PROBLEM SET 6

Problem 1. Let L_1, L_2 be complex line bundles on X. Show that $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$.

Problem 2. For a complex line bundle L, denote by $\overline{L}$ the conjugate bundle, i.e. the same total space but with new $\mathbb{C}$-action $a * x = (\bar{a})x$. Show that $L \otimes \overline{L} \simeq \mathbb{C}$ and deduce that $c_1(\overline{L}) = -c_1(L)$.

Problem 3. Problem 1, the Whitney sum formula and the splitting principle imply that if V, W are complex vector bundles, then $c_i(V \otimes W)$ can be expressed in terms of the Chern classes of V and W (in a universal way).

(1) Do this explicitly for V, W of rank ≤ 2.
(2) Compute the Chern classes of the tangent bundle of $Gr_2(\mathbb{C}^4)$ in terms of the Chern classes of the tautological bundle.

Problem 4. If X is a manifold admitting a nowhere-zero vector field, show that the mod 2 Euler characteristic of X must be zero.

Deduce that any vector field on $\mathbb{CP}^n, \mathbb{RP}^{2n}, S^{2n}$ has a zero.

Problem 5. The Stiefel-Whitney classes can be viewed as characteristic classes for complex vector bundles. Show that w_i (for complex vector bundles) can be written as a polynomial in the Chern classes modulo 2. What is this polynomial?

Problem 6. For an oriented $(2k+1)$-dimensional vector bundle E, show that $e(E) = \beta w_{2k}(E)$, where $\beta : H^{2k}(X, \mathbb{Z}/2) \to H^{2k+1}(X, \mathbb{Z})$ is the boundary map in the long exact sequence of cohomology groups associated with the short exact sequence $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2$ of coefficient groups.