MATH 18.966: HOMEWORK 3

DUE THURSDAY, APRIL 4, 2019

1. Cohomology

This section will discuss the notion of cohomology on a manifold. Recall that, on a smooth manifold M, we have the de Rham differential $d : C^\infty(M, \Lambda^p T^*M) \to C^\infty(M, \Lambda^{p+1}T^*M)$, where $C^\infty(M, \Lambda^p T^*M)$ denotes the space of smooth p-forms on M. We define the de Rham cohomology of M by

$$H^p_{dR}(M, \mathbb{R}) = \frac{\{\mathrm{Ker} d : C^\infty(M, \Lambda^p T^*M) \to C^\infty(M, \Lambda^{p+1}T^*M)\}}{\{\mathrm{Im} d : C^\infty(M, \Lambda^{p-1}T^*M) \to C^\infty(M, \Lambda^p T^*M)\}}$$

where, by convention, $C^\infty(M, \Lambda^0 T^*M) = C^\infty(M, \mathbb{R})$, the space of smooth functions on M and $C^\infty(M, \Lambda^{-1}T^*M) = \emptyset$.

(0) Show that $H^p_{dR}(M, \mathbb{R})$ naturally has the structure of a vector space over $\mathbb{R}$.

(1) Prove that if $f : M \to N$ is a diffeomorphism, then we have an isomorphism $H^p_{dR}(M, \mathbb{R}) \cong H^p_{dR}(N, \mathbb{R})$. In particular, the cohomology groups are (at the very least) invariants of the smooth structure of M (in fact, they topological invariants).

(2) Consider the case of $S^1 = \mathbb{R}/\mathbb{Z}$. Show that

$$H^0(S^1, \mathbb{R}) = \mathbb{R}, \qquad H^1(S^1, \mathbb{R}) = \mathbb{R}$$

(3) More generally, consider the n-torus $T^n = S^1 \times \cdots \times S^1$. Compute $H^1(T^n, \mathbb{R})$. Briefly explain how you could similarly compute all the cohomology groups of T^n.

2. The Hodge Laplacian

In the following two sections we will explain the central result of Hodge theory, which says that the topological invariants constructed in the previous section can also be understood analytically. We will begin by studying the Hodge laplacian.

Let (M, g) be an orientable, compact Riemannian manifold, and let $C^\infty(M, \Lambda^p T^*M)$ denote the space of smooth p-forms on M. The metric g induces a natural metric on $\Lambda^p T^*M$.

(1) We define the Hodge-$*$ operator by the following formula; for two p-forms α, β, we define $*\beta \in \Lambda^{n-p}T^*M$ by

$$\langle \alpha, \beta \rangle_g d\mathrm{Vol}_g = \alpha \wedge *\beta.$$

In fact, this is just a construction from linear algebra, and makes sense for any vector space with an inner product. Describe how to compute $*\alpha$ by reducing to, and then computing, $*(e_{i_1} \wedge \cdots \wedge e_{i_p})$ where $\{e_i\}$ an orthonormal basis of 1-forms. Show, in general, that $*^2\alpha = (-1)^{p(n-p)}\alpha$. Note that when $n = 2$, then $* : T^*M \to T^*M$ and $*^2\alpha = -\alpha$. This is a special case of a *complex structure*. Note also when $n = 4$, then $* : \Lambda^2T^*M \to \Lambda^2T^*M$ has $*^2\alpha = \alpha$, and so Λ^2T^*M decomposes into ± 1 eigenspaces; these are called the self-dual, and anti-self dual forms, and play an important role in the study of gauge theory and topology on 4-manifolds.

(2) We can define a natural L^2 inner-product on $C^\infty(M, \Lambda^pT^*M)$ by

$$\langle \alpha, \beta \rangle_{L^2} = \int_M \langle \alpha, \beta \rangle_g d\mathrm{Vol}_g = \int_M \alpha \wedge *\beta$$

Using this formula, together with the covariant derivative on (M, g), define $W^{k,2}(M, \Lambda^pT^*M)$, the space of $W^{k,2}$ sections of Λ^pT^*M.

(3) The next step is to construct an adjoint operator to d. Namely, recall that from the "Cohomology" section we have

$$d : C^\infty(M, \Lambda^pT^*M) \to C^\infty(M, \Lambda^{p+1}T^*M).$$

This descends to an operator

$$d : W^{k,2}(M, \Lambda^pT^*M) \to W^{k-1,2}(M, \Lambda^{p+1}T^*M)$$

We define the adjoint operator $d^* : W^{k,2}(M, \Lambda^{p-1}T^*M) \to W^{k-1,2}(M, \Lambda^pT^*M)$ by the following equation: for a p-form β we define $d^*\beta$ by requiring that, for ever $p - 1$ form α, we have

$$\langle d\alpha, \beta \rangle_{L^2} = \langle \alpha, d^*\beta \rangle_{L^2}$$

