WHAT IS JACKSON'S CONSTANT?

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Abstract. We prove a refinement of Jackson's theorem on the approximation of Lipschitz functions by trigonometric polynomials. Our result precisely characterizes the leading error term associated with Jackson's construction. We do the same for a related construction commonly used in the kernel polynomial method for spectral density estimation, which is slightly better than Jackson's construction in this respect.

1. Overview and main results

Theorems established by Dunham Jackson in the early 1900s regarding the approximation of Lipschitz functions by trigonometric and algebraic polynomials have undergone a resurgence in recent decades. This renewed interest is due in large part to their utility in large-scale eigenvalue computations for Hermitian matrices, most notably the kernel polynomial method for spectral density estimation and polynomial filtering in the Eigenvalues Slicing Library (EVSL) [WWAF06, LSY16, LXES19]. Indeed, Jackson's theorems and variants of his construction have been used repeatedly in recent research developing and analyzing algorithms for spectral density estimation [BKM22, MMRS24, CTU25, MMS$^+$26]. In this note, we provide sharp error estimates for the constants associated with Jackson's original construction [Jac12] and the more recent variant described by Rivlin [Riv81] and Weiße et al. [WWAF06]. We find that the constant in Jackson's original construction is about 9% larger than that of the more recent variants, suggesting that the latter possesses a slight edge for practical applications.

Throughout this note, we assume the given function $f : \mathbb{R} \to \mathbb{R}$ is 2π-periodic and L-Lipschitz continuous, i.e. there exists a constant $L > 0$ such that

$$|f(x) - f(y)| \le L|x - y| \quad \text{for all } x, y \in \mathbb{R}.$$

A function is a *trigonometric sum of degree n* if it is in the linear span of the functions

$$\{1, \cos(x), \sin(x), \cdots, \cos(nx), \sin(nx)\}.$$

Jackson's main theorems from [Jac12] state the following.[1]

Theorem 1.1 (Jackson). *For each $n \ge 1$, there exists a trigonometric sum p of degree at most n satisfying*

$$\max_{x\in[-\pi,\pi]} |p(x) - f(x)| \le \frac{2.9L}{n}. \tag{1.1}$$

Jackson's proof has several attractive properties for applications: it is *constructive*, i.e. he shows explicitly how to convert a given function f into a trigonometric sum p; it is *positivity-preserving*, so that if $f(x) \ge 0$ everywhere then $p(x) \ge 0$ everywhere; and it is *mass-preserving* in the sense that

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[1] Jackson also includes results analogous to Theorem 1.1 on functions whose $(k-1)$th derivative is L-Lipschitz, showing that the order of convergence is in general $O(n^{-k})$, but this is beyond the scope of this note.

$\int f(x)\,\mathrm{d}x = \int p(x)\,\mathrm{d}x$. A related but distinct construction of Rivlin [Riv81], derived independently by Weiße et al. [WWAF06], is also positivity-preserving, mass-preserving, and satisfies a guarantee similar to (1.1) for some other constant in place of 2.9.[2] We emphasize these latter two properties of positivity and mass preservation since they undergird the aforementioned application to the kernel polynomial method for approximating probability distributions. Positivity-preservation and mass-preservation ensure in particular that if f defines a probability density function on $[-\pi, \pi]$, then p will as well.

Jackson also showed that the constant 2.9 appearing in (1.1) could not be pushed lower than $\pi/2$ for any construction, irrespective of the positivity and normalization requirements, though he did not show that $\pi/2$ was achievable. This was shown later for a different construction by Favard and independently by Akhiezer and Krein. More specifically, one can construct an optimal approximation satisfying the improved error bound

$$\max_{x\in[-\pi,\pi]} |p(x) - f(x)| \le \frac{\pi L}{2(n+1)},$$

which precisely matches Jackson's lower bound. An exposition of this result is given, for example, in [DL93]. Although Favard's construction is optimal, it is not positivity-preserving, which is significant, for example, in the approximation of probability densities.

