18.965: HOMEWORK 5

DUE: TUESDAY, NOVEMBER 19

(1) Let $(M, g|_M) \hookrightarrow (N, g)$ be an isometrically embedded submanifold. Let R^M denote the Riemann curvature tensor of M, R^N denote the Riemann curvature tensor of N, and let A denote the second fundamental form of M in N. Prove the Gauss-Codazzi equation: for $X, Y, Z, W \in TM$ we have

$$R^N(X,Y,Z,W) = R^M(X,Y,Z,W) + \langle A(X,Z), A(Y,W)\rangle - \langle A(Y,Z), A(X,W)\rangle$$

Note, in particular, that if (N, g) has constant sectional curvature, then the curvature of M is determined by the second fundamental form.

(2) Suppose now that $\dim M = n$, and $\dim N = n+1$ so that M has codimension 1. Let ν be a unit normal vector to M defined on an open subset $U \subset M$. If M is orientable, we may assume that ν is globally defined, but in general the existence of a global unit normal is *equivalent* to an orientation. Recall that for $p \in U$ the shape operator is the map $S_\nu : T_pM \to T_pM$ defined by

$$S_\nu(X) = -\pi^{TM}(\nabla^N_X \nu)$$

where π^{TM} denotes the orthogonal projection to $T_pM \subset T_pN$, and ∇^N is the Levi-Civita connection of (N, g). The shape operator satisfies

$$\langle A(X,Y), \nu\rangle = \langle S_\nu(X), Y\rangle$$

Since $A(X,Y) = A(Y,X)$ we see that $S_\nu : T_pM \to T_pM$ is a self-adjoint endomorphism. Therefore, there is an orthonormal basis $X_1, \ldots, X_n$ of T_pM consists of eigenvectors for S_ν with eigenvalues $S_\nu X_i = \lambda_i X_i$, for $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Suppose there is a point $p \in M$ where either $0 < \lambda_i$ for all i, for $0 > \lambda_i$ for all i. Show that at p the sectional curvatures of M satisfy

$$K^M(\sigma) < K^N(\sigma)$$

for all 2-planes $\sigma \subset T_pM$.

(3) Consider a (local) graph in $\mathbb{R}^{n+1}$. That is, if $(x_1, \ldots, x_n, y) = (x, y)$ are coordinates on $\mathbb{R}^{n+1}$, $U \subset \mathbb{R}^n$ is open and $f : U \to \mathbb{R}$ is a smooth function, consider the locally defined surface

$$y = f(x_1, \ldots, x_n).$$

Without loss of generality, we may perform a rotation and a translation so that the the tangent space of the surface $\{y = f(x)\}$ at $x = 0$ agrees with the plane $\{y = 0\}$. That is, without loss of generality we may assume that $f(0) = 0$ and $\nabla f(0) = 0$. Compute the second fundamental form, and the mean curvature of the graph. Note in particular how the formulas simplify at $x = 0$. Since any hypersurface is locally a graph, this should give some insight into what the second fundamental form and mean curvature measure.

(4) Using exercises $(1), (3)$, prove that the sphere $S^n = \{x_1^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$ has constant sectional curvature.

(5) Show that the hyperbolic space has constant sectional curvature.

(6) Suppose $M \subset \mathbb{R}^{n+1}$ is a compact manifold of codimension 1. Show that there is a point $p \in M$ where either $0 < \lambda_i$ for all i, for $0 > \lambda_i$ for all i (which case occurs depends on your choice of normal vector). Conclude, in particular that : (i) there are no compact minimal hypersurfaces in $\mathbb{R}^n$, and (ii) no compact flat manifold (eg. the flat torus) can be isometrically embedded in $\mathbb{R}^{n+1}$ as a hypersurface.

