PROBLEM SET 3

Problem 1. Let G be a compact Lie group acting smoothly and freely on a smooth manifold X. Show that $X \to X/G$ is a principal G-bundle.

Problem 2. Let G be a Lie group, H a closed subgroup, N a closed normal subgroup.

(a) $G \to G/H$ is a principal bundle with fiber H by Problem 1, hence yields a map $G/H \to BH$. Show that $G \to G/H \to BH$ is a fibration sequence.
(b) Show that there is a fibration sequence $G/H \to BH \to BG$, where $BH \to BG$ is induced by the inclusion $H \to G$.
(c) Show that there is a fibration sequence $BN \to BG \to B(G/N)$, where $BG \to B(G/N)$ is induced by $G \to G/N$.

Problem 3. Construct fiber bundles

$$O(n-1) \to O(n) \to S^{n-1}$$
$$U(n-1) \to U(n) \to S^{2n-1}$$

and deduce that the sequences

$$\pi_k(U(1)) \to \pi_k(U(2)) \to \pi_k(U(3)) \to \dots$$
$$\pi_k(O(1)) \to \pi_k(O(2)) \to \pi_k(O(3)) \to \dots$$

eventually stabilize.

Problem 4. Show that the inclusions $O(n) \hookrightarrow GL(n, \mathbb{R})$ and $U(n) \hookrightarrow GL(n, \mathbb{C})$ are homotopy equivalences.

Problem 5. If V is an inner product space, then the tautological bundle $\gamma_n(V) \subset Gr_n(V) \times V$ has a fiberwise orthogonal complement $\gamma_n^\perp(V)$. Show that $\gamma_n^\perp(V)$ is a vector bundle and express the tangent bundle of $Gr_n(V)$ in terms of γ and $\gamma^\perp$.

Problem 6. For simplicial sets X, Y, show that $|X \times Y|$ is homeomorphic to $|X| \times |Y|$ (in k-spaces!) as follows. First consider the case where $X = \Delta^n, Y = \Delta^m$. Then show that X and Y can be written as colimits of simplices and conclude.