HAMILTON POWERS OF EULERIAN DIGRAPHS

ENRICO COLÓN AND JOHN URSCHEL

Abstract. In this note, we prove that the $\left\lceil \frac{1}{2} \sqrt{n} \log_2^2 n \right\rceil$th power of a connected $n$-vertex Eulerian digraph is Hamiltonian, and provide an infinite family of digraphs for which the $\left\lfloor \sqrt{n}/2 \right\rfloor$th power is not.

1. Preliminaries

The $k$th power of a (directed or undirected) graph $G$, denoted $G^k$, is the graph on the vertices of $G$ in which there is an edge from vertex $u$ to vertex $v$ if there exists a $uv$-path in $G$ of length at most $k$. It is well-known that the cube of any connected undirected graph is Hamiltonian (see [6, 11], also [3, Ex 10-14]). In 1974, Fleischner proved that the square of any two-connected undirected graph is Hamiltonian, solving the Plummer-Nash-Williams conjecture [4] (see [5] for a much simpler proof). Unfortunately, strongly-connected directed graphs (digraphs) may require the $\left\lceil n/2 \right\rceil$th power to be Hamiltonian; even $k$-strong connectedness is only sufficient for guaranteeing that the $\left\lceil n/(2k) \right\rceil$th power is Hamiltonian [10]. For a general survey on Hamilton cycles in digraphs, we refer the reader to [7]. Interestingly, results for Eulerian digraphs are not nearly so bleak.

Through the study of minimally Eulerian digraphs (connected Eulerian digraphs with no proper connected Eulerian subgraph), we prove that

**Theorem 1.1.** The $\left\lceil \frac{1}{2} \sqrt{n} \log_2^2 n \right\rceil$th power of any $n$-vertex connected Eulerian digraph is Hamiltonian.

In fact, we prove an even stronger result (in Theorem 2.1) that, given a minimally Eulerian digraph $G = (V, A)$, specifies an ordering $v_1, \ldots, v_n$ of $V$ and an edge-disjoint directed path (dipath) decomposition $P_1, \ldots, P_n$ of $G$, such that each $P_i$ is a $v_i v_{i+1}$-dipath ($v_{n+1} := v_1$) of length at most $\left\lceil \frac{1}{2} \sqrt{n} \log_2^2 n \right\rceil$. In addition, we provide an infinite family of minimally Eulerian digraphs for which the $\left\lfloor \sqrt{n}/2 \right\rfloor$th power is not Hamiltonian (Example 2.2). For details regarding the importance of minimally Eulerian digraphs and their connection to the traveling salesman problem, we refer the reader to [2, 8].

1.1. Notation, Definitions, and Basic Results. Let $G = (V, A)$ be a simple digraph. If $G$ contains a spanning directed cycle (dicycle), then $G$ is Hamiltonian. If $G$ contains an Euler circuit (a circuit containing every edge), then $G$ is Eulerian. If $G$ is connected, this is equivalent to the condition that, for every vertex $v \in V$, the indegree $d^-(v)$ equals the outdegree $d^+(v)$. If $G$ is a connected Eulerian digraph and contains no proper connected Eulerian subgraph on the vertices of $G$, then $G$ is minimally Eulerian; equivalently, a connected Eulerian digraph $G$ is minimally Eulerian if, for any dicycle $C$ of $G$, the graph $G - C := (V, A - A(C))$ is disconnected. If $G$ contains no dicycle,

---

1991 Mathematics Subject Classification. Primary 05C20, 05C45.

1The first notable example of a class of digraphs requiring a “non-trivial” (say, $o(n)$) Hamiltonicity exponent are cacti, see [9] for details.
then $G$ is acyclic. For more details regarding graph theoretic definitions and notation, we refer the reader to \[\mathbb{I}\]. Let

$$p_\#(G) := \frac{1}{2} \sum_{u \in V} |d_+(u) - d_-(u)|,$$

a measure of how "close" to Eulerian a digraph is, and a key ingredient in our proof. The quantity $p_\#(G)$ is exactly the minimal number of dipaths required in an edge-disjoint decomposition of $G$ into dipaths and dicycles. The size of an acyclic digraph $G$ is immediately bounded above by $p_\#(G) (|V| - 1)$, and an even tighter estimate can be obtained relatively quickly:

