(1) Let \((M_k, g_k)\) be the \(n\)-dimensional manifold of constant sectional curvature \(k\). Show that there is a radius \(R = R(k)\), and a function \(\rho_k(r)\) so that \((M_k, g_k)\) is isometric to \(B_R(k)(0)\), with the metric

\[ dr^2 + \rho_k(r)^2 g_{S^{n-1}} \]

where \(g_{S^{n-1}}\) is the standard round metric on the \(n-1\) sphere, and \(r^2 = \sum_i x_i^2\) is the radial coordinate. Find \(\rho_k(r)\), and the maximal \(R(k)\) so that this holds.

(2) Let \((N, g_N)\) be a smooth Riemannian manifold, and fix a positive function \(\rho(r) : (0, a) \to \mathbb{R}_{>0}\), where \(a \in \mathbb{R}_{>0}\). A warped product manifold is

\[ (M, g_M) = (M \times (0, a), dr^2 + \rho(r)^2 g_N =: N \times_\rho (0, a)). \]

Special cases of the warped product construction are:
- the product (take \(a = \infty, \rho = 1\)),
- the metric cone (take \(\rho = r^2\)),
- the metric suspension (take \(a = \pi, \rho = \sin(r)\))

Draw some pictures of these (you do not have to submit the pictures)!

Define a function \(f(r) = \int_0^r \rho(s)ds\). Show that

\[ df \neq 0, \quad \text{Hess}(f) = \frac{\partial \rho}{\partial r} g. \]

What is the corresponding result for the spaces of constant curvature \(k\)?

(3) Here is a converse to the previous problem. Suppose \((M, g)\) admits a non-trivial smooth function \(u\) with \(du \neq 0\), and \(\text{Hess}u = hg\) for some function \(h\). Then \((M, g)\) is isometric to a warped product with \(\rho = cu'\) for some constant \(c\). Here is an outline for how to show this.

(i) Suppose \(u\) is as above. Show that for any vector field \(X\) we have

\[ X \left(\frac{1}{2} |\nabla u|^2\right) = hg(X, \nabla u). \]

Conclude that \(|\nabla u|^2\) is locally constant along the level surfaces \(u^{-1}(s)\) (which are smooth since \(du \neq 0\)).
(ii) More generally, show that $d\left(\frac{1}{2}|\nabla u|^2\right) = 2hdu$. In particular, conclude that
\[
d \left( \frac{du}{|du|} \right) = 0.
\]

(iii) Let $N = u^{-1}(s*)$, show that there is a function $r : M \to \mathbb{R}$, vanishing on a connected component of $N$, and with $dr = \frac{du}{|du|}$ (hint: Integrate along curves, and explain why this is well-defined). Show that $u = u(r)$.

(iv) Calculate that $\nabla u = u'(r)\nabla r$, $\text{Hess}(u) = u''dr^2 + u'\text{Hess}(r)$ and conclude that $h = u''$, and
\[
\text{Hess}(r) \bigg|_{(\nabla r)^\perp} = \frac{u''}{u}g \bigg|_{(\nabla r)^\perp}.
\]

(v) Let $g = dr^2 + g_r$ where $g_r$ is the induced metric on the level sets of $r$. Then
\[
\mathcal{L}_{\nabla r}g_r = \mathcal{L}_{\nabla r}g = 2\text{Hess}(r) = 2\frac{u''}{u}g_r.
\]
It follows that $(cu')^{-2}g_r$ is independent of $r$. So $g_r = (cu'(r))^2g_N$ where $g_N$ is the induced metric on $N = \{r = 0\}$, and $c$ is chosen so that $cu'(0) = 1$. Thus the metric is $dr^2 + (cu'(r))^2g_N$ and $M$ is a warped product.

(4) Consider the warped product $M = N \times_{\rho}(0,a)$. For any $0 < c < d < a$ consider the annulus $A_{c,d} := r^{-1}((c,d)) \subset M$. Show that
\[
\text{Vol}(A_{c,d}) = \text{Vol}(N,g_N) \cdot \int_c^d \rho(r)^{n-1}dr.
\]
In particular, if $M$ is a metric cone, then if we let $B_s(0) = r^{-1}(0,s) \subset M$, then
\[
\frac{\text{Vol}(B_s(0))}{\omega_n s^n} = \text{const.}
\]
where $\omega_n = \text{Vol}(S^{n-1})$. Argue that the metric cones obtained by taking $(N,g_N) = (S^{n-1}, \lambda^2 g_{S^{n-1}})$, and $\rho(r) = r$ can be extended over $0$ to obtain a smooth manifold if and only if $\lambda = 1$. When $n = 2$, draw a picture of these cones.
(5) Suppose that \( M = N \times \rho(0, a) \). Let \( H \) denote the mean curvature of \( r^{-1}(s) \subset M \). Show that
\[
H = (n - 1) \frac{\rho'}{\rho}
\]
\[
\text{Ric}(\partial_r, \partial_r) = -(n - 1) \frac{\rho''}{\rho}
\]

(6) State and prove a version of the Bishop-Gromov theorem, together with a corresponding rigidity theorem, of the following type. Let \((\tilde{M}, g)\) be a Riemannian manifold, and \( \tilde{r} = d(p, \cdot) \) be the distance from a point \( p \in M \). Suppose that the mean curvature \( \tilde{H} \) of \( r^{-1}(c) \) satisfies
\[
\tilde{H} \leq (n - 1) \frac{\rho'(a)}{\rho(a)}
\]
and, for all \( c < \tilde{r} < d \) we have \( \text{Ric} \geq -(-n - 1) \frac{\rho''(\tilde{r})}{\rho(\tilde{r})} \). Then
\[
\frac{\text{Vol}_\tilde{M}(c < \tilde{r} < d)}{\int_c^d \rho^{n-1}(r) dr} \leq \frac{\text{Vol}_\tilde{M}(\tilde{r}^{-1}(c))}{\rho^{n-1}(c)}
\]
with equality if and only if \( \tilde{M} \) is isometric to a warped product space. In particular, if the volume of \( \tilde{M} \) agrees with the volume function of a cone on some annulus, then on that annulus, \( \tilde{M} \) is isometric to a cone.

(7) Prove the following linear algebra lemma. Suppose \( B \) is a symmetric bilinear form on \( \mathbb{R}^n \). Then
\[
\frac{1}{\text{Vol}(S^{n-1})} \int_{v \in S^{n-1}} B(v, v) = \frac{1}{n} \text{Tr}(B)
\]
In particular, explain why this suggests that a manifold with \( \text{Ric}(g) \geq (n - 1)\kappa \) should behave “on average” like a manifold with sectional curvature bounded below by \( \kappa \).