(7) Recall that we have defined the lie derivative $\mathcal{L}$ on vector fields in the following way. If X, V are vector fields on M, and φ_t denotes the time t-flow of X then

$$\mathcal{L}_X V(p) = \lim_{t \to 0} \frac{d\varphi_{-t} V - V}{t}$$

where $d\varphi_{-t} : T_{\varphi_t(p)} M \to T_p M$. This can be extended to tensor fields in an obvious way. Namely, if T is a section of $TM^{\otimes r} \otimes T^*M^{\otimes s}$ then

$$\mathcal{L}_X T = \lim_{t \to 0} \frac{(\varphi_{-t})_* T - T}{t}$$

where $(\varphi_{-t})_* = d\varphi_{-t} : T_{\varphi_t(p)} M \to T_p M$ is the push-forward on tangent vectors, and $(\varphi_{-t})_* = \varphi_t^* : T^*_{\varphi_t(p)} M \to T^*_p M$ is the pull-back. Prove the following basic properties.

(a) If f is a *function* (which we think of as a 0-form), then with this definition we have

$$\mathcal{L}_X f = Xf = df(X)$$

(b) If T, S are tensors then the Lie derivative satisfies the product rule.

$$\mathcal{L}_X T \otimes S = \mathcal{L}_X T \otimes S + T \otimes \mathcal{L}_X S.$$

(c) Since we know how to compute the Lie derivative of functions and vector fields, it suffices to compute the Lie derivative of a 1-form by (b). Show that if α is a 1-form and V is any vector field then we have

$$(\varphi_{-t})_*(\alpha(V)) = ((\varphi_{-t})_*\alpha)\left[((\varphi_{-t})_*V)\right].$$

Using the product rule, deduce that

$$X(\alpha(V)) = (\mathcal{L}_X\alpha)(V) + \alpha([X,V])$$

(d) Define the *interior product* on k-forms in the following way. Give a k-form β, and a vector field X we define a $k-1$-form $\iota_X\beta$ called the interior product of X and β by the formula

$$\iota_X\beta(Y_1,\ldots,Y_{k-1}) = \beta(X,Y_1,\ldots,Y_{k-1}).$$

If $k=0$ we just define $\iota_X\beta = 0$. Prove that the interior product satisfies: if β is a k-form and η another form, then

$$\iota_X(\beta\wedge\eta) = \iota_X\beta\wedge\eta + (-1)^k\beta\wedge\iota_X\eta.$$

(e) Using the formula for the de Rham differential that you proved in Homework 3, together with the (b), (c), (d) show that the Lie derivative of a k-form β is given by *Cartan's magic formula*

$$\mathcal{L}_X\beta = d\iota_X\beta + \iota_X d\beta$$

(f) Prove that the Riemannian volume form $\nu := \sqrt{\det g}dx_i\wedge\cdots\wedge dx_n$ satisfies

$$\mathcal{L}_X\nu = \mathrm{div}(X)\nu$$

(g) We can now give an easy proof of Gauss' theorem for manifolds. Suppose (M,g) is an oriented, compact manifold (without boundary), and suppose that X is a vector field on M. Let φ_t denote the flow of X, which is a diffeomorphism of M to itself, and let $\nu := \sqrt{\det g}dx_i\wedge\cdots\wedge dx_n$ be the Riemannian volume form. By the change of variables formula we have

$$\int_M \nu = \int_{\varphi_t(M)}\nu = \int_M \varphi_t^*\nu$$

differentiating with respect to t gives

$$\int_M \mathrm{div}(X)\nu = 0$$

which is Gauss theorem for manifolds without boundary.

(8) We are now going to give a few applications of this result. We say that a function $u : M \to \mathbb{R}$ is *harmonic* if $\Delta u = 0$. Using the previous exercise, show that if M is compact, orientable without boundary, then there are no non-constant harmonic functions. (**Hint:** $u\Delta u = u\mathrm{div}\nabla u = \frac{1}{2}\mathrm{div}\nabla(u^2) - |\nabla u|^2$). The same argument shows that if M is compact, orientable without boundary then there are no non-constant functions satisfying $\Delta u \geqslant 0$; such functions are called *sub-harmonic*.