The purpose of this note is to identify the exact error constants involved in the positivity-preserving constructions of Jackson and Rivlin. Before stating our theorems, we first state what these constructions are. Define the following kernel functions:

$$J_m(t) = \left(\frac{\sin(mt/2)}{m\sin(t/2)}\right)^4 \cdot \mathbf{1}_{[-\pi,\pi]}(t), \qquad R_n(t) = \left(\frac{1}{2} + \sum_{j=1}^{n} \rho_{j,n}\cos(jt)\right) \cdot \mathbf{1}_{[-\pi,\pi]}(t),$$

$$\rho_{j,n} = \frac{(n+2-j)\cos\left(\frac{j\pi}{n+2}\right) + \sin\left(\frac{j\pi}{n+2}\right)\cot\left(\frac{\pi}{n+2}\right)}{n+2},$$

$$\hat{J}_m(t) = \frac{1}{\int_{\mathbb{R}} J_m(x)\,\mathrm{d}x} J_m(t), \qquad \hat{R}_n(t) = \frac{1}{\int_{\mathbb{R}} R_n(x)\,\mathrm{d}x} R_n(t).$$

The kernel on the left, J_m, we call Jackson's kernel and the kernel on the right, R_n, we call Rivlin's kernel. The derivation of the coefficients $\rho_{j,n}$ is provided in [WWAF06]. The convolution of two functions is defined as

$$(f * g)(x) = \int_{\mathbb{R}} f(x-t)g(t)\,\mathrm{d}t.$$

Jackson's construction sets $p = f * \hat{J}_m$ for $m = \lfloor n/2 \rfloor + 1$ and Rivlin's sets $p = f * \hat{R}_n$. When restricted to $[-\pi, \pi]$, R_n is an even trigonometric sum of order n, and J_m can be written as the square of an even trigonometric sum of order $(m-1)$, see (2.2), so that J_m is an even trigonometric sum of order $2(m-1) \le n$. Since p is formed by convolution, in either case it will be a trigonometric sum of order n.

Our main results provide precise control on leading terms of Jackson and Rivlin's approximations.

Theorem 1.2 (Control of Jackson's kernel)**.** *Let $p = f * \hat{J}_m$ for $n \ge 2$ even and $m = (n/2) + 1$. Then*

$$\sup_{x\in\mathbb{R}} |f(x) - p(x)| \le \frac{12\log 2}{\pi} \cdot \frac{L}{n+2} + O\left(\frac{\log n}{n^3}\right).$$

Note

$$\frac{12\log(2)}{\pi} = 2.6476\cdots.$$

Moreover, equality is obtained when $f(x) = L|x|$ periodically extended from $[-\pi, \pi]$ to the real line.

[2] We note that Weiße et al. refer to the kernel derived in their work as "Jackson's kernel" although technically the kernel is not identical to that used by Jackson or as presented in [DL93]. It is, however, identical to the kernel derived in [Riv81] in proving a variant of Jackson's theorem.

Proof. This is a direct consequence of Lemma 2.1 and Lemma 2.2, proven in the second part of this note. Note that when the kernel J_m is used, it produces an approximation of degree $n = 2(m-1)$. This causes the doubling from $6\log(2)/\pi$ in Lemma 2.2 to $12\log(2)/\pi$ in the present result, as well as the use of $n+2$ in the denominator. If n is odd, the result remains true with the substitution $n+2 \to n+1$ in the denominator and the choice $m = \lfloor n/2 \rfloor + 1$. $\square$

Theorem 1.3 (Control of Rivlin's kernel)**.** *Let $p = f * \hat{R}_n$. Then*

$$\sup_{x\in\mathbb{R}} |f(x) - p(x)| \leq \left(2\int_0^\pi \frac{\sin u}{u}\,\mathrm{d}u - \frac{4}{\pi}\right)\frac{L}{n} + O\left(\frac{1}{n^2}\right).$$

Note

$$2\int_0^\pi \frac{\sin u}{u}\,\mathrm{d}u - \frac{4}{\pi} = 2.4306\cdots.$$

Moreover, equality is obtained when $f(x) = L|x|$ periodically extended from $[-\pi,\pi]$ to the real line.

