PROBLEM SET 1

Problem 1. Show that any (small) limit can be written as an equalizer between two products.

Here an *equalizer* means a limit of a diagram of the form $A \rightrightarrows B$ and a *product* means a limit of a discrete diagram, i.e. the indexing category has only identity morphisms.

Problem 2. Suppose that $F, F' : \mathcal{C} \to \mathcal{D}$ are both left adjoint to $G : \mathcal{D} \to \mathcal{C}$. Construct a natural isomorphism $F \Rightarrow F'$ and discuss its uniqueness.

Problem 3. Let I be a small category and $\mathcal{C}$ any category such that all I-indexed colimits in $\mathcal{C}$ admit colimits. Denote by $\Delta : \mathcal{C} \to \mathrm{Fun}(I, \mathcal{C})$ the "constant diagram" functor, sending $c \in \mathcal{C}$ to the diagram $I \ni i \mapsto c, \alpha : i \to j \mapsto \mathrm{id}_c$.

Construct a left adjoint of $\mathcal{C}$.

Problem 4. Let $S^\infty = \mathrm{colim}_i\, S^i$, where the transition maps come from the evident inclusions $\mathbb{R}^i \to \mathbb{R}^{i+1}$.

Show that S^∞ is contractible.

Problem 5. Let $i : A \hookrightarrow B$, $j : A' \hookrightarrow B'$ be CW pairs, where A, A' are CW complexes. Show that $i \times j : A \times A' \hookrightarrow B \times B'$ is a CW pair.

Problem 6. Consider the "quasi-circle" X obtained by taking the graph of $\sin(x^{-1})$ on $(0, 1)$ and adding a path connecting $(0, 0)$ to $(1, 0)$ without touching the graph. Compute the homotopy groups of X. Deduce that X is weakly contractible. Is X contractible?