**Proposition 1.2.** Let $G = (V, A)$ be an acyclic digraph. Then $|A| \leq \sqrt{2p_\#(G)} |V|$.

*Proof.* Let $p_\#(G) > 2$ ($p_\#(G) = 0, 1, 2$ is straightforward), $V = \{v_1, ..., v_n\}$ be a topological sorting of $G$ (i.e., $v_i v_j \in A$ implies that $i < j$), $k \in \mathbb{N}$ be the smallest number such that $p_\#(G) \leq \binom{k}{2}$, $\ell = \left\lceil n/k \right\rceil$, and $V_i = \{v_{i(k-1) + 1}, ..., v_{ik}\}$, $i = 1, ..., \ell - 1$, $V_\ell = \{v_{(\ell - 1)k + 1}, ..., v_n\}$. There are at most $\binom{k}{2}$ edges within each of the subsets $V_i$, $i = 1, ..., \ell - 1$, and at most $(n-k(\ell-1))$ within the subset $V_\ell$. Our digraph $G$ can be decomposed into $p_\#(G)$ edge-disjoint dipaths, and, by the topological sorting of $V$, each of the aforementioned $p_\#(G)$ dipaths has at most $\ell - 1$ edges between the subsets $V_1, ..., V_\ell$. Therefore, there are at most $(\ell - 1)p_\#(G)$ total edges between the subsets $V_1, ..., V_\ell$. Combining these estimates gives

$$|A| \leq \left\lceil \frac{n}{k} \right\rceil - 1 \left( \binom{k}{2} + p_\#(G) \right) + \frac{(n-k(\left\lceil n/k \right\rceil - 1))}{2}.$$ 

Dividing by $\sqrt{p_\#(G)} n$ and noting that the right hand side is convex w.r.t. $p_\#(G)$ and maximized when $p_\#(G)$ is as small as possible, we obtain

$$\frac{|A|}{\sqrt{p_\#(G)} n} \leq \frac{n}{k} \left( \frac{k}{2} \right) \left( \sqrt{\frac{p_\#(G)}{n}} \right) + \frac{\left( n-k(\left\lceil n/k \right\rceil - 1) \right)}{2} \left( k^{-1} \right)^{1/2}.$$ 

The right hand side is a convex quadratic function in the term $\left\lceil n/k \right\rceil$ (treating $\left\lceil n/k \right\rceil$ as a variable independent of $n$ and $k$), and achieves its maximum at the right endpoint of the interval $[n/k, n/k + 1]$. Replacing $\left\lceil n/k \right\rceil$ by $n/k + 1$, we obtain

$$|A| < \frac{(k - 1)^2}{k} \sqrt{\frac{p_\#(G)}{n}} \leq \sqrt{2p_\#(G)} n,$$

as $k \geq 3$ (recall, $p_\#(G) > 2$).

From Proposition 1.2, we immediately obtain a bound (tight up to a multiplicative constant; see Example 2.2) on the maximum size of a minimally Eulerian digraph:

**Proposition 1.3.** Let $G = (V, A)$ be a minimally Eulerian digraph. Then

$$|A| \leq \sqrt{2(|V| - 1) |V| + |V| - 1}.$$ 

*Proof.* $G$ is an Eulerian digraph, so it admits a spanning arborescence $T$. That is, $G$ admits a rooted, directed subgraph of $G$ in which the path from the root to any other vertex of $T$ is unique. Every dicycle must intersect an edge of $T$, as the removal of any
dicycle from a minimally Eulerian graph disconnects it. Therefore, $G - T$ is acyclic, and by Proposition 1.2 $|A| \leq |A(G - T)| + |A(T)| \leq \sqrt{2(|V| - 1)} |V| + |V| - 1$. 

2. A Proof of Theorem 1.1 and A Lower Bound

To prove Theorem 1.1 we show an even stronger statement regarding minimally Eulerian digraphs.

**Theorem 2.1.** Let $G = (V, A)$, $|V| = n > 1$, be a minimally Eulerian graph. Then there exists an ordering $v_1, ..., v_n$ of $V$ and an $n$-dipath edge-disjoint decomposition $P_1, ..., P_n$ of $G$ such that each $P_i$ is a $v_i, v_{i+1}$-dipath ($v_{n+1} := v_1$) of length at most $[f(n) \sqrt{n}]$, where

$$f(n) = (\log_2 n)^{\log_3/2 + o(1)} \leq \frac{1}{2} \log_2 n.$$

**Proof.** It suffices to consider $n \geq 6388$, as $G^{n-1}$ is always Hamiltonian and $[\frac{1}{2} \sqrt{n} \log_2 n] \geq n - 1$ for $n = 1, ..., 6387$. Let $v_1, ..., v_n$ and $P_1, ..., P_n$ be an ordering and decomposition of $G$ into edge-disjoint $v_i, v_{i+1}$-dips $P_i$ that lexicographically minimizes the elements of the set $\{|A(P_1)|, ..., |A(P_n)|\}$ (i.e., minimizes the length of the longest dipath, minimizes the length of the 2nd longest dipath conditional on the minimality of the longest dipath, etc). Let $P_j$ be the longest dipath, with length $|A(P_j)| = \alpha n^{1/2}$ for some $\alpha \geq \frac{1}{2} \log_2 n$. We aim to build a sequence of subgraphs $H_0 := P_j \subset H_1 \subset H_2 \subset ...$, lower bound the order of each subgraph using the lexicographic minimality of path lengths, and conclude that if $\alpha$ is too large then some $H_t$ contains too many vertices, reaching a contradiction.

