MATH 263: PROBLEM SET 2: PSH FUNCTIONS, HORMANDER'S ESTIMATES AND VANISHING THEOREMS

1. Plurisubharmonic functions and currents

The first part of the homework studies some properties of PSH functions.

(1) Let $\varphi, \psi \in PSH(\Omega)$. Prove that $\max\{\varphi, \psi\}$ is plurisubharmonic.

(2) More generally, suppose $\chi(t_1, \ldots, t_p) : \mathbb{R}^p \to \mathbb{R}$ is a convex function, which is non-decreasing in each t_j. Extend χ by continuity to $[-\infty, +\infty)^p \to [-\infty, +\infty)$. Show that if $\varphi_1, \ldots, \varphi_p \in PSH(\Omega)$, then $\chi(\varphi_1, \ldots, \varphi_p) \in PSH(\Omega)$. Use this to show that if $f_1, \ldots, f_p$ are holomorphic functions on Ω, then

$$\log\left(\sum_{i=1}^{p} |f_i|^{\alpha_i}\right)$$

is PSH provided $\alpha_i \geqslant 0$ for all i. (**Hint**: Write χ as the supremum of its supporting hyperplanes).

(3) Suppose $\varphi_k \in PSH(\Omega)$, and $\varphi_k \geqslant \varphi_{k+1}$. Show that $\varphi = \lim_k \varphi_k$ is PSH and that $i\partial\overline{\partial}\varphi_k$ converges to $i\partial\overline{\partial}\varphi$ as distributions.

(4) Suppose φ is C^2, and for all $a \in \Omega \subset \mathbb{C}^n$, has the sub-mean value property

$$\varphi(a) \leqslant \frac{1}{2\pi}\int_0^{2\pi} \varphi(a + \xi e^{i\theta})d\theta$$

for an $\xi \in \mathbb{C}^n$ so that $d(a, \partial\Omega) > |\xi|$. Show that the Hermitian matrix $\partial\overline{\partial}\varphi \geqslant 0$. Prove the converse: namely, if φ is C^2, and $\partial\overline{\partial}\varphi \geqslant 0$, then φ has the sub-mean value property.

(5) Consider the function z on $\mathbb{C}$. Show that $\partial\overline{\partial}\log|z| = c\delta_0$ in the sense of distributions, where $c > 0$ is a dimensional constant, and δ_0 is the Dirac delta function at the origin. More generally, suppose $f : \mathbb{C}^n \to \mathbb{C}$ is holomorphic, and $\mathrm{div}(f) = \sum_j m_j Z_j$

is the divisor of f. Then

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log|f|^2 = \sum_j m_j[Z_j],$$

his is called the Poincaré-Lelong formula. If you want, try to prove this when $j = 1$ and $Z = Z_1$ is smooth.

(6)

2. Hörmander's estimates

The goal of this section is to prove a more robust version of Hörmander's L^2 estimates for singular potentials. The exercises here follow the paper [1].

Let (X, ω) be a compact Kähler manifold, and h a smooth metric on L. Let

$$\Theta(h) = \frac{-\sqrt{-1}}{2\pi}\partial_j\partial_{\bar{k}}\log(h)d\bar{z}^k \wedge dz^j$$

be the curvature $(1,1)$ form. Recall that $\Theta \in c_1(L)$, the first Chern class of L. We will sometimes suppress the dependence of Θ on h, for simplicity. We begin with some linear algebra.

