18.786. (Spring 2014) Problem set # 10 (due Thu May 8) - last one!

Let Q and other notation be as in [Gee, 5.5].

1. The following has been used when studying objects on the Galois side.

   (a) Show that $\rho_Q^{\mathrm{univ}}|_{G_{F_v}}$ is isomorphic to a direct sum of two characters χ_α and χ_β such that the two characters modulo $\mathfrak{m}_{R_Q^{\mathrm{univ}}}$ map Frob_v to $\overline{\alpha}_v$ and $\overline{\beta}_v$, respectively. (Feel free to use Exercise 3.34, cf. [Gee, p.37].)

   (b) Check that there is a natural $\mathcal{O}$-algebra isomorphism $(R_Q^{\mathrm{univ}})_{\Delta_Q} \simeq R_\emptyset^{\mathrm{univ}}$.

2. Let Δ be a finite ℓ-group and $\mathcal{O}$ the integer ring in an ℓ-adic field L. Let S be a finite free $\mathcal{O}[\Delta]$-module. Then construct an $\mathcal{O}$-module[1] isomorphism $S_\Delta \simeq S^\Delta$, where S_Δ (resp. S^Δ) is Δ-coinvariant quotient (resp. Δ-invariant subspace).[2]

3. Let $R := R_\emptyset^{\mathrm{univ}}$, $S := S_\emptyset$, R_∞, and S_∞ be as in the last diagram of [Gee, p.44]. As usual R acts on S through the surjection $R \twoheadrightarrow \mathbb{T} := \mathbb{T}_{U,\mathfrak{m}}$ onto a suitable Hecke algebra. Suppose that there is an $\mathcal{O}$-algebra surjection $R \twoheadrightarrow R'$, that an $\mathcal{O}$-algebra map $f : R \to \mathcal{O}$ factors through $R \twoheadrightarrow R'$, and that the image of $\mathrm{Spec}\, R' \hookrightarrow \mathrm{Spec}\, R_\infty$ is contained in $\mathrm{Supp}_{R_\infty}(S_\infty)$.

   (a) Show that $\mathrm{Supp}_{R'}(S_\infty \otimes_{R_\infty} R') = \mathrm{Spec}\, R'$.

   (b) Deduce that f factors through the surjection $R \twoheadrightarrow \mathbb{T}$.

   (Here is a long note on how this exercise is used to complete the proof of minimal ALT: Recall that our patching argument a priori shows only that $\mathrm{Supp}_{R_\infty}(S_\infty)$ is a union of irreducible components of $\mathrm{Spec}\, R_\infty$ and falls short of proving that $\mathrm{Supp}_{R_\infty}(S_\infty) = \mathrm{Spec}\, R_\infty$; the latter is not even expected because we start with a coarse local deformation problem at $v \in T_r$. To achieve our goal of showing that $f_\rho : R \to \mathcal{O}$ arising from ρ factors through $R \twoheadrightarrow \mathbb{T}$, the main idea is to consider a refined global deformation problem

   $$\mathscr{S}' := (F, T, \overline{\rho}, \chi, \{\mathcal{D}'_v\}_{v\in T})$$

   where

   - for $v \in T_\ell$, $\mathcal{D}'_v = \mathcal{D}_v$ collects crystalline reps of prescribed weights $\{H_\tau\}$ but
   - for $v \in T_r$, $\mathcal{D}'_v$ corresponds to the unique irreducible component $\mathcal{C}$ in $R^\square_{\overline{\rho}_v,\chi_v}[1/\ell]$ containing $\rho_{0,v}$ and ρ_v (the two are in the same component by the assumption of minimal ALT; recall that $\mathcal{D}_v$ consists of all liftings in the original problem $\mathscr{S}$.).[3][4]

   Set $R' := R^{\mathrm{univ}}_{\mathscr{S}'}$. Then one easily verifies that $\mathrm{Spec}\, R' \hookrightarrow \mathrm{Spec}\, R_\infty$ is contained in $\mathrm{Supp}_{R_\infty}(S_\infty)$ from the fact that $\mathrm{Supp}_{R_\infty}(S_\infty)$ is a union of irreducible components. Now this exercise tells you that f_ρ factors through $R \twoheadrightarrow \mathbb{T}$, so ρ is automorphic as desired. Compare all this with the argument in the last two paragraphs in the proof of [Tho12, Thm 6.8], which deals with a higher dimensional case.)

[1] Brian pointed out the typo: I mistakenly wrote "an $\mathcal{O}$-algebra isomorphism" in an earlier version.

[2] Applying this to $S = S_Q$ and $\Delta = \Delta_Q$, one obtains $(S_Q)_{\Delta_Q} \simeq S_Q^{\Delta_Q} = S(U_{Q,0}, \mathcal{O})_{\mathfrak{m}_Q}$ (in fact there is a natural isomorphism induced by the "trace map" in our setup), which together with Prop 5.8 yields an isomorphism $(S_Q)_{\Delta_Q} \simeq S_\emptyset$, compatibly with the actions of the Galois deformation rings. It also follows that $(S_Q^\square)/\mathfrak{a}_Q \simeq S_\emptyset$, cf. [Gee, p.41].

[3] More precisely $\mathcal{D}'_v$ corresponds to $\mathcal{I}(\mathcal{D}'_v) = \ker(R^\square_{\overline{\rho}_v,\chi_v} \to R^\square_{\overline{\rho}_v,\chi_v,\mathcal{C}})$ where the last ring is the maximal quotient which is ℓ-torsion free, reduced, and whose characteristic 0 points are in $\mathcal{C}$.

[4] To ensure that each of $\rho_{0,v}$ and ρ_v is indeed in a unique component (i.e. does not lie in the intersection of multiple components) one appeals to the local-global compatibility of the Langlands correspondence and the fact that the local components of every cuspidal automorphic representation of $GL_2(\mathbb{A}_F)$ are generic (=have Whittaker models) at all places. This is a technical remark, which you may ignore until you get familiar.

References

[Gee] Toby Gee, *Modularity lifting theorems – Notes for Arizona winter school*, draft, http://www2.imperial.ac.uk/~tsg.

[Tho12] Jack Thorne, *On the automorphy of l-adic Galois representations with small residual image*, J. Inst. Math. Jussieu **11** (2012), no. 4, 855–920, With an appendix by Robert Guralnick, Florian Herzig, Richard Taylor and Thorne. MR 2979825