Proof. The result follows directly from Lemma 2.1 and Lemma 2.3. $\square$

2. Proofs

Our first lemma establishes that the periodic extension of $f(x) = L|x|$ from $[-\pi,\pi]$ to all of $\mathbb{R}$ is the worst-case function for an upper bound on $\max_{x\in\mathbb{R}}|f(x) - p(x)|$, and that the maximum is achieved for $x = 0$.

Lemma 2.1. *Let $K \not\equiv 0$ be any non-negative kernel supported on $[-\pi,\pi]$ and set*

$$\hat{K}(t) = \frac{1}{\int K(t)\,\mathrm{d}t} K(t).$$

Let $f_0(x)$ be the periodic extension of $L|x|$ from $[-\pi,\pi]$ to all of $\mathbb{R}$. Then

$$\sup_{x\in\mathbb{R}} |(f * \hat{K})(x) - f(x)| \leq \left|(f_0 * \hat{K})(0) - f_0(0)\right| = L\frac{\int_{-\pi}^{\pi}|t|K(t)\,\mathrm{d}t}{\int_{-\pi}^{\pi} K(t)\,\mathrm{d}t}. \tag{2.1}$$

Equality is obtained at $x = 0$ for the triangle wave $f(x) = L|x|$ periodically extended from $[-\pi,\pi]$ to the real line.

Proof.

$$\begin{aligned}\left|(f * \hat{K})(x) - f(x)\right| &= \left|\int_{-\pi}^{\pi} (f(x-t) - f(x))\hat{K}(t)\,\mathrm{d}t\right| \\ &\leq \int_{-\pi}^{\pi} |f(x-t) - f(x)|\hat{K}(t)\,\mathrm{d}t \\ &\leq L\int_{-\pi}^{\pi} |t|\hat{K}(t)\,\mathrm{d}t.\end{aligned}$$

$\square$

Our goal is therefore to compute the ratio of $\int_{-\pi}^{\pi}|t|K\,\mathrm{d}t$ and $\int_{-\pi}^{\pi} K\,\mathrm{d}t$ for $K \in \{J_m, R_n\}$, which the next lemmas do.

Lemma 2.2.

$$\frac{\int_{-\pi}^{\pi}|t|J_m(t)\,\mathrm{d}t}{\int_{-\pi}^{\pi} J_m(t)\,\mathrm{d}t} = \frac{6\log(2)}{\pi}\cdot\frac{1}{m} + O\left(\frac{\log m}{m^3}\right).$$

Proof. We will set

$$A = m^4 \int_0^\pi tJ_m(t)\,\mathrm{d}t, \quad B = m^4 \int_0^\pi J_m(t)\,\mathrm{d}t.$$

Using the evenness of $J_m(t)$ and $|t|J_m(t)$ and canceling common factors in the numerator and denominator, the desired quantity is A/B.

We first compute B. We will use the identity

$$\frac{1}{2m}\left(\frac{\sin(mt/2)}{\sin(t/2)}\right)^2 = \frac{1}{2} + \sum_{k=1}^{m-1}\left(1-\frac{k}{m}\right)\cos(kt) \tag{2.2}$$

found in [DL93, Page 203].[3] Along with orthogonality of $\cos(kx)$, we have

$$\begin{aligned} B = \int_0^\pi \left(\frac{\sin(mx/2)}{\sin(x/2)}\right)^4 \mathrm{d}x &= 4m^2\left(\frac{\pi}{4} + \frac{\pi}{2}\sum_{k=1}^{m-1}\left(1-\frac{k}{m}\right)^2\right) \\ &= 4m^2\left(\frac{\pi}{4} + \frac{\pi}{2}\frac{(m-1)(2m-1)}{6m}\right) \\ &= \frac{\pi}{3}(2m^3+m). \end{aligned}$$