(1) Let (V, h) be a complex vector space with a Hermitian metric, and $A : V \to V$ a hermitian endomorphism with non-negative eigenvalues. Define the "Cauchy-Schwartz norm"

$$|v|_A^2 := \inf\left\{t > 0 \text{ such that } \forall w \in V \text{ we have } |\langle v, w\rangle|^2 \leqslant t^2 \cdot \langle Aw, w\rangle\right\}.$$

and we put $|v|_A = +\infty$ if the set on the right hand side is empty. Show that if A has positive eigenvalues, and is hence invertible, then

$$|v|_A^2 = \langle A^{-1}v, v\rangle.$$

In general, show that $|v|_A$ is finite if and only if v is orthogonal the kernel of A. Show that if the eigenvalues of A are bounded below by $\varepsilon > 0$, then we have

$$|v|_A^2 \leqslant \frac{1}{\varepsilon}|v|^2.$$

Furthermore, show that if $\tilde{A}$ is another non-negative hermitian endomorphism, then

$$|v|_{A+B}^2 \leqslant |v|_A^2.$$

(2) Returning to the setting of $L \to X$, recall that $L \otimes \bigwedge^{n,q}$ has a hermitian metric induced by h and g. Suppose that the hermitian endomorphism

$$A_q := [\Theta, \Lambda_\omega] : L \otimes \bigwedge^{n,q} \to L \otimes \bigwedge^{n,q}$$

has non-negative eigenvalues. This is the case, for example, if $\Theta(h) \geqslant 0$, by our computations in class. Then for a section σ we can define $|\sigma|_{A_q}$ as above. Note that if A_q is positive definite (e.g. if $\Theta(h) \geqslant \varepsilon\omega$), then the previous exercise shows

$$|\sigma|^2_{A_q} = \langle A_q^{-1}\sigma, \sigma\rangle$$

which is precisely the term that appears in Hörmanders L^2 estimates using the Kodaira-Akizuki-Nakano formula. This exercise and the next will prove

Theorem 2.1. *Let (X, ω) be a Kähler manifold of dimension n which admits a complete Kähler metric $\tilde{\omega}$. Let $L \to X$ be a holomorphic line bundle with a metric h, so that A_q is semi-positive. Given $g \in \Gamma(X, L \otimes \bigwedge^{n,q})$ such that $\overline{\partial} g = 0$, and*

$$\int_X |g|^2 + \int_X |g|^2_{A_q} < +\infty$$

there exists $f \in \Gamma(X, L \otimes \bigwedge^{n,q-1})$ solving $\overline{\partial} f = g$ and

$$\int_X |f|^2 \leqslant \int_X |g|^2_{A_q}$$

For simplicity you can assume that ω is itself complete. I will use $L_{n,q}$ to denote the L valued (n, q) forms, which will have more (smooth) or less (L^2) regularity depending on the context. You can remove the completeness assumption using the same trick we used in class– considering $\omega + \varepsilon\tilde{\omega}$. Our main interest is when X is compact, in which case completeness is automatic. Let $\overline{\partial}$ be the usual operator, $\overline{\partial}^\dagger$ the formal adjoint, and $\overline{\partial}^*$ the Von Neumann adjoint. We already know the following stuff:

- By completeness we have $\overline{\partial}^* = \overline{\partial}^\dagger$, extended to $\mathrm{Dom}(\overline{\partial}^*) = \mathrm{Dom}(\overline{\partial}^\dagger)$ by acting by distributional derivatives.

- By the Kodaira-Akizuki-Nakano formula we have, for all $u \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^\dagger) \subset L_{n,q}$

$$\|\overline{\partial}\sigma\|^2 + \|\overline{\partial}^\dagger\sigma\|^2 \geqslant \int_X \langle A_q u, u\rangle$$

(3) Use the Cauchy-Schwarz inequality, the definition of $|\sigma|_{A_q}$, and the KAN estimate to prove

$$\left|\int_X \langle g, \sigma\rangle\right|^2 \leqslant \int_X |g|^2_{A_q} \cdot \left(\|\overline{\partial}\sigma\|^2 + \|\overline{\partial}^\dagger\sigma\|^2\right)$$

(4) Argue as in class to show that we can decompose $\sigma = \sigma_1 + \sigma_2$ with $\sigma_1 \in Ker\overline{\partial}$ and $\sigma_2 \in (KerS)^{\perp} \subset Ker\overline{\partial}^{\dagger}$. Then, since $g \in Ker\overline{\partial}$

$$\left|\int_X \langle g, \sigma\rangle\right|^2 = \left|\int_X \langle g, \sigma_1\rangle\right|^2 \leqslant \int_X |g|^2_{A_q} \cdot \|\overline{\partial}^{\dagger}\sigma\|^2.$$

Now use Hahn-Banach to argue that there exists $f \in L_{n,q-1}$ with

$$\|f\|^2 \leqslant \int_X |g|^2_{A_q}$$

and $\overline{\partial} f = g$ weakly.