Let $H_0 = P_j$, and $H_\ell$, $\ell > 0$, be the union of all $P_i$ with either start- or end-vertex in $H_{\ell-1}$ and $|A(P_i)| \geq \alpha n^{1/2}/2^{\ell}$. Let $n_\ell$, $m_\ell$, and $k_\ell$ be the number of vertices, edges, and dipaths $P_i$ in $H_\ell$. We have $n_0 = \alpha n^{1/2} + 1$, $m_0 = \alpha n^{1/2}$, $k_0 = 1$, and, by construction, $m_\ell \geq k_\ell m_0 / 2^\ell$ for all $\ell \geq 0$. Since $H_\ell$ is the union of dipaths of lengths at least $m_0 / 2^{\ell}$, by lexicographic minimality, every vertex of $H_\ell$ is either the start- or end-vertex of a dipath $P_i$ of length at least $m_0 / 2^{\ell+1}$, and so $k_{\ell+1} \geq n_\ell / 2$ for all $\ell \geq 0$. The graph $H_\ell$ can be decomposed into the edge-disjoint union of two graphs $H_{\ell,a}$ and $H_{\ell,e}$, where $H_{\ell,a}$ is acyclic with $p(\#H_{\ell,a}) \leq k_\ell$ and $H_{\ell,e}$ is the vertex-disjoint union of minimally Eulerian graphs $H_{\ell,e}^{(1)}, ..., H_{\ell,e}^{(p)}$. By Proposition 1.2, $H_{\ell,a}$ has at most $\sqrt{2k_\ell n_\ell}$ edges. By Proposition 1.3, $H_{\ell,e}$ has at most

$$\sum_{i=1}^{p} \left( \sqrt{2(n_\ell(i) - 1)n_\ell(i) + n_\ell(i) - 1} \right) \leq \sqrt{2(n_\ell - 1)n_\ell + n_\ell - 1}$$

edges, where $n_\ell(i) := |V(H_{\ell,e}^{(i)})|$, $i = 1, ..., p$. Therefore,

$$m_\ell \leq \sqrt{2k_\ell n_\ell} + \sqrt{2(n_\ell - 1)n_\ell + n_\ell - 1}.$$

Combining this inequality with the bound $m_\ell \geq k_\ell m_0 / 2^\ell$, we have

$$k_\ell m_0 / 2^\ell \leq \sqrt{2k_\ell n_\ell} + \sqrt{2(n_\ell - 1)n_\ell + n_\ell - 1}. \quad (1)$$

Using Inequality (1), we produce a recursive lower bound on $n_\ell$ that gives an upper bound on $\alpha$. In particular, we aim to show that

$$n_\ell \geq \left( \frac{n_{\ell-1} m_0}{5 \times 2^\ell} \right)^{2/3} \quad \text{for all } \ell \leq \log_2(5^2\alpha). \quad (2)$$
If \( n_\ell \geq \sqrt{2k_\ell m_0/2^\ell} \), then Inequality (2) immediately holds, as
\[
n_\ell \geq \frac{\sqrt{2k_\ell m_0}}{2^\ell} = \left[ \left( \frac{n_{\ell-1}m_0}{5 \times 2^\ell} \right)^2 \left( \frac{(2k_\ell)^{3/2}n^{1/2}}{n_{\ell-1}^2} \right) \left( \frac{5^2 \alpha}{2^\ell} \right) \right]^{1/3} \geq \left( \frac{n_{\ell-1}m_0}{5 \times 2^\ell} \right)^{2/3}
\]
for \( \alpha \geq 2^\ell/5^2 \). Now, suppose that \( n_\ell \leq \sqrt{2k_\ell m_0/2^\ell} \). Then \( k_\ell m_0/2^\ell - \sqrt{2k_\ell n_\ell} \) is monotonically non-decreasing with respect to \( k_\ell \). Combining this fact with the bound \( k_\ell \geq n_{\ell-1}/2 \) and Inequality (1), we obtain
\[
n_{\ell-1}m_0/2^{\ell+1} - \sqrt{n_{\ell-1}} n_\ell \leq k_\ell m_0/2^\ell - \sqrt{2k_\ell n_\ell} \leq \sqrt{2(n_\ell - 1)} n_\ell + n_\ell - 1.
\]
This implies that
\[
n_{\ell-1}m_0/2^{\ell+1} \leq 2(n_\ell - 1) n_\ell + \sqrt{n_{\ell-1}} n_\ell + n_\ell - 1 < \frac{5}{2} n_{\ell+1}^{3/2},
\]
for \( n \geq 6388 \) and so the claim holds in this case as well.

