MATH 18.966: GEOMETRY OF MANIFOLDS II

DUE TUESDAY, FEBRUARY 26, 2019

(1) Let (M_k, g_k) be the simply connected n-dimensional manifold of constant sectional curvature k. Show that there is a radius $R = R(k)$, and a function $\rho_k(r)$ so that any ball of radius R in (M_k, g_k) is isometric to $B_{R(k)}(0)$, with the metric

$$dr^2 + \rho_k(r)^2 g_{S^{n-1}}$$

where $g_{S^{n-1}}$ is the standard round metric on the $n-1$ sphere, and $r^2 = \sum_i x_i^2$ is the radial coordinate. Find $\rho_k(r)$, and the maximal $R(k)$ so that this holds.

(2) Let (N, g_N) be a smooth Riemannian manifold, and fix a positive function $\rho(r) : (0, a) \to \mathbb{R}_{>0}$, where $a \in \mathbb{R}_{>0}$. A *warped product* manifold is

$$(M, g_M) = (M \times (0, a), dr^2 + \rho(r)^2 g_N =: N \times_\rho (0, a).$$

Special cases of the warped product construction are:
- the product (take $a = \infty, \rho = 1$),
- the metric cone (take $\rho = r^2$),
- the metric suspension (take $a = \pi$, and $\rho = \sin(r)$)

Draw some pictures of these (you do not have to submit the pictures)! Define a function $f(r) = \int_0^r \rho(s)ds$. Show that

$$df \neq 0, \quad \mathrm{Hess}(f) = \frac{\partial \rho}{\partial r} g.$$

What is the corresponding result for the spaces of constant curvature k?

(3) Here is a converse to the previous problem. Suppose (M, g) is a complete Riemannian manifolds admitting a non-trivial smooth function u with $du \neq 0$, and $\mathrm{Hess} u = hg$ for some function h. Then (M, g) is isometric to a warped product with $\rho = cu'$ for some constant c. Here is an outline for how to show this.

(i) Suppose u is as above. Show that for any vector field X we have

$$X\left(\frac{1}{2}|\nabla u|^2\right) = hg(X, \nabla u).$$

Conclude that $|\nabla u|^2$ is locally constant along the level surfaces $u^{-1}(s)$ (which are smooth since $du \neq 0$).

(ii) More generally, show that $d(\frac{1}{2}|\nabla u|^2) = 2h du$. In particular, conclude that

$$d\left(\frac{du}{|du|}\right) = 0.$$

(iii) Let $N = u^{-1}(s_*)$, show that there is a function $r : M \to \mathbb{R}$, vanishing on N, and with $dr = \frac{du}{|du|}$ (hint: For any point p consider the flow along the vector field $\frac{\nabla u}{|\nabla u|}$). Show that $u = u(r)$.

(iv) Calculate that

$$\nabla u = u'(r)\nabla r \qquad \text{Hess}(u) = u'' dr^2 + u' \text{Hess}(r)$$

and conclude that $h = u''$, and

$$\text{Hess}(r)\Big|_{(\nabla r)^\perp} = \frac{u''}{u'} g\Big|_{(\nabla r)^\perp}.$$

(v) Let $g = dr^2 + g_r$ where g_r is the induced metric on the level sets of r. Then

$$\mathcal{L}_{\nabla r} g_r = \mathcal{L}_{\nabla r} g = 2\text{Hess}(r) = 2\frac{u''}{u'} g_r.$$

It follows that $(cu')^{-2} g_r$ is independent of r. So $g_r = (cu'(r))^2 g_N$ where g_N is the induced metric on $N = \{r = 0\}$, and c is chosen so that $cu'(0) = 1$. Thus the metric is

$$dr^2 + (cu'(r))^2 g_N$$

and M is a warped product.

(4) Consider the warped product $M = N \times_\rho (0, a)$. For any $0 < c < d < a$ consider the annulus $A_{c,d} := r^{-1}((c, d)) \subset M$. Show that

$$\text{Vol}(A_{c,d}) = \text{Vol}(N, g_N) \cdot \int_c^d \rho(r)^{n-1} dr.$$

In particular, if M is a metric cone, then if we let $B_s(0) = r^{-1}(0, s) \subset M$, then

$$\frac{\text{Vol}(B_s(0))}{\omega_n s^n} = \text{const}.$$

where $\omega_n = \text{Vol}(S^{n-1})$. Argue that the metric cones obtained by taking $(N, g_N) = (S^{n-1}, \lambda^2 g_{S^{n-1}})$, and $\rho(r) = r$ can be extended over 0 to obtain a smooth manifold if and only if $\lambda = 1$. When $n = 2$, draw a picture of these cones.

(5) Suppose that $M = N \times_\rho (0, a)$. Let H denote the mean curvature of $r^{-1}(s) \subset M$. Show that

$$H = (n-1)\frac{\rho'}{\rho}$$
$$\mathrm{Ric}(\partial_r, \partial_r) = -(n-1)\frac{\rho''}{\rho}$$

(6) State and prove a version of the Bishop-Gromov theorem, together with a corresponding rigidity theorem, of the following type. Let $(\tilde{M}, g)$ be a Riemannian manifold, and $\tilde{r} = d(p, \cdot)$ be the distance from a point $p \in M$. Suppose that the mean curvature $\tilde{H}$ of $r^{-1}(c)$ satisfies

$$\tilde{H} \leqslant (n-1)\frac{\rho'(a)}{\rho(a)}$$

and, for all $c < \tilde{r} < d$ we have $\mathrm{Ric} \geqslant -(-n-1)\frac{\rho''(\tilde{r})}{\rho(\tilde{r})}$. Then

$$\frac{\mathrm{Vol}_{\tilde{M}}(c < \tilde{r} < d)}{\int_c^d \rho^{n-1}(r)dr} \leqslant \frac{\mathrm{Vol}_{\tilde{M}}(\tilde{r}^{-1}(c))}{\rho^{n-1}(c)}$$

with equality if and only if $\tilde{M}$ is isometric to a warped product space. In particular, if the volume of $\tilde{M}$ agrees with the volume function of a cone on some annulus, then on that annulus, $\tilde{M}$ is isometric to a cone.

(7) Prove the following linear algebra lemma. Suppose B is a symmetric bilinear form on $\mathbb{R}^n$. Then

$$\frac{1}{\mathrm{Vol}(S^{n-1})}\int_{v \in S^{n-1}} B(v, v) = \frac{1}{n}\mathrm{Tr}(B)$$

In particular, explain why this suggests that a manifold with $\mathrm{Ric}(g) \geqslant (n-1)\kappa$ should behave “on average” like a manifold with sectional curvature bounded below by κ.