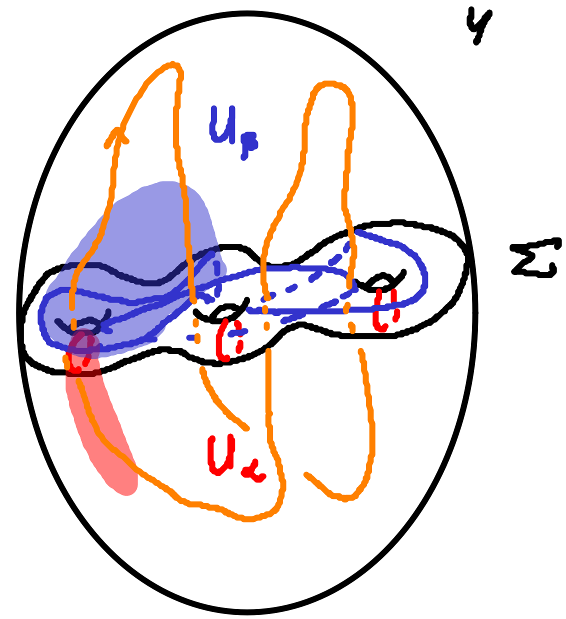
Thm (Ozsváth - Szabó) Max grading is the genus

Thm (Ghiggini, Ng) Detects fibered knots

(Ng - Ozsváth - Thurston, Lisca - Ozsváth - Stipsicz - Szabó)
Gives effective Legendrian / transverse knot invariants

Generalization





Heegaard decomposition of S^3 $\longrightarrow$ Heegaard decomposition of Y

α-curve bound discs in U_α $\longrightarrow$ α-curves on Σ

β-curve bound disc in U_β $\longrightarrow$ β-curves on Σ

X + intersection of K w/ T^2 $\longrightarrow$ one X and one O in each comp of $\Sigma - \alpha$ & $\Sigma - \beta$

O + intersection of K w/ T^2 $\longrightarrow$

generators: N-tuples of α & β curves; one on each $\longrightarrow$

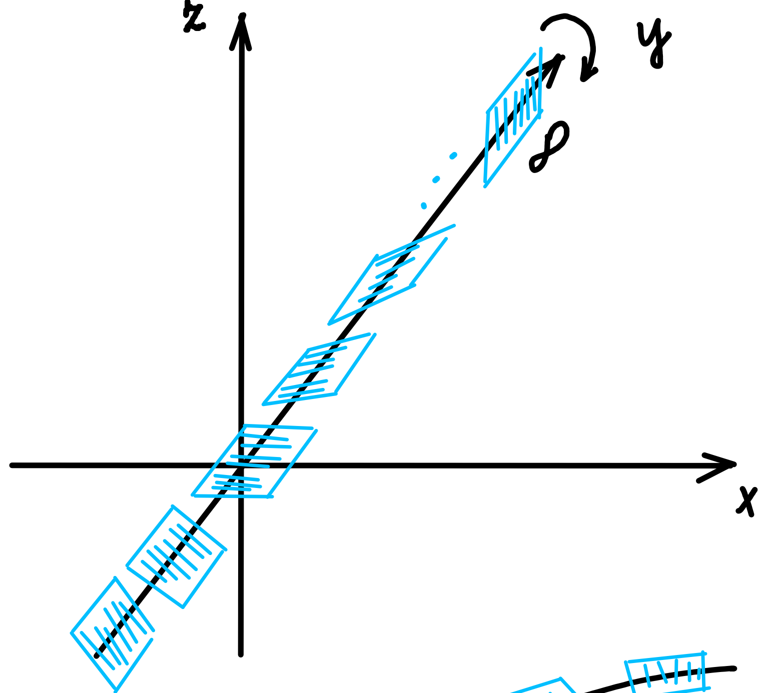
boundary map: rectangles - - - - - -> holomorphic curves in $\Sigma \times D^2$

Thm (Ozsváth-Szabó): the homology of gives an invariant $\widehat{HFK}(Y, K)$

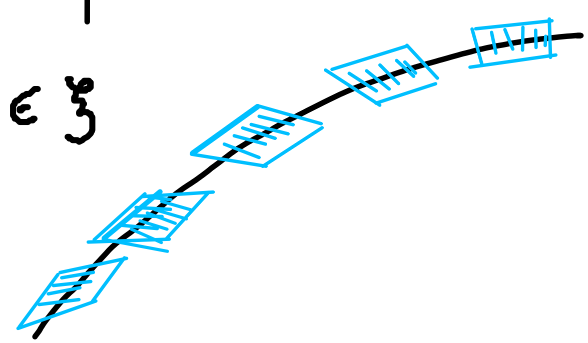
Legendrian Knots

Def: a contact structure is a totally nonintegrable planefield ξ

e.g. $(\mathbb{R}^3, \xi_{st} = \ker(dz - y\,dx))$



Def: a knot is Legendrian if $TK \in \xi$



Note: in $(\mathbb{R}^3, \xi_{st})$: $\quad TK \in \xi_{st} = \ker(dz - y\,dx) \iff y = \dfrac{dz}{dx}$

Thm (Lisca-Ozsváth-Stipsicz-Szabó): there is an invariant for Legendrian knots $\hat{\mathfrak{L}}(L) \in \widehat{HFK}(Y,K)$

PROPERTIES OF THESE NVARIANTS

Def L loose if its complement is OT
L is exceptional or non-loose otherwise

		$c(L)$	$\mathfrak{L}(L)$	$\lambda(L)$
		in $SHF(-Y\setminus N(L),\Gamma)$	in $HFK(-Y,K)$	in $HFK(-S^3,K)$
L loose		0 HKM	0 LOSS	N.A.
complement of L contains Giroux torsion		0 HKM	0 Vela-Vick, Stipsicz-V	N.A
stabilisation	L^+	$SHF(-Y\setminus N(L),\Gamma) \downarrow SHF(-Y\setminus N(L^\pm),\Gamma)$	0	0
	L^-		$\mathfrak{L}(L)$ LOSS	$\lambda(L)$ O-Sz-T

Cor: Both $\mathfrak{L}(L)$ and $\lambda(L)$ defines a transverse invariant by:

$$\vartheta(T)=\lambda(L) \qquad \text{if } T=T(L)$$
$$\tau(T)=\mathfrak{L}(L)$$

So Far...

Thm (V) $\lambda(L_1 \# L_2) = \lambda(L_1) \otimes \lambda(L_2)$

construction of infinitely many transversaly nonsimple knots

Thm (Stipsicz - V) there are maps:

$$\begin{tikzcd} SHF(-Y\setminus N(L),\Gamma) \arrow[r, bend left, "+"] \arrow[r, bend right, "-"'] & HFK(-Y,K) \end{tikzcd}$$

$$c(L) \mapsto \mathcal{L}(L), \qquad c(L) \mapsto \mathcal{L}(\overline{L})$$

$\mathcal{L}(L)$ vanishes if its complement contains Giroux torsion

Thm (Baldwin - Vela-Vick - V)

$\lambda(L) = \mathcal{L}(L)$ for any Legendrian knot in (S^3, ξ_{st})

Thm (Etnyre - Ng - V) Complete Legendrian classification of twist knots

Plans

- define a gluable version of $\widehat{HFK}$ for tangles
- Legendrian classification of positive braids, 3-braids
- understand Legendrian classification of braid satellites

$\vdots$