Using the initial bound \( n_0 > m_0 \) and Inequality (2), we obtain
\[
n \geq n_\ell > \left( \frac{16}{25} \right)^{1-(2/3)^\ell} \frac{m_0^{2-(2/3)^\ell}}{2^{2\ell}} > \frac{16 m_0^{2-(2/3)^\ell}}{25 \times 2^{2\ell}} = \frac{16 \alpha^{2-(2/3)^\ell} n^{1-(2/3)^\ell}}{25 \times 2^{2\ell}}
\]
for \( \ell \leq \log_2(5^2 \alpha) \). Taking the logarithm of both sides, we obtain the inequality
\[
\log_2 \alpha < \frac{1}{2 - (2/3)^\ell} \left( \log_2(25/16) + 2\ell + \frac{1}{2} (2/3)^\ell \log_2 n \right).
\]
Setting
\[
\ell = \left\lfloor \log_{3/2} \left( \frac{3}{11} \log_2 n \right) \right\rfloor \leq \log_2 \left( \frac{5^2}{4} \log_2 \log_3^{1/2} n \right) \leq \log_2 (5^2 \alpha)
\]
gives
\[
\log_2 \alpha < \frac{1}{1 - \frac{11}{6 \log_2 n}} \left[ \log_2 (5/2) + \log_3^{1/2} \left( \frac{3}{11} \log_2 n \right) + \frac{11}{12} \right].
\]
Taking the (base two) exponential of both sides, we obtain
\[
\alpha < 2^{\frac{\log_2 (5/2) - \log_3^{1/2}(11/3) + 11/12}{1 - 11/6 \log_2 n}} \frac{\log_3^{1/2} n} {\log_2 n} \leq .46 \log_2 n^{1.9995}.
\]
This completes the proof. \[\square\]

Finally, we give the following infinite class of digraphs to illustrate that Theorem 1.1 is tight up to a logarithmic factor.

**Example 2.2.** Let \( G_k = (V_k, A_k) \), \( k \in \mathbb{N}, k \geq 4 \), where \( V_k = \{v_1, \ldots, v_{\ell-1}, v_1, \ldots, v_{\ell}\} \), \( \ell := k(k+1)/2 \), and \( u_i u_j \in A_k \) for \( 0 < j - i \leq k \), and \( u_i \phi(i) v_i, v_i u_{\phi(i)} \in A_k \) for all \( i = 1, \ldots, \ell \), where \( \phi(i) \) is the smallest number \( p \in \mathbb{N} \) such that \( \sum_{j=1}^p (k+1-j) \geq i \). This digraph is minimally Eulerian, as every dicycle contains some vertex \( v_i \) and \( d^+(v_i) = d^-(v_i) = 1 \) for all \( i \). There are \( n = k^2 + k - 1 \) vertices and \( k(k^2 + 2k - 1)/2 \) edges (i.e., about \( n^{3/2}/2 \)). The graph distance between any pair \( v_i, v_j \) is at least \( \left[ (\ell + 1)/k \right] = \left[ k/2 \right] + 1 \geq \left[ \sqrt{n}/2 \right] + 1 \). In any Hamiltonian dicycle of a power of \( G_k \), some pair \( v_i, v_j \) must be adjacent, and so at least the \( \left[ \sqrt{n}/2 \right] + 1 \)th power is required. See Figure 1 for a visual example for \( k = 4 \).
Figure 1. The minimally Eulerian graph $G_k$ from Example 2.2 for $k = 4$.

Acknowledgements

This manuscript is the result of an undergraduate research project supported by Professor Michel Goemans through the MIT undergraduate research opportunities program (UROP), and we are thankful for his support. This project is also supported by the Lord Foundation through the MIT UROP program, and we are thankful for their support. This material is based upon work supported by the Institute for Advanced Study and the National Science Foundation under Grant No. DMS-1926686. The authors are grateful to Louisa Thomas for improving the style of presentation.

References

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA USA.
Email address: ecolon@mit.edu

Department of Mathematics, Harvard University, Cambridge, MA USA.
Email address: urschel@mit.edu