PROBLEM SET 2

Problem 1. For a pointed space X, denote by ΣX its suspension

$$X \times I / * \times I \cup X \times \{0,1\}.$$

(1) Show that $\Sigma : \mathcal{S}\mathrm{pc}_* \leftrightarrows \mathcal{S}\mathrm{pc}_* : \Omega$ is an adjunction.
(2) Show that this descends to an adjunction $\Sigma : Ho(\mathcal{S}\mathrm{pc}_*) \leftrightarrows Ho(\mathcal{S}\mathrm{pc}_*) : \Omega$.

Problem 2. Show that the exponential map $\mathbb{R} \to S^1$ is a fibration. Use this to compute $\pi_*(S^1)$.

Problem 3 (the Hopf fibration). Consider the map $\eta : S^3 \subset \mathbb{C}^2 \setminus 0 \to \mathbb{CP}^1 = S^2$.

(1) Show that η is a fibration with fiber S^1.
(2) Deduce that $\pi_2(S^2) = \mathbb{Z}$.
(3) Assuming that $\pi_n(S^n) = \mathbb{Z}$ (generated by the identity), deduce that $\pi_3(S^2) = \mathbb{Z}$ generated by η.
(4) Construct elements in $\pi_7(S^4)$ and $\pi_{15}(S^8)$ of infinite order.

Problem 4. Let $f : X \to Y \in \mathcal{S}\mathrm{pc}_*$. Devise an action of ΩY on $F(f)$. Deduce that there is a canonical map $\Omega Y \times F(f) \to F(f) \times_X F(f)$. (Why is this fiber product homotopically meaningful?) Show that this map is a homotopy equivalence.

Problem 5. Consider the category $\mathcal{C}$ of pairs (π, G) where π is a group and G is a group acted on by π.

(1) Construct binary products in $\mathcal{C}$.
(2) Suppose that (π, G) is given a unital multiplication. Show that both groups are abelian and the action is trivial.
(3) Deduce that path-connected H-spaces (objects with unital multiplication in $Ho(\mathcal{S}\mathrm{pc})$) are simple.

Problem 6. Show that $[K(A,n), K(B,n)]_* \simeq \mathrm{Hom}(A,B)$. Deduce that the category $K_n \subset Ho(CW_*)$ of connected spaces with only homotopy group in degree n is equivalent to a familiar category. Use this to give another proof of the uniqueness of Eilenberg-MacLane spaces.