In essence, this is nothing but integration by parts. Compute, explicitly, a local expression for $d^*\beta$ for β a p-form, $p = 0, 1, 2$. Note that, in general, by Stoke's theorem (since $\partial M = \emptyset$) we have we have

$$\begin{aligned}\langle d\alpha, \beta \rangle_{L^2} &= \int_M d\alpha \wedge *\beta = \int_M d(\alpha \wedge *\beta) + (-1)^{p-1}\alpha \wedge d * \beta \\ &= (-1)^p \int_M \alpha \wedge d * \beta\end{aligned}$$

From this formula, and part (1), we see that

$$d^*\beta = (-1)^{n(n-p)+1} * d * \beta.$$

(4) We define the Hodge Laplacian by

$$\Box\alpha = -(dd^* + d^*d)\alpha$$

This defines a map $\Box : W^{k,2}(M, \Lambda^p T^*M) \to W^{k-2,2}(M, \Lambda^p T^*M)$. Show that

$$\langle \Box\alpha, \beta\rangle_{L^2} = -\langle d\alpha, d\beta\rangle_{L^2} - \langle d^*\alpha, d^*\beta\rangle_{L^2} = \langle \alpha, \Box\beta\rangle_{L^2}$$

In particular, show that $\Box\alpha = 0$ if and only if $d\alpha = 0$ and $d^*\alpha = 0$.

(5) Find an explicit expression for $\Box$ on 1-forms in local coordinates, and check that $\Box$ is elliptic, and of divergence form with respect to the volume form on (M, g).

(6) Define the rough laplacian on p forms by

$$\Delta\alpha = g^{ij}\nabla_i\nabla_j\alpha = \nabla_i(g^{ij}\nabla_j\alpha)$$

and note that, by integration by parts, we have

$$\langle \Delta\alpha, \beta\rangle = -\langle \nabla\alpha, \nabla\beta\rangle_{L^2}$$

Using your solution to the previous problem, find an expression for $\Box - \Delta$ acting on 1-forms in terms of curvature. This is the *Bochner formula.*

3. Hodge Theory

We are now going to use the elliptic theory we have developed in class to prove the Hodge theorem. We will make some use of Hilbert spaces. If you are unfamiliar with Hilbert spaces, it may be worth taking a moment to familiarize yourself with the definition, and some simple examples– particularly the Sobolev spaces $W^{k,2}$, keeping in mind problem (2) from the last section.

In what follows we will need the following three results.

Theorem 3.1 (Elliptic regularity in Sobolev spaces). *Suppose $k \geqslant 2$, and $\alpha \in W^{k,2}(M, \Lambda^p T^*M)$ is a p-form, satisfying $\Box\alpha = \beta$, where $\beta \in W^{k,2}$. Then $\alpha \in W^{k+2,2}(M, \Lambda^p T^*M)$, and has*

$$\|\alpha\|_{W^{k+2,2}} \leqslant C_1 \left(\|\beta\|_{W^{k,2}} + \|\alpha\|_{W^{k,2}}\right)$$

where C_1 is a constant depending on (M, g) and k, but not on α or β.

This result is a direct consequence of the elliptic theory we covered in class, together with the existence of partitions of unity.

Theorem 3.2 (The Sobolev Imbedding Theorem). *If $2k > n$, then the Sobolev spaces embed into the Schauder spaces*

$$W^{k,2}(M, \Lambda^p T^*M) \subset C^{k-1-[\frac{n}{2}],\mu}(M, \Lambda^p T^*M)$$

where $[\frac{n}{2}]$ is the integer part of n/p, and $\mu = [\frac{n}{2}] + 1 - (n/p)$ is n/p is not an integer, and $\mu \in (0, 1)$ otherwise.

Theorem 3.3 (Rellich's Theorem)**.** *The natural inclusion $W^{k,2}(M, \Lambda^p T^*M) \hookrightarrow W^{p,2}(M, \Lambda^p T^*M)$ for $k > p$ is compact. That is, if $\{\alpha_j\}$ is a bounded sequence in $W^{k,2}(M, \Lambda^p T^*M)$, then it contains a convergent subsequence in $W^{p,2}(M, \Lambda^p T^*M)$.*

You can think of this as a version of Arzela-Ascoli: you pay a derivative to get a convergent subsequence.

(0) Consider the Hodge laplacian as a continuous (ie. bounded) linear operator $\Box : W^{2,2}(M, \Lambda^p T^*M) \to L^2(M, \Lambda^p T^*M)$. Suppose $\alpha \in W^{2,2}(M, \Lambda^p T^*M)$ satisfies

$$\Box\alpha = \lambda\alpha.$$

Show that $\lambda \leqslant 0$, and α is smooth. **: Hint:** For the first part, use the integration by parts formula from the previous section. For the second part, "bootstrap", by repeatedly applying the elliptic regularity theorem, then apply the Sobolev Imbedding theorem.

(1) Consider the Hodge laplacian $\Box : W^{2,2}(M, \Lambda^p T^*M) \to L^2(M, \Lambda^p T^*M)$. Let Ker$\Box$ denote the kernel of this map. Show that Ker$\Box$ is finite dimensional. **Hint:** Suppose not. Let $\{\alpha_\ell\}$ be an infinite, orthonormal basis (this exists, since the Hilbert spaces we're studying are separable). Then

$$\|\alpha_\ell - \alpha_j\|_{W^{2,2}} = \sqrt{2}$$

Using elliptic regularity and Rellich's theorem, obtain a contradiction. Note that this argument also works to prove that, for each $\lambda \in \mathbb{R}$, the space $\{\alpha \in W^{2,2} : \Box\alpha = \lambda\alpha\}$ is either empty, or finite dimensional.