We now turn our attention to the numerator. First use the substitution $u = mt$ and factor to obtain

$$A = m^2 \int_0^{m\pi} u\left(\frac{\sin(u/2)}{u/2}\right)^4\left(\frac{u/2m}{\sin(u/2m)}\right)^4 \mathrm{d}u.$$

First using $|x/\sin(x)| \geq 1$ and then $|\sin(x)/x| \leq 1/|x|$,

$$\begin{aligned} A &\geq m^2 \int_0^{m\pi} u\left(\frac{\sin(u/2)}{u/2}\right)^4 \mathrm{d}u \\ &= m^2\left(4\log 2 - \int_{m\pi}^\infty u\left(\frac{\sin(u/2)}{u/2}\right)^4 \mathrm{d}u\right) \\ &\geq m^2\left(4\log 2 - \int_{m\pi}^\infty \frac{16}{u^3}\,\mathrm{d}u\right) \\ &= m^2\left(4\log 2 - \frac{8}{\pi^2 m^2}\right). \end{aligned}$$

For an upper bound on A, use $|x/\sin(x)| \leq (1+6x^2)^{1/4}$ for $|x| \leq \pi/2$:

$$A \leq m^2 \int_0^{m\pi} u\left(\frac{\sin(u/2)}{u/2}\right)^4\left(1+\frac{3u^2}{2m^2}\right)\mathrm{d}u \leq m^2\cdot 4\log(2) + \frac{3}{16}\int_0^{m\pi}\frac{\sin^4(u/2)}{u}\,\mathrm{d}u. \tag{2.3}$$

The final integral may be bounded by

$$\int_0^{m\pi}\frac{\sin^4(u/2)}{u}\,\mathrm{d}u \leq \int_0^{2\pi}\frac{(u/2)^4}{u}\,\mathrm{d}u + \int_{2\pi}^{m\pi}\frac{1}{u}\,\mathrm{d}u = \frac{\pi^4}{4} + \log(m/2). \tag{2.4}$$

In particular, $A = m^2\cdot 4\log(2) + O(\log m)$. Finally, we take the quotient:

$$\frac{A}{B} = \frac{\int_0^\pi tJ_m(t)\,\mathrm{d}t}{\int_0^\pi J_m(t)\,\mathrm{d}t} = \frac{4\log(2)m^2 + O(\log m)}{\frac{\pi}{3}(2m^3+m)} = \frac{6\log 2}{\pi}\cdot\frac{1}{m} + O\left(\frac{\log m}{m^3}\right).$$

□

[3]The formula in [DL93, Page 203] has a typo that includes $k = 0$ in the sum. We have corrected that here.

Lemma 2.3.
$$\frac{\int_{-\pi}^{\pi}|t|R_n(t)\,\mathrm{d}t}{\int_{-\pi}^{\pi}R_n(t)\,\mathrm{d}t} = \left(2\int_0^\pi \frac{\sin t}{t}\,\mathrm{d}t - \frac{4}{\pi}\right)\cdot\frac{1}{n} + O\left(\frac{1}{n^2}\right).$$

Proof. R_n is even, so it suffices to integrate over 0 to π instead. The denominator can be computed exactly:
$$\int_{-\pi}^{\pi} R_n(t)\,\mathrm{d}t = \pi$$
since $\int_{-\pi}^{\pi} 1\,\mathrm{d}t = 2\pi$ and $\int_{-\pi}^{\pi}\cos(jt)\,\mathrm{d}t = 0$ for $j \geq 1$. Integrating the numerator term by term gives
$$\int_0^\pi tR_n(t)\,\mathrm{d}t = \frac{\pi^2}{4} - 2\sum_{\substack{1\leq j\leq n\\ j\text{ odd}}}\frac{\rho_{j,n}}{j^2} = 2\sum_{\substack{1\leq j\leq n\\ j\text{ odd}}}\frac{1-\rho_{j,n}}{j^2} + 2\sum_{\substack{j>n\\ j\text{ odd}}}\frac{1}{j^2},$$
where the first equality follows from direct integration of $tR_n(t)$ according to its series definition and the second equality uses the fact that the sum of the odd reciprocal squares is equal to $\pi^2/8$.

