(1) In this exercise you will prove the following

**Lemma 0.1.** Let \((M, g)\) be a Riemannian manifold, and suppose that the sectional curvature satisfies \(K(g) \leq \kappa\). Fix a point \(p \in M\), then

\[
\text{inj}(p) \geq \min\left\{ \pi \sqrt{\frac{1}{\kappa}}, \frac{1}{2} \text{ length of shortest geodesic loop passing through } p \right\}
\]

In particular,

\[
\text{inj}(M, g) \geq \min\left\{ \pi \sqrt{\frac{1}{\kappa}}, \frac{1}{2} \text{ length of shortest geodesic loop in } (M, g) \right\}
\]

As usual, we use the convention that if \(\kappa \leq 0\), then we just ignore the symbol \(\pi \sqrt{\frac{1}{\kappa}}\). This estimate is called Klingenberg’s estimate.

(a) First recall that we proved that the distance to the conjugate locus of \(p\) is no less than \(\pi \sqrt{\frac{1}{\kappa}}\) as a consequence of Rauch’s theorem, (or our proof of Bishop-Gromov volume comparison, for example). So we just need to show that if \(q\) is in the cut locus of \(p\), \(\text{inj}(p) = d(p, q)\) but \(q\) is not a conjugate point, then there is a geodesic loop containing \(p\) and \(q\). To do this let \(v_1, v_2 \in T_p M\) be unit length vectors so that \(\gamma_1(t) := \exp_p tv_1\) and \(\gamma_2(t) := \exp_p tv_2\) have \(\gamma_1(1) = \gamma_2(1) = q\). It suffices to show that \(\gamma_1'(1) = -\gamma_2'(1)\), since implies that \(\gamma_1, \gamma_2\) can be combined to form a closed geodesic loop. Suppose not. Argue that there is \(w \in T_q M\) such that \(\langle w, \gamma_1'(1) \rangle < 0\) and \(\langle w, \gamma_2'(1) \rangle < 0\).

(b) For \(s \in (0, \varepsilon)\) consider the points \(q(s) := \exp_q(sw)\). Show that \(d(p, q(s)) < d(p, q)\) for \(s > 0\).

(c) Now argue that there are distinct tangent vectors \(v_1(s), v_2(s)\) such that \(\exp_p(v_1(s)) = \exp_p(v_2(s))\). Explain why this contradicts the fact that \(d(p, q) = \text{inj}(p)\).

(2) Next you will prove a lemma relating volume non-collapsing and the injectivity radius, under a bound for the sectional curvature. This result is originally due to Cheeger with a different argument.
Lemma 0.2. Given $n \geq 2$, $\nu, \kappa > 0$, there is a constant $R = R(\nu, \kappa, n) > 0$ such that any compact manifold $(M^n, g)$ with $|K(g)| \leq \kappa$, and $\text{Vol}(B(p, 1)) > \nu$ for all $p \in M$ has $\text{inj}(M, g) \geq R$.

The argument is by contradiction. Assume the result is false, so that there is a sequence $(M_i, g_i)$ with $n$ fixed, sectional curvature bounded by $\kappa$, and $\text{Vol}(B_i(p, 1)) > \nu$ for any $p \in M_i$.

(a) Rescale the manifolds to get $(M_i, \bar{g}_i)$ with $\text{inj}(M, \bar{g}_i) = 1$. What happens to the sectional curvature?

(b) Choose points $p_i \in M_i$ achieving the injectivity radius. Show that the sequence $(M_i, \bar{g}_i, p_i)$ converges in the pointed $C^{1,\alpha}$ topology to a flat manifold $(M_\infty, g_\infty, p_\infty)$. By applying Klingen-berg’s lemma to $(M_i, \bar{g}_i)$, argue that there must be a geodesic loop of length 2 in $(M_\infty, g_\infty)$ passing through $p_\infty$. In particular, $\text{inj}(M_\infty, g_\infty) \leq 1$

(c) Using the volume comparison theorem, prove the following claim:

Claim: There is a constant $\nu' > 0$ depending only on $\nu, K, n$ such that

$$\text{Vol}(B_\infty(p_\infty, r)) \geq \nu' r^n$$

for all $r > 0$.

In order to obtain a contradiction, note that by (b) it suffices to prove that $(M_\infty, g_\infty) = (\mathbb{R}^n, g_{\text{Euc}})$. To do this, you will prove the following lemma

Lemma 0.3. Suppose $(M, g)$ is a flat manifold which is not simply connected. Then, for any $p \in M$ we have

$$\lim_{r \to \infty} \frac{\text{Vol}(B(p, r))}{r^{n-1}} < +\infty$$

As a first step toward proving this result recall that by the classification of manifolds with constant sectional curvature, $(M, g)$ is the quotient of $(\mathbb{R}^n, g_{\text{Euc}})$ by a group of isometries $\Gamma$ acting totally discontinuously (in particular without fixed points). The goal is to argue that if $\Gamma$ acts non-trivially, and totally discontinuously and $\pi : \mathbb{R}^n \to \mathbb{R}^n/\Gamma = M$ is a covering map and a local isometry, then the volume of any ball $B(p, R)$ in $M$ grows at most like $R^{n-1}$ for $R \gg 0$. To build some intuition, consider the case when $\Gamma$ acts by translation along a fixed vector!

(d) Begin by recalling that any isometry of $\mathbb{R}^n$ is given by $h(v) = Av + w$ for some $A \in O(n)$, and $w \in \mathbb{R}^n$. It follows that $0, w$ are identified under the quotient map $\pi$. Argue that the
curve $\gamma(t) = tw$ descends to a closed geodesic $\tilde{\gamma} \subset (M, g)$ with $\tilde{\gamma}(0) = \tilde{\gamma}(1)$ and $\tilde{\gamma}'(0) = \tilde{\gamma}'(1)$.

(e) Next we argue that $Aw = w$. If not, then the curve $h(tw) = w + tAw$ descends to $\tilde{\gamma}(t)$ under the projection map. Argue that this implies that the covering map $\pi$ cannot be a local diffeomorphism (Hint: Consider $d\pi|_w : \mathbb{R}^n \to T_{\pi(p)}M$).

(f) By a rotation we can assume $w = (a, 0, \ldots, 0)$ for some $a > 0$. Identify $(0, x_2, \ldots, x_n)$ with $\mathbb{R}^{n-1}$. Show that $A$ has the block diagonal form

$$
\begin{pmatrix}
1 & 0 \\
0 & A'
\end{pmatrix}
$$

where $A' \in O(n-1)$. Conclude that $[0, a) \times \mathbb{R}^{n-1}$ covers $(M, g)$. Argue that this implies the lemma. (Note: This is an ad-hoc argument to understand a part of the group of deck transformations acting on the universal cover).

(g) Find a counterexample to Lemma 0.1 if we drop the assumption $\text{Vol}(B(p, 1)) > \nu$ for all $p \in M$.

(3) Use Lemma 0.1 together with results proved in class to prove Cheeger’s finiteness theorem.

**Theorem 0.4 (Cheeger’s finiteness theorem).** Let $n \geq 2$, and fix constants $\kappa, D, \nu > 0$. The class of Riemannian manifolds $(M^n, g)$ satisfying the bounds

$$
|K(g)| \leq \kappa \\
\text{diam}(M, g) \leq D \\
\text{Vol}(M, g) \geq \nu
$$

contains only finitely many diffeomorphism types.