3. Hörmander with singular potentials

We will now investigate how to extend this result to the case of singular potentials. So suppose $he^{-2\varphi}$ is a singular Kähler metric on $L \to X$, so that

$$\Theta + \frac{\sqrt{-1}}{\pi}\partial\overline{\partial}\varphi \geqslant \gamma$$

in the sense of distributions, for some smooth $(1,1)$ form γ– for our purposes, $\gamma = \varepsilon\omega$, where ω is the Kähler metric, and $\varepsilon \geqslant 0$. We have seen in class that, locally, it is always possible to approximate singular metrics with positive curvature by smooth metrics with curvature bounded below by γ. However, finding global approximations with this property is in general not possible (we will discuss this in class). Instead, we have the following approximation result of Demailly

Theorem 3.1. *There exists a sequence of smooth functions φ_j decreasing to φ pointwise so that*

$$\Theta(he^{-2\varphi_j}) \geqslant \gamma - \lambda_j\omega$$

where $\lambda_j : X \to \mathbb{R}$ are positive, continuous functions decreasing to 0 almost everywhere. Furthermore,

$$\Theta(he^{-2\varphi_j}) \to \Theta(he^{-2\varphi})$$

almost everywhere on X.

In fact, the limit of λ_j as $j \to \infty$ is explicitly computed in terms of the singularities of φ. The main challenge in extending Hörmander's L^2 estimates to the functions φ_j is to cope with the negativity introduced by the approximation. The idea is to suitably modify the Hörmander argument to deal with the negative contributions, and then use the fact that the negativity disappears (in an L^2 sense) in the limit as $j \to +\infty$. From now on, we will always assume that X is compact (everything can be extended to the complete case). We are going to prove

Theorem 3.2. *Let $L \to (X, g)$ be a holomorphic line bundle equipped with a singular metric $h_\varphi := he^{-2\varphi}$ satisfying $\Theta(h_\varphi) \geqslant 0$ in the sense of distributions. Suppose $\psi \in L \otimes \bigwedge^{n,q}$ has $\overline{\partial}\psi = 0$ and*

$$\int_X |\psi|^2_h e^{-2\varphi} + \int_X |\psi|^2_{\Theta(h_\varphi)} e^{-2\varphi} < +\infty.$$

Then there exists $\sigma \in L \otimes \bigwedge^{n,q-1}$ *solving* $\overline{\partial}\sigma = \psi$ *and with*

$$\int_X |\psi|_h^2 e^{-2\varphi} \leqslant \frac{1}{q} \int_X |\psi|^2_{\Theta(h_\varphi)} e^{-2\varphi}$$

First, let's make sense of the norm $|\psi|_{\Theta(h_\varphi)}$ introduced above.

(1) Argue that we can decompose the curvature $\Theta(h_\varphi)$ as

$$\Theta(h_\varphi) = \Theta(h_\varphi)_{ac} + \Theta(h_\varphi)_{sing}$$

where $\Theta(h_\varphi)$ is a $(1,1)$ form whose coefficients are absolutely continuous measures. In particular, the coefficients of $\Theta(h_\varphi)_{ac}$ are L^1 functions (this is just the Radon-Nikodym theorem). Now for almost every point $p \in X$ we can define

$$|\psi|^2_{\Theta(h_\varphi)} := |\psi|^2_{\Theta(h_\varphi)_{ac}}$$

using the "Cauchy-Schwartz" norm introduced in the previous problem– here the hermitian inner produce on $L \otimes \wedge^{n,q}$ is the one induced by the smooth metrics h, g. In particular, it makes sense to integrate the positive function $|\psi|^2_{\Theta(h_\varphi)} e^{-2\varphi}$ over X.