We first claim that the first term may be rewritten
$$2\sum_{\substack{1\leq j\leq n\\ j\text{ odd}}}\frac{1-\rho_{j,n}}{j^2} = \frac{1}{n+2}\int_0^1 \psi(x)\,\mathrm{d}x + O\left(\frac{1}{n^2}\right), \tag{2.5}$$
where
$$\psi(x) = \frac{1}{x^2}\left(1-(1-x)\cos(\pi x) - \frac{\sin(\pi x)}{\pi}\right).$$
To see this, observe that
$$\frac{\cot\left(\dfrac{\pi}{n+2}\right)}{n+2} = \frac{1}{\pi} + O\left(\frac{1}{n^2}\right)$$
implies
$$1-\rho_{j,n} = 1 - \left(1-\frac{j}{n+2}\right)\cos\left(\frac{\pi j}{n+2}\right) - \frac{\sin\left(\dfrac{\pi j}{n+2}\right)}{\pi} + O\left(\frac{1}{n^2}\right).$$
Setting
$$\phi(x) = 1-(1-x)\cos(\pi x) - \frac{\sin(\pi x)}{\pi},$$
we then have
$$1-\rho_{j,n} = \phi\left(\frac{j}{n+2}\right) + O\left(\frac{1}{n^2}\right).$$
Finally, we interpret the summation as a Riemann sum. Recalling the definition of $\psi(x)$, which we note extends smoothly to $[0,1]$ since $\phi(0) = \phi'(0) = 0$, we write
$$2\sum_{\substack{1\leq j\leq n\\ j\text{ odd}}}\frac{\phi\left(\dfrac{j}{n+2}\right)}{j^2} = \frac{2}{(n+2)^2}\sum_{\substack{1\leq j\leq n\\ j\text{ odd}}}\psi\left(\frac{j}{n+2}\right) = \frac{1}{n+2}\int_0^1\psi(x)\,\mathrm{d}x + O\left(\frac{1}{n^2}\right).$$
Integration by parts yields
$$\int_0^1 \psi(x)\,\mathrm{d}x = \pi\int_0^\pi \frac{\sin u}{u}\,\mathrm{d}u - 3.$$

This provides control on the first term. The second term may be estimated by

$$2\sum_{\substack{j>n\\ j\text{ odd}}} \frac{1}{j^2} = \frac{1}{n} + O\left(\frac{1}{n^2}\right) = \frac{1}{n+2} + O\left(\frac{1}{n^2}\right).$$

As a result, we obtain

$$\int_0^\pi tR_n(t)\,\mathrm{d}t = \frac{1}{n}\left(\pi\int_0^\pi \frac{\sin u}{u}\,\mathrm{d}u - 2\right) + O\left(\frac{1}{n^2}\right).$$

Consequently, the quotient satisfies

$$\frac{\int_0^\pi tR_n(t)\,\mathrm{d}t}{\int_0^\pi R_n(t)\,\mathrm{d}t} = \frac{\frac{1}{n}\left(\pi\int_0^\pi \frac{\sin u}{u}\,\mathrm{d}u - 2\right) + O\left(\frac{1}{n^2}\right)}{\pi/2} = \left(2\int_0^\pi \frac{\sin u}{u}\,\mathrm{d}u - \frac{4}{\pi}\right)\frac{1}{n} + O\left(\frac{1}{n^2}\right).$$

□

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