(2) Prove the following "improved" version of elliptic regularity. There is a constant $C > 0$ such that, if $\alpha \in W^{2,2}(M, \Lambda^p T^*M)$, and α is orthogonal to Ker$\Box$, then

$$\|\alpha\| \leqslant C\|\Box\alpha\|_{L^2}.$$

Hint: Suppose not, then there are $W^{2,2}$ p-forms α_n orthogonal to Ker$\Box$ such that $\|\alpha_n\|_{W^{2,2}} \geqslant n\|\alpha_n\|_{L^2}$. Define $\tilde{\alpha}_n = \frac{\alpha_n}{\|\alpha_n\|_{W^{2,2}}}$, use elliptic regularity and Rellich's lemma to obtain a contradiction.

(3) Define Range$(\Box) = \Box(W^{2,2})$ to be the range of $\Box$. Prove that Range$(\Box)$ is closed; that is, if $\beta_n = \Box\alpha_n$, and $\beta_n \to \beta$ in L^2, then $\beta = \Box\alpha$ for some $\alpha \in W^{2,2}$. **Hint:** Use the estimate from the last problem.

(4) Since Range$(\Box)$ is closed, we can define an orthogonal decomposition

$$L^2 = \text{Range}(\Box) \oplus \text{Range}(\Box)^\perp$$

Concretely, for any $\beta \in L^2$ we can write

$$\beta = \Box\alpha + \beta_0 \tag{3.1}$$

for $\alpha \in W^{2,2}$, and β_0 orthogonal (in L^2) to Range($\Box$). We claim that there is a natural identification

$$\mathrm{Range}(\Box)^{\perp} \longleftrightarrow \mathrm{Ker}(\Box)$$

To see this, note that for $\beta \in \mathrm{Range}(\Box)^{\perp}$, then for any smooth p-form φ we have

$$\langle \beta, \Box\varphi \rangle_{L^2} = 0$$

We noted (but didn't prove) in class that, for elliptic operators of divergence form, this type of formula implies β is in fact smooth, and satisfies $\Box\beta = 0$. Conversely, if $\beta \in W^{2,2}$ has $\Box\beta = 0$, show that β is orthogonal to Range($\Box$) in L^2. Show that this implies an orthogonal decomposition

$$L^2 = \mathrm{Ker}\Box \oplus \mathrm{Range}(d) \oplus \mathrm{Range}(d^*)$$

where

$$\mathrm{Range}(d) = \{d\eta : \eta \in W^{2,1}(M, \Lambda^{p-1}T^*M)\}$$
$$\mathrm{Range}(d^*) = \{d^*\eta : \eta \in W^{2,1}(M, \Lambda^{p+1}T^*M)\}$$

(5) Show that the last result implies that

$$H^p_{dR}(M, \mathbb{R}) \cong \left\{\mathrm{Ker}(\Box) \subset W^{2,2}(M, \Lambda^p T^*M)\right\}$$

and in particular, $H^p_{dR}(M, \mathbb{R})$ is finite dimensional. This is the fundamental result of Hodge theory. It says that cohomology is detected by the kernel of a certain Laplace operator. This is a very general story, that works in a wide variety of situations.

(6) Finally, in the notation of (3.1), define $G : L^2 \to W^{2,2}$ by $G\beta = \alpha$; this is called the *Green's function*, it "inverts the Laplacian". Show that

$$\Box G\beta = (1 - \pi)\beta$$
$$G\Box\alpha = \alpha$$

where π is the orthogonal projection in L^2 to Ker($\Box$). Show also that

$$\|G\beta\|_{W^{2,2}} \leqslant C\|\beta\|_{L^2}$$

so G is a bounded linear operator from $L^2 \to W^{2,2}$.

(7) Finally, we arrive at the punchline. Combine Problem (6) above, with Problem (6) from the last section to show that if Ric > 0, then $H^1_{dR}(M, \mathbb{R}) = 0$. Note that this implies, for example, that the torus cannot admit a metric of positive Ricci curvature. Furthermore, show that if Ric $\geqslant 0$, then $\dim_{\mathbb{R}} H^1_{dR}(M, \mathbb{R}) \leqslant n$, by observing

that Bochner's formula implies harmonic forms are parallel. In fact, you can show (see, for example Petersen, Corollary 9.2.5) that if $\dim_{\mathbb{R}} H^1_{dR}(M, \mathbb{R}) = n$, then (M, g) is isometric to a flat n-torus. **Recall** that a flat $n-$torus is the product of n circles $S^1 = \mathbb{R}/\mathbb{Z}$ each equipped with the metric adx^2 for some $a > 0$, and x is the coordinate on $\mathbb{R}$. So a flat n-torus is

$$S^1 \times \cdots \times S^1, \qquad g = a_1 dx_1^2 + \cdots a_n dx_n^2.$$