For the rest of the problem we are going to be approximating the singular metric by smooth metrics $h_j = he^{-2\varphi_j}$. We will thus have a lot of different norms and curvatures floating around. Let's introduce some notation to keep them all straight. First, we will have the L^2 inner product and norm induced by g, h_j,

$$\langle g, f\rangle_j = \int_X \langle g, f\rangle e^{-2\varphi_j}$$

where $\langle g, f\rangle$ is the inner-product induced by the smooth metrics g, h. Given h_j, we then also get a curvature operator on $\bigwedge^{n,q}$.

$$A_q^j = [\Theta(h_j), \Lambda_\omega] : \bigwedge^{n,q} \to \bigwedge^{n,q}$$

Since the regularization doesn't preserve positivity, this operator will not be positive in general. We perturb the curvature operator to a positive operator by introducing

$$\tilde{A}_q^{j,\nu} = [\Theta(h_j) + \lambda_j \omega + \frac{1}{q\nu}\omega, \Lambda_\omega]$$

Note that this is a positive operator, and indeed has eigenvalues bounded below by $\frac{1}{\nu}$. We can therefore introduce the "Cauchy-Schwartz" norm, which is the norm that will appear in the KAN formula:

$$\|g\|_{\tilde{A}_q^{j,\nu}} = \int_X |g|_{\tilde{A}_j^{q,\nu}} e^{-2\varphi_j}$$

where $|g|_{\tilde{A}_j^{q,\nu}}$ is the pointwise "Cauchy-Schwartz" norm induced by $\tilde{A}_q^{j,\nu}$ where we take the inner product on the fibers of $L \otimes \bigwedge^{n,q}$ induced by h, g. I will also allow $j = \infty$ in these formulas, by which I mean the corresponding quantity computed using $he^{-2\varphi}$– as discussed above, this involves taking the absolutely continuous part!. Finally, note that

$$\int_X |\sigma|_j^2 = \int_X |\sigma|_h^2 e^{-2\varphi_j} \leqslant \int_X |\sigma|_h^2 e^{-2\varphi}$$

(2) Show that if $\psi \in L \otimes \bigwedge^{n,q}$ has

$$\int_X |\psi|_h^2 e^{-2\varphi} + \int_X |\psi|^2_{\Theta(h_\varphi)} e^{-2\varphi} < +\infty$$

then

$$\limsup_{\nu\to\infty} \limsup_{j\to\infty} \int_X |\psi|^2_{\tilde{A}_q^{j,\nu}} e^{-2\varphi_j} \leqslant \int_X |\psi|^2_{\Theta(h_\varphi)} e^{-2\varphi} < +\infty.$$

Here is a "hint". By the definition of $\tilde{A}_q^{j,\nu}$, and exercise (1) from section 2, we have

$$|\psi|^2_{\tilde{A}_q^{j,\nu}} e^{-2\varphi_j} \leqslant \nu|\psi|^2 e^{-2\varphi_j} \leqslant \nu|\psi|^2 e^{-2\varphi} \in L^1(X)$$

so by Fatou's lemma (really the "reverse" Fatou's lemma) and exercise (1) from section 2 we have

$$\limsup_{j\to\infty} \int_X |\psi|^2_{\tilde{A}_q^{j,\nu}} e^{-2\varphi_j} \leqslant \int_X |\psi|^2_{\tilde{A}_q^{\infty,\nu}} e^{-2\varphi} \leqslant \int_X |\psi|^2_{\Theta(he^{-2\varphi})} e^{-2\varphi} < +\infty$$

This is uniform in ν. So we can take $\limsup_\nu$ so conclude.