(9) Let $(M, g|_M) \hookrightarrow (N, g)$ be an isometrically embedded submanifold. Let $u : N \to \mathbb{R}$ be a smooth function. Let $p \in M$, $U \subset M$ an open set, and let $\nu_1, \dots, \nu_k$ be a set of smooth orthonormal vector fields spanning $TM^\perp$ at each point of U. Show that, for any vectors $X, Y \in T_pM$ we have

$$\mathrm{Hess}^M(u)(X,Y) = \mathrm{Hess}^N(u)(X,Y) + \sum_{i=1}^{k} \nu_i(u)\langle A(X,Y), \nu_i\rangle$$

where $\mathrm{Hess}^M, \mathrm{Hess}^N$ denote the Riemannian hessians in M, N respectively. Conclude that if M is minimal, then

$$\Delta_M u = \sum_j \mathrm{Hess}^N(u)(e_j, e_j)$$

where e_j form an orthonormal basis for T_pM. Conclude that if (N, g) admits a function $u : N \to \mathbb{R}$ with $\mathrm{Hess}^N u = 0$ (ie. a generalized linear function), then any compact minimal surface $M \subset N$ without boundary must be contained in a level set of u. Using this show that there are no compact, orientable minimal surfaces in $\mathbb{R}^n$ (**Hint**: There are many linear functions!). In fact, by using the strong maximum principle instead of the integration-by-parts formula we can even drop the assumption of orientability.

(10) **This problem is totally optional! It's just for your interest and enjoyment. You don't have to hand it in, or even read it, if you don't want to.**

We discussed in class the construction of the hyperbolic space from the Minkowski space. Let's briefly recall this. Consider the space $\mathbb{R}^{n,1}$, which is $\mathbb{R}^{n+1}$ equipped with the Minkowski metric

$$g_M = (dx_1)^2 + (dx_2)^2 + \cdots + (dx_n)^2 - (dt)^2$$

where $(x_1, \dots, x_n, t)$ are coordinates on $\mathbb{R}^{n+1}$. The hyperbolic space is then a connected component of a level set of this quadratic form:

$$\mathcal{H} := \{x_1^2 + \cdots + x_n^2 = t^2 - 1, \quad t > 1\}$$

We showed that the induced metric on $\mathcal{H}$ is in fact Riemannian. In order to study gravity and Einsteins equations, we typically want an n-dimensional space with a Lorentzian metric (ie. a metric with signature $(n-1,1) = (+,\cdots,+,-)$). There is a natural generalization of the construction of the hyperbolic space which yields this. Consider $R^{n-1,2}$ equipped with the quadratic form

$$g_{n-1,2} = (dx_1)^2 + (dx_2)^2 + \cdots + (dx_{n-1})^2 - (dt_1)^2 - (dt_2)^2$$

where $(x_1,\ldots,x_{n-1},t_1,t_2)$ are coordinates on $\mathbb{R}^{n+1}$. We then consider the level set of this quadratic form, which yields the *Anti de Sitter space* of dimension n

$$AdS_n := \{x_1^2 + \cdots + x_{n-1}^2 = t_0^2 + t_1^2 - 1\}$$

One can check directly that the induced metric has signature $(n-1,1)$. Furthermore, this space carries an isometric action by the group $O(n-1,2)$, and hence will have constant curvature. We take the connected component $SO(n-1,2)$ consisting of elements with determinant 1, which also acts transitively. What's interesting is that the group $SO(n-1,2)$ is isomorphic to the group of *conformal transformations* of the Minkowski space $\mathbb{R}^{n-2,1}$, which has dimension $n-1$. This is observation is connected with the AdS/CFT correspondence, which says that gravity in AdS_5 is intimately connected with conformal field theory in $\mathbb{R}^{3,1}$.