18.965: Homework 1

Due: Tuesday, September 24

1. (Stereographic projection) Let

$$S^n := \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} | x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$$

be equipped with the subset topology. That is, a set $V \subset S^n$ is open if $V = S^n \cap U$ for an open set $U \subset \mathbb{R}^{n+1}$. Let $N = (0, \dots, 0, 1)$ be the North pole, and $S = (0, \dots, 0, -1)$ be the south pole. Define $\pi_1 : S^n - \{N\} \to \mathbb{R}^n$ (resp. $\pi_2 : S^n - \{S\} \to \mathbb{R}^n$) so that $(\pi_1(p), 0)$ (resp. $(\pi_2(p), 0)$) is the point where the Line passing through N (resp. S) and p intersects the hyperplane $\{x_{n+1} = 0\}$.

   (a) Prove that $\Phi := \{(S^n - \{N\}, \pi_1), (S^n - \{S\}, \pi_2)\}$ is a C^∞ atlas on S^n.

   (b) Prove that (S^n, Φ) is a smooth submanifold on $\mathbb{R}^{n+1}$. That is, the smooth structure defined by the Φ coincides with the smooth structure induced on S^n as a submanifold of $\mathbb{R}^{n+1}$.

2. Suppose X is a connected topological space. Assume that X is Hausdorff, and locally euclidean of dimension n; that is, X can be covered by charts (U_α, ϕ_α) such that

$$\phi_\alpha : U_\alpha \to \mathbb{R}^n$$

is a homeomorphism. The following three properties are equivalent

   (a) X is second countable. That is, there is a countable collection of open sets $\{U_i\}_{i \in \mathbb{N}}$ such that, for an open set W we can write

$$W = \bigcup U_{i_k}$$

   for some i_k. For example, $\mathbb{R}^n$ is second countable, where the U_i can be taken to be open balls centered on rational points, and with rational radii.

   (b) X is paracompact.

   (c) There exist compact sets $\{K_i\}_{i \in \mathbb{N}}$ such that $K_i \subset \operatorname{int}(K_{i+1})$ and $X = \cup_i K_i$. That is, X has a compact exhaustion.

Prove that (b) and (c) are equivalent. **Here is a "hint"**. To prove $(b) \Rightarrow (c)$, cover X by open sets which are preimages, under ϕ_α of open balls (with compact closure). By paracompactness, you can take a locally finite refinement $\{V_\alpha\}_{\alpha \in A}$ all of which have compact closure. Use these sets to construct K_i iteratively. To prove $(c) \Rightarrow (b)$, let $\{V_\alpha\}$ be any open cover. Since X is Hausdorff, compact sets are closed, and so $E_{i,j} := K_i - \operatorname{int}(K_j)$ is compact for $j < i$. Take a finite subcover of the $\{V_\alpha\}$ covering $E_{i+1,i}$, and set

$$W_{\alpha,i} = V_\alpha \cap \operatorname{int}(E_{i+2,i-1}).$$

Show that the resulting collection $\{W_{\alpha,i}\}$ is a locally finite refinement. For fun, prove the equivalence of $(a)/(b)$ and (c).

3. If M, N are connected, smooth manifolds, then the product $M \times N$ can be made into a smooth manifold using **the product manifold** structure. Given patches (U, ϕ) on M and (V, ψ) on N we use $(U \times V, \phi \times \psi)$ as a patch on $M \times N$. Show that this makes $M \times N$ into a smooth manifold. To show $M \times N$ is paracompact, use the preceding problem.

4. Prove the following lemma stated in class: Suppose $f : M_1^{m+k} \to M_2^m$ is a smooth map. Suppose $q \in M_2$ is a regular value of f. Then $f^{-1}(q)$ is a smooth submanifold of M_1 dimension k.

5. Let (x, y, z) be coordinates on $\mathbb{R}^3$. Let Y_r be the set of points in $\mathbb{R}^3$ at distance $r > 0$ from the circle
$$C = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = 1, z = 0\}$$
   (a) Let $A = \{r \in (0, \infty) | Y_r \text{ is a submanifold of } \mathbb{R}^3\}$. Find A.
   (b) Let S^1 be equipped with the smooth structure given by stereographic projection (see (1)), and let $S^1 \times S^1$ be equipped with the product manifold structure (see below). Prove that Y_r is diffeomorphic to $S^1 \times S^1$ for any $r \in A$.

6. Prove Hadamard's Lemma. If $F : \mathbb{R}^n \to \mathbb{R}$ is smooth, then there are smooth functions $H_1(x), \ldots, H_n(x)$ so that
$$F = F(0) + \sum_{i=1}^{n} x_i H_i(x)$$
with $H_i(0) = \frac{\partial F}{\partial x_i}(0)$. (**Hint**: Use the fundamental theorem of calculus).

7. Let M be a smooth manifold, and $p \in M$ a point. Let $(x_1, \ldots, x_n)$ be local coordinates near p. Show that every derivation D at p is given by
$$D = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\bigg|_p$$
for $a_i \in \mathbb{R}$. (**Hint**: Use Hadamard's lemma).

8. Let $p(x_1, \ldots, x_k)$ be a homogeneous polynomial of degree $m \geq 2$. That is,
$$p(tx_1, \ldots, tx_k) = t^m p(x_1, \ldots, x_k).$$
   (a) Prove that if $a \neq 0$, and $p^{-1}(a)$ is not empty, then $X_a := \{p(x) = a\}$ is a smooth, $k-1$ dimensional submanifold of $\mathbb{R}^k$.
   (b) Prove that X_a is diffeomorphic to X_1 if $a > 0$, and X_a is diffeomorphic to X_{-1} if $a < 0$, provided a is in the range of p.

9. Let $M_n(\mathbb{R})$ be the space of $n \times n$ matrices with real entires. Assume $n \geq 2$, and define $f : M_n(\mathbb{R}) \to \mathbb{R}$ to be $f(A) = \det(A)$.

(a) Recall that the adjoint of A has entry in the i-th row and j-th column

$$(\text{adj}A)_{ij} = (-1)^{i+j} \det A(j|i)$$

where $A(j|i) \in M_{n-1}(\mathbb{R})$ is the matrix obtained by removing the j-th column and the i-th row. Show that the differential of f at A is given by

$$df_A : M_n(\mathbb{R}) \to \mathbb{R}, \qquad df_A(B) = \text{Tr}((\text{adj}A)B)$$

(b) Use the fact that $A(\text{adjA}) = (\det A)I$ to prove the follwing formula for the differential of f

$$df_A = (\det A)\text{Tr}(A^{-1}B)$$

whenever $\det A \neq 0$.

(c) Conclude that $SL(n, \mathbb{R}) := \{A \in M_n(\mathbb{R}) : \det A = 1\}$ is a smooth submanifold of $M_n(\mathbb{R})$.

10. Let X, Y, Z be the vector fields on $\mathbb{R}^3$ defined by

$$X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad Y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad Z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

(a) Compute the flow of the vector field X.

(b) The map $\mathbb{R}^3 \ni (a, b, c) \mapsto aX + bY + cZ$ injects onto its image which is a subspace of the space of smooth vectorfields on $\mathbb{R}^3$. Show that, under this map, the bracket of vector fields induces the cross product on $\mathbb{R}^3$.