(3) By what we showed in class, we can assume that $\mathrm{Dom}(\overline{\partial}^\dagger) = \mathrm{Dom}(\overline{\partial}^*)$, and that $\overline{\partial}^\dagger = \overline{\partial}^*$. Note that all the adjoints here depend on j, since the norms do! I am going to suppress this dependence. By the KAN identity, for every $\sigma \in \mathrm{Dom}(\overline{\partial}^\dagger) \cap \mathrm{Dom}(\overline{\partial}) \subset L_{n,q}$ we have

$$\|\overline{\partial}^\dagger \sigma\|_j^2 + \|\overline{\partial}\sigma\|_j^2 \geqslant \langle A_q^j \sigma, \sigma\rangle_j.$$

Show that

$$\begin{aligned}\langle \tilde{A}_q^{j,\nu}\sigma, \sigma\rangle &\leqslant \langle A_q^j\sigma, \sigma\rangle + \left(\lambda_j + \frac{1}{q\nu}\right)\langle[\omega, \Lambda_\omega]\sigma, \sigma\rangle \\ &\leqslant \|\overline{\partial}^\dagger\sigma\|_j^2 + \|\overline{\partial}\sigma\|_j^2 + \left(q\lambda_j + \nu^{-1}\right)\|\sigma\|_j^2.\end{aligned}$$

and hence, for any $\sigma \in \mathrm{Dom}(\overline{\partial}^\dagger) \cap \mathrm{Dom}(\overline{\partial})$ we have

$$|\langle \sigma, g\rangle|^2 \leqslant B_{j,\nu}\left(\|\overline{\partial}^\dagger\sigma\|_j^2 + \|\overline{\partial}\sigma\|_j^2 + (q\lambda_j + \nu^{-1})\|\sigma\|_j^2\right)$$

where

$$B_{j,\nu} = \int_X |g|^2_{\tilde{A}_q^{j,\nu}} e^{-2\varphi_j}$$

is the "Cauchy-Schwartz" norm introduced in the previous section.

(4) We now show that Hahn-Banach theorem implies that there are $f_{j,\nu}, v_{j,\nu} \in L_{n,q}$ so that

$$\langle g, \sigma\rangle_j = \langle f_{j,\nu}, \overline{\partial}^{\dagger}\sigma\rangle_j + (q\lambda_j + \nu^{-1})^{1/2}\langle v_j, \sigma\rangle_j$$

and

$$\|f_{j,\nu}\|_j^2 + \|v_{j,\nu}\|_j^2 \leqslant B_{j,\nu}.$$

in particular $\overline{\partial} f_{j,\nu} + (q\lambda_j + \nu^{-1})^{1/2} v_{j,\nu} = g$ weakly. Now argue as we did in class to take limits as $j \to \infty$, and then $\nu \to \infty$ and pass to weak limits to conclude.

4. Multiplier ideals and Vanishing Theorems

This section of the homework concerns some basic computations with multiplier ideals and applications to vanishing theorems. Throughout this section X is a smooth projective variety. Whenever we write a $\mathbb{Q}$-divisor

$$D = \sum a_i D_i$$

D_i will be irreducible prime divisors. Recall that for a divisor D we can associate its fist Chern class $c_1(D) \in H^{1,1}(X, \mathbb{Z})$. The map

$$D \longmapsto c_1(D)$$

is in general not an isomorphism, though the exponential sequence in sheaf cohomology shows that it is surjective. The divisors (eq. line bundles) in the kernel of the map $D \mapsto c_1(D)$ are said to be numerically trivial. To mod out by the kernel we introduce an equivalence relation $D_1 \equiv_{num} D_2$ if $c_1(D_1) = c_1(D_2)$. In particular, if D_1, D_2 are numerically equivalent, then they are not distinguished by intersection theory (this is often the definition of numerical equivalence). We denote the divisors modulo numerical equivalence as $N^1(X)$, and call this the Neron-Severi cone.

Note that this map extends by linearity to $\mathbb{Q}$ and $\mathbb{R}$ divisors by sending

$$\sum a_i D_i \longmapsto c_1(D) := \sum a_i c_1(D_i)$$

and so we get also $N^1(X)_{\mathbb{R}} = N^1(X) \otimes \mathbb{R}$ and $N^1(X)_{\mathbb{Q}} = N^1(X) \otimes \mathbb{Q}$. Kodaira's theorem says that a divisor D is ample if $c_1(D) \in H^{1,1}(X, \mathbb{Z})$ is a Kähler class. It therefore makes sense to define a $\mathbb{R}$ or $\mathbb{Q}$ divisor D, to be ample if $c_1(D)$ contains a Kähler metric, and D to be nef if $c_1(D)$ lies on the closure of the cone in $H^{1,1}(X, \mathbb{R})$ generated by classes of Kähler metrics. Recall that a $\mathbb{Q}$ divisor D is big if mD is big for some $m \in \mathbb{N}$ (ie. clear denominators).

(1) (Rounding divisors). Given a $\mathbb{Q}$ or $\mathbb{R}$ divisor, write $D = \sum a_i D_i$ where D_i are irreducible prime divisors. We define the round up $\lceil D \rceil$, and the round down $\lfloor D \rfloor$ by

$$\lceil D \rceil = \sum \lceil a_i \rceil D_i, \qquad \lfloor D \rfloor = \sum \lfloor a_i \rfloor D_i$$

Show by example that round up and round down **DO NOT** commute with numerical equivalence or pull-back. For the former, consider $\frac{1}{2}C$, where C is an irreducible conic in $\mathbb{P}^2$, and for the latter, consider a line L tangent to C.

(2) Prove the following general version of Kawamata-Viehweg vanishing, which generalizes what we did in class

Theorem 4.1. *If N is an integral divisor in X, and*

$$N \equiv_{num} B + \Delta$$

where B is a nef and big $\mathbb{Q}$ divisor and $\Delta = \sum a_i \Delta_i$ is a $\mathbb{Q}$-divisor with simple normal crossing support and fractional coefficients

$$0 \leqslant a_i < 1 \qquad \forall i.$$

Then $H^q(X, K_X + N) = 0$ for all $q \geqslant 1$.

Here is a sketch– fill in the details! Since B is a big and nef $\mathbb{Q}$-divisor, we can choose m large and divisible so that mB is integral, nef and big and $m\Delta$ is an integral effective divisor. Arguing as we did in class, this means you can construct singular metrics with positive curvature on mB with arbitrarily small singular part. Let s_i be global holomorphic sections of $\mathcal{O}_X(\Delta_i)$ with $\{s_i = 0\} = \Delta_i$, and let h_i be metrics on $\mathcal{O}_X(\Delta_i)$. Then argue that

$$h_{mN} = h_{triv} h_{mB} \prod_i h_i^{ma_i} e^{-ma_i \log |s_i|^2_{h_i}}$$

defines a singular metric on mN, where h_{triv} is a metric on some numerically trivial line bundle, with $\Theta(h_{triv}) = 0$. Show that this metric has positive curvature if h_{mB} does. Thus we get a positively curved singular metric

$$h_N = \left(h_{triv} h_{mB} \prod_i h_i^{ma_i} \right)^{1/m} e^{-\sum_i a_i \log |s_i|^2_{h_i}}$$

on N. Now argue that since the Δ_i have simple normal crossings, and $0 \leqslant a_i < 1$, we can choose the metric on mB in such a way that the resulting multiplier ideal sheaf is trivial. Use Nadel vanishing to conclude the cohomology vanishing.

(3) Let $\mathfrak{a} = (x, y^4)$. Describe $\mathcal{J}(c \cdot a)$ for all c. In particular, compute

$$lct(\mathfrak{a}) = \sup\{c : \mathcal{J}(c \cdot \mathfrak{a}) = \mathcal{O}_{\mathbb{C}^2}\}.$$

References

[1] J.-P. Demailly, *Estimations L^2 pour l'opérateur ∂ d'un fibré vectoriel holomorphe semi-positif au-dessus d'un variété Kählerienne complète*, Ann. Sci. d L'